Degrees of Freedom of MIMO Wireless Networks with General Message Sets: Channel Decomposition, Message Splitting and Beamformer Allocation

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Degrees of Freedom of MIMO Wireless Networks with General Message Sets: Channel Decomposition, Message Splitting and Beamformer Allocation

by

Yao Wang

B.S., Tsinghua University, 2009
M.S., Tsinghua University, 2012

A thesis submitted to the
Faculty of the Graduate School of the
University of Colorado in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Department of Electrical, Computer and Energy Engineering

2017
This thesis entitled:
Degrees of Freedom of MIMO Wireless Networks with General Message Sets: Channel
Decomposition, Message Splitting and Beamformer Allocation
written by Yao Wang
has been approved for the Department of Electrical, Computer and Energy Engineering

_____________________________

Prof. Mahesh K. Varanasi

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Prof. Youjian Liu

Date _______________

The final copy of this thesis has been examined by the signatories, and we find that both the
content and the form meet acceptable presentation standards of scholarly work in the above
mentioned discipline.
The physical layer of a wireless network can be used in various settings distinguished by the specific set of messages transmitted in the network. In this thesis, rather than considering each setting in isolation as has been the traditional approach, we study the unified setting of **general message sets** in which any subset (including all) of messages can be transmitted simultaneously in a network. The total number of possible messages is exponential in the size of the network and the number of settings simultaneously studied is double-exponential. For this reason, the problem quickly would become difficult or even impossible to solve for large networks unless some structure is found in smaller network settings.

In this thesis, we begin this journey by settling the approximate capacity regions in the form of exact degrees of freedom (DoF) or the linear DoF regions of selected small, but representative, multiple-input multiple-output (MIMO) wireless broadcast and interference networks. The aim is to not only develop novel approaches to fully obtain the DoF-optimal solutions through tight inner and outer bounds but also to suggest approaches that could be potentially useful in generalizing the results herein to a broader class of problems that may include larger networks and/or channel uncertainty models.

In developing novel achievable schemes, we propose a methodology that combines the idea of **message splitting** and **channel decomposition**, which notably simplifies the construction of the achievable region for the network. Using channel decomposition, the transmitter beamformer space is partitioned into several linearly independent subspaces, each of which has special properties and is easier to analyze. Message splitting involves expanding the number of message types beyond the original ones by splitting each message into several independent types according to their differ-
ent impacts to the receivers or which beamformer subspaces they are transmitted through. This enlarges the dimensions needed to specify the achievable DoF region in split-message space. Interestingly though, it also simplifies the analysis and provides what is effectively a high-dimensional description of the achievable DoF region. When projected to the desired dimensions, the achievable region is specified.
Dedication

To my wife and parents,
for their unwavering support and patience.

To my lovely son,
for the happiness he brought into my life.
Acknowledgements

I would never have been able to finish my dissertation without the guidance of my thesis advisor, help from friends, and support from my family and wife.

Firstly, I would like to express my deepest gratitude to my thesis advisor Dr. Mahesh K. Varanasi for the continuous support of my Ph.D study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I am also thankful for his high standard in research and academic writing, always pushing deep and exploring the hidden essence of the problems, all of which inspired me a lot.

I would like to also thank the rest of my thesis committee members – Dr. David Grant, Dr. Dejan S. Filipovic, Dr. Peter Mathys and Dr. Youjian (Eugene) Liu – for taking time to serve and for their encouragement.

Special thanks are also in order for the friendly and helpful staff in the Department of Electrical, Computer and Energy Engineering, and my fellow lab mates in the Communications and Signal Processing group, who gave me lots of support and brought memorable fun and happiness to my stressful Ph.D study.

Last but not least, I would like to thank my family for always supporting me and encouraging me with their best wishes, especially to my wife Yunzhi Xu, who was always there cheering me up and stood by me through the good times and bad.
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Chapter 1

Introduction

1.1 Motivation

Traditional models of network communication consider only private (multiple unicast) messaging or universal multicast messaging. The former sets up a competition for channel resources between the messages awhile ignoring the broadcast nature of wireless communication while the latter exploits the broadcast nature but without providing much flexibility or privacy of targeting messages to specific groups of users.

In order to design communication systems that can flexibly and efficiently handle the concurrent groupcasting signaling requirements of futuristic applications, such as simultaneous and distinct mobile live TV broadcast to multiple groups of subscribers or in the delivery phase of caching systems over wireless interference channels, it may be necessary to offer multiple physical layer modes that allow for the transmission of some or all of multiple unicast, multiple multicast, multiple broadcast (i.e., $X$-channel), and/or cooperative/cognitive/common messages. However, little is known in the literature on how to satisfy such diverse demands of messages by different user groups from the network. Novel and sophisticated resource identification and data allocation strategies are needed to maximally eliminate the undesired interference in the network and reserve the limited channel resources to desired messages. With this motivation, we study the information-theoretic limits in terms of approximate capacity metrics of selected wireless networks with general message sets.

More specifically, we establish the degrees of freedom (DoF) and linear degrees of freedom
(LDoF) region for the multiple-input multiple-output (MIMO) two-transmitter, two-receiver (2 × 2) interference network (IN) and the three-user broadcast channel (BC), respectively. For each model, outer bounds on the DoF/LDoF region are established by developing novel techniques and then shown to be tight. In our achievable schemes, with an emphasis on vector space beamforming through various novel manipulations of signaling and interference subspaces, the precise roles played by transmit zero-forcing, interference alignment, random beamforming, symbol extensions and asymmetric complex signaling (ACS) are delineated.

We also study the MIMO two-user broadcast channel with general messages under hybrid channel state information at transmitter (CSIT) models. Nine different channel state knowledge assumptions are considered wherein the transmitter has either perfect/instantaneous (P), delayed (D) or no (N) channel state information (CSI) from each of the two receivers. As the key to the converse proofs of the LDoF region of the MIMO BC-CM under such hybrid CSIT assumptions, we show that, when only considering linear encoding strategies, the channel state information from the receiver with more antennas does not help if there is no channel state information available from the receiver with fewer antennas.

A key difficulty in the analysis of a communication model with general message sets is the large number of different messages within the network. A beamformer allocation approach that could maximally eliminate the undesired interference in the network and reserve the limited channel resources to desired messages is developed that leads to optimal use of channel resources available in the form of signaling subspaces and their inter-relationships. In this thesis, we propose a methodology that combines the idea of message splitting and channel decomposition, which notably simplifies the construction of achievable region of the entire system. By channel decomposition, the entire transmitter beamformer space is partitioned into several linearly independent subspaces, each of which has special properties and easier to analyze. Message splitting involves splitting each message to several independent parts according to their different impacts to the receivers or which beamformer subspaces they are transmitted through. The splitting of messages introduces multiple auxiliary variables to describe the individual DoF of the split messages to the system and
enlarges the number of independent messages. However, it also simplifies the analysis and provides another form of higher-dimensional description of the achievable DoF region. By projecting the higher-dimensional region to the desired dimensions through techniques such as Fourier-Motzkin elimination (FME), we construct the achievable region of the system we need.

1.2 Message splitting and channel decomposition: a simple Illustration

We illustrate the idea of rate splitting and channel decomposition with a simple example.

Consider the two-user MIMO \((M, N_1, N_2)\) broadcast channel, in which the transmitter has \(M\) antennas and the two receivers have \(N_1\) and \(N_2\) antennas, respectively. The transmitter has two messages \(W_1\) and \(W_2\), intended for the two receivers, respectively. Denote the channel between the transmitter and receiver \(i\) \((i \in \{1, 2\})\) as matrix \(H_i\).

When perfect channel state information is available at the transmitter, the DoF region of the system is well-known to be given by

\[
D = \{(d_1, d_2) \in \mathbb{R}_+^2 : d_1 \leq N_1, \quad d_2 \leq N_2, \quad d_1 + d_2 \leq M \}. 
\]

The converse proof is trivial since all constraints in \(D\) are cut-set bounds. Now we use both message splitting and channel decomposition method to construct the achievable DoF region and show that \(D\) is achievable.

In the following analysis, we assume that \(M \leq N_1 + N_2\) because if \(M > N_1 + N_2\), there are redundant antennas at the transmitter, and the system has the same DoF region even if we reduce the number of antennas at the transmitter to \(N_1 + N_2\).
1.2.1 Constructing the achievable region via message splitting

Since \( M \leq N_1 + N_2 \), the null-space of the concatenated channel \( \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \) does not exist and zero-forcing a transmit beamformer at both receivers simultaneously is not possible, nor is it desirable.

We split the \( d_1 \) symbols of message \( W_1 \) into two linearly independent parts based on the impact of the associated beamformers at the two receivers. More specifically, the first part is received by receiver \( R_1 \) but zero-forced at receiver \( R_2 \), the second part is received by both receivers.

Denote the number of symbols in each part as \( d^Z_1 \) and \( d^R_1 \), respectively. We have that \( d^Z_1 + d^R_1 = d_1 \).

In order to zero-force a symbol at receiver \( R_2 \), the transmitter needs to transmit the symbol via a beamformer chosen from the nullspace of channel \( H_2 \), the dimension of which is equal to \( (M - N_2)^+ \).

In other words, there can be at most \( (M - N_2)^+ \) such symbols that can be zero-forced at receiver \( R_2 \), and we get the constraint that \( d^Z_1 \leq (M - N_2)^+ \). For \( d^R_1 \), we simply transmit these symbols via random beamforming, which will almost surely be received by both receivers. It is worth noting that this kind of splitting is both intuitive and comprehensive in this setting.

Similarly, we split message \( W_2 \) into two linearly independent parts and have that \( d_2 = d^Z_2 + d^R_2 \), where \( d^Z_2 \) denotes the number of \( W_2 \) symbols that will be zero-forced at receiver \( R_1 \), and \( d^R_2 \) denotes the number of \( W_2 \) symbols that will be received by both receivers. We get the similar constraint that \( d^Z_2 \leq (M - N_1)^+ \).

Now, consider the signals received by receiver \( R_1 \). The total number of independent beamformers received by receiver \( R_1 \) is equal to \( d^R_1 + d^Z_1 + d^R_2 \). Since all these beamformers are generated from \( \mathcal{N}(H_2) \) or randomly, they are all independent with channel matrix \( H_1 \). Consequently, they will be almost surely independent with each other at receiver \( R_1 \) and thus decodable if the number of beamformers is no greater than the available dimensions at receiver \( R_1 \), i.e., \( d^R_1 + d^Z_1 + d^R_2 \leq N_1 \).

Similarly, if \( d^R_1 + d^Z_2 + d^R_2 \leq N_2 \), these beamformers will almost surely be decodable at receiver \( R_2 \).

Thus, we obtain the following achievable region of the split messages

\[
\mathbb{D}^1_{in} = \{(d^R_1, d^Z_1, d^R_2, d^Z_2) \in \mathbb{R}^4_+ : \quad \}
\]
\[ d_1^R + d_2^Z + d_2^R \leq N_1 \]
\[ d_1^R + d_2^Z + d_2^R \leq N_2 \]
\[ d_1^Z \leq (M - N_2)^+ \]
\[ d_2^Z \leq (M - N_1)^+ \]
\[ d_1^R + d_1^Z + d_2^R + d_1^R \leq M \].

where the last constraint comes from the fact that there can be at most \( M \) beamformers.

This gives us another description of the achievable region of the model, which is in the 4-dimensional \((d_1^R, d_1^Z, d_2^R, d_2^Z)\) split message space. In order to obtain the achievable region in the desired 2-dimensional \((d_1, d_2)\) space. First, we replace \(d_1^R\) with \(d_1 - d_1^Z\) and replace \(d_2^R\) with \(d_2 - d_2^Z\), and we obtain the following achievable region

\[
\mathbb{D}_{in}^2 = \{(d_1, d_2, d_1^Z, d_2^Z) \in \mathbb{R}_+^4 : \\
\quad d_1 + d_2 - d_1^Z \leq N_1 \\
\quad d_1 + d_2 - d_2^Z \leq N_2 \\
\quad d_1^Z \leq (M - N_2)^+ \\
\quad d_2^Z \leq (M - N_1)^+ \\
\quad d_1 + d_2 \leq M \\
\quad d_1 - d_1^Z \geq 0 \\
\quad d_2 - d_2^Z \geq 0 \}
\]

where the last two inequalities come from the requirement that \(d_1^R \geq 0\) and \(d_2^R \geq 0\).

Then, we project the 4-dimensional \((d_1, d_2, d_1^Z, d_2^Z)\) polytope to the first two dimensions by doing the Fourier-Motzkin elimination (FME) of the variable \(d_1^Z\) and \(d_2^Z\). It can be easily shown that the projection is exactly the same with \(\mathbb{D}\).

One can observe that the \(d_1^R\) symbols of message \(W_1\) and \(d_2^R\) symbols of message \(W_2\) are both decodable at both receivers. In this sense, we are actually using beamformers that can transmit
common messages to send out one of the private messages. If we move one further step and add an additional common message $W_0$, which is desired by both receivers, to the system, we could obtain the following achievable DoF region using the same approach.

$$\{(d_1, d_2, d_0) \in \mathbb{R}^3_+ :$$

$$d_1 + d_0 \leq N_1,$$

$$d_2 + d_0 \leq N_2,$$

$$d_1 + d_2 + d_0 \leq M\}.$$ 

which is exactly the DoF region of the 2-user MIMO broadcast channel with common message (BC-CM).

1.2.2 Constructing the achievable region via channel decomposition

Now, we construct the achievable region via another approach and introduce the idea of channel decomposition.

Since the transmitter has $M$ antennas, the beamformer space at the transmitter is a $M$-dimensional vector space, namely, $\mathcal{M}$. We decompose $\mathcal{M}$ into two linearly independent subspaces as follows.

First, consider the nullspaces of channel $H_1$ and $H_2$. Since we assume $M \leq N_1 + N_2$, we have that $\mathcal{N}(H_1)$ is linearly independent with $\mathcal{N}(H_2)$ almost surely, otherwise there will exist a beamformer that can be zero-forced at both receivers. We define the first subspace as $\mathcal{V}_1$, and let $\mathcal{V}_1$ be the direct sum of nullspaces $\mathcal{N}(H_1)$ and $\mathcal{N}(H_2)$, i.e., $\mathcal{V}_1 = \mathcal{N}(H_1) \oplus \mathcal{N}(H_2)$. We define the second subspace as $\mathcal{V}_2$, and let $\mathcal{V}_2$ be the orthogonal complement of $\mathcal{V}_1$ in $\mathcal{M}$.

It can be easily observe that the rank of $\mathcal{V}_1$ is equal to $r_1 = (M - N_1)^+ + (M - N_2)^+$. Among these $r_1$ beamformers, there are $(M - N_1)^+$ beamformers that can only be received by receiver $R_2$, thus we can use them to transmit message $W_2$ without bringing interference to receiver $R_1$. The rest $(M - N_2)^+$ beamformers can only be received by receiver $R_1$, and we can use them to transmit message $W_1$ without bringing interference to receiver $R_2$. The rank of subspace $\mathcal{V}_2$ is equal to
\[ r_2 = M - r_1 = M - (M - N_1)^+ + (M - N_2)^+. \]

Each of these \( r_2 \) beamformers can be received by both receivers, and thus can be used to transmit either a \( W_1 \) symbol or a \( W_2 \) symbol while causing interference to the unintended receiver. It can be easily proven that the transformation of \( V_1 \) and \( V_2 \) are also linearly independent with each other at both receivers.

Now, consider the case that we only communicate through subspace \( V_1 \). We can at most send \( (M - N_2)^+ \) symbols of \( W_1 \) and \( (M - N_1)^+ \) symbols of \( W_2 \). In other words, we can achieve the following DoF region

\[
\mathcal{D}^3_{in} = \{(d_1, d_2) \in \mathbb{R}_+^2 : \\
\quad d_1 \leq (M - N_2)^+ \\
\quad d_2 \leq (M - N_1)^+ \}.
\]

Next, consider the case that we only communicate through subspace \( V_2 \). Each of the \( r_2 \) beamformers can be used to transmit either a symbol of message \( W_1 \) or a symbol of message \( W_2 \), but not both. Thus, we easily obtain the following achievable region.

\[
\mathcal{D}^4_{in} = \{(d_1, d_2) \in \mathbb{R}_+^2 : \\
\quad d_1 + d_2 \leq M - (M - N_1)^+ - (M - N_2)^+ \}.
\]

![Figure 1.1: Minkowski-sum of region \( \mathcal{D}^3_{in} \) and \( \mathcal{D}^4_{in} \)](image)

Since both the beamformers in \( V_1 \) and \( V_2 \), and their transformations at the receivers, are linearly independent with each other, the communication through subspace \( V_1 \) and \( V_2 \) are actually independent with each other and can proceed independently. The system works as two parallel
subsystems. Consequently, the Minkowski-sum of region \( D_3^{1} \) and \( D_4^{1} \) is achievable by the entire system, which can be calculated as

\[
\{ (d_1, d_2) \in \mathbb{R}_+^2 : \\
\quad d_1 \leq N_1, \\
\quad d_2 \leq N_2, \\
\quad d_1 + d_2 \leq M \}.
\]

which is exactly equal to outer bound \( \mathcal{D} \). In other words, transmitting in parallel in the two subspaces is DoF-optimal for the two-user MIMO broadcast channel.

The extension of the result to the case that common message \( W_0 \) is transmitted is also straightforward. In this case, the two sub-region will be

\[
\{ (d_1, d_2, d_0) \in \mathbb{R}_+^3 : \\
\quad d_1 + d_0 \leq (M - N_2)^+ \\
\quad d_2 + d_0 \leq (M - N_1)^+ \}
\]

and

\[
\{ (d_1, d_2, d_0) \in \mathbb{R}_+^3 : \\
\quad d_1 + d_2 + d_0 \leq M - (M - N_1)^+ - (M - N_2)^+ \}.
\]

The Minkowski-sum of these two regions is exactly the same as the DoF region of the two-user MIMO BC-CM.

**1.2.3 Generalization of the ideas**

Both message splitting and channel decomposition are very efficient ways in the analysis of the achievable capacity/DoF/LDoF region of a wireless network. The two-user MIMO BC is relatively a simple model, and one may easily obtain the achievable region even without using any of these two approaches. However, for networks with more complex structures, it is beneficial to
use a systematic way to analyze the problem. The advantage of using message splitting and/or channel decomposition will be much more significant.

Actually, message splitting and channel decomposition are just two complementary ways to start analyzing the problem. For example, in the MIMO 2-user BC model, after the channel decomposition, we do parallel transmissions through two linearly independent subspaces. This is equivalent to the idea of splitting the rate or DoF of the message into two parts, each corresponding to the DoF in one of the two parallel transmissions. Although the fundamental idea behind the two approaches are similar, it may be easier to solving a certain problem following the methodology of one of them than the other. Also, as it can be shown in the later chapter, we may need to use both of them together in a same problem. For example, after decomposing the channel into several parallel channels, we may still need to use the idea of message splitting to solve the achievable region of transmitting via some of the sub-channels.

1.3 Organization and Original Contributions

In this thesis, we aim to explore how to take the most advantage of the idea of message/rate/DoF splitting and channel decomposition. We show that these two ideas are indeed powerful by successfully solving three open problems in network information theory. The results obtained on two of those problems will appear in two papers in the IEEE Transaction on Information Theory [1, 2] and the results on the third problem will be submitted for publication to the same journal.

1.3.1 MIMO $2 \times 2$ interference network with general message sets

In Chapter 2, we establish the degrees of freedom (DoF) region for the multiple-input multiple-output (MIMO) two-transmitter, two-receiver ($2 \times 2$) interference network with a general message set, consisting of nine messages, one for each pair of a subset of transmitters at which that message is known and a subset of receivers where that message is desired. An outer bound on the general nine-message $2 \times 2$ interference network is obtained and it is shown to be tight,
establishing the DoF region for the most general antenna setting wherein all four nodes have an arbitrary number of antennas each. The DoF-optimal scheme is applicable to the MIMO $2 \times 2$ interference network with constant channel coefficients, and hence, a fortiori, to time/frequency varying channel scenarios. The DoF region is thus settled for the various instances out of the $2^9 - 1$ cases of the $2 \times 2$ interference network for which it was not known previously in which some subset of all possible messages is to be transmitted. An important special case is the four private message MIMO X channel with four unicast messages, the DoF region of which had remained an open problem despite several previous studies.

Such special cases include the well-known settings of the two-unicast interference channel, the cognitive interference channel in which one of the transmitters knows both messages, the simultaneous broadcast of two unicast messages from each transmitter, also known as the X-channel, as well as new settings including, for instance, the six-message cognitive X-channel with four unicast X channel messages along with two more messages, each known to both transmitters and desired by each of the two receivers. Many more new settings are possible as well, including, possibly simultaneously, multiple multicasting where each transmitter has a message for both receivers and a common message known to both transmitters and desired at both receivers.

In particular, a linear precoding scheme is proposed that can achieve all the DoF tuples in the DoF region. In it, the precise roles played by transmit zero-forcing, interference alignment, random beamforming, symbol extensions and asymmetric complex signaling (ACS) are delineated. For instance, we identify a class of antenna settings in which ACS is required to achieve the fractional-valued corner points. Evidently, the DoF regions of all previously unknown cases of the $2 \times 2$ interference network with a subset of the nine-messages are newly established as special cases of the general result of this paper. In particular, the DoF region of the well-known four-message (and even three-message) MIMO X channel is established. This problem had remained open despite previous studies which had found inner and outer bounds that were not tight in general. Hence, the DoF regions of all special cases obtained from the general DoF region of the nine-message $2 \times 2$ interference network of this work that include at least three of the four X channel messages are
new, among many others. Our work sheds light on how the same physical $2 \times 2$ interference network could be used by a suitable choice of message sets to take most advantage of the channel resource in a flexible and efficient manner.

### 1.3.2 MIMO 3-user broadcast channel with general message sets

In Chapter 3, we study the optimum linear coding scheme over the 3-user multiple-input multiple-output (MIMO) broadcast channel (BC). General message sets are considered, in which there can be at most seven different types of messages classified by which receivers the messages are intended for. To begin with, we do the subspace resource identification of the system and study the various nullspaces of the channels. The entire $m$-dimensional transmitter beamformer space is decomposed into nine linearly independent subspaces according to their diverse impacts on the three receivers. Then, a systematic way to assign the beamformers from these subspaces to different messages is proposed, such that the interference received by each receiver is minimized. Furthermore, an outer bound on the linear degrees of freedom (LDoF) region of the 3-user broadcast channel with general message sets is given for the symmetric antenna case, i.e., $(m,n,n,n)$ setting., which is proven to be tight using the linear coding scheme associated with the decomposed subspaces. The channel decomposition provides a fresh perspective of understanding the 3-user broadcast channel. Our result also provides the first example of how interference alignment can be helpful in the broadcast channels with general message sets and with perfect CSIT.

### 1.3.3 MIMO 2-user broadcast channel with common message under hybrid CSIT models

In Chapter 4, we study the degrees of freedom (DoF) regions of the two-user multiple-input multiple-output (MIMO) broadcast channel with a general message set (BC-CM) —that includes private and common messages —under fast fading. Nine different channel state knowledge assumptions —collectively known as hybrid CSIT models —are considered wherein the transmitter has either perfect/instantaneous (P), delayed (D) or no (N) channel state information (CSI) from each
of the two receivers. General antenna configurations are addressed wherein the three terminals have arbitrary numbers of antennas. The DoF regions are established for the five hybrid CSIT models in which either both channels are unknown at the transmitter or each of the two channels is known perfectly or with delay. In the four remaining cases in which exactly one of the two channels is unknown at the transmitter, the DoF regions under the restriction of linear encoding strategies —also known as the linear DoF (LDoF) regions —are established. As the key to the converse proofs of the LDoF region of the MIMO BC-CM under such hybrid CSIT assumptions, we show that, when only considering linear encoding strategies, the channel state information from the receiver with more antennas does not help if there is no channel state information available from the receiver with fewer antennas. This result is conjectured to be true even without the restriction on the encoding strategies to be linear. If true, the LDoF regions obtained for the four hybrid CSIT cases herein will also be the DoF regions for those cases.

Many of the results of this work when specialized to even the two-message problems are new. These include the LDoF regions of the MIMO BC-CM (when one of the two channels is not known) when specialized to the MIMO BC with private messages. They also include the DoF/LDoF regions for all the hybrid CSIT models obtained by specializing the corresponding regions for the MIMO BC-CM to the case with degraded messages.

First, we establish the DoF region of the two-user MIMO BC with only private messages. Then, by employing the approach of loosening decoding requirement of the common message, the common message can be subsumed into any one of these two private messages, and the BC-CM problem degenerates into the BC with private messages problem. Two groups of outer bounds for the two-user BC-CM can thus be obtained from DoF region results of the two-user BC with only private messages. Interestingly, it is shown via the development of linear precoding schemes, that the outer bounds obtained in this way each of the nine hybrid CSIT assumptions are tight for the two-user BC-CM system under the matching hybrid CSIT assumption for all nine cases. Consequently, the complete DoF regions of two-user MIMO BC-CM are established under the CSIT assumptions of type ‘NN’, ‘DD’, ‘PP’, ‘PD’ and ‘DP’. For the ‘PN’, ‘DN’, ‘NP’ and ‘ND’ cases, the
linear DoF regions are established, and conjectured to be the DoF regions.
Chapter 2

Degrees of Freedom Region of the MIMO $2 \times 2$ Interference Network with
General Message Sets

2.1 Introduction

In order to design network communication systems that can flexibly and efficiently handle the complex signaling requirements of futuristic applications it may be necessary to offer multiple physical layer modes that allow for the transmission of some or all of multiple unicast, multiple multicast, multiple broadcast (i.e., $X$-channel), and/or cooperative/cognitive/common messages. In this chapter, rather than considering each such transmission mode in isolation, we study the unified setting in which all (or any subset of) such messages can be transmitted simultaneously over the MIMO $2 \times 2$ interference network. For this simple network, depending on the subset of the two transmitters at which a message is known, and the subset of the two receivers where it is desired, there are nine possible messages in the general message set. For this fully general message set, the associated nine-dimensional DoF region of the MIMO $2 \times 2$ interference network is established herein.

The most studied and also the best understood setting of the $2 \times 2$ interference network is the two-unicast setting, referred to in the literature as the interference channel [3], in which each transmitter has a private message for its single distinct intended receiver (cf. [4, 5, 6] and the references therein). In particular, the DoF region of the two-user MIMO interference channel was found in [4] and more refined characterizations in terms of generalized degrees of freedom and constant bit-gap to capacity were found in [5] and [6], respectively.
The four private message case, which can be thought of as a two-broadcast network, more commonly known as the X channel, allows for the transmission of a private message to each of the two receivers from each transmitter. The now well-known, and more broadly applicable, linear precoding technique known as interference alignment is needed to achieve the DoF in some cases. With its use, the MIMO X channel was shown in [7, 8] to achieve higher sum DoF than the MIMO interference channel. For example, when all transmitters and receivers are equipped with the same number, \( M \), of antennas, the two-user MIMO interference channel has a sum DoF of \( M \), while the MIMO X channel can achieve a sum DoF of \( \frac{4}{3}M \) for \( M > 1 \), achievable with interference alignment. The key idea — when \( M \) is a multiple of 3 — is that by aligning undesired signals (i.e., interference) from the two transmitters into the same subspace at a receiver, one can maximize the desired signal dimensions at that receiver.

In [8], an outer bound on the DoF region of the MIMO X channel is given based on the sum rate outer bound of the embedded MAC, BC and Z channels in the X channel. Moreover, [8] gives an achievability scheme based on interference alignment and presents an achievable DoF region that is given as the convex hull of all integer-valued degrees of freedom within that outer bound region. But those inner and the outer bounds are not identical. However, using interference alignment over multi-letter extensions of the MIMO X channel, it was shown that the outer bound is tight (including non-integer corner points) when all nodes have equal number of antennas \( M \), when \( M > 1 \). In the context of the general MIMO X channel with an arbitrary numbers of antennas at the four terminals, [8] claims that the DoF outer bound region obtained therein is tight in “most cases”, but a precise statement and proof of this claim is not provided. Later, the authors of [9] introduced a novel technique named asymmetric complex signaling (ACS). By allowing the inputs to be complex but not circularly symmetric and using an alternative representation of the channel models in terms of only real quantities, the problem is transformed to delivering real messages over channels with real-valued coefficients. Consequently, it was shown that the 2-user single-input, single-output (SISO) X channel with constant channel coefficients achieves the outer bound of \( \frac{4}{3} \) DoF.
However, it remained an open problem as to whether the outer bound of [8] is tight for any of the multiple antenna cases. For instance, the problem remained open as to whether there are other scenarios in which ACS is required in addition to multi-letter extensions, as did the problem of identifying cases in which just multi-letter extensions suffice to achieve the outer bound. More recently, it was shown in [10] that the outer bound on the sum DoF for the MIMO $X$ channel with generic channel coefficients derived in [8] is tight for any antenna configuration. The work in [10] proposes a linear precoding method based on the generalized singular value decomposition (GSVD), and with the aid of computational experiments, the authors of [10] offer a conjecture that the outer bound region obtained in [8] is also tight. The general DoF region result of this chapter for the MIMO $2 \times 2$ interference network with nine distinct messages, when specialized to the four private-message MIMO $X$ channel, settles this conjecture in the affirmative. It therefore also expands on, and makes precise, the claim in [8]. In particular, it establishes that the outer bound on the DoF region of [8] is indeed tight.

Besides the aforementioned MIMO interference and $X$ channels (and its embedded MAC and/or BC), in which only private messages are considered (see also [5, 6, 11]), the $2 \times 2$ interference network can work in various other modes if common messages, multicast messages and/or transmitter cognition are allowed. For example, if both transmitters share the same three messages, and each receiver demands one of the first two messages while both demand the third, we have what is known as the broadcast channel with private and common messages (BC-CM) [12, 13]. On the other hand, if each transmitter has a private message, and both receivers demand both of the messages, the system works as a compound multiple access channel (C-MAC) [14]. If there are two private messages as in the interference channel and there is a common message known by both transmitters and also demanded by both receivers, the network is known as the interference channel with common message (IC-CM) [15, 16]. The network is referred to as a cognitive $X$ channel in [8] if there are four independent messages to be sent as in the $X$ channel, but with one of the four messages known at both transmitters. A new three-message setting could be defined in which one transmitter has 2 messages, each intended for a distinct receiver, and a third shared
message that is known to both transmitters and desired at one of the receivers. Interpreting the second transmitter as a relay, such a setting could be described as a broadcast channel with a partially cognitive relay (BC-PCR). A six-message cognitive $X$ channel could be defined as having the four private messages as in the $X$ channel as well as two more messages that are known to both transmitters with each desired at a distinct receiver. Evidently, based on different message sets, the $2 \times 2$ interference network can represent many different settings and potential applications, one for each of the $2^9 - 1 = 511$ non-empty subsets of the nine messages.

**Notation:** $\text{co}(A)$ is the convex hull of set $A$, $\mathbb{R}_+^n$ and $\mathbb{Z}_+^n$ denote the set of non-negative $n$-tuples of real numbers and integers, respectively. $(x)^+$ represents the larger of the two numbers, $x$ and 0. $A \otimes B$ denotes the Kronecker product of matrix $A$ and $B$. $[A \; B]$ means the horizontal concatenation of matrix $A$ and $B$, and $[A; B]$ or $\begin{bmatrix} A \\ B \end{bmatrix}$ means the vertical concatenation of matrix $A$ and $B$. $\text{Re}(A)$ and $\text{Im}(A)$ denote the real part and imaginary part of complex matrix $A$, respectively. $\mathcal{N}(A)$ denotes the null space of the linear transformation $A$. $\text{Span}(V)$ denotes the subspace spanned by the column vectors of matrix $V$.

### 2.2 System Model

![Figure 2.1: The $2 \times 2$ Interference Network with General Message Set. The two transmitters are equipped with $M_1$, $M_2$ antennas respectively, and the two receivers are equipped with $N_1$, $N_2$ antennas respectively. The network has nine possible messages, one each for a subset of transmitters where it is known and a subset of receivers where it is desired.](image-url)
We consider the complex Gaussian network with two transmitters and two receivers, as it is shown in Figure 2.1. The two transmitters are equipped with $M_1$, $M_2$ antennas respectively, and the two receivers are equipped with $N_1$, $N_2$ antennas respectively. We denote the channel between transmitter $t$ and receiver $r$ as the $N_r \times M_t$ complex matrix $H_{rt}$ and assume all channels to be generic, i.e., all the channel coefficient values are drawn independently from a continuous probability distribution. The channel is assumed to be constant over the duration of communication and all channel coefficients are perfectly known at all transmitters and receivers. The received signal at receiver $r$ ($r = 1, 2$) is given by $Y_r = H_{r1}X_1 + H_{r2}X_2 + Z_r$, where $X_t$ ($t = 1, 2$) is the $M_t \times 1$ input vector at transmitter $t$, $Z_r$ is the $N_r \times 1$ additive white Gaussian noise (AWGN) vector at receiver $r$.

General message sets are considered in this chapter. For $2 \times 2$ interference network, there are at most nine possible messages classified by different sources and destinations. We index them as $W_{11}$, $W_{12}$, $W_{21}$, $W_{22}$, $W_0$, $W_1$ and $W_2$, as shown in Figure 2.1. $W_{rt}$ ($r, t = 1, 2$) is a private message sent from transmitter $t$ to receiver $r$; $W_0r$ ($r = 1, 2$) is a common message transmitted cooperatively from both transmitters to receiver $r$; $W_t$ ($t = 1, 2$) is a multicast message transmitted from transmitter $t$ and demanded by both receivers simultaneously; $W_0$ is a common multicast message transmitted cooperatively from both transmitters and demanded by both receivers.

Assume the total power across all transmitters to be equal to $\rho$ and indicate the message set size by $|W(\rho)|$. For codewords occupying $t_0$ channel uses, the rates $R(\rho) = \frac{\log |W(\rho)|}{t_0}$ are achievable if the probability of error for all nine messages can simultaneously be made arbitrarily small by choosing appropriately large $t_0$. The capacity region $C(\rho)$ of the MIMO $2 \times 2$ interference network with general message sets is the set of all achievable rate-tuples $R(\rho) = (R_{11}(\rho), R_{12}(\rho), ..., R_0(\rho))$. Define the degrees of freedom region $D$ for MIMO $2 \times 2$ interference network with general message sets as

$$D \triangleq \left\{ (d_{11}, d_{12}, ..., d_0) \in \mathbb{R}_+^E : \forall (\omega_{11}, \omega_{12}, ..., \omega_0) \in \mathbb{R}_+^E \right. \right.$$

$$\left. \sum_{x \in E} \omega_x d_x \leq \limsup_{\rho \to \infty} \left[ \sup_{R(\rho) \in C(\rho)} \frac{\sum_{x \in E} \omega_x R_x(\rho)}{\log(\rho)} \right] \right\}.$$
where \( \mathbf{E} = \{11, 12, 21, 22, 01, 02, 1, 2, 0\} \).

This definition is the general message set counterpart of the one provided in [8] for the MIMO X channel. Note that \( \mathcal{D} \) is a closed convex set.

### 2.2.1 Organization

In Section 2.3, we consider first the previously studied MIMO X channel, for which the best inner and outer bounds of [8, 9, 10] known to date are not coincident in general. The MIMO X channel provides the context in which to introduce the notation used in this chapter and all the relevant linear precoding techniques, namely, zero-forcing, interference alignment, symbol extension, and ACS.

The key results of this work on closing the gap between the known inner and outer bounds on the MIMO X channel are given in Section 2.3.4 where we provide a class of antenna configurations for which, among linear schemes, ACS is required and is sufficient, along with multi-letter extensions and the other linear precoding techniques, to achieve all fractional DoF corner points for those antenna configurations. More generally, we show that the use of linear precoding techniques including symbol extensions and ACS, whether ACS is required or not, are sufficient to achieve any corner point of the DoF region regardless of the antenna configuration.

The main results of this work on closing the gap between the known inner and outer bounds on the MIMO X channel are given in Section 2.3.4 where we provide a class of antenna configurations for which, among linear schemes, ACS is required and is sufficient, along with multi-letter extensions and the other linear precoding techniques, to achieve all fractional DoF corner points for those antenna configurations. More generally, we show that the use of linear precoding techniques including symbol extensions and ACS, whether ACS is required or not, are sufficient to achieve any corner point of the DoF region regardless of the antenna configuration.

The main result of this chapter, namely the DoF region of the general nine-message problem, is established in Sections 2.4, 2.5 and 2.6, with Sections 2.4 and 2.6 dealing with achievability and Section 2.5 establishing the converse.

A reader not familiar with [8] and/or not immediately concerned with how the gap between the inner and outer bounds on DoF of [8] can be closed may want to defer reading Section 2.3.4 until after the proof of Lemma 4 in Section 2.4. However, Sections 2.3.1, 2.3.2 and 2.3.3 serve to not only provide an overview of the ingredients of the achievable scheme for the MIMO X channel but also serve to lay the ground work for the achievable scheme for the more general nine-message MIMO \( 2 \times 2 \) interference network detailed in Sections 2.4 and 2.6.

Finally, in Section 2.7, we provide a few specializations of our result to recover some previously
known cases such as the DoF regions of the interference channel, the IC-CM and the cognitive interference channel and provide DoF regions for two three-message networks we call the generalized cognitive interference channel and the broadcast channel with partially cognitive relay. We then provide some sharp results on sum-DoF for the X channel and the cognitive X channel under certain settings.

Section 2.8 concludes the chapter.

2.3 The MIMO X Channel

The MIMO X channel is an important special case of the 2 × 2 interference network in which only the four private messages, namely, $W_{11}$, $W_{12}$, $W_{21}$, $W_{22}$, are present. Hence the message index set in this case is $E = \{11, 12, 21, 22\}$. Each of these four messages is intended for one of the two receivers and is a source of interference to the other receiver.

We start by stating the DoF region of the MIMO X channel.

**Theorem 1.** The DoF region of the MIMO X channel with constant generic channel coefficients is (with probability one)

$$\mathbb{D}_X = \{(d_{11}, d_{21}, d_{12}, d_{22}) \in \mathbb{R}_+^4 :$$

\[
\begin{align*}
    d_{11} + d_{12} + d_{21} &\leq \max(M_1, N_1), \\
    d_{11} + d_{12} + d_{22} &\leq \max(M_2, N_1), \\
    d_{21} + d_{22} + d_{11} &\leq \max(M_1, N_2), \\
    d_{21} + d_{22} + d_{12} &\leq \max(M_2, N_2), \\
    d_{11} + d_{12} &\leq N_1, \quad d_{21} + d_{22} \leq N_2, \\
    d_{11} + d_{21} &\leq M_1, \quad d_{12} + d_{22} \leq M_2. 
\end{align*}
\]

That the above DoF region is an outer bound for the DoF region of the MIMO X channel is proved in Theorem 2 of [8]. The outer bounding inequalities result, respectively, from the embedded
multiple-access channel, broadcast channel and Z channels, in the MIMO X channel. The readers
can refer to [8] for details. These outer bounds are generalized to the general nine-message problem
in Section 2.5.

The authors of [8] also provide a constructive achievability proof to show that the convex
hull of all the integer-valued DoF-tuples in $\mathbb{D}_X$ is achievable. The techniques used in the achievable
scheme are zero-forcing, interference alignment and random beamforming. Since these techniques
are among the techniques used in our $2 \times 2$ interference network with general message sets problem,
we provide a succinct account of them in Section 2.3.1. The techniques of symbol extension and
ACS are described in Sections 2.3.2 and 2.3.3.

2.3.1 Zero-forcing and interference alignment

Consider message $W_{11}$ as an example. If $M_1 > N_2$, the null space of channel $H_{21}$ is not
empty. By transmitting some symbols of message $W_{11}$ using the beamformers chosen from the null
space $\mathcal{N}(H_{21})$, we can zero-force these symbols at receiver $R_2$ and thus introduce no interference
to it. The maximum number of such symbols that can be zero-forced is $(M_1 - N_2)^+$, which
is equal to the rank of $\mathcal{N}(H_{21})$. Similarly, we can transmit, at most, $(M_2 - N_2)^+$ symbols of
message $W_{12}$ via the nullspace of channel $H_{22}$ and zero-force them all at their unintended receiver,
$R_1$. Note that we can construct the basis vectors of the null space of the concatenated channel
$[H_{21} \ H_{22}]$ from the basis vectors of null space $\mathcal{N}(H_{22})$ and $\mathcal{N}(H_{21})$ by padding zeros before and
after them respectively, to make the length of the vectors equal to $M_1 + M_2$. Transmitting a
symbol via $\mathcal{N}(H_{21})$ at transmitter 1 or via $\mathcal{N}(H_{22})$ at transmitter 2 will equivalently consume
a beamformer from $\mathcal{N}([H_{21} \ H_{22}])$. The remaining dimension of $\mathcal{N}([H_{21} \ H_{22}])$ is equal to
$A = (M_1 + M_2 - N_2)^+ - (M_1 - N_2)^+ - (M_2 - N_2)^+$. By choosing beamformers for message $W_{11}$ and $W_{12}$
jointly from the rest of the subspace of $\mathcal{N}([H_{21} \ H_{22}])$, it is possible to align this part of message
$W_{11}$ and $W_{12}$ into the same subspace and thus reserve more dimensions for the desired messages
at receiver $R_2$, and the maximum number of such pairs of streams is equal to $A$. If there are more
symbols of message $W_{11}$ left, they can be transmitted using random beamforming, which would
create unavoidable interference at its unintended receiver.

Since the technique of zero-forcing is more efficient in terms of reducing interference than interference alignment, it is given the highest priority when constructing precoding beamformers. Following that, interference alignment is used to the extent possible, and following which all of the remaining symbols are sent using random beamforming. The beamformers for each private message is hence divided into three linearly independent parts based on the precoding technique used. Here we use superscript 'Z' to indicate a message is zero-forced at its unintended receiver, 'A' to indicate a message is aligned with another interference at their commonly unintended receiver, and 'R' to indicate the remainder of a certain message that is transmitted using random beamforming. Hence a message $W_x$ for $x \in E$ (recall $E = \{11, 12, 21, 22\}$ for the MIMO X channel) is split in general into three components or sub-messages, denoted $W_x^Z$, $W_x^A$ and $W_x^R$, with the number of symbols (dimensions) in each denoted as $d_x^Z$, $d_x^A$ and $d_x^R$, respectively. In general, we use the notation $W_x^y$ and $d_x^y$ with $x \in E$ and $y \in \{Z, A, R\}$ for the component messages and dimensions, respectively. Similarly, the precoding matrix for any sub-message $W_x^y$ is denoted as $V_x^y$. Thus we have that $d_{ij} = d_{ij}^Z + d_{ij}^A + d_{ij}^R$ and let $V_{ij}$ denote the horizontally concatenated matrix $V_{ij} = [V_{ij}^Z V_{ij}^A V_{ij}^R]$, where $i, j = 1, 2$. It was shown in [8] that any integer-valued DoF-tuple within the outer bound can be divided into three such parts within the decoding ability of the channels. It is thus achievable.

2.3.2 Symbol extensions

When a corner point of $\mathbb{D}_X$ is not integer-valued, it is rational-valued. It is therefore natural to consider a multi-letter extension of the channels to obtain a larger but equivalent system with the corresponding corner point of the DoF region being integer-valued. The length of symbol extensions can be chosen to be the least common multiple of the denominators of all the fractional values. To this time-extended channel, the techniques of zero-forcing, alignment and random beamforming can be applied as described in the previous section. This was proposed in [8].

Consider $T$ symbol extensions of the $X$ channel with complex and constant (across time) channel coefficients. We have the equivalent $\tilde{N}_i \times \tilde{M}_j$ channel matrix $\tilde{H}_{ij}$, in which $i, j = 1, 2,$
\[ \hat{M}_i = T \cdot M_i, \quad \hat{N}_i = T \cdot N_i, \] and

\[ \hat{H}_{ij} = I_T \otimes H_{ij}, \]

\[
\begin{bmatrix}
H_{ij} & 0 & \ldots & 0 \\
0 & H_{ij} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & H_{ij}
\end{bmatrix}.
\]

Hence, we effectively have an $X$ channel with $\hat{M}_j$ antennas at the $j$ th transmitter and $\hat{N}_i$ antennas at the $i$ th receiver and channel matrices $\hat{H}_{ij} \in \mathbb{C}^{\hat{N}_i \times \hat{M}_j}$. To achieve a degrees of freedom tuple $\vec{d}$ for the original system, we need to achieve $T \cdot \vec{d}$ for this equivalent system, and we can use the exact same precoding scheme designed for integer-valued corner points.

However, the equivalent channel matrices after symbol-extension are unlike those for their original counterparts (with $T = 1$) in that they are block-diagonal. The primary question that arises is whether the channel matrices of the time-extended channel continue to yield the linear independence results of the single-letter generic unstructured channels in spite of their special non-generic structure. If they do, then it can be asserted that multi-letter extensions are sufficient to achieve all fractional DoF tuples of $\mathbb{D}_X$.

However, this is not the case in general. Indeed, as it was observed in [8] the symbol extension technique is not sufficient even for the SISO $X$ channel. Interestingly, on the other hand, it is shown in [8] that, in the symmetric MIMO case, where all nodes have equal number, $M$, of antennas, and $M > 1$, the symbol extension technique is sufficient.

Nevertheless, the authors of [8] claim, based on a few examples, that the DoF outer bound region obtained therein is tight in “most cases”, and give the SISO case as an exception. But it is not clear if there are other cases that are also such exceptions, and if so, whether they can indeed be seen as exceptions, i.e., it is unclear as to how commonly these exceptions arise, in which just symbol extensions are not enough to achieve all the fractional corner points of $\mathbb{D}_X$. This brings us to the next section.
2.3.3 Asymmetric Complex Signaling

As stated previously, since the equivalent channel matrices after symbol-extension will be block-diagonal, many nice properties of the original generic channels can be lost. It is shown in [8] that, in the SISO case, the precoding scheme provided previously (with three symbol extensions) fails to achieve the important integer-valued corner point \((1, 1, 1, 1)\) which achieves sum-DoF, because of the block diagonal structure in the extended channel matrices.

In response to this phenomenon, the authors of [9] introduced a new technique named asymmetric complex signaling (ACS). The key idea of ACS is to allow the inputs to be complex but not circularly symmetric and use an alternative representation of the channel models in terms of only real quantities. Hence, all dimensions of the new system are doubled and all channel coefficients, beamformers, inputs and outputs are real-valued. With \(H_{ij} (i, j = 1, 2)\) being the original complex channel matrices, their alternative real representations have the following forms

\[
\hat{H}_{ij} = \begin{bmatrix}
\text{Re}(H_{ij}) & -\text{Im}(H_{ij}) \\
\text{Im}(H_{ij}) & \text{Re}(H_{ij})
\end{bmatrix}. \tag{2.2}
\]

In order to transmit \(T \cdot \vec{d}\) complex-valued streams over the original system, we need to transmit \(2T \cdot \vec{d}\) real-valued streams over the equivalent real channels.

It is shown in [9] that using ACS, the outer bound of \(\frac{4}{3}\) degrees of freedom is achievable for the SISO \(X\) channel. In particular, with a three-symbol extension and ACS, all equivalent channel matrices are of size \(6 \times 6\), and using the same precoding scheme as used in the other MIMO cases, two real-valued symbols can be transmitted via the real channels. The unintended linear dependencies in the previous complex-valued transmission disappear almost surely in this new model. Thus the sum-DoF of \(4/3\) is achievable (and hence also the DoF region). The readers are referred to [9] for further details.
2.3.4 Closing the gap

The important question as to whether there are MIMO antenna configurations for which, among linear schemes including symbol extensions, ACS is necessary, remains open. The question is also open about whether ACS, along with the other linear techniques, is sufficient for MIMO antenna configurations to achieve all fractional DoF-tuples in $\mathbb{D}_X$. If so, for what antennas configurations is it sufficient? Are there DoF-tuples and antenna configurations for which linear precoding schemes including time extensions and ACS are not sufficient?

In this section, all of the above questions are definitively answered. In particular, a class of antenna configurations (that includes the SISO case) are identified that require ACS among linear schemes; i.e., in which just employing symbol extensions alone does not suffice. More generally, it is shown that ACS along with the other linear schemes is sufficient to achieve any fractional corner points of the DoF region $\mathbb{D}_X$ of the MIMO $X$ channel for any antenna configuration.

**Lemma 1.** In the case that $M_1 + M_2 = N_1 + N_2$ and $\min(M_1, M_2, N_1, N_2) = 1$, if interference alignment is needed\(^1\) to achieve any fractional DoF-tuple in $\mathbb{D}_X$, then the achievability scheme in 2.3.1, applied to the $T$-symbol extended $2 \times 2$ interference network, fails to make the corresponding symbols distinguishable at the receiver where they are desired. In particular, if $M_1$ or $N_2 = 1$, then $\text{span}\left(\tilde{H}_{21}\tilde{V}_A^{21}\right) \subseteq \text{span}\left(\tilde{H}_{22}\left[\tilde{V}_Z^{22} \tilde{V}_A^{22}\right]\right)$; if $M_2$ or $N_1 = 1$, then $\text{span}\left(\tilde{H}_{12}\tilde{V}_A^{12}\right) \subseteq \text{span}\left(\tilde{H}_{11}\left[\tilde{V}_Z^{11} \tilde{V}_A^{11}\right]\right)$.

**Proof.** We give the proof of Lemma 1 in the case that $M_1$ or $N_2 = 1$, and the validity for the case that $M_2$ or $N_1 = 1$ follows in the same way.

First, consider the situation when $N_2 = 1$, and we have that $N_1 = M_1 + M_2 - 1 \geq \max(M_1, M_2)$. Consequently, zero-forcing any symbol of message $W_{21}$ and $W_{22}$ at receiver $R_1$ is not possible, i.e., $\tilde{V}_Z^{22} = \tilde{V}_Z^{21} = \emptyset$. However, since $M_1 + M_2 - N_1 = 1$, there exists a one-dimensional null space of the concatenated channel $[H_{11} H_{12}]$. Thus, it is possible to align one symbol.

---

\(^1\) Whether or not interference alignment is needed depends on the antenna configuration and the DoF-tuple to be achieved. For example, consider the SISO case. Interference alignment is required to achieve corner point $(d_{11}, d_{12}, d_{21}, d_{22}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, but it is not required for corner point $(1,0,0,0)$.  

of message $W_{21}$ with one symbol of message $W_{22}$ at receiver $R_1$. When $T$ channel extensions are used, the available dimension for interference alignment is equal to $T$. Suppose the basis vector of the null space of $\mathcal{N}([H_{11} \ H_{12}])$ is given by

$$
\begin{bmatrix}
V_{a,M_1 \times 1} \\
V_{b,M_2 \times 1}
\end{bmatrix}_{(M_1+M_2) \times 1}
$$

(2.3)

Then one set of basis vectors of $T$-dimensional subspace after symbol extension will be the column vectors of matrix

$$
\begin{bmatrix}
V_{a,M_1 \times 1} & 0 & \ldots & 0 \\
0 & V_{a,M_1 \times 1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & V_{a,M_1 \times 1} \\
V_{b,M_2 \times 1} & 0 & \ldots & 0 \\
0 & V_{b,M_2 \times 1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & V_{b,M_2 \times 1}
\end{bmatrix}_{(M_1T+M_2T) \times T}.
$$

(2.4)

All beamformers generated from this basis should be of the form

$$
[\alpha_1 V_a; \alpha_2 V_a; \ldots; \alpha_T V_a; \alpha_1 V_b; \alpha_2 V_b; \ldots; \alpha_T V_b],
$$

(2.5)

where $\alpha_1, \alpha_2, \ldots, \alpha_T \in \mathbb{C}$ are $T$ random scalars. Since $N_2 = 1$, $H_{21}V_a$ and $H_{22}V_b$ will be scalars. $\tilde{H}_{21} \tilde{V}_{21}^A$ and $\tilde{H}_{22} \tilde{V}_{22}^A$ will have the following form

$$
\tilde{H}_{21} \tilde{V}_{21}^A = \begin{bmatrix}
H_{21} & 0 & \ldots & 0 \\
0 & H_{21} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & H_{21}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 V_a \\
\alpha_2 V_a \\
\vdots \\
\alpha_T V_a
\end{bmatrix}
$$

$^2$ The dimensions of matrices will be specified in a subscript when such dimensions have to be emphasized or defined for the first time.
\[
\tilde{H}_{21}V_{21}^A = (H_{21}V_a) \cdot \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_T
\end{bmatrix},
\]

(2.6)

and

\[
\tilde{H}_{22}V_{22}^A = \begin{bmatrix}
H_{22} & 0 & \ldots & 0 \\
0 & H_{22} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & H_{22}
\end{bmatrix} \cdot \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_T
\end{bmatrix} = (H_{22}V_b) \cdot \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_T
\end{bmatrix},
\]

(2.7)

Hence \(\tilde{H}_{21}V_{21}^A\) is also aligned with \(\tilde{H}_{22}V_{22}^A\) at their commonly desired destination, i.e., \(\text{span}(\tilde{H}_{21}V_{21}^A) = \text{span}(\tilde{H}_{22}V_{22}^A)\). This makes \(\tilde{W}_{21}^A\) and \(\tilde{W}_{22}^A\) indistinguishable at receiver \(R_2\).

Next, consider the situation when \(M_1 = 1\). In this case, \(M_1 \leq N_1\) and \(M_2 = N_1 + N_2 - 1 \geq N_1\). In other words, the null space of channel \(H_{11}\) does not exist, and the null space of channel \(H_{12}\) may exist. Recall that we only do interference alignment after zero-forcing of more symbols is not possible. Thus, when interference alignment is used, \(M_2 - N_1\) streams of the message \(W_{22}\) have already been zero-forced at receiver \(R_1\). Since \(\max(d_{22}) = N_2 > M_2 - N_1\), it is still possible to transmit another symbol of \(W_{22}\). The dimension of the null-space of the concatenated channel \([H_{11} \ H_{12}]\) is equal to \(M_1 + M_2 - N_1\). Letting vector \(v_{21}\) be in the subspace of null space \(\mathcal{N}(H_{12})\), we have that \(\begin{bmatrix}0 \\ v_{21}\end{bmatrix}\) will belong to \(\mathcal{N}([H_{11} \ H_{12}])\). In other words, \(M_2 - N_1\) dimensions of the null space \(\mathcal{N}([H_{11} \ H_{12}])\) are already occupied when doing zero-forcing of message \(W_{22}\). The remaining dimension of null space \(\mathcal{N}([H_{11} \ H_{12}])\) is equal to \((M_1 + M_2 - N_1) - (M_2 - N_1) = M_1 = 1\). Thus, 1 dimension of interference alignment is possible at receiver \(R_1\) for messages \(W_{21}\) and \(W_{22}\). When
T channel extensions are applied, the available dimension for interference alignment is equal to \( T \), and the dimension of zero-forcing subspace is equal to \( T \cdot (M_2 - N_1) \). The beamformers \( \tilde{V}_{21}^A \) and \( \tilde{V}_{22}^A \) are also in the form of (2.3)-(2.5). However, since \( N_2 \) can be greater than 1 in this case, \( H_{21}V_a \) and \( H_{22}V_b \) are no longer scalars, and we don’t have the desirable result that \( \tilde{H}_{21}\tilde{V}_{21}^A \) is aligned with \( \tilde{H}_{22}\tilde{V}_{22}^A \) any more.

To prove that span\( \left( \tilde{H}_{21}\tilde{V}_{21}^A \right) \subseteq \text{span} \left( \tilde{H}_{22}\left[ \tilde{V}_{22}^Z \tilde{V}_{22}^A \right] \right) \), we instead prove that \( \tilde{H}_{21}\tilde{v}_{21,i} \in \text{span} \left( \tilde{H}_{22}\left[ \tilde{v}_{22,i}^Z \tilde{v}_{22,i}^A \right] \right) \), where \( \tilde{v}_{21,i}^A, \tilde{v}_{22,i}^A \) are any pair of alignment vectors drawn from the same beamformer from the null space of \( \mathcal{N} \left( \begin{bmatrix} H_{11} & H_{12} \end{bmatrix} \right) \). Then, we will have that \( \cup_i \text{span} \left( \tilde{H}_{21}\tilde{v}_{21,i} \right) \subseteq \cup_i \text{span} \left( \tilde{H}_{22}\left[ \tilde{v}_{22,i}^Z \tilde{v}_{22,i}^A \right] \right) \), which is the desired result. Let \( \tilde{v}_{21,i}^A = [\alpha_1 V_a; \alpha_2 V_a; \ldots; \alpha_T V_a] \) and \( \tilde{v}_{22,i}^A = [\alpha_1 V_b; \alpha_2 V_b; \ldots; \alpha_T V_b] \), we have that

\[
\tilde{H}_{21}\tilde{V}_{21}^A = \begin{bmatrix} H_{21}V_a & 0 & \ldots & 0 \\ 0 & H_{21}V_a & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & H_{21}V_a \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_T \end{bmatrix}
\]

and

\[
\tilde{H}_{22}\tilde{V}_{22}^A = \begin{bmatrix} H_{22}V_b & 0 & \ldots & 0 \\ 0 & H_{22}V_b & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & H_{22}V_b \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_T \end{bmatrix}
\]

Let column vectors of \( \tilde{V}_{22}^Z \) be a basis of the nullspace \( \mathcal{N} \left( H_{12} \right) \). We have that

\[
\tilde{H}_{22}\tilde{V}_{22}^Z = \begin{bmatrix} H_{22}V_{22}^Z & 0 & \ldots & 0 \\ 0 & H_{22}V_{22}^Z & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & H_{22}V_{22}^Z \end{bmatrix}
\]

Note that the \( T \cdot N_2 \) dimensional space at receiver \( R_2 \) can be partitioned into \( T \) linearly independent subspaces according to different symbol extension slot index. In order to prove that \( \tilde{H}_{21}\tilde{v}_{21,i}^A \in \)
span(\(\tilde{H}_{22} \left[ \tilde{V}_{22} \tilde{v}_{22,i}^A \right]\)), it is sufficient to prove that \(H_{21}V_a\alpha_i \in \text{span}\left(H_{22}[\tilde{V}_{22} V_b\alpha_i]\right)\) for all \(i = 1, ..., T\). Since \(\alpha_i\) here are all scalars, we only need to show that \(H_{21}V_a \in \text{span}\left(H_{22}[\tilde{V}_{22} V_b]\right)\).

Because \(V_{22}\)'s is generated from the nullspace of channel \(H_{12}\), it is independent with channel matrix \(H_{22}\). Consequently, \(H_{22}V_{22}\) will almost surely reserve the column rank of \(V_{22}\)'s, since \(H_{22}\) is a generic full matrix whose rank is greater than \(V_{22}\)'s. In other words, \(\text{rank}(H_{22}[V_{22} V_b]) = M_2 - N_1 = N_2 - 1\) almost surely. Since \(V_b\) is linearly independent with the column vectors of \(V_{22}\), \(H_{22}V_b\) will also be linear independent with the column vectors of \(H_{22}V_{22}\) almost surely. Thus, \(\text{rank}(H_{22}[V_{22} V_b]) = (N_2 - 1) + 1 = N_2\). In other words, the column vectors of \(H_{22}[V_{22} V_b]\) would span the entire \(N_2\)-dimensional subspace at receiver \(R_2\). Since vector \(H_{21}V_a\) also belongs to the same subspace, we have that \(H_{21}V_a \in \text{span}\left(H_{22}[V_{22} V_b]\right)\), which leads to that \(\tilde{H}_{21} \tilde{v}_{21,i}^A \in \text{span}\left(\tilde{H}_{22} \left[ \tilde{V}_{22} \tilde{v}_{22,i}^A \right]\right)\). Thus, message \(\tilde{W}_{21}^A\) and \(\tilde{W}_{22}^A\) are indistinguishable at receiver \(R_2\).

Remark 1. Lemma 1 points out the possibility that the desired signals can also be aligned together and would thus be indistinguishable at their intended receiver when interference alignment and symbol extensions are utilized simultaneously under certain antenna settings. However, it is not easy to explicitly describe all the cases when interference alignment will be necessary. If interference alignment is not necessary to achieve a certain DoF tuple, such a problem does not arise.

**Lemma 2.** By using the technique of ACS together with symbol extensions, the problem of unexpected alignment of desired messages is avoided.

**Proof.** The equivalent channel matrices, when doing \(T\)-symbol extension and ACS, are given as \(\tilde{H}_{ij} = I_{T \times T} \otimes \hat{H}_{ij}\), where \(\hat{H}_{ij}\) is given in equation 2.2. We need to transmit \(2T \cdot d\) real-valued streams over the equivalent real channels.

Consider again the independence of \(\tilde{H}_{21} \tilde{V}_{21}^A\) and \(\tilde{H}_{22} \tilde{V}_{22}^A\) for the cases in Lemma 1. If \(N_2 = 1\), when doing asymmetric complex signaling, the dimension of \(V_a\) and \(V_b\) in (2.3) will be \(2M_1 \times 2\) and
\(2M_2 \times 2\), respectively. \(\bar{H}_{21} \bar{V}_{21}^A\) and \(\bar{H}_{22} \bar{V}_{22}^A\) will instead have the following form

\[
\bar{H}_{21} \bar{V}_{21}^A = \begin{bmatrix}
\hat{H}_{21} V_a & 0 & \ldots & 0 \\
0 & \hat{H}_{21} V_a & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \hat{H}_{21} V_a
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{2T}
\end{bmatrix}
\]

and

\[
\bar{H}_{22} \bar{V}_{22}^A = \begin{bmatrix}
\hat{H}_{22} V_b & 0 & \ldots & 0 \\
0 & \hat{H}_{22} V_b & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \hat{H}_{22} V_b
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{2T}
\end{bmatrix}
\]

where \(\alpha_1, \alpha_2, \ldots, \alpha_{2T} \in \mathbb{R}^1\) are \(2T\) random real scalars. Now, \(\hat{H}_{21} V_a\) and \(\hat{H}_{22} V_b\) are both \(2 \times 2\) real matrices rather than scalars as in (2.6) and (2.7). Each diagonal block of \(\hat{H}_{21} V_a\) or \(\hat{H}_{22} V_b\) works as if it is to rotate a random \(2 \times 1\) real vector with a certain degree. However, the randomness of \([\alpha_1; \ldots; \alpha_{2T}]\) makes the projections in different symbol extension slots independent with each other. Thus, \(\bar{H}_{21} \bar{V}_{21}^A\) and \(\bar{H}_{22} \bar{V}_{22}^A\) will be linearly independent almost surely.

Consider again the case that \(M_1 = 1\), the column rank of \(\hat{H}_{22} [V_{22}^Z V_b]\) will be equal to \(2(N_2 - 1) + 1 = N_2 - 1\). In other words, there is still 1 dimension left in the receiver subspace. Thus, \(\hat{H}_{21} V_a\) is independent with \(\hat{H}_{22} [V_{22}^Z V_b]\) almost surely. In the situation that the column dimension of \(V_a\) and \(V_b\) are \(n\), which is greater than 1, i.e., there are multiple pairs of symbols to be aligned, the column size of \([\hat{H}_{22} [V_{22}^Z V_b] \hat{H}_{21} V_a]\) will be equal to \(2(N_2 - 1) + n + n > N_2\) and thus the columns of \(\hat{H}_{22} [V_{22}^Z V_b]\) are linear dependent with columns of \(\hat{H}_{21} V_a\). However, since the coefficients required for dependence for \(\hat{H}_{21} V_a\) are different almost surely in different time slots, the \(2 \cdot T\) dimensional \(\hat{H}_{22} [V_{22}^Z V_{22}^A]\) will still be linearly independent with \(\hat{H}_{12} \bar{V}_{12}^A\) almost surely, so long as \(d_{22}^Z + d_{22}^A + d_{12}^A \leq T \cdot N_2\).

In summary, the desired messages are still linearly independent with each other at both receiver.
Remark 2. The authors of [9] introduced ACS in the context of the SISO X channel and showed that the total DoF of $\frac{4}{3}$ can be achieved in that channel. In this chapter, we provided a new and simplified perspective on how ACS works. In particular, it transforms the previous scalar multiplication to a local vector rotation, thus obviating the unexpected linear dependences among all the beamformers. Using this we broaden its applicability to MIMO X channel, and more generally, in a later section, to the nine-message MIMO 2×2 interference network.

Remark 3. For all the other antenna settings not included in the cases given in Lemma 1, there is no unexpected loss of independence of desired messages when doing symbol extensions. Thus, ACS is not necessary in those cases.

2.4 Main Result

Now, let us consider the general MIMO 2 × 2 interference network with nine messages.

The following theorem gives the nine-dimensional DoF region of the MIMO 2 × 2 Gaussian interference network with general message sets.

**Theorem 2.** The degrees of freedom region of the MIMO 2 × 2 Gaussian interference network with the general message set is $\mathbb{D} =$

$$\{(d_{11}, d_{21}, d_{12}, d_{22}, d_1, d_2, d_{01}, d_{02}, d_0) \in \mathbb{R}_+^E :$$

$$d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} + d_{21} \leq \max(M_1, N_1) \quad (2.8)$$

$$d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} + d_{22} \leq \max(M_2, N_1) \quad (2.9)$$

$$d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22} + d_{11} \leq \max(M_1, N_2) \quad (2.10)$$

$$d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22} + d_{12} \leq \max(M_2, N_2) \quad (2.11)$$

$$d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} \leq N_1 \quad (2.12)$$

$$d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22} \leq N_2 \quad (2.13)$$

$$d_1 + d_{11} + d_{21} \leq M_1 \quad (2.14)$$

$$d_2 + d_{12} + d_{22} \leq M_2 \quad (2.15)$$
\[ d_1 + d_2 + d_0 + d_{01} + d_{02} + d_{11} + d_{21} + d_{12} + d_{22} \leq \min(M_1 + M_2, N_1 + N_2) \]  

(2.16)

**Proof.** The proof of \( \mathbb{D} \) being an outer bound is given in Section 2.5. The inner bound is given in Lemmas 3 and 4 in this section.

**Lemma 3.** An inner bound to the degrees of freedom region of the MIMO \( 2 \times 2 \) interference network with general message set is \( \mathbb{D}_{in} = \text{co} (\mathbb{D} \cap \mathbb{Z}^9) \), i.e., all the integer-valued degrees of freedom in \( \mathbb{D} \) as well as their convex hull are achievable.

**Outline of Proof:** In this outline, we will describe a method to construct the transmit beamformers for various messages. It will be shown later in Section 2.6 that using this scheme the DoF region \( \mathbb{D}_{in} \) can be achieved.

To achieve any integer-valued nine-dimensional DoF tuple \( \overrightarrow{d} = (d_{11}, d_{21}, d_{12}, d_{22}, d_1, d_2, d_{01}, d_{02}, d_0) \) within \( \mathbb{D} \), we use the following precoding scheme.

Consider linear beamforming. Expressing received signals at receive \( r \) \((r=1,2)\) in the form of different messages, we have

\[
Y_r = H_{r1} \cdot (V_{11}S_{11} + V_{21}S_{21} + V_1S_1) \\
+ H_{r2} \cdot (V_{12}S_{12} + V_{22}S_{22} + V_2S_2) \\
+ [H_{r1} \ H_{r2}] \cdot (V_{01}S_{01} + V_{02}S_{02} + V_0S_0) + Z_r,
\]

where \( S_x \) and \( V_x \) denote the symbols and the corresponding precoding matrices for the message with index \( x \in \mathcal{E} \). Let the column size of \( V_x \) is equal to \( d_x \).
Table 2.1: Message Grouping and Corresponding Precoding Methods

<table>
<thead>
<tr>
<th>Group</th>
<th>Messages</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(W_{11}, W_{12}, W_{21} \text{ and } W_{22})</td>
<td>(d_{11}) independent symbols. (d_{11}^Z) of them are zero-forced at receiver (R_2), (d_{11}^A) of them are aligned with part of (W_{12}) at receiver (R_2). The remaining (d_{11}^R = d_{11} - d_{11}^Z - d_{11}^A) symbols are transmitted using random beamforming.</td>
</tr>
<tr>
<td>2</td>
<td>(W_{01} \text{ and } W_{02})</td>
<td>(d_{01}) independent symbols. (d_{01}^Z) of them are zero-forced at receiver (R_2), and the remaining (d_{01}^R = d_{01} - d_{01}^Z) symbols are transmitted using random beamforming.</td>
</tr>
<tr>
<td>3</td>
<td>(W_1, W_2 \text{ and } W_0)</td>
<td>Random beamforming is used for all symbols of this group.</td>
</tr>
</tbody>
</table>

The techniques used here are transmit zero-forcing, interference alignment and random beamforming.

The nine messages are divided into three groups as shown in Table 2.1. Group 1 consists of the four point-to-point private or X-channel messages \(\{W_{11}, W_{12}, W_{21}, W_{22}\}\), Group 2 consists of the cognitive and common messages which are known to both transmitters, namely, \(\{W_{01}, W_{02}\}\). Group 3 consists of the remaining three multicast messages \(\{W_1, W_2, W_0\}\). The transmission of Group 1 messages is done in the exact same way as in the MIMO X channel. Then, the other two groups are transmitted via the channel resources still available. Recall that, for Group 1, message
$W_{ij} (i, j = 1, 2)$ is partitioned into three linearly independent parts, i.e., $W^Z_{ij}, W^A_{ij}$ and $W^R_{ij}$. Here, for Group 2, message $W_{0i} (i = 1, 2)$ is partitioned into two linearly independent parts, namely, $W^Z_{0i}$ and $W^R_{0i}$. For Group 3, all messages are transmitted using random beamforming, since no interference elimination is necessary for them. Thus, message $W_k (k = 0, 1, 2)$ is all classified as $W^R_k$. We have that

$$d_{ij} = d^Z_{ij} + d^A_{ij} + d^R_{ij}$$
$$d_{0i} = d^Z_{0i} + d^R_{0i}$$
$$d_k = d^R_k$$

and

$$V_{ij} = [V^Z_{ij} V^A_{ij} V^R_{ij}]$$
$$V_{0i} = [V^Z_{0i} V^R_{0i}]$$
$$V_k = V^R_k$$

where $i, j = 1, 2$ and $k = 0, 1, 2$. The dimensions of different parts of each message are given as follows

$$d^Z_{ij} = \min(d_{ij}, (M_j - N_i^+))$$
(2.17)
$$d^A_{i1} = d^A_{i2} = \min(d_{i1} - d^Z_{i1}, d_{i2} - d^Z_{i2}, (M_1 + M_2 - N_i^+ - d^Z_{i1} - d^Z_{i2})^+)$$
(2.18)
$$d^R_{ij} = d_{ij} - d^Z_{ij} - d^A_{ij}$$
(2.19)
$$d^Z_{0i} = \min(d_{0i}, (M_1 + M_2 - N_i^+ - d^Z_{i1} - d^Z_{i2} - d^A_{i1})^+)$$
(2.20)
$$d^R_{0i} = d_{0i} - d^Z_{0i}$$
(2.21)
$$d^R_k = d_k,$$
(2.22)

where $i, j = 1, 2$, $\widehat{i} = 3 - i$, and $k = 0, 1, 2$. To make the expressions more succinct, we define
following auxiliary variables:

\[ Z_{ij} \equiv d_{ij}^2 \tag{2.23} \]
\[ A_i \equiv d_{1i}^4 = d_{2i}^4 \tag{2.24} \]
\[ Z_{0i} \equiv d_{0i}^2 \tag{2.25} \]

These values are pre-determined according to the value of the DoF tuple to be achieved and the system antenna setting. They naturally follow from the fact that the numbers of beamformers transmitted using zero-forcing or interference alignment cannot exceed the corresponding available null space dimensions. For the four private messages, if zero-forcing is possible, use zero-forcing first. If there are more streams that must be sent, use interference alignment next. If there are still more streams after running out of the possibility of doing alignment, use random beamforming. For the two cognitive and common messages, if there are residual available null space dimensions, transmit their symbols using zero-forcing; otherwise, just use random beamforming. For three multicast messages, all streams are transmitted using random beamforming.

The key to using zero-forcing or interference alignment is to appropriately utilize the beamformers picking from the null space of corresponding channels. For a generic channel matrix \( H_{n \times m} \) (\( n < m \)), the dimension of its nullspace is equal to \( m - n \). To obtain a basis of \( \mathcal{N}(H) \), we can do a singular value decomposition (SVD) of matrix \( H \) while arranging the singular values in non-increasing order. Then, the last \( m - n \) right-singular column vectors, which are corresponding to singular value 0, will form a basis of \( \mathcal{N}(H) \). We construct matrix \( \Phi(H) \) such that its column vectors are equal to these basis vectors of \( \mathcal{N}(H) \). Let matrix \( X_{(m-n) \times a} \) denote a randomly generated \((m - n) \times a\) matrix, whose columns are generated independently from a uniform distribution on a \( m - n \) dimensional sphere of radius 1. Then, \( \Phi(H) \cdot X_{(m-n) \times b} \) will generate \( b \) random linear combinations of these basis vectors. If \( b \leq m - n \), these \( b \) vectors will be linearly independent of each other almost surely.

Now, construct the beamformers for all 9 messages according to the equations (2.26)-(2.32).
listed below.

\[ V_{ij}^Z = \Phi(H_{ij}) \cdot X_{ij,(M_j-N_i)} \times d_{ij}^Z \]  
\[ \begin{bmatrix} V_{i1}^A \\ V_{i2}^A \end{bmatrix} = \Phi([H_{i1} \ H_{i2}]) \cdot X_{i,(M_1+M_2-N_i)} \times d_{i1}^A \]  
\[ V_{ij}^R = X_{ij,M_j} \times d_{ij}^R \]  
\[ V_{0i}^Z = \Phi([H_{i1} \ H_{i2}]) \cdot X_{0i,(M_1+M_2-N_i)} \times d_{0i}^Z \]  
\[ V_{0i}^R = X_{0i,(M_1+M_2)} \times d_{0i}^R \]  
\[ V_i^R = X_{i,M_i} \times d_i^R \]  
\[ V_0^R = X_{0,(M_1+M_2)} \times d_0^R \]

where \( i, j = 1, 2 \) and \( \tilde{i} = 3 - i \). The beamformers used for \( V_{ij}^Z, V_{ij}^A \) and \( V_{0i}^Z \) come from the nullspace of the channels or concatenated channels, and the beamformers used for \( V_{ij}^R, V_{0i}^R, V_i^R \) and \( V_0^R \) are just generated randomly as described previously. It’s shown later in Section 2.6 that, if the DoF tuple \( \vec{d} \) is in the region of \( \mathbb{D}_{in} \), then using the above precoding beamformers, all messages are decodable at their intended receivers with probability one. Hence, the DoF tuple \( \vec{d} \) is achievable and \( \mathbb{D}_{in} \) is an achievable DoF region.

**Remark 4.** In linear beamforming, to achieve \( \mathbb{D}_{in} \), only the techniques of zero-forcing, interference alignment and random beamforming are required. Furthermore, as is shown later in Section 2.6, interference alignment is needed only among the four private \( X \) channel messages, i.e., aligning \( W_{11} \) with \( W_{12} \) at receiver \( R_2 \) or aligning \( W_{21} \) with \( W_{22} \) at receiver \( R_1 \). Somewhat surprisingly perhaps, it is not necessary to align interference due to any part of \( W_{01} \) with that due to \( W_{11} \) or \( W_{12} \) at receiver \( R_2 \), or to align interference due to any part of \( W_{02} \) with that due to \( W_{21} \) or \( W_{22} \) at receiver \( R_1 \).

**Remark 5.** In the construction of \( V_{ij}^Z, V_{ij}^A \) and \( V_{0i}^Z \), we use the random linear combinations of the basis vectors of the nullspace of corresponding channels, instead of directly picking beamformers from those basis vectors obtained through an SVD. The advantage is that it avoids picking a same
basis vector repetitively in following procedures and potentially leading to unexpected dependence among the beamformers.

**Lemma 4.** The fractional numbers at the boundary of $\mathbb{D}$, i.e., the gap between $\mathbb{D}_{\text{in}}$ and $\mathbb{D}$, can be achieved using appropriate length of symbol extension. In the case that $M_1 + M_2 = N_1 + N_2$ and $\min(M_1, M_2, N_1, N_2) = 1$, ACS is required in addition to symbol extension.

*Proof.* To achieve a DoF tuple $\vec{d}$ with fractional values, we use a $T$-symbol extension of the channel such that $T \cdot \vec{d}$ is integer-valued. The problem of unexpected dependencies, which is brought on by the structured channel matrices after symbol extensions, also exists here in the nine-message problem. The random beamforming part of messages in Groups 2 and 3, i.e., $W_{0i}^R$ and $W_{k}^R$ ($i = 1, 2$, $k = 0, 1, 2$), cause no problem; they behave in the same way as do $W_{ij}^R$ ($i, j = 1, 2$) from Group 1 in terms of independence results. Since the beamformers for the zero-forcing part of Group 2 messages, i.e., $W_{0i}^Z$ ($i = 1, 2$), are generated from the null space of corresponding concatenated channels, they face the same situation as the interference alignment beamformer pairs (of the private messages) do. Since all the zero-forcing and interference alignment beamformers are derived from the same source but belong to different messages, their behaviors are actually equivalent when considering independence results. The analyses of when ACS is necessary and how ACS works which were detailed in Section 2.3 for the MIMO $X$ channel are the same as in the MIMO $X$ channel problem as well.

In summary, Lemmas 3 and 4 establish that $\mathbb{D}$ is an inner bound to the DoF region of the $2 \times 2$ interference network. Together with the proof of the outer bound in Section 2.5, this completes the proof of Theorem 2.

### 2.5 Outerbound on the degrees of freedom region

In this section, we prove the converse part of Theorem 2, i.e., that the region $\mathbb{D}$ is an outer bound for the DoF region of the $2 \times 2$ interference network.
First, the outer bound (2.16) comes from the MIMO point-to-point channel outer bound when cooperation between transmitters and receivers are both allowed.

Second, consider the embedded multiple-access channel which only contains transmitters \( T_1 \) and \( T_2 \) and receiver \( R_1 \). In this situation, message \( W_{02}, W_{21}, W_{22} \) are irrelevant and set to \( \emptyset \) to avoid interference. The original message \( W_1 \) will degenerate to \( W_{11} \), since we don’t require \( W_1 \) to be decoded by receiver \( R_2 \). Similarly, \( W_2 \) will degenerate to \( W_{12} \), and \( W_0 \) will degenerate to \( W_{01} \).

The cut-set bound for multiple-access channel with common message is \( \hat{d}_{01} + \hat{d}_{11} + \hat{d}_{12} \leq N_1 \). Hence in this scenario, we get the equivalent outer bound \( (d_0 + d_{01}) + (d_1 + d_{11}) + (d_2 + d_{12}) \leq N_1 \), which is outer bound (2.12). In the same way, we get outer bound (2.13) by considering the embedded multiple-access channel which only contains transmitter \( T_1 \) and \( T_2 \) and receiver \( R_2 \).

Third, consider the embedded broadcast channel which only contains transmitter \( T_1 \) and receivers \( R_1 \) and \( R_2 \). In this situation, message \( W_{12}, W_{22} \) and \( W_2 \) are irrelevant and set to \( \emptyset \) to avoid interference. We also set \( W_0, W_{01}, W_{02} \) to \( \emptyset \) and loosen the requirement for transmitter \( T_1 \) by not requiring it to help in transmitting \( W_0, W_{01} \) and \( W_{02} \). Then we get the outer bound from the result of broadcast channel with common message \( d_1 + d_{11} + d_{21} \leq M_1 \), which is outer bound (2.14). Similarly, by considering the embedded broadcast channel with transmitter \( T_2 \), we get outer bound (2.15).

Figure 2.2: MIMO \( Z_{21}^* \) channel with general message sets (a) complete (b) reduced (c) only private
Next we prove outer bound (2.9). Outer bounds (2.8), (2.10) and (2.11) can be similarly inferred. Consider the channel depicted in Figure 2.2.(a), in which there is no communication link between transmitter $T_1$ and receiver $R_2$. Since channel is the MIMO $2 \times 2$ interference network with channel matrix $H_{21} = 0$, we refer to it as the MIMO $Z_{21}$ channel. The reduced message sets shown in Figure 2.2.(a) contains all five possible messages for this channel. Thus, Figure 2.2.(a) depicts the $Z_{21}$ channel with fully general message sets. Here we use * to indicate considering fully general message sets and rename Figure 2.2.(a) as $Z^*_{21}$ channel.

We show that the outer bound on the total DoF of the MIMO $Z^*_{21}$ channel is also an outer bound of the sum-DoF in the outer bound (2.9) for the original MIMO $2 \times 2$ interference network with general message sets, i.e.,

$$
\max_{D^{2x2}} (d_0 + d_{01} + d_1 + d_{11} + d_{12} + d_2 + d_{22})
\leq \max_{D^{2x2}} (d_{01} + d_{11} + d_{12} + d_2 + d_{22}).
$$

(2.33)

Suppose we have a coding scheme that is able to achieve $(d_0, d_{01}, d_1, d_{11}, d_{12}, d_2, d_{22})$ on the nine-message MIMO $2 \times 2$ interference network. Now, suppose, in place of message $W_{21}$ and $W_{02}$ we use two known sequences that are available to all transmitters and receivers a priori. Also, a genie provides $W_{11}, W_1, W_0$ and $W_{01}$ to receiver $R_2$. Thus receiver $R_2$ knows all the information available to transmitter $T_1$ and can subtract transmitter $T_1$’s signal from its received signal. This is equivalent to $H_{21} = 0$. Since receiver $R_2$ already knows $W_1$, transmitter $T_1$ only needs to make sure that receiver $R_1$ can successfully decode $W_1$, so that $W_1$ degenerates to $W_{11}$. Similarly, $W_0$ degenerates to $W_{01}$. The resulting $2 \times 2$ interference network becomes identical to the $Z^*$ channel with the general message set as depicted in Figure 2.2.(a). Since neither setting $W_{21}$ and $W_{02}$ to known sequences nor the assistance of genie to receiver $R_2$ can deteriorate the performance of the coding scheme, the same degrees of freedom $d_{01,Z^*_{21}} = d_{01} + d_0$, $d_{11,Z^*_{21}} = d_1 + d_{11}$, $d_{12,Z^*_{21}} = d_{12}$, $d_{21,Z^*_{21}} = d_2$, $d_{22,Z^*_{21}} = d_{22}$ are achievable on the $Z^*$ channel as well. This proves inequality (2.33).

The argument here is similar to the proof of Lemma 1 in [8], in which $Z_{21}$ channel with only private messages is considered.
In the $Z_{21}^*$ channel depicted in Figure 2.2.(a), message $W_2$ is sent out from transmitter 2 and desired at both receivers, $R_1$ and $R_2$. If we loosen this requirement and only demand receiver $R_2$ to be able to decode this message, the degrees of freedom of the new system will be no less than that of the original system, since reducing decoding requirement cannot hurt. In this case, $W_2$ actually plays the same role as $W_{22}$ does. As a result, we can combine them together and the system reduces to Figure 2.2.(b).

The system in Figure 2.2.(c) is the ordinary MIMO $Z$ channel, which only contains private messages $W_{11}, W_{12}, W_{22}$. An outer bound of ordinary MIMO $Z$ channel is given in Corollary 1 of [8], which is

$$\max(d_{11} + d_{12} + d_{22}) \leq \max(N_1, M_2).$$

The idea of the proof therein is to show the sum capacity of $Z_{21}$ channel (Figure 2.2.(c)) is bounded above by the MAC with $M_2$ receive antennas if $N_1 < M_2$ and bounded above by the MAC with $N_1$ receive antennas if $N_1 \geq M_2$. The multiplexing gain of a MAC cannot be greater than the total number of receive antennas. Therefore, we have $\max(d_{11} + d_{12} + d_{22}) \leq \max(N_1, M_2)$ for Figure 2.2.(c). Now, consider the $Z_{21}$ channel in Figure 2.2.(b), in which one additional common message $W_{01}$ is applied. Following the exact same argument as in [8], we get that the sum capacity of $Z_{21}$ channel (Figure 2.2.(b)) is bounded above by corresponding MAC with common message, whose multiplexing gain is also no greater than its total number of receive antennas, i.e.,

$$\max(d_{01} + d_{11} + d_{12} + d_{22}) \leq \max(N_1, M_2).$$

Including common message or not doesn’t affect the relationship and transformation between $Z$ channel and corresponding MAC channel in the proof. The reader can refer to [8] for more details.

So far we obtained an outer bound for the MIMO $Z^*$ channel with general message sets in Figure 2.2.(a), which is

$$\max(d_{01} + d_{11} + d_{12} + (d_{2} + d_{22})) \leq \max(N_1, M_2).$$
According to inequality (2.33), we have that an outer bound for the MIMO $2 \times 2$ interference network with general message sets is

$$d_0 + d_{01} + d_1 + d_{11} + d_{12} + d_2 + d_{22} \leq \max(N_1, M_2),$$

which is the outer bound (2.9).

Similarly, we obtain outer bounds (2.8), (2.10) and (2.11) from the MIMO $Z_2^*$, $Z_{12}^*$, $Z_{11}^*$ channel respectively. The general message set for the MIMO $Z_{ij}^* (i, j \in \{1, 2\})$ channel consists of message $W_{ij}$, $W_{ij}, W_{ij}^*, W_0^i$ and $W_{ij}^*$, where $\hat{i} = 3 - i$, $\hat{j} = 3 - j$.

### 2.6 Achievability of the inner bound

We have already described the precoding scheme and given the expressions for all the beamformers for all nine messages in the outline of proof of Lemma 3. In this section, we continue the proof and show that, using this scheme, the inner bound $D_{in} = \text{co} (D \cap Z_+^9)$ is achievable.

First, it is shown that all the desired messages are distinguishable, at their intended receivers; and then, we show that the region achievable is identical to $D_{in}$.

#### 2.6.1 Independence requirements

The messages received by each receiver can be divided into two groups based on whether they are desired or undesired messages. The undesired messages are also potentially sources of interference. For receiver $R_1$, desired messages contain $W_{D1}=(W_{11}, W_{12}, W_{01}, W_1, W_2, W_0)$, and undesired messages contain $W_{U1}=(W_{21}, W_{22}, W_{02})$. For receiver $R_2$, desired messages contain $W_{D2}=(W_{21}, W_{22}, W_{02}, W_1, W_2, W_0)$, and undesired messages contain $W_{U2}=(W_{11}, W_{12}, W_{01})$. Let $D_i$ denote the matrix of received vectors associated with the desired messages at receiver $i$, and $U_i$ denote the matrix of directions of the receive beamformers associated with the undesired messages at receiver $i$. We thus have

$$D_i = \begin{bmatrix} H_{11}V_{11} \mid H_{12}V_{12} \mid [H_{11} \ H_{12}] V_{01} \mid \cdots \end{bmatrix}$$
\[
\cdots H_{11}V_1 | H_{12}V_2 | [H_{11} H_{12}]V_0
\]

\[
D_2 = \begin{bmatrix}
H_{21}V_{21} | H_{22}V_{22} | [H_{21} H_{22}]V_{02} & \cdots & H_{21}V_1 | H_{22}V_2 | [H_{21} H_{22}]V_0
\end{bmatrix}
\]

\[
\cdots H_{21}V_1 | H_{22}V_2 | [H_{21} H_{22}]V_0
\]

\[
U_1 = \begin{bmatrix}
H_{11}V_{21} | H_{12}V_{22} | [H_{11} H_{12}]V_{02}
\end{bmatrix}
\]

\[
U_2 = \begin{bmatrix}
H_{21}V_{11} | H_{22}V_{12} | [H_{21} H_{22}]V_{01}
\end{bmatrix}
\]

For successful communication, each receiver needs to be able to decode all its own desired messages. In order to take the most advantage of channel resource, we allocate as much resource as possible to desired messages to minimize the resource consumed by undesired messages, i.e., by interference.

**Lemma 5.** If all the channels are generic and the following constraints are satisfied

\[
d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} + d_{21} + d_{22} + d_{02}
\]

\[
-Z_{21} - Z_{22} - A_1 - Z_{02} \leq N_1 \quad (2.34)
\]

\[
d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22} + d_{11} + d_{12} + d_{01}
\]

\[
-Z_{11} - Z_{12} - A_2 - Z_{01} \leq N_2 \quad (2.35)
\]

\[
d_1 + d_{11} + d_{21} \leq M_1 \quad (2.36)
\]

\[
d_2 + d_{12} + d_{22} \leq M_2 \quad (2.37)
\]

\[
d_1 + d_2 + d_0 + d_{01} + d_{02} + d_{11} + d_{21} + d_{12} + d_{22}
\]

\[
\leq \min(M_1 + M_2, N_1 + N_2), \quad (2.38)
\]

using the precoding scheme described in the outline of proof of Lemma 3 in Section 2.4, we have the following independence results

\[
\text{rank}(U_1) = (d_{21} - Z_{21}) + (d_{22} - Z_{22}) - A_1 + (d_{02} - Z_{02}) \quad (2.39)
\]

\[
\text{rank}(U_2) = (d_{11} - Z_{11}) + (d_{12} - Z_{12}) - A_2 + (d_{01} - Z_{01}) \quad (2.40)
\]

\[
\text{rank}(D_1) = d_{11} + d_{12} + d_{01} + d_1 + d_2 + d_0 \quad (2.41)
\]

\[
\text{rank}(D_2) = d_{21} + d_{22} + d_{02} + d_1 + d_2 + d_0, \quad (2.42)
\]
\[ \text{rank}([D_1 \ U_1]) = \text{rank}(D_1) + \text{rank}(U_1) \] (2.43)

\[ \text{rank}([D_2 \ U_2]) = \text{rank}(D_2) + \text{rank}(U_2). \] (2.44)

These independence results together ensure that all desired messages are distinguishable, and thus decodable, at their intended receivers.

Note that, constraints (2.34) and (2.35) imply that the total independent number of received beamformers at receiver \( R_i \) will be no greater than \( N_i \), the number of its antennas; constraints (2.36) and (2.37) imply that the number of independent streams sent out by transmitter \( T_i \) are restricted to be no greater than \( M_i \); constraint (2.38) implies that the number of all independent streams two transmitters sent out together will be no greater than their total number of antennas.

Regarding the independence result, equations (2.39) and (2.40) give the dimension of the subspace spanned by the received beamformers associated with the undesired messages, i.e., interference; equations (2.41) and (2.42) show that the directions of the received beamformers associated with the desired messages at each receiver are linearly independent of each other; equations (2.43) and (2.44) indicate that the subspace occupied by the desired messages is linearly independent of that of the interference.

Proof. We only give the proof of (2.39), (2.42) and (2.43), since the other three follow in the same way.

First consider equation (2.39). According to the expressions of beamformers provided in equations (2.26), (2.28) and (2.30), we have that \( V^Z_{21} \) and \( V^Z_{22} \) are drawn from the nullspace of \( H_{11} \) and \( H_{12} \), respectively, and \( V^Z_{02} \) is generated from the nullspace \( \mathcal{N}([H_{11} \ H_{12}]) \). Thus, they will all be zero-forced at receiver \( R_1 \), i.e.,

\[ H_{11}V^Z_{21} = 0 \]
\[ H_{12}V^Z_{22} = 0 \]
\[ [H_{11} \ H_{12}]V^Z_{02} = 0. \]
Consequently, we have
\[
\begin{align*}
\text{rank}(H_{11}V_{21}) &= \text{rank}(H_{11}[V_{21}^Z V_{21}^A V_{21}^R]) \\
&= \text{rank}(H_{11}[V_{21}^A V_{21}^R]) \\
\text{rank}(H_{12}V_{22}) &= \text{rank}(H_{12}[V_{22}^Z V_{22}^A V_{22}^R]) \\
&= \text{rank}(H_{12}[V_{22}^A V_{22}^R]) \\
\text{rank}([H_{11} H_{12}]V_{02}) &= \text{rank}([H_{11} H_{12}]V_{02}^R) \\
&= \text{rank}([H_{11} H_{12}]V_{02}^R).
\end{align*}
\]

Furthermore, from equation (2.27), we have
\[
\begin{align*}
[H_{11} H_{12}][V_{21}^A] &= 0 \\
\Rightarrow \quad H_{11}V_{21}^A + H_{12}V_{22}^A &= 0,
\end{align*}
\]
which indicates that the subspace spanned by $H_{11}V_{21}^A$ is aligned with the subspace spanned by $H_{12}V_{22}^A$ at receiver $R_1$. So, we have
\[
\begin{align*}
\text{rank}([H_{11} V_{21}^A H_{12} V_{22}^A]) &= \text{rank}(H_{11}V_{21}^A) \\
&= \text{rank}(H_{12}V_{22}^A).
\end{align*}
\]
One can observe that the nullspace of $H_{11}$ and $H_{12}$ is closely related to the nullspace of $[H_{11} H_{12}]$. In particular, since $H_{11}\Phi(H_{11}) = 0$ and $H_{12}\Phi(H_{12}) = 0$, we have that
\[
\begin{align*}
[H_{11} H_{12}][\Phi(H_{11})] &= 0 \\
[H_{11} H_{12}][0 & \Phi(H_{12})] &= 0,
\end{align*}
\]
which means that the column vectors of \( \begin{bmatrix} \Phi(H_{11}) \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ \Phi(H_{12}) \end{bmatrix} \) are both in $\mathcal{N}([H_{11} H_{12}])$. 
Since beamformer \[
\begin{bmatrix}
V_{21}^A \\
V_{22}^A
\end{bmatrix}
\] is obtained as random linear combinations of the null space basis vectors \(\Phi([H_{11} \ H_{12}])\), the probability that it belongs to the subspace spanned only by column vectors of \[
\begin{bmatrix}
\Phi(H_{11}) \\
0
\end{bmatrix}
\] and \[
\begin{bmatrix}
0 \\
\Phi(H_{12})
\end{bmatrix}
\] is zero. In other words, \([H_{11}V_{21}^A]\) and \([H_{12}V_{22}^A]\) will have full column rank almost surely, since none of the column vectors of \(V_{21}^A\) or \(V_{22}^A\) will be accidentally zero-forced at receiver \(R_1\). This is one benefit of using random linear combinations, as mentioned in Remark 5.

Beamformers \(V_{21}^R\), \(V_{22}^R\) and \(V_{02}^R\) are generated randomly, they will all have full column rank almost surely. Their transformations at the receivers will be linearly independent of each other unless they can’t be. According to constraint (2.34)-(2.38), the total number of beamformers transmitted in any channel is always no greater than the channel dimension, so there will be no loss of column ranks. As a result, we have \(\text{rank}(U_1)\)

\[
= \text{rank}([H_{11}V_{21}^A \ H_{11}V_{21}^R \ H_{12}V_{22}^R \ [H_{11} \ H_{12}][V_{02}^R])
\]

\[= A_1 + (d_{21} - Z_{21} - A_1) + (d_{22} - Z_{22} - A_1)
\]

\[+ (d_{02} - Z_{02})
\]

which proves equation (2.39). Similarly, we have equation (2.40).

Next, consider equation (2.42). We have just shown that sending a symbol of \(W_{21}^Z\) or \(W_{22}^Z\) or \(W_{02}^Z\), or a pair of symbols of \(W_{21}^A\) and \(W_{22}^A\) will consume 1 dimension of the subspace of \([H_{11} \ H_{12}]\). From the dimension of each part given in equations (2.17), (2.18) and (2.20), we have that

\[Z_{21} \leq (M_1 - N_1)^+ \] \hspace{1cm} (2.45)

\[Z_{22} \leq (M_2 - N_1)^+ \] \hspace{1cm} (2.46)

\[Z_{21} + Z_{22} + A_1 + Z_{02} \leq (M_1 + M_2 - N_1)^+, \] \hspace{1cm} (2.47)

which means the total numbers of beamformers do not exceed the dimensions of corresponding nullspaces. Since we generate all the beamformers as random linear combinations of the entire basis of the respective nullspaces, column vectors of \(V_A = \begin{bmatrix}
V_{21}^Z & 0 & V_{21}^A \\
0 & V_{22}^Z & V_{22}^A \\
0 & V_{02}^Z
\end{bmatrix}\) will be linearly
independent of each other almost surely. Meanwhile, they will also be linearly independent of the random column vectors of $V_B = \begin{bmatrix} V_{21}^R & 0 \\ 0 & V_{22}^R \end{bmatrix}$. Since all of these beamformers in $V_A$ and $V_B$ are derived from $[H_{11} H_{12}]$ or generated randomly, they are independent of channel matrix $[H_{21} H_{22}]$. Since $H_{21}$ and $H_{22}$ are both full rank matrices with generic elements, the column vectors of $[H_{21} H_{22}] [V_A V_B]$ will be linearly dependent only if they have to be linearly dependent.

Because we have the constraint (2.35), which indicates $d_{21} + d_{22} + d_{02} + d_1 + d_2 + d_0 \leq N_2$, $[H_{21} H_{22}] [V_A V_B]$ will have rank $d_{21} + d_{22} + d_{02} + d_1 + d_2 + d_0$ almost surely. So, we have equation (2.42). Similarly, we have equation (2.41).

Finally, consider equation (2.43). Since the beamformers associated with $D_1$ are independent of the beamformers associated with $U_1$, the subspace spanned by $D_1$ and the subspace spanned by $U_1$ will be linearly dependent only if they have to be linearly dependent. According to constraint (2.34), rank($D_1$)+rank($U_1$) $\leq N_1$. Consequently, rank([$D_1 U_1$]) will be equal to rank($D_1$)+rank($U_1$) almost surely. So we have equation (2.43). Similarly, we have equation (2.44).

In Lemma 5, we show that if inequalities (2.34)-(2.38) are satisfied, all desired messages will be distinguishable at their respective intended receivers. In other words, DoF tuples that satisfy (2.34)-(2.38) are achievable. In the next section, we explicitly characterize this achievable DoF region.

### 2.6.2 The achievability of inner bound

According to the analysis in Lemma 5 of the precoding scheme described in Section 2.4, we have shown the achievability of the integer-valued points in $\mathbb{D}_{eq}$, which is defined as

$$\mathbb{D}_{eq} \triangleq \{(d_{11}, d_{21}, d_{12}, d_{22}, d_1, d_2, d_{01}, d_{02}, d_0) \in \mathbb{R}_+^E :$$

$$d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} + d_{21} + d_{22} + d_{02} - Z_{21} - Z_{22} - A_1 - Z_{02} \leq N_1$$

$$d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22} + d_{11} + d_{12} + d_{01} \leq N_2 \}$$

(2.48)
\[-Z_{11} - Z_{12} - A_2 - Z_{01} \leq N_2\]  \hspace{1cm} (2.49)
\[d_1 + d_{11} + d_{21} \leq M_1\]  \hspace{1cm} (2.50)
\[d_2 + d_{12} + d_{22} \leq M_2\]  \hspace{1cm} (2.51)
\[d_1 + d_2 + d_0 + d_{01} + d_{02} + d_{11} + d_{21} + d_{12} + d_{22} \leq \min(M_1 + M_2, N_1 + N_2)\]  \hspace{1cm} (2.52)

are satisfied for some
\[\{(Z_{11}, Z_{12}, Z_{21}, Z_{22}, A_1, A_2, Z_{01}, Z_{02}) \in \mathbb{R}_+^A : \]
\[Z_{21} + Z_{22} + A_1 + Z_{02} \leq (M_1 + M_2 - N_1)^+\]  \hspace{1cm} (2.53)
\[Z_{21} \leq (M_1 - N_1)^+\]  \hspace{1cm} (2.54)
\[Z_{22} \leq (M_2 - N_1)^+\]  \hspace{1cm} (2.55)
\[Z_{21} + A_1 \leq d_{21}\]  \hspace{1cm} (2.56)
\[Z_{22} + A_1 \leq d_{22}\]  \hspace{1cm} (2.57)
\[Z_{02} \leq d_{02}\]  \hspace{1cm} (2.58)
\[Z_{11} + Z_{12} + A_2 + Z_{01} \leq (M_1 + M_2 - N_2)^+\]  \hspace{1cm} (2.59)
\[Z_{11} \leq (M_1 - N_2)^+\]  \hspace{1cm} (2.60)
\[Z_{12} \leq (M_2 - N_2)^+\]  \hspace{1cm} (2.61)
\[Z_{11} + A_2 \leq d_{11}\]  \hspace{1cm} (2.62)
\[Z_{12} + A_2 \leq d_{12}\]  \hspace{1cm} (2.63)
\[Z_{01} \leq d_{01}\}\]  \hspace{1cm} (2.64)

where set \(A\) contains all the auxiliary variables. Inequalities (2.53)-(2.64) on the auxiliary variables are obtained from equations (2.17)-(2.22).

To prove the inner bound, we need to find the connection between \(D\) and \(D_{eq}\). Remarkably, it is shown that these two regions are identical. Note that \(D_{eq}\) is obtained from a 17-dimensional polyhedron in \(\mathbb{R}_+^E \times \mathbb{R}_+^A\) defined via 17 inequalities which include eight auxiliary variables. The
problem is to project this polyhedron onto the nine dimensional positive orthant $\mathbb{R}^9_+$. The standard technique to perform this projection is via the Fourier-Motzkin Elimination wherein the auxiliary variables are eliminated one at a time but by creating a large number of inequalities of $O(m^2)$ starting with $m$ inequalities and then eliminating redundant inequalities [3]. Such a technique is clearly infeasible for the size of the problem at hand here. Instead, we use the special structure of the inequalities that define $\mathcal{D}_{eq}$ to prove that it is equivalent to $\mathcal{D}$ in the following lemma.

**Lemma 6.** The 9-dimensional region $\mathcal{D}$ is equal to $\mathcal{D}_{eq}$.

*Proof.* First show any vector in $\mathcal{D}$ is also in $\mathcal{D}_{eq}$, and then show any vector in $\mathcal{D}_{eq}$ is also in $\mathcal{D}$. The detailed proof is given in Appendix A.1. \qed

Thus, we prove that the inner bound $\mathcal{D}_{in} = \text{co}(\mathcal{D} \cap \mathbb{Z}^9_+)$ is achievable.

### 2.6.3 No interference alignment is needed for $W_{01}$ and $W_{02}$

In our precoding scheme, interference alignment is used only among the four private messages. Only zero-forcing is used for the cognitive and common messages $W_{01}$ and $W_{02}$. In this section, we demonstrate why.

Consider $W_{02}$, for instance. If $M_1 + M_2 > N_1$, transmit zero-forcing of $W_{02}$ is possible. We can choose beamformers for $W_{02}$ from the null space $\mathcal{N}([H_{11} H_{12}])$. It is worth noting that $\mathcal{N}([H_{11} H_{12}])$ has already been used to generate $V_{Z21}^Z, V_{Z22}^Z$ and $(V_{A21}^Z, V_{A22}^Z)$ pairs. To transmit a data symbol in $W_{02}$, we cannot choose a vector in the span of the column vectors in

$$
\begin{bmatrix}
V_{21}^Z \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
V_{22}^Z
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
V_{A21}^Z \\
V_{A22}^Z
\end{bmatrix},
$$

otherwise the data symbol of $W_{02}^Z$ will not be distinguishable with part of $W_{21}^Z, W_{22}^Z$ and $(W_{A21}^Z, W_{A22}^Z)$ at receiver $R_2$. As a result, $V_{02}^Z$ can be only chosen from the unoccupied subspace of $\mathcal{N}([H_{11} H_{12}])$. This is also why the dimension available for transmit zero-forcing of $W_{02}^Z$ is at most $M_1 + M_2 - N_1 - d_{21}^Z - d_{22}^Z - A_1$ in equation (2.20).

Next, consider the possibility of aligning the beamformer, denoted as $V_{02}^A$, of data symbol in
$W_{02}$ with the existing interference due to $W_{21}^R$, $W_{22}^R$ or $(W_{21}^A, W_{22}^A)$. Take $(W_{21}^A, W_{22}^A)$ for example. If vector $[H_{11} H_{12}]v_{02}^A$ aligns with $(H_{11}v_{21}^A, H_{12}v_{22}^A)$, where $v_{21}^A$ and $v_{22}^A$ are some column vectors lie in $\text{span}(V_{21}^A)$ and $\text{span}(V_{22}^A)$, respectively, it is easy to see that

$$v_{02}^A = \begin{bmatrix} \alpha v_{21}^A \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta v_{22}^A \end{bmatrix} + \gamma v_0$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $v_0$ is a column vector in the null space $\mathcal{N}([H_{11} H_{12}])$. To make $W_{02}^A$ distinguishable at receiver $R_2$, $v_0$ must be linearly independent of the already used subspace of $\mathcal{N}([H_{11} H_{12}])$.

Hence, if we transmit a data symbol in $W_{02}^A$ by having its direction lie in the subspace spanned by the directions associated with data symbols in $W_{21}^A$ and $W_{22}^A$, we consume one dimension in $\mathcal{N}([H_{11} H_{12}])$. A similar result holds in attempting to align with existing interference $H_{11}V_{21}^R$ or $H_{12}V_{22}^R$ at receiver $R_1$.

In summary, for each $W_{02}$ stream, both transmit zero-forcing and interference alignment consume one more available dimension of $\mathcal{N}([H_{11} H_{12}])$. In other words, either strategy costs the same in terms of using the remaining subspace (if any) of $\mathcal{N}([H_{11} H_{12}])$. As a practical matter, one might choose transmit zero-forcing since it easier to compute the corresponding beamformer.

### 2.7 Some special cases

In this section, we specify the DoF regions for certain special cases of Theorem 2 to illustrate its scope. In Section 2.7.1 we specialize Theorem 2 to three settings for which the DoF regions are known from previous works. In Section 2.7.2 we provide DoF regions for two settings with three messages each that haven’t been considered in the literature before. In Section 2.7.3 we explore the implications of Theorem 1 on the MIMO X-channel in some detail, obtaining novel results on sum-DoF and on the necessity of time extensions in particular settings. In Section 2.7.4 we specialize Theorem 2 to the so-called cognitive MIMO X channel in which one of the four messages is known to both transmitters and explore this DoF region further as in the case of the MIMO X channel.
2.7.1 Known Results as Special Cases

Figure 2.3: Special cases of the MIMO $2 \times 2$ interference networks (a) IC (b) IC-CM (c) cognitive IC

2.7.1.1 Case 1. IC (Figure 2.3.a)

There are only two messages in the interference channel, i.e., $W_{11}, W_{22}$. By eliminating all absent variables in $D$, we get the degrees of freedom region for two-user interference channel as

\[
D_{IC} = \left\{ (d_{11}, d_{22}) \in \mathbb{R}_+^2 : 
\begin{align*}
    d_{11} &\leq \min(M_1, N_1), \\
    d_{22} &\leq \min(M_2, N_2), \\
    d_{11} + d_{22} &\leq \min (\max(M_2, N_1), \max(M_1, N_2))
\end{align*}
\right\}
\]

Hence, Theorem 2 reduces to the well-known result in [4]. We can follow the precoding scheme shown in Section 2.4 and skip the parts that are not applicable. In this case, we only need to consider $[V_{11}^{Z} V_{11}^{R}]$ and $[V_{22}^{Z} V_{22}^{R}]$. Only transmit zero-forcing is possible here. Interference alignment is not applicable since there is only one source of interference at each receiver.
2.7.1.2 Case 2. IC-CM (Figure 2.3.b)

Specializing Theorem 2 to the case where only messages $W_{11}$, $W_{22}$ and $W_0$ are present as depicted in Fig. 2.3.b (and eliminating absent variables), we have

$$
\mathbb{D}_{IC-CM} = \left\{ (d_{11}, d_{22}, d_0) \in \mathbb{R}_+^3 : \\
\quad d_{11} \leq M_1, \quad d_{22} \leq M_2, \\
\quad d_0 + d_{11} \leq N_1, \quad d_0 + d_{22} \leq N_2, \\
\quad d_0 + d_{11} + d_{22} \leq \\
\quad \min(M_1 + M_2, \max(M_2, N_1), \max(M_1, N_2)) \right\}.
$$

In this case, $W_{11}$ and $W_{22}$ are transmitted using the same scheme as in IC along with random beamforming for $W_0$ in the remaining channel dimensions that are still available.

2.7.1.3 Case 3. Cognitive IC (Figure 2.3.c)

Theorem 2, when specialized to the degraded message set depicted in Fig. 2.3.c, results in the DoF region of the Cognitive IC, which is

$$
\mathbb{D}_{co-IC} = \left\{ (d_{01}, d_{22}) \in \mathbb{R}_+^2 : \\
\quad d_{01} \leq N_1, \quad d_{22} \leq \min(M_2, N_2), \\
\quad d_{01} + d_{22} \leq \min(M_1 + M_2, \max(M_2, N_1)) \right\}.
$$

This DoF region matches with the result of [17] in the same cognitive message sharing scenario. In this case, we only need zero-forcing and random beamforming to achieve any vertex of the DoF region. The dimensions of symbols of $W_{01}$ and $W_{22}$ that are transmitted using zero-forcing are $\min(d_{01}, (M_1 + M_2 - N_2)^+)$ and $\min(d_{22}, (M_2 - N_1)^+)$, respectively.
2.7.2 Examples of New Results

**Figure 2.4**: Special cases of the MIMO $2 \times 2$ interference networks (a) generalized cognitive IC (b) BC-PCR

### 2.7.2.1 Case 4. Generalized Cognitive IC (Figure 2.4.a)

Consider the generalized cognitive IC, in which there are three messages $W_{21}$, $W_{01}$ and $W_{22}$.

In this model, the two transmitters send one message each, i.e., $W_{21}$ and $W_{22}$, respectively, to Receiver 2 along with another message, i.e., $W_{01}$, cooperatively to the Receiver 1. Specializing Theorem 2 to this model, we have the following DoF region result

$$
\mathbb{D}_{g-co-IC} = \{(d_{21}, d_{22}, d_{01}) \in \mathbb{R}^3_+ : \\
d_{01} \leq N_1, \ d_{21} \leq M_1, \ d_{22} \leq M_2, \\
d_{21} + d_{22} \leq N_2, \\
d_{01} + d_{21} \leq \max(M_1, N_1), \\
d_{01} + d_{22} \leq \max(M_2, N_1), \\
d_{01} + d_{21} + d_{22} \leq M_1 + M_2 \}.
$$

Both zero-forcing and interference alignment, if possible, are used to mitigate the impact of two private messages $W_{21}$ and $W_{22}$ on their common unintended receiver, i.e., receiver $R_1$; while zero-forcing, if possible, is used to reduce the interference received by receiver $R_2$ due to message $W_{01}$. 
2.7.2.2 Case 5. Broadcast Channel with Partially Cognitive Relay (BC-PCR)
(Figure 2.4.b)

Consider the model depicted in Figure 2.4.b. Transmitter 1 broadcasts two private messages $W_{11}$ and $W_{21}$ to two receivers, respectively, while it simultaneously cooperates with transmitter 2 (the PCR) to send another message $W_{01}$ to receiver $R_1$. From Theorem 2, we can deduce the DoF region of BC-PCR as

$$\mathbb{D}_{BC-PCR} = \{(d_{21}, d_{11}, d_{01}) \in \mathbb{R}^3_+ :$$

$$d_{21} \leq N_2, \quad d_{01} + d_{11} \leq N_1, \quad d_{11} + d_{21} \leq M_1,$$

$$d_{01} + d_{11} + d_{21} \leq \min (M_1 + M_2, \max (M_1, N_1))\}. $$

From the analysis in Section 2.6.3.B.(2), we know that using zero-forcing, if possible, is enough for transmitting message $W_{01}$. There is no need to additionally attempt to align the symbols of $W_{11}$ and $W_{01}$ together at receiver $R_2$, since interference alignment and zero-forcing costs the same in terms of using the null space of $[H_{21} \ H_{22}]$.

2.7.3 Further results on the MIMO $X$ channel

In this section, which can be seen as a continuation of Section 2.3, we discuss several novel observations/results about the MIMO $X$ channel, the DoF region for which was established in Theorem 1.

**Corollary 1.** In the symmetric $(M,M,N,N)$ antenna setting, the maximum sum DoF of the MIMO $X$ channel is given by

$$2M, \quad \text{if} \quad 0 < \frac{M}{N} \leq \frac{2}{3},$$

$$\frac{4N}{3}, \quad \text{if} \quad \frac{2}{3} < \frac{M}{N} \leq 1,$$

$$\frac{4M}{3}, \quad \text{if} \quad 1 < \frac{M}{N} \leq \frac{3}{2},$$

$$2N, \quad \text{if} \quad \frac{3}{2} < \frac{M}{N}$$
Corollary 1 can be obtained by specializing Theorem 1 to the symmetric antenna setting. In terms of sum-DoF performance, there are hence redundant antennas at the transmitters if $\frac{2}{3} < \frac{M}{N} \leq 1$ or $\frac{M}{N} > \frac{3}{2}$, and there are redundant antennas at the receivers if $0 < \frac{M}{N} < \frac{2}{3}$ or $1 \leq \frac{M}{N} < \frac{3}{2}$. In the case that $M = N$, the redundancy exists both at the transmitters and at the receivers. For example, the three antenna settings of $(3,3,3,3)$, $(3,3,2,2)$ and $(2,2,3,3)$ all have the same maximum sum-DoF of 4.

Interestingly, for the equal-antenna case of $(3,3,3,3)$, one can easily achieve the DoF-tuple of $(1,1,1,1)$ by turning off one antenna at each receiver and then transmitting all four symbols of the private messages using zero-forcing beamforming in each of the one-dimensional null space of the remaining channel matrices. No explicit interference alignment is actually needed to achieve the optimal sum-DoF. Given that explicit interference alignment was first discovered in the context of the symmetric three-antenna MIMO $X$ channel as being the key ingredient [8] needed to achieve DoF-optimality, this observation is surprising. To the best of the authors’ knowledge, this is the first time this simple result has been noted. Shutting down the redundant antenna at each receiver could however be seen as implicitly aligning interference in a subspace that would only be seen by that antenna and then discarding that subspace.

**Corollary 2.** For the special cases given in Lemma 1, in which ACS is required to achieve the maximum sum-DoF, the maximum sum-DoF is equal to $C - \frac{2}{3}$, where $C = M_1 + M_2 = N_1 + N_2$. The DoF tuple to achieve the maximum sum-DoF is given by $d_{ij} = \min(M_j, N_i) - \frac{2}{3}$.

**Proof.** We give the proof for the case that $M_1 = \min(M_1, M_2, N_1, N_2) = 1$ here. The other cases follow in the same way.

Since $M_1 = 1$ and $M_1 + M_2 = N_1 + N_2 = C$, we have that $\max(M_1, N_1) = N_1$, $\max(M_1, N_2) = N_2$, $\max(M_2, N_1) = M_2$ and $\max(M_2, N_1) = M_2$. Adding the first 4 inequalities in $D_X$ given by Theorem 1 together, we have that

$$3(d_{11} + d_{12} + d_{21} + d_{22}) \leq N_1 + N_2 + 2M_2 = 3C - 2$$
Thus, the sum-DoF is bounded by $C - \frac{2}{3}$. It is easy to verify that the DoF tuple $d_{ij} = \min(M_j, N_i) - \frac{2}{3}$ ($i, j = 1, 2$) achieves the optimal sum DoF and is within the DoF region $D_X$. Thus, the maximum sum DoF is equal to $C - \frac{2}{3}$.

A symbol extension of length 3, together with ACS, is required to achieve this corner point.

Corollary 3. In the case that only three private messages are transmitted in the channel, all the corner points will be integer-valued. Thus, neither symbol extension nor ACS is necessary.

Proof. Since the channel is isotropic with respect to any message, we can assume without loss of generality that the three private messages are $W_{11}$, $W_{12}$ and $W_{21}$. By deleting $d_{22}$ from the DoF region $D_X$ of Theorem 1 and removing the redundant inequalities, we obtain the following 3-dimensional DoF region.

$$D' = \{(d_{11}, d_{21}, d_{12}) \in \mathbb{R}_+^3 :$$

$$d_{11} + d_{12} + d_{21} \leq \max(M_1, N_1), \quad (2.65)$$

$$d_{11} + d_{12} \leq N_1, \quad (2.66)$$

$$d_{21} + d_{11} \leq M_1 \quad (2.67)$$

$$d_{21} + d_{12} \leq \max(M_2, N_2), \quad (2.68)$$

$$d_{21} \leq N_2, \quad (2.69)$$

$$d_{12} \leq M_2\}. \quad (2.70)$$

Each corner point of this 3-D region will be the intersection of three of the nine facets describing the polytope. Observing the constraints, it is easy to verify that the only possible combination of facets that can have a fractional intersection are (2.66), (2.67) and (2.68), and the corresponding
vertex is

\[
\begin{align*}
  d_{11} &= \frac{M_1 + N_1 - \max(M_2, N_2)}{2} \\
  d_{12} &= \frac{N_1 + \max(M_2, N_2) - M_1}{2} \\
  d_{21} &= \frac{M_1 + \max(M_2, N_2) - N_1}{2},
\end{align*}
\]

These three values will be all integers or all non-integer fractions which are an odd-multiple of \(\frac{1}{2}\).

From constraint (2.65), we have that

\[
\frac{M_1 + N_1 + \max(M_2, N_2)}{2} \leq \max(M_1, N_1).
\]

Otherwise, this corner point will be outside the DoF region. Consequently, one of \(d_{12}\) and \(d_{21}\) will be \(\leq 0\). If it is less than zero, this corner point is outside the DoF region and therefore irrelevant; if it is equal to 0, then the other two values will be integers.

The intersection of all other combinations of facets will be integer-valued, thus, all the corner points of \(\mathbb{D}'\) are integer-valued, and neither symbol extension nor ACS are necessary to achieve them.

Corollary 4. For the MIMO X channel of an arbitrary antenna setting, if there are fractional-valued corner points and symbol extension is required to achieve this corner point, the length of symbol extension will be at most 3.

Proof. Again, each corner point of the 4-dimensional DoF region is the intersection of four of the facets describing the polytope. Since the coefficients of any facet are either 0 or 1, any selected 4-by-4 coefficient matrix will be a binary matrix. According to the Hadamard maximal determinant problem [18], the determinant of an order 4 binary matrix can at most be 3. Consequently, the inverse of any 4-by-4 coefficient matrix, if it exists, can at most have a denominator of 3. Thus, for any non-integer valued corner points, the denominator will be at most 3. Thus, the length of symbol extension will be at most 3.

More specifically in this problem, it is shown that there is only one corner point whose
denominator is 3, and this corner point is the intersection of the four facets corresponding to the 
first four constraints in $D_X$.

\[ \square \]

### 2.7.4 Cognitive MIMO $X$ channel

If one of the four private messages in the MIMO $X$ channel, for example $W_{11}$, is made 
available non-causally at the other transmitter, the channel is named cognitive MIMO $X$ channel. 
It is shown in [8] that the sum DoF of the cognitive MIMO $X$ channel with equal number, $M$, 
of antennas at each terminal is equal to $\frac{3}{2}M$, which is greater than the sum DoF of $\frac{4}{3}M$ of the 
symmetric $X$ channel. So, cognitive message sharing helps increase sum DoF in this case. We 
discuss more general properties of the cognitive MIMO $X$ channel here.

**Theorem 3.** The degrees of freedom region of the cognitive MIMO $X$ channel with message $W_{21}$, 
$W_{12}$, $W_{22}$ and $W_{01}$ is given by

\[
D_{co-X} = \left\{ (d_{01}, d_{21}, d_{12}, d_{22}) \in \mathbb{R}^4_+ : 
\begin{align*}
    d_{01} + d_{12} + d_{21} &\leq \max(M_1, N_1), \\
    d_{01} + d_{12} + d_{22} &\leq \max(M_2, N_1), \\
    d_{21} + d_{22} + d_{12} &\leq \max(M_2, N_2), \\
    d_{01} + d_{12} &\leq N_1, \quad d_{21} + d_{22} \leq N_2, \\
    d_{21} &\leq M_1, \quad d_{12} + d_{22} \leq M_2, \\
    d_{01} + d_{21} + d_{12} + d_{22} &\leq M_1 + M_2
\end{align*}
\right\}
\]

Theorem 3 follows directly from our main result of the 9-dimensional DoF region of the MIMO 
$2 \times 2$ Gaussian interference network with general message sets given in Section 2.4. When the above 
DoF region is specialized to the symmetric, equal-antenna case, all but the first three bounds are 
redundant, and it is easy to see that the DoF-tuple $(d_{01} = M/2, d_{12} = 0, d_{21} = M/2, d_{22} = M/2)$, 
the achievability of which was shown in [8] for $M > 1$ (using two-symbol extensions), is a maximum 
sum-DoF corner point of $D_{co-X}$ for any $M \geq 1$. 

More generally, the DoF region of cognitive MIMO X channel is greater than that of the MIMO X channel. For example, consider the case of $M_1 = 3, M_2 = 4, N_1 = 5, N_2 = 6$. When $d_{12}, d_{21}$ and $d_{22}$ are all set to be equal to 1, $d_{11}$ can be at most 2 in the MIMO X channel, whereas $d_{01}$ can be up to 3 in the cognitive MIMO X channel. Even the cognition of one message among the transmitters can significantly improve the maximum achievable DoF.

**Corollary 5.** In the symmetric $(M,M,N,N)$ antenna setting, the maximum sum DoF of the cognitive MIMO X channel is given by

$$\begin{align*}
2M, & \quad \text{if } 0 < \frac{M}{N} \leq \frac{3}{4} \\
\frac{3N}{2}, & \quad \text{if } \frac{3}{4} < \frac{M}{N} \leq 1 \\
M + \frac{N}{2}, & \quad \text{if } 1 < \frac{M}{N} \leq \frac{3}{2} \\
2N, & \quad \text{if } \frac{3}{2} < \frac{M}{N}
\end{align*}$$

Corollary 5 can be obtained from Theorem 3. Comparing with the result of MIMO X channel, the sum DoF of the cognitive MIMO X is strictly greater than that of the MIMO X when $\frac{2}{3} < \frac{M}{N} < \frac{3}{2}$. When $\frac{M}{N} \leq \frac{2}{3}$ or $\frac{M}{N} \geq \frac{3}{2}$, there are redundant antennas at the transmitters or the receivers, and message cognition does not help in improving the sum DoF of the system.

**Corollary 6.** In the case that $M_1 + M_2 = N_1 + N_2$ and $\min(M_1, M_2, N_1, N_2) = 1$, among linear strategies, ACS is required to achieve the DoF region of the cognitive MIMO X channel.

**Proof.** The reason that ACS is necessary for the cognitive MIMO X channel is the same as the reason it is necessary for the MIMO X channel in Lemma 1. We omit the details for brevity.

**Corollary 7.** For the special cases given in Corollary 6, in which ACS is required to achieve the maximum sum-DoF of the cognitive MIMO X channel, the maximum sum-DoF is equal to $C - \frac{1}{2}$, where $C = M_1 + M_2 = N_1 + N_2$. The DoF tuple to achieve the maximum sum-DoF is given by $(d_{01}, d_{21}, d_{12}, d_{22}) = (\min(M_1, N_1) - \frac{1}{2}, \min(M_1, N_2) - \frac{1}{2}, \min(M_1 + M_2, N_1) - \min(M_1, N_1), \min(M_2, N_2) - \frac{1}{2})$ or $(\min(M_1 + M_2, N_1) - \frac{1}{2}, \min(M_1, N_2) - \frac{1}{2}, 0, \min(M_2, N_2) - \frac{1}{2})$. 


Proof. Adding the 1st, 2nd and 5th inequalities in $D_{co-X}$ together, we have that $2d_{sum} \leq \max(M_1, N_1) + \max(M_2, N_1) + N_2$, which is always equal to $2C - 1$. Thus, the sum DoF is upper bounded by $C - \frac{1}{2}$. One can easily verify that the two given DoF tuples are both within $D_{co-X}$ and achieve the maximum sum-DoF.

A symbol extension of length 2, together with ACS, is required to achieve this corner point.

There can be two non-integer-valued corner points which achieve the maximum sum-DoF. However, when $\min(M_1 + M_2, N_1) = \min(M_1, N_1)$, or equivalently $M_1 \geq N_1$, these two corner points are the same. If they are different, we can get one of them from the other by just regarding the non-zero $d_{12}$ symbols of message $W_{12}$ as part of message $W_{01}$.

**Corollary 8.** For the cognitive MIMO X channel of arbitrary antenna setting, if there are any fractional-valued corner point and symbol extensions are required to achieve this corner point, the length of symbol extension will be at most 2.

**Proof.** Although the determinant of an arbitrary 4-by-4 binary matrix can be at most 3, it is easy to verify that the maximum determinant of any 4-by-4 coefficient matrix generating from any four facets given in $D_{co-X}$ is equal to 2. Thus, the length of symbol extension will be at most 2.

### 2.8 Conclusion

The degrees of freedom region for the nine-message MIMO $2 \times 2$ interference network is established. Each of the nine messages is uniquely identified based on the transmitter(s) it is known to and the receiver(s) at which it is desired and therefore include broadcast/multiple-access/multicast/cognitive/common messages. The DoF region for a setting that involves any subset of the nine messages can thus be derived as a special case. In particular, the DoF region of the MIMO X channel, a problem that remained open despite previous studies, is completely settled.
The achievability scheme uses (a) transmit zero-forcing, a well-known technique known to be sufficient for the MIMO IC [3], interference alignment and symbol extensions the necessity (but not sufficiency) for which was discovered in the context of the constant-coefficient MIMO X channel in [8], and finally, asymmetric complex signaling which was discovered in the context of the constant-coefficient SISO X channel in [9], but whose benefit (necessity or sufficiency) in the MIMO (i.e., non-SISO) X channel remained unclear despite [8, 9]. The achievability scheme in this chapter combines the principles of transmit zero-forcing, interference alignment, symbol extensions and ACS in a novel way that allows not only the complete characterization of the DoF of the four-message MIMO X channel – thereby proving that they are both necessary and sufficient in general for the constant-coefficient MIMO X channel – but also the precise DoF region of the much more general nine-message, constant-coefficient MIMO 2 × 2 network considered in this chapter.

In considering some interesting subsets of the general message set (including the 9-message case) for the 2 × 2 MIMO interference network, and making simplifying assumptions on the channel models if needed, future work could include the discovery of new encoding and decoding principles inspired by the goal of characterizing information theoretic metrics that are finer than the degrees of freedom, such as, for instance, the generalized degrees of freedom, as was done for the two-user MIMO interference channel in [5]. There is also the potential for the discovery of hitherto unknown encoding schemes tailored for various models of channel uncertainty, as has been done for the MIMO interference and the MIMO X channels in [19, 20] under delayed CSIT.
Chapter 3

Matrix Decomposition and Linear Degrees of Freedom of the 3-user MIMO Broadcast Channel with General Message Sets

3.1 Introduction

The multiple-input multiple-output (MIMO) broadcast channel (BC) with only private messages has been studied extensively during the past several decades in [21, 22, 23] etc. A simple extension of the broadcast channel with only private messages is to add an extra common message which is desired by all receivers. The two-user broadcast channel with common message (BC-CM) is studied in [24] and [25] under the prefect and hybrid channel state information at the transmitter (CSIT) assumption, respectively. Furthermore, if we restrict that the private message be kept perfectly secret from the other user, the problem turns to the broadcast channel with common and confidential messages. The two-user case is studied in [26] and the three-user case in studied in [27]. Although the physical structures of the channels are not changed in these problems, associating different possible message sets with the model helps us to explore the system in different perspectives and understand the fundamental characteristics of the system more thoroughly.

There are also other examples of general message sets considered in the literature. The author of [28] considers the $K \times J$ interference network, in which each transmitter has one and only one independent message, and each receiver can request an arbitrary set of messages from multiple transmitters. The message amount is fixed, however, there are various match-up possibilities, which gives the problem lots of diversity. The X network is another example of more general message demands [29]. Although only private messages are considered, there is independent message
transmitting from each transmitter to each receiver, which makes the communication more flexible than traditional one-one mapping mode, such as the interference channel (IC) [3]. Even more general cases are that to consider all the possible messages that can be transmitted via a certain networks. For example, [30] studies the $K$-user multiple access channel (MAC) with arbitrary sets of common messages, whose amount increases exponentially with $K$. In [31, 32], the author studies the degrees of freedom (DoF) region of the MIMO $2 \times 2$ interference network with all 9 possible messages, from which all possible communication scenarios based on $2 \times 2$ interference network can be regarded as special cases. The study of general message sets allows a more flexible way to design communication systems and handle the complex signaling requirements that are necessary in modern applications.

The broadcast channel with general message sets is also referred to as multicast channel in the literature. In [33, 34], the multicast achievable rate region of broadcast channel is studied in the deterministic channel model. However, instead of directly finding the overall achievable rate region, the intrinsic rate region is given. More specifically, given an achievable rate tuple, the intrinsic achievable rate region generated from this rate tuple is characterized. This intrinsic rate region of multicast channel is further studied in [35] and referred to as the latent capacity region. These works shed light on the understanding of the structure of the channel and capturing underlying constraints. However, since the intrinsic or latent capacity region relies on the generator rate tuple or the seed, it is still not clear what the exact rate region is.

In this chapter, we consider the optimal linear coding scheme for the 3-user MIMO broadcast channel with general message sets. The transmitter has $m$ antennas and could have independent message intended for any subset of the three receivers. For example, message $W_1$ is intended for receiver 1, message $W_{23}$ is required by both receiver 2 and receiver 3, message $W_{123}$ is desired by all three receivers, etc. Based on different combinations of destinations, there can be at most $2^3 - 1 = 7$ independent messages in the 3-user network. We first provide an outer bound on the 7-dimensional degrees of freedom region of the system under the restriction of linear encoding strategies. Then, the outer bound is shown to be tight using our proposed encoding schemes.
The basic principle of any successful coding scheme is to try to maximally eliminate the interference at the receivers and reserve as many channel resources as possible to the desired messages. The first technique to be considered is zero-forcing. By picking a beamformer from the nullspace of a channel, the beamformer will be zero-forced at the corresponding receiver and thus bring no interference to it. For example, if we could transmit a symbol of $W_1$ via the nullspace of concatenated channel to receiver $R_2$ and $R_3$, then this symbol will be zero-forced at receiver $R_2$ and $R_3$ simultaneously and thus introduce no interference to the system. Similarly, we would like to send a $W_{23}$ symbol via beamformers chosen from the nullspace of the channel to receiver $R_1$ such that it is only received by receiver $R_2$ and $R_3$, but zero-forced at receiver $R_1$. Zero-forcing is a non-interference introducing technique and is very efficient in interference elimination. Besides zero-forcing, another popular interference elimination technique in the literature is interference alignment. In the case that the interferences are inevitable at a receiver, if we could let the interferences be mapped to the same subspace at the receiver, i.e., align their images at the receiver together at the same subspace, they would occupy and waste fewer dimensions. Interference alignment is shown to be a very useful technique in communication networks such as interference channels [36], however, no current result about broadcast channel has even used interference alignment as an interference elimination technique. It is worth noting that both the techniques of zero-forcing and interference reference utilize the nullspaces of the channels or concatenated channel such that the dimension of the overall interference at the receivers can be reduced.

Based on this fact, in the proof of achievability, we introduce the idea of channel decomposition and messages splitting. We begin with studying the various nullspaces of the 3-user broadcast channel and decompose the entire $m$-dimensional transmit beamformer vector space into nine\(^1\) linearly independent subspaces based on their different impacts on the three receivers. Beamformers chosen from each subspace are only received by a subset of receivers. Then, by systematically assigning the beamformers from different subspaces to corresponding best matching messages, the dimension of overall interference subspace at each receiver is minimized via the zero-forcing and

\(^1\) The dimensions of certain subspaces can be zero, and not all nine subspaces need to exist at the same time.
interference alignment techniques. By doing channel decomposition, we study the intrinsic properties of the beamformer subspaces and do the subspace resource identification of the system. Then, based on the similarity of the subspaces, we divide them into several groups, such that the images of the groups at the receivers are also linearly independent with each other. Consequently, we decompose the entire system to multiple parallel subsystems, the achievable region of which are much easier to analyze due to their special properties. Then, we split the message to sub messages transmitted parallelly in the sub systems. In this way, we obtain an inner bound on the achievable region of the entire system as the Mincowskii-sum of the achievable regions of the sub-channels. The proposed coding scheme associated with the decomposed beamformer subspaces is shown to be optimal in the symmetric antenna setting, i.e., all three receivers have equal number, $n$, of antennas. More specifically, an outer bound on the linear degrees of freedom region of the 3-user MIMO broadcast channel with general message sets is first given and then shown to be tight using proposed coding scheme. For the asymmetric antenna setting, i.e., three receivers have $n_1$, $n_2$ and $n_3$ antennas, respectively, the general outer bound becomes much more involved and are not fully described yet. It is also relatively harder to provide a closed form achievable LDoF region using our proposed scheme. However, once a DoF tuple is identified to be within the LDoF region for arbitrary antenna setting, one can readily construct the beamformers for each message from the decomposed beamformer subspace to achieve the specific DoF tuple.

The rest of this chapter is organized as follows. In Section 3.2, the necessary mathematical preliminaries about vector space analysis and matrix manipulations are introduced. The system model of the 3-user MIMO broadcast channel with general message is also defined. Then, we introduce the idea of channel decomposition using the example of two-user broadcast channel. The overview of main results as well as the converse proof is given in Section 3.3. The decomposition of beamformer vector space is introduced in Section 3.4. A further illustration of the different beamformer types and the associated precoding scheme are given in Section 3.5. The achievability proof of the outer bound is given in Section 3.6. Finally, we give several example of how to allocate the beamformers to achieve certain DoF tuples in Section 3.7.
3.2 Mathematical preliminaries and notations

In this section, we give a brief introduction of the key mathematical preliminaries that will be used in the calculations in our precoding scheme. Since we focus on linear coding scheme, vector space analysis and matrix manipulation are the main techniques closely related to our scheme. We denote a vector spaces or sets of vectors by Calligraphic letters (e.g. $V$, $U$), matrices by upper case letters (e.g. $V$, $U$), column vectors by bold lower case letters (e.g. $v$, $u$). For any scalar $x$, $(x)^+$ denotes the greater one between $x$ and 0, i.e., $\max(x, 0)$. Span($V$) denotes the subspace spanned by the column vectors of matrix $V$. $V^\dagger$ denotes the conjugate transpose of matrix $V$.

3.2.1 Definitions

Here, we list several fundamental definitions in the topic of vector space and linear transformation, which are frequently used throughout this chapter.

- A **basis** in a vector space $V$ is a set $\mathcal{X}$ of linearly independent vectors such that every vector in $V$ is a linear combination of elements of $\mathcal{X}$. A vector space $V$ is finite-dimensional if it has a finite basis. We denote the dimension of a vector space $V$ by $\dim(V)$.

- A **basis-matrix** of a vector space $V$ is a matrix whose column vectors together form a basis of $V$.

- If $S$ is any set of vectors in a vector space $V$, then the **subspace**, $\mathcal{M}$, spanned by $S$ is the set of all linear combinations of elements of $S$. If all elements of $S$ are linearly independent with each other, then $S$ is a basis of subspace $\mathcal{M}$.

- Let $\mathcal{V}$ be a vector space and $\mathcal{M}$ be a subspace of $\mathcal{V}$, then, the **orthogonal complement** of $\mathcal{M}$ in $\mathcal{V}$ is the set of vectors $\mathbf{z}$ such that $\mathbf{z}$ is orthogonal to all vectors in $\mathcal{M}$. We denote it by $\mathcal{M}_\mathcal{V}^\perp$. Let $\mathcal{V}$ and $\mathcal{M}$ be a basis-matrix of vector space $\mathcal{V}$ and $\mathcal{M}$, respectively, we denote a basis-matrix of $\mathcal{M}_\mathcal{V}^\perp$ by $M_\mathcal{V}^\perp$. 
• The intersection of two subspaces \( U, V \) of \( W \) is the set, denoted by \( U \cap V \), consisting of all the vectors \( x \), such that \( x \in U \) and \( x \in V \). If \( U \cap V = \{0\} \), where \( \{0\} \) denotes the set contains only the all-zero vector, then \( U \) and \( V \) are said to be linearly independent with each other.

• Let \( U \) and \( V \) be two subspaces of \( W \), we say \( U \) and \( V \) are orthogonal to each other, if for any vector \( u \) and \( v \) chosen from \( U \) and \( V \), respectively, we have that \( u \) and \( v \) are orthogonal. We denote it as \( U \perp V \), or equivalently, \( U \perp V \).

• The sum of two subspaces \( U, V \) of \( W \) is the set, denoted by \( U + V \), consisting of all the vectors \( u + v \), for any \( u \in U \), \( v \in V \). For brevity, we use operator \( \sum \) to denote the sum of multiple subspaces. For example, \( \sum_{i=1,2,3} V_i = V_1 + V_2 + V_3 \). If here \( U \) and \( V \) are linearly independent, we say \( W \) is the direct sum of \( U \) and \( V \), denoted as \( U \oplus V \).

• If \( A \) is a linear transformation on a vector space \( V \), the range of \( A \) is the set \( \mathcal{R}(A) = AV \), and the null-space of \( A \) is the set \( \mathcal{N}(A) \) of all vectors \( x \in V \) for which \( Ax = 0 \). We denote a basis-matrix of the null-space of \( A \) as \( N(A) \).

• Let \( V \) a set of vectors, the image of \( V \) through transformation \( A \) is define as the set of vectors \( \{y | y = Ax, x \in V\} \).

• The rank, \( \text{Rank}(A) \), of a linear transformation \( A \) on a finite-dimensional vector space is the dimension of \( \mathcal{R}(A) \), and the nullity, \( \text{Nullity}(A) \), is the dimension of \( \mathcal{N}(A) \).

### 3.2.2 Theorems

Here, we list some basic theorems in vector space analysis without providing proofs, since they are all easy to obtain. Readers can refer to [37] for more background knowledge.

• Any subspace of a vector space is also a vector space.

• The range and null-space of a linear transformation are both vector spaces.
• If $V$ is a basis-matrix of vector space $V$, we have that $V = \text{Span}(V)$, and for each vector $x \in V$, there exist an unique $n \times 1$ vector $y$ such that $x = V \cdot y$.

• Given any $m$-dimensional subspace $M$ in an $n$-dimensional vector space $V$, we can find a basis $\{x_1, \cdots, x_m, x_{m+1}, \cdots, x_n\}$ in $V$ such that $x_1, \cdots, x_m$ are in $M$ and forms a basis of $M$, and $x_{m+1}, \cdots, x_n$ are in $M_V^\perp$ and forms a basis of $M_V^\perp$. It is always the case that $\dim(M) + \dim(M_V^\perp) = \dim(V)$.

• The sum, $U + V$, of two subspaces $U, V$ of $W$ is also a subspace of $W$. Let $U = \text{Span}(U)$ and $V = \text{Span}(V)$, then $U + V = \text{Span}([U \ V])$.

• If $A$ is a linear transformation on an $n$-dimensional vector space, then $\text{Rank}(A) + \text{Nullity}(A) = n$.

• The rank of a generic transformation matrix $A_{m \times n}$ is equal to $\min(m, n)$, while its nullity is $\max(0, n - m)$.

3.2.3 Useful matrix manipulation algorithms

In this section, we introduce some useful matrix or vector space manipulations in linear coding scheme. It is worth noting that there are different approaches to solve each of these problems and we only provide one feasible way here.

3.2.3.1 Algorithm 1: Computing a basis matrix for the null-space of a linear transformation

Let $A_{n \times m}$ ($n < m$) be a linear transformation on vector space $C^m$, and let $A$ have full row rank, i.e., the row vectors of $A$ are linearly independent with each other. It’s easy to see that the rank of $A$ is equal to $n$ and the nullity of $A$ is equal to $m - n$. In order to obtain a basis of $N(A)$, we can do a singular value decomposition (SVD) of matrix $A$ while arranging the singular values in non-increasing order. Then, the last $m - n$ right-singular column vectors, which are corresponding to singular value 0, will form a basis of $N(A)$. 
3.2.3.2 Algorithm 2: Computing a basis matrix of the intersection of two subspaces

Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be two intersecting subspaces of a finite vector space \( \mathcal{V} \), \( M_1 \) and \( M_2 \) be the basis matrices of \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), respectively. Suppose a non-zero vector \( v \in \mathcal{M}_1 \cap \mathcal{M}_2 \), we have \( v \in \mathcal{M}_1 \) and \( v \in \mathcal{M}_2 \). Then, we can express \( v \) in both forms of \( v = M_1 \cdot x \) and \( v = -M_2 \cdot y \), where \( x, y \) are two non-zero column vectors. Consequently, we have \( [M_1 \ M_2] \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \), which means the concatenated column vector \( \begin{bmatrix} x \\ y \end{bmatrix} \) is in the null-space of \( [M_1 \ M_2] \). Using the method provided in Section 3.2.3.1, we can obtain a basis of \( \mathcal{N}([M_1 \ M_2]) \). Construct a matrix \( X \) such that its column vectors have length of \( \text{rank}(M_1) \) and are equal to the top part of the basis vectors of \( \mathcal{N}([M_1 \ M_2]) \), respectively. Then, the column vectors of \( M_1 \cdot X \) will form a basis of \( \mathcal{M}_1 \cap \mathcal{M}_2 \).

3.2.3.3 Algorithm 3: Computing a basis matrix of the orthogonal complement of a subspace in a vector space

Let \( \mathcal{M} \) be a subspace in the finite-dimensional vector space \( \mathcal{V} \), and let matrix \( M \) be a basis matrix of \( \mathcal{M} \). By definition, the orthogonal complement of \( \mathcal{M} \) in \( \mathcal{V} \), namely \( \mathcal{M}_\perp \), is equal to the set \( \{ v | M^\dagger v = 0, v \in \mathcal{V} \} \). Since the set \( \{ v | M^\dagger v = 0 \} \) is equal to the nullspace of transformation \( M^\dagger \), i.e., \( \mathcal{N}(M^\dagger) \). We have that \( \mathcal{M}_\perp = \mathcal{N}(M^\dagger) \cap \mathcal{V} \). Consequently, we can easily obtain the orthogonal complement of \( \mathcal{M} \) in \( \mathcal{V} \) by finding the intersection of \( \mathcal{N}(M^\dagger) \) and \( \mathcal{V} \).

3.2.4 System model

In this chapter, we consider the 3-user broadcast channel, in which the transmitter has \( m \) antennas and the receivers have \( n_1, n_2 \) and \( n_3 \) antennas, respectively. We denote the channel between the transmitter and the \( i \) th (\( i = 1, 2, 3 \)) receiver as the \( n_i \times m \) complex matrix \( H_i \), and
for the sake of simplicity, we also define

\[ H_{123} = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \]

and

\[ H_{ij} = \begin{bmatrix} H_i \\ H_j \end{bmatrix}, \]

where \( i, j = 1, 2, 3 \) and \( i \neq j \). We assume all channels to be generic, i.e., all the channel coefficient values are drawn independently from a continuous probability distribution. The channel can be either constant or time-varying over the duration of communication, and we assume perfect channel state information at the transmitter (CSIT). The received signal at receiver \( i \) is given by \( Y_i = H_i S + Z_i \), where \( S \in \mathbb{C}^{m \times 1} \) is the \( m \times 1 \) input vector at the transmitter, \( Z_i \) is the \( n_i \times 1 \) complex additive white Gaussian noise (AWGN) vector at receiver \( i \).

Under the perfect CSIT setting, without loss of generality, we assume in this chapter that \( m \leq n_1 + n_2 + n_3 \), since one can easily observe that the transmitter will have redundant antennas if \( m > n_1 + n_2 + n_3 \), and the cut-set bound will be optimal. We also assume in this chapter that \( m \geq \max(n_1, n_2, n_3) \), otherwise one or more receivers have redundant antennas. For example, if receiver 1 has more antennas than the transmitter does, i.e., \( n_1 > m \), it can turn off \( n_1 - m \) antennas and still be able to decode all the messages the transmitter sends. In other words, the degrees of freedom region of the system is unchanged even if we reduce \( n_1 \) to be equal to \( m \). In this chapter, we focus on the non-trivial case that \( \max(n_1, n_2, n_3) \leq m \leq n_1 + n_2 + n_3 \).
General message sets are considered in the chapter, in which there can be independent messages intended simultaneously for any subset $S \subseteq \{1, 2, 3\}$ of the 3 users. Classified by the different destinations, all the possible messages of the 3-user system can be divided to 7 types. As it is shown in Figure 3.1, $W_S (S \subseteq \{1, 2, 3\})$ denotes the message intended for all receivers in $S$, where we abbreviate $\{1, 2, 3\}$ as 123, $\{1, 2\}$ as 12, and so on so forth.

We restrict ourselves to linear coding strategies as defined in [25, 38, 39], in which the degrees of freedom simply indicates the dimension of the linear subspace of transmitted signal. More specifically, consider a linear coding scheme with block length $T$. The transmitter will send out $m_s(T)$, where $s \in E$ and $E = \{1, 2, 3, 12, 23, 13, 123\}$, independent symbols of message $W_s$ in each block. The equivalent overall channel matrix from the transmitter to receiver $r$ ($r = 1, 2, 3$) will be the block diagonal matrix given by

$$H_r^{(T)} = \begin{bmatrix}
H_r(1) & 0 & \cdots & 0 \\
0 & H_r(2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_r(T)
\end{bmatrix},$$

where $H_r(t) (t = 1, \ldots, T)$ is the channel matrix from the transmitter to receiver $r$ at time slot $t$. 
The equivalent overall precoding matrix of message $W_s$ of the entire block is given by

$$U_s^{(T)} = \begin{bmatrix}
U_s(1) \\
U_s(2) \\
\vdots \\
U_s(T)
\end{bmatrix},$$

where $U_s(t)$ $(t = 1, ..., T)$ denotes the precoding matrix for message $W_s$ at time slot $t$ in the block.

The column size of matrix $U_s^{(T)}$ is equal to $m_s^{(T)}$, i.e., the number of independent information symbols of message $W_s$ that will be transmitted in the entire $T$ time slots. The signal transmitted by the transmitter at time slot $t$ can be written as $S(t) = \sum_{s \in E} U_s(t)x_s^{(T)}$, where $x_s^{(T)} \in \mathbb{C}^{m_s^{(T)} \times 1}$ contains the entire $m_s^{(T)}$ information symbols.

At receiver 1, for example, the corresponding signal subspace will be $\text{Span}(H_1^{(T)}[U_1^{(T)} U_{12}^{(T)} U_{13}^{(T)} U_{123}^{(T)}])$, the interference subspace will be $\text{Span}(H_1^{(T)}[U_2^{(T)} U_3^{(T)} U_{23}^{(T)}])$. In order to decode the information symbols correctly, it requires that the signal subspace and interference subspace be linearly independent with each other and the signal subspace reserves the full column rank. In other words, the following two constraints need to be satisfied for receiver 1.

$$\text{Rank}(H_1^{(T)}U_{all}) = \text{Rank}(H_1^{(T)}[U_1^{(T)} U_{12}^{(T)} U_{13}^{(T)} U_{123}^{(T)}]) + \text{Rank}(H_1^{(T)}[U_2^{(T)} U_3^{(T)} U_{23}^{(T)}])$$ (3.1)

$$\text{Rank}(H_1^{(T)}[U_1^{(T)} U_{12}^{(T)} U_{13}^{(T)} U_{123}^{(T)}]) = m_1^{(T)} + m_{12}^{(T)} + m_{13}^{(T)} + m_{123}^{(T)}$$ (3.2)

where $U_{all} = [U_1^{(T)} U_2^{(T)} U_3^{(T)} U_{12}^{(T)} U_{13}^{(T)} U_{23}^{(T)} U_{123}^{(T)}]$. Similarly, we have the following constraints at receiver 2 and 3.

$$\text{Rank}(H_2^{(T)}U_{all}) = \text{Rank}(H_2^{(T)}[U_2^{(T)} U_{12}^{(T)} U_{23}^{(T)} U_{123}^{(T)}]) + \text{Rank}(H_2^{(T)}[U_1^{(T)} U_3^{(T)} U_{13}^{(T)}])$$ (3.3)

$$\text{Rank}(H_2^{(T)}[U_2^{(T)} U_{12}^{(T)} U_{23}^{(T)} U_{123}^{(T)}]) = m_2^{(T)} + m_{12}^{(T)} + m_{23}^{(T)} + m_{123}^{(T)}$$ (3.4)

$$\text{Rank}(H_3^{(T)}U_{all}) = \text{Rank}(H_3^{(T)}[U_3^{(T)} U_{13}^{(T)} U_{23}^{(T)} U_{123}^{(T)}]) + \text{Rank}(H_3^{(T)}[U_1^{(T)} U_2^{(T)} U_{12}^{(T)}])$$ (3.5)

$$\text{Rank}(H_3^{(T)}[U_3^{(T)} U_{13}^{(T)} U_{23}^{(T)} U_{123}^{(T)}]) = m_3^{(T)} + m_{13}^{(T)} + m_{23}^{(T)} + m_{123}^{(T)}$$ (3.6)
Based on this setting, we now define the linear degrees of freedom of 3 user MIMO broadcast channel with general message sets.

**Definition 1.** The DoF 7-tuple \((d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123})\) is linearly achievable if there exists a sequence of linear encoding strategies with block length of \(T\), such that for each \(T\) and the choice of \(m_s^{(T)}(s \in E), U_s^{(T)}\) satisfy the decodability condition of (3.1) - (3.6) with probability 1, and

\[
d_s = \lim_{T \to \infty} \frac{m_s^{(T)}}{T}
\]

holds for all \(s \in E\). We also define the linear degrees of freedom region, \(\mathbb{D}\), as the closure of the set of all achievable 7-tuples.

### 3.2.5 Channel decomposition of 2-user BC

In this section, we use a simple example to introduce the idea of channel decomposition and show how it may be helpful in analyzing the achievable DoF/LDoF region of our problem.

Consider the 2-user MIMO broadcast channel with only two private messages \(W_1\) and \(W_2\). The transmitter has \(m\) antennas and the two receivers have \(n_1\) and \(n_2\) antennas, respectively. Similarly, we assume that \(m \leq n_1 + n_2\), otherwise the transmitter has redundant antennas and the problem is trivial. When perfect channel state information is available at the transmitter, the DoF region of the system is well-known to be given by

\[
\mathbb{D}_{2BC} = \{(d_1, d_2) \in \mathbb{R}_+^2 : d_1 \leq n_1, \quad d_2 \leq n_2, \quad d_1 + d_2 \leq m\}
\]

The converse proof is trivial since all constraints in \(\mathbb{D}_{2BC}\) are cut-set bounds. The proof of achievability is also not hard. However, here we introduce a novel approach which uses the idea of channel decomposition.
Since the transmitter has \( m \) antennas, the beamformer space at the transmitter is a \( m \)-dimensional vector space, namely, \( \mathcal{M} \). We decompose \( \mathcal{M} \) into two linearly independent subspaces as follows.

First, consider the nullspaces of channel \( H_1 \) and \( H_2 \). Since we assume \( m \leq n_1 + n_2 \), we have that \( \mathcal{N}(H_1) \) is linearly independent with \( \mathcal{N}(H_2) \) almost surely, otherwise there will exist a beamformer that can be zero-forced at both receivers. We define the first subspace as \( \mathcal{V}_A \), and let \( \mathcal{V}_A \) be the direct sum of nullspaces \( \mathcal{N}(H_1) \) and \( \mathcal{N}(H_2) \), i.e., \( \mathcal{V}_A = \mathcal{N}(H_1) \oplus \mathcal{N}(H_2) \). The rank of \( \mathcal{V}_A \) is equal to \( r_A = \text{Rank}(\mathcal{N}(H_1)) + \text{Rank}(\mathcal{N}(H_2)) = (m - n_1)^+ + (m - n_2)^+ \). Among these \( r_A \) beamformers, there are \( (m - n_1)^+ \) beamformers that can only be received by receiver \( R_2 \), thus we can use them to transmit message \( W_2 \) without bringing interference to receiver \( R_1 \). The rest \( (m - n_2)^+ \) beamformers can only be received by receiver \( R_1 \), and we can use them to transmit message \( W_1 \) without bringing interference to receiver \( R_2 \). Consider the case that we only communicate through subspace \( \mathcal{V}_A \). We can at most send \( (m - n_2)^+ \) symbols of \( W_1 \) and \( (m - n_1)^+ \) symbols of \( W_2 \). In other words, we can achieve the following DoF region

\[
\mathbb{D}_A = \{(d_1, d_2) \in \mathbb{R}_+^2 : \\
\quad d_1 \leq (m - n_2)^+ \\
\quad d_2 \leq (m - n_1)^+ \}. 
\]

Figure 3.2: \( \mathcal{M} \) is decomposed to two linearly independent parts: \( \mathcal{V}_A \) and \( \mathcal{V}_B \), where \( \mathcal{V}_A = \mathcal{N}(H_1) \oplus \mathcal{N}(H_2) \)
Next, we define the second subspace as $\mathcal{V}_B$, and let $\mathcal{V}_B$ be a randomly generated $r_B = m - r_A$ dimensional subspace in $\mathcal{M}$. Since the total rank of $\mathcal{V}_A$ and $\mathcal{V}_B$ is equal to $M$, which is no greater than the available dimensions at the transmitter, $\mathcal{V}_B$ will be almost surely linearly independent with $\mathcal{V}_A$. In other words, we have $\mathcal{V}_A \oplus \mathcal{V}_B = \mathcal{M}$ as is shown in Figure 3.2. Since $r_B \leq n_1$ and $r_B \leq n_2$, each of the $r_B$ random beamformers in $\mathcal{V}_B$ is almost surely received by both receivers, and thus can be used to transmit either a $W_1$ symbol or a $W_2$ symbol while causing interference to the unintended receiver. Consider the case that we only communicate through subspace $\mathcal{V}_B$. We can at most send $r_B$ symbols of messages, with each symbol be either $W_1$ or $W_2$. In other words, we can achieve the following DoF region

$$\mathbb{D}_B = \{(d_1, d_2) \in \mathbb{R}^2_+ : d_1 + d_2 \leq m - (m - n_1)^+ - (m - n_2)^+\}.$$ 

![Figure 3.3: Minkowski-sum of region $\mathbb{D}_A$ and $\mathbb{D}_B$ is equal to $\mathbb{D}_{2BC}$](image)

Since $r_B + (m - n_2)^+ \leq n_1$ and $r_B + (m - n_1)^+ \leq n_2$, the image of $\mathcal{V}_B$ will almost surely be linearly independent to the image of $\mathcal{N}(H_2)$ at receiver $R_1$, and also almost surely be linearly independent to the image of $\mathcal{N}(H_1)$ at receiver $R_2$. In other words, the communication through subspace $\mathcal{V}_1$ and $\mathcal{V}_2$ are independent with each other and can be proceeded independently. The system works as two parallel sub systems. Consequently, the Minkowski-sum of region $\mathbb{D}_A$ and $\mathbb{D}_B$ is achievable by the entire system, which can be calculated as

$$\mathbb{D}_A \oplus \mathbb{D}_B = \{(d_1, d_2) \in \mathbb{R}^2_+ : d_1 \leq n_1, \quad d_2 \leq n_2\}.$$
\[ d_2 \leq n_2, \]
\[ d_1 + d_2 \leq m \}

and is exactly equal to outer bound \( \mathbb{D}_{2BC} \). In other words, transmitting parallelly in the two subspaces is DoF-optimal for the two-user MIMO Broadcast channel. An illustration of \( \mathbb{D}_A \) and \( \mathbb{D}_B \), as well as their Minkowski-sum, is given in Figure 3.3.

Remark 6. By channel decomposition, the entire transmitter beamformer space is partitioned into several linearly independent subspaces, each of which has special properties and easier to analyze. The Minkowski-sum of these achievable sub-regions provides an inner bound on the achievable region of the entire system. In the case, e.g., the 2-user BC, that transmitting parallelly is DoF-optimal, the obtained Minkowski-sum inner bound will be tight. The idea of channel decomposition provides a possible way to study a system by breaking it down to several sub-systems which are easier to deal with than the entire system.

3.3 Overview of main results

The main insight offered in this chapter is the idea of subspace resource identification. Since most linear interference elimination techniques, such as zero-forcing and interference alignment, are utilizing the nullspaces of the channels, it would be beneficial if we have a detailed knowledge of the relationships among various nullspaces. Based on the subspace resource identification, we decompose the entire \( m \)-dimensional transmitter beamformer vector space, \( \mathcal{M} \), to 9 linearly independent subspaces, each of which has its own preferred messages to transmit. Then, the linear coding scheme is simplified to how to allocate beamformers from each subspace to corresponding best matching messages.

The second contribution of this chapter is establishing a 7-dimensional outer bound on the linear degrees of freedom (LDoF) region of the 3-user MIMO broadcast channel with general message sets for the symmetric antenna setting, i.e., \((m, n, n, n)\) case. The outer bound is further shown to be tight using aforementioned precoding scheme associated with the channel decomposition. Thus,
it is also the LDoF region.

3.3.1 Channel matrix decomposition

Figure 3.4: Nullspace $\mathcal{N}(H_{ij})$ is subspace of $\mathcal{N}(H_i)$ and $\mathcal{N}(H_j)$, where $i, j \in \{1, 2, 3\}$ and $i < j$. The direction of the arrow indicates the source is a subspace of the destination.

Similar as what we have done in Section 3.2.5 on the 2-user BC, we would like to study the nullspaces of the 3-user BC and discover whether we could obtain similar parallel sub-channels. In the 3-user BC, we have three order-1\(^2\) nullspaces, i.e., $\mathcal{N}(H_1)$, $\mathcal{N}(H_2)$ and $\mathcal{N}(H_3)$, and three order-2 nullspaces, i.e., $\mathcal{N}(H_{12})$, $\mathcal{N}(H_{13})$ and $\mathcal{N}(H_{23})$. Since we assume $m \leq n_1 + n_2 + n_3$, the order-3 nullspace $\mathcal{N}(H_{123}) = \{0\}$. In other words, no non-zero beamformer can be zero-forced at all three receivers simultaneously. Unlike in the 2-user BC, where the two nullspaces are linearly independent with each other, the 6 nullspaces in the 3-user BC have more complicated relationships. The three order-2 nullspaces are linearly independent with each other since otherwise $\mathcal{N}(H_{123}) \neq \{0\}$. However, as shown in Figure 3.4, $\mathcal{N}(H_{ij})$ is subspace of $\mathcal{N}(H_i)$ and $\mathcal{N}(H_j)$, where $i, j \in \{1, 2, 3\}$ and $i < j$, and thus the order-1 nullspaces are not linearly independent. Consequently, we need a more clever way to do the channel decomposition of the 3-user BC.

Here, we give a brief overview of the channel matrix decomposition of the 3-user broadcast channel. More details are given later in Section 3.4.

\(^{2}\) Here we use order-$k$ ($k = 1, 2, 3$) to indicate that the nullspace is to $k$ of the three receivers
As is shown in Figure 3.5, the overall $m$-dimensional transmitter vector space is decomposed into 9 linearly independent subspaces. The first 7 subspaces are $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_{12R}, \mathcal{V}_{13R}, \mathcal{V}_{23R}$ and $\mathcal{V}_{123}$, while the last 2 subspaces can be any two of $\mathcal{V}_{12X}, \mathcal{V}_{13X}$ and $\mathcal{V}_{23X}$.

First, we choose $\mathcal{V}_1 = \mathcal{N}(H_{23}), \mathcal{V}_2 = \mathcal{N}(H_{13})$ and $\mathcal{V}_3 = \mathcal{N}(H_{12})$. This follows from the fact that each of them are zero-forced at two of the receivers. As it is shown later in Lemma 9, $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_3$ are linearly independent with each other almost surely, and their images at the receivers are orthogonal to each other, i.e., $H_{123}\mathcal{V}_1 \perp H_{123}\mathcal{V}_2 \perp H_{123}\mathcal{V}_3$.

Note that both $\mathcal{N}(H_{13})$ and $\mathcal{N}(H_{12})$, or equivalently $\mathcal{V}_2$ and $\mathcal{V}_3$, are subspaces of $\mathcal{N}(H_{1})$. Then, from $\mathcal{N}(H_{1})$, we construct an auxiliary subspace $\mathcal{V}_{23}$, such that $\mathcal{V}_{23}$ is linearly independent with $\mathcal{V}_2 \oplus \mathcal{V}_3$ and

$$\mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_{23} = \mathcal{N}(H_{1}).$$

(3.7)
Furthermore, similar to the fact that $H_{123} V_2 \perp H_{123} V_3$, as we have mentioned previously, $V_{23}$ is selected such that

$$H_{123} V_{23} \perp H_{123} [V_2 \ V_3]. \quad (3.8)$$

In the same way, we construct another two auxiliary subspaces $V_{12}$ and $V_{13}$ such that

$$V_1 \oplus V_3 \oplus V_{13} = \mathcal{N}(H_2) \quad (3.9)$$
$$H_{123} V_{13} \perp H_{123} [V_1 \ V_3] \quad (3.10)$$

and

$$V_1 \oplus V_2 \oplus V_{12} = \mathcal{N}(H_3) \quad (3.11)$$
$$H_{123} V_{12} \perp H_{123} [V_1 \ V_2] \quad (3.12)$$

It is shown later in Lemma 10 and Remark 7 that subspaces $V_{23}$, $V_{13}$ and $V_{12}$ are pair-wise linearly independent with each other, i.e., $V_{12} \cap V_{23} = V_{12} \cap V_{13} = V_{13} \cap V_{23} = \{0\}$. However, it is further shown in Section 3.4.3 that any one of them can be linearly dependent with the sum of the other two, i.e., $V_{13} \cap (V_{23} \oplus V_{12})$, $V_{12} \cap (V_{23} \oplus V_{13})$ and $V_{23} \cap (V_{12} \oplus V_{13})$ are not necessarily equal to $\{0\}$. Thus, we construct $V_{23X}$, $V_{13X}$ and $V_{12X}$ to be equal to the following intersections

$$V_{23X} = V_{23} \cap (V_{12} \oplus V_{13}) \quad (3.13)$$
$$V_{13X} = V_{13} \cap (V_{23} \oplus V_{12}) \quad (3.14)$$
$$V_{12X} = V_{12} \cap (V_{23} \oplus V_{13}), \quad (3.15)$$

where the subscript 'X' indicates intersection. Interestingly, as it is shown later in Section 3.4.3, these aforementioned three intersections are closely related with each other. In particular, we have that $V_{12X} + V_{13X} + V_{23X} = V_{12X} \oplus V_{13X} = V_{12X} \oplus V_{23X} = V_{13X} \oplus V_{23X}$. Consequently, we have that $V_{12X} \subset V_{13X} \oplus V_{23X}$, $V_{13X} \subset V_{12X} \oplus V_{23X}$ and $V_{23X} \subset V_{12X} \oplus V_{13X}$. In other words, each one of $V_{12X}$, $V_{13X}$ and $V_{23X}$ is implicitly contained in the sum of the other two (A 2-D illustration is given in Figure 3.6). We name this sum $V_{12X} + V_{13X} + V_{23X}$ as $V_X$, as shown in Figure 3.5. We also use intersection part to refer to subspace $V_X$ in the following part of the chapter.
Next, we construct subspace $\mathcal{V}_{23}^R$, $\mathcal{V}_{13}^R$, $\mathcal{V}_{12}^R$ from $\mathcal{V}_{23}$, $\mathcal{V}_{13}$, $\mathcal{V}_{12}$, respectively, such that they are linearly independent with $\mathcal{V}_{23}^X$, $\mathcal{V}_{13}^X$, $\mathcal{V}_{12}^X$, respectively, and

$$\mathcal{V}_{23}^R \oplus \mathcal{V}_{23}^X = \mathcal{V}_{23}$$
$$\mathcal{V}_{13}^R \oplus \mathcal{V}_{13}^X = \mathcal{V}_{13}$$
$$\mathcal{V}_{12}^R \oplus \mathcal{V}_{12}^X = \mathcal{V}_{12}.$$

One can check that subspace $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, $\mathcal{V}_{23}^R$, $\mathcal{V}_{13}^R$, $\mathcal{V}_{12}^R$ and $\mathcal{V}_X$ together span the entire nullspaces $\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3$. In the case that $\text{rank}(\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3) < m$, i.e., the nullspaces together do not cover the entire $\mathcal{M}$, we randomly generate another subspace $\mathcal{V}_{123}$ with rank equal to $m - \text{rank}(\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)$. Since all three channels are generic, beamformers chosen from $\mathcal{V}_{123}$ will almost surely be received by all three receivers.

In summary, the $m$-dimensional beamformer space is decomposed into nine linearly independent subspaces. The dimension of each subspace is determined by the antenna configuration. It is worth noting that, under certain antenna settings, the dimension of some subspaces will be equal to zero. In fact, there is always at least one subspace whose rank is 0. In other words, all of the 9 linearly independent subspaces do not exist.

Define the following auxiliary variables as

$$a = (m - n_2 - n_3)^+$$

(3.16)
\[ b = (m - n_1 - n_3)^+ \quad (3.17) \]
\[ c = (m - n_1 - n_2)^+ \quad (3.18) \]
\[ d = (m - n_2)^+ - (m - n_1 - n_2)^+ - (m - n_2 - n_3)^+ \quad (3.19) \]
\[ e = (m - n_3)^+ - (m - n_1 - n_3)^+ - (m - n_2 - n_3)^+ \quad (3.20) \]
\[ f = (m - n_1)^+ - (m - n_1 - n_2)^+ - (m - n_1 - n_3)^+ \quad (3.21) \]
\[ x = (a + b + c + d + e + f - m)^+ \quad (3.22) \]
\[ r = (m - a - b - c - d - e - f)^+, \quad (3.23) \]

then, as is proven later in Section 3.4, the dimensions of the 9 precoding beamformer subspaces, i.e., \( \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_{13R}, \mathcal{V}_{12R}, \mathcal{V}_{23R}, \mathcal{V}_{13X}, \mathcal{V}_{12X}^3 \) and \( \mathcal{V}_{123} \), are equal to \( a, b, c, d - x, e - x, f - x, x, x \) and \( r \), respectively.

Let \( V \) be the concatenation of these nine subspaces, i.e., \( V = [V_1 V_2 V_3 V_{13R} V_{12R} V_{23R} V_{13X} V_{12X} V_{123}] \), then we have \( H_{123}V = U \), such that \( U = \)

\[
\begin{bmatrix}
A_1 & 0 & 0 & A_{13R} & A_{12R} & 0 & A_{13X} & A_{12X} & A_{123} \\
0 & B_2 & 0 & 0 & B_{12R} & B_{23R} & 0 & B_{12X} & B_{123} \\
0 & 0 & C_3 & C_{13R} & 0 & C_{23R} & C_{13X} & 0 & C_{123}
\end{bmatrix}
\]

where \( i \in \{1, 2, 3, 13R, 12R, 23R, 13X, 12X, 123\} \), \( A_i = H_1V_i \), \( B_i = H_2V_i \) and \( C_i = H_3V_i \). The three rows in \( U \) denote the signal received at three receivers, respectively. The zero elements in \( U \) indicates that beamformers chosen from corresponding subspace/column are zero-forced at the corresponding receiver/row.

\[^3\text{In the following part of the chapter, we use } \mathcal{V}_{13X} \text{ and } \mathcal{V}_{12X} \text{ to denote the intersection part.}\]
Figure 3.7 gives an illustration of matrix $U$ and how the 9 beamformer subspaces will impact three receivers, respectively. From left to right, the nine columns denote the beamformer subspace $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, $\mathcal{V}_{13R}$, $\mathcal{V}_{12R}$, $\mathcal{V}_{23R}$, $\mathcal{V}_{13X}$, $\mathcal{V}_{12X}$ and $\mathcal{V}_{123}$, respectively. The number under each column denotes the dimension of the corresponding subspace. It is worth noting that the 2nd and 3rd columns from right are the intersection subspace $\mathcal{V}_{13X}$ and $\mathcal{V}_{12X}$, and as is mentioned previously, $\mathcal{V}_{23X}$ is contained implicitly in subspace $\mathcal{V}_{13X} \oplus \mathcal{V}_{12X}$, and thus $\mathcal{V}_{23X}$ is not a new ingredient and not shown in the figure. One can further observe the fact that $\mathcal{V}_{13X}$ and $\mathcal{V}_{12X}$ are mapped into the same signal space at receiver 1 (their images at receiver 1 share the same row in the figure).
As is shown later, all A’s, B’s and C’s, when they exist, are matrices with full column ranks. Furthermore, they have the following properties.

(1) Counting vertically the signal dimensions at each receiver, we can check that the following three equalities hold.

\[
\text{Rank} \left( \begin{bmatrix} A_1 & A_{13} & A_{12} & A_{123} \end{bmatrix} \right) = a + d + e - x + r = n_1
\]

\[
\text{Rank} \left( \begin{bmatrix} B_2 & B_{23} & B_{12} & B_{123} \end{bmatrix} \right) = b + e + f - x + r = n_2
\]

\[
\text{Rank} \left( \begin{bmatrix} C_3 & C_{23} & C_{13} & C_{123} \end{bmatrix} \right) = c + d + f - x + r = n_3
\]

In other words, by construction, the total dimension of the received signal at each receiver is equal to the number of antennas at that receiver, i.e., the dimension of available signal space.

(2) Counting horizontally the total dimension of all the nine subspaces, we can check that the following equality holds

\[
a + b + c + (d - x) + (e - x) + (f - x) + x + x + r = m.
\]

which shows that, by construction, the total number of independent beamformers from all subspaces is equal to the number of transmit antennas, i.e., the maximum number of independent beamformers the transmitter can send out at once.

### 3.3.2 Coding scheme and the LDoF region of \((m, n, n, n)\) case

Following the basic principle of minimizing interference to unintended receivers, it is straightforward to allocate beamformers from each subspace to different messages. The idea is simply that, if possible, transmit message \(W_s\) using beamformers chosen from subspace \(\mathcal{V}_s\), where \(s \in \mathcal{E}\). In the case that there is more demand for certain messages, we choose from sub-optimal subspaces. For example, if subspace \(\mathcal{V}_1\) is already exhausted and there are still more symbols of message \(W_1\) to be transmitted, we can choose from either subspace \(\mathcal{V}_{12}\) or \(\mathcal{V}_{13}\) or even \(\mathcal{V}_{123}\). The drawback of choosing from sub-optimal subspaces is that it causes interference at one or two of the other unintended
receivers, and this kind of interference is unavoidable due to the high demand of a certain message in the system.

Given any 7-tuple of linear degrees of freedom that is achievable, we specify a way to allocate the \( m \) beamformers to all messages. However, it is hard to explicitly establish the closed form achievable region using our precoding scheme, since there are too many possible antenna configurations. However, if we restrict that there are equal antennas at the receivers, i.e., considering the \((m,n,n,n)\) case, a closed form linear degrees of freedom region is available and given as follows.

**Theorem 4.** Under the \((m,n,n,n)\) antenna setting, the linear DoF region of the three-user broadcast channel with general message sets is given by

\[
\mathcal{D} = \{(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) \in \mathbb{R}_+^7 : \\
\begin{align*}
    d_1 + d_{12} + d_{13} + d_{123} &\leq n \quad (3.24) \\
    d_2 + d_{12} + d_{23} + d_{123} &\leq n \quad (3.25) \\
    d_3 + d_{13} + d_{23} + d_{123} &\leq n \quad (3.26) \\
    d_1 + d_2 + d_3 + d_{12} + d_{13} + d_{23} + d_{123} &\leq m \quad (3.27) \\
    d_1 + d_2 + d_3 + 2d_{12} + d_{13} + d_{23} + 2d_{123} &\leq \max(m, 2n) \quad (3.28) \\
    d_1 + d_2 + d_3 + d_{12} + 2d_{13} + d_{23} + 2d_{123} &\leq \max(m, 2n) \quad (3.29) \\
    d_1 + d_2 + d_3 + d_{12} + d_{13} + 2d_{23} + 2d_{123} &\leq \max(m, 2n) \quad (3.30) \\
    2d_1 + 2d_2 + 2d_3 + 2d_{12} + 2d_{13} + 2d_{23} + 3d_{123} &\leq \max(2m, 3n) \quad (3.31)
\end{align*}
\]

*Proof.* The proof of achievability of region \( \mathcal{D} \) is provided in Section 3.6. Here, we prove the converse.

The first four outer bounds, i.e., (3.24) - (3.27), are trivial cut-set bounds.

Next, consider outer bound (3.28). Suppose we loosen the decoding requirement of message \( W_{13}, W_{23}, W_{123} \) and don’t require receiver 3 to be able to decode them, these three messages will degenerate to message \( W_1, W_2 \) and \( W_{12} \), respectively. Since reducing decoding requirement cannot hurt, the outer bound on the LDoF region of the new system is also an outer bound of the original system. From Lemma 7 given later in this section, we obtain an outer bound on the LDoF of the
original system as \((d_1 + d_{13}) + (d_2 + d_{23}) + d_3 + 2(d_{12} + d_{123}) \leq \max(m, 2n)\), which is inequality (3.28). Similarly, we obtain outer bound (3.29) and (3.30).

Finally, consider outer bound (3.31). Again, we loosen the decoding requirement of message \((W_{12}, W_{13}, W_{23})\), and only require them to be decodable at one of their two intended receivers. Thus, these three messages will all degenerate to private messages. From Lemma 8 given later in this section, we obtain an outer bound on the LDoF region of the original system as inequality (3.31).

The following two lemmas give two key outer bounds on the linear DoF region of the three-user broadcast channel with selected message sets. We only state these two lemmas here and their proofs are provided later in Appendix B.1.

**Lemma 7.** Under the \((m, n, n, n)\) antenna setting, an outer bound on the linear DoF region of the three-user broadcast channel with message sets \((W_1, W_2, W_3, W_{12})\) is given by

\[ d_1 + d_2 + d_3 + 2d_{12} \leq \max(m, 2n). \]  

**Lemma 8.** Under the \((m, n, n, n)\) antenna setting, an outer bound on the linear DoF region of the three-user broadcast channel with message sets \((W_1, W_2, W_3, W_{123})\) is given by

\[ 2d_1 + 2d_2 + 2d_3 + 3d_{123} \leq \max(2m, 3n). \]  

### 3.4 Decomposition of the Precoding Matrix

Since the transmitter has in total \(m\) antennas, there can be at most \(m\) independent transmit beamformers. In this section, we study the relationship among nullspaces and show that this \(m\)-dimensional beamformer vector space can be decomposed into nine linearly independent subspaces, i.e., \(V_1, V_2, V_3, V_{13R}, V_{12R}, V_{23R}, V_{13X}, V_{12X}\) and \(V_{123}\), according to their different impacts on the three receivers. The description of each subspace is given previously in Section 3.3.1. We also provide the specific ways to calculate the basis matrix of each subspace.
We denote the rank of precoding matrix $V_x$ as $r_x$, where $x$ is the subscript matching $V$ and $r$.

Recall that we assume $m \leq n_1 + n_2 + n_3$ in this chapter, and $\mathcal{N}(H_{123}) = \{0\}$.

### 3.4.1 $V_1$, $V_2$ and $V_3$

As was mentioned in Section 3.3.1, we choose $V_1 = \mathcal{N}(H_{23})$, $V_2 = \mathcal{N}(H_{13})$ and $V_3 = \mathcal{N}(H_{12})$, such that beamformers chosen from $V_i$ ($i \in \{1, 2, 3\}$) will be only received by receiver $R_i$. Their dimensions $r_1$, $r_2$ and $r_3$ are equal to the dimension of corresponding nullspaces, which are equal to $(m - n_2 - n_3)^+$, $(m - n_1 - n_3)^+$ and $(m - n_1 - n_2)^+$, respectively.

Since $V_1$ is obtained from the null-space of $H_{23}$, the columns of $V_1$ will be linearly dependent of the rows of $H_1$ only if they have to be. Because $H_1$ is a full matrix with generic elements and notice that we have $r_1 \leq n_1$, according to Lemma 18 given later in the Appendix B.2, we have that $H_1 V_1$ have full column rank almost surely. Similarly, $H_2 V_2$ and $H_3 V_3$ also have full column ranks almost surely. In other words, $V_1$, $V_2$ and $V_3$ are decodable almost surely at receiver 1, 2 and 3, respectively.

From the aforementioned description, we have that

$$H_{123}V_1 = \begin{bmatrix} (H_1 V_1)_{n_1 \times r_1} \\ 0_{n_2 \times r_1} \\ 0_{n_3 \times r_1} \end{bmatrix}$$ \hspace{1cm} (3.34)

$$H_{123}V_2 = \begin{bmatrix} 0_{n_1 \times r_2} \\ (H_2 V_2)_{n_2 \times r_2} \\ 0_{n_3 \times r_2} \end{bmatrix}$$ \hspace{1cm} (3.35)

$$H_{123}V_3 = \begin{bmatrix} 0_{n_1 \times r_3} \\ 0_{n_2 \times r_3} \\ (H_3 V_3)_{n_3 \times r_3} \end{bmatrix}$$ \hspace{1cm} (3.36)

where the subscripts denote the dimension of the sub-matrices.
Lemma 9. Subspaces $V_1$, $V_2$ and $V_3$ are linearly independent with each other, i.e., $\dim(V_1 + V_2 + V_3) = \sum_{i=1,2,3} \dim(V_i)$. Their images at the receivers are orthogonal to each other, i.e., $H_{123}V_1 \perp H_{123}V_2 \perp H_{123}V_3$.

Proof. The orthogonality of the images follows directly from the fact that beamformers from $V_1$, $V_2$ and $V_3$ are received by three different receivers. The linear independence follows from the orthogonality of their images. \qed

3.4.2 $V_{13}$, $V_{23}$ and $V_{12}$

Consider $V_{13}$ first.

In the case that $m - n_2 > 0$, the null-space of channel matrix $H_2$ exists. By sending the message using beamformers choose from $\mathcal{N}(H_2)$, we can zero-force the message at receiver 2. The maximum number of such beamformers is equal to the nullity of $H_2$, i.e., $m - n_2$. Let $\mathcal{N}(H_2)$ denote any basis-matrix of $\mathcal{N}(H_2)$.

Figure 3.8: nullspace $\mathcal{N}(H_2)$ are partitioned into 3 linearly independent subspaces, i.e., $V_1$, $V_3$ and $V_{13}$.

Recall that in Section 3.4.1, we have already extracted subspace $V_1$ and $V_3$ from the entire transmitter beamformer vector space. Furthermore, both $V_1$ and $V_3$ are subspaces of $\mathcal{N}(H_2)$. Besides $V_1$ and $V_3$, there are still $\text{nullity}(H_2) - r_1 - r_3 = (m-n_2)^+ - (m-n_1-n_2)^+ - (m-n_2-n_3)^+$ independent dimensions left in $\mathcal{N}(H_2)$. Now, as is shown in Figure 3.8, we can construct a new beamformer subspace, $V_{13}$, from the remainder of $\mathcal{N}(H_2)$, such that it is linearly independent with
$\mathcal{V}_1$ and $\mathcal{V}_3$, and their direct sum is equal to $\mathcal{N}(H_2)$. Beamformers chosen from $\mathcal{V}_{13}$ will be zero-forced at receiver 2 and be received by receivers 1 and 3.

Since the images of $\mathcal{V}_1$ and $\mathcal{V}_3$ at the receivers are orthogonal to each other, i.e., $H_{123} \mathcal{V}_1 \perp H_{123} \mathcal{V}_3$, similarly, we construct $\mathcal{V}_{13}$ such that its image is orthogonal to the images of $\mathcal{V}_1$ and $\mathcal{V}_3$, i.e., $H_{123} \mathcal{V}_{13} \perp H_{123} [\mathcal{V}_1 \mathcal{V}_3]$ and we have

$$H_{123} \mathcal{V}_{13} = (H_{123} [V_1 V_3])^\perp_{H_{123} \mathcal{N}(H_2)}. \tag{3.37}$$

We can further obtain that $H_1 \mathcal{V}_{13} \perp H_1 \mathcal{V}_1$, $H_3 \mathcal{V}_{13} \perp H_3 \mathcal{V}_3$, since $H_2 \mathcal{V}_1 = 0$ and $H_2 \mathcal{V}_3 = 0$.

Because channel matrix $H_{123}$ is a generic matrix and its row size $n_1 + n_2 + n_3$ is no less than its column size $m$, the product matrix $H_{123}^\dagger H_{123}$ is almost surely a full rank square matrix and thus invertible. Then, we can obtain a feasible choice of $\mathcal{V}_{13}$ as

$$\mathcal{V}_{13} = (H_{123}^\dagger H_{123})^{-1} H_{123}^\dagger (H_{123} [V_1 V_3])^\perp_{H_{123} \mathcal{N}(H_2)}. \tag{3.38}$$

Since $H_{123} \mathcal{V}_{13}$ is orthogonal to $H_{123} [V_1 V_3]$, $\mathcal{V}_{13}$ is guaranteed to be linearly independent with $[V_1 V_3]$. Consequently, the direct sum of the obtained $\mathcal{V}_{13}$, $\mathcal{V}_1$ and $\mathcal{V}_3$ is equal to null space $\mathcal{N}(H_2)$.

Following the same way, we could construct similar $\mathcal{V}_{23}$ and $\mathcal{V}_{12}$ from $\mathcal{N}(H_1)$ and $\mathcal{N}(H_3)$, respectively, and feasible choices of their basis matrices are given by

$$\mathcal{V}_{12} = (H_{123}^\dagger H_{123})^{-1} H_{123}^\dagger (H_{123} [V_1 V_2])^\perp_{H_{123} \mathcal{N}(H_3)} \tag{3.39}$$

and

$$\mathcal{V}_{23} = (H_{123}^\dagger H_{123})^{-1} H_{123}^\dagger (H_{123} [V_2 V_3])^\perp_{H_{123} \mathcal{N}(H_1)} \tag{3.40}$$

respectively. The rank of $\mathcal{V}_{12}$ is equal to $\text{Nullity}(H_3) - r_1 - r_2 = (m-n_3)^+ - (m-n_2-n_3)^+$, and we have that $H_1 \mathcal{V}_{12} \perp H_1 \mathcal{V}_1$, $H_2 \mathcal{V}_{12} \perp H_2 \mathcal{V}_2$. The rank of $\mathcal{V}_{23}$ is equal to $\text{Nullity}(H_1) - r_2 - r_3 = (m-n_1)^+ - (m-n_1-n_3)^+ - (m-n_1-n_2)^+$, and we have that $H_2 \mathcal{V}_{23} \perp H_2 \mathcal{V}_2$, $H_3 \mathcal{V}_{23} \perp H_3 \mathcal{V}_3$.

**Lemma 10.** Subspace $\mathcal{V}_{12}$, $\mathcal{V}_{13}$ and $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ are linearly independent with each other, i.e.,

$$\text{dim}\left(\sum_{s=1,2,3,12,13} \mathcal{V}_s\right) = \sum_{s=1,2,3,12,13} \text{dim}(\mathcal{V}_s). \tag{3.41}$$

In other words, the column vectors of $\mathcal{V}_{12}$ and $\mathcal{V}_{13}$ are linearly independent with that of $[V_1 V_2 V_3]$. 


\textbf{Proof.} Lemma 9 has already shown that $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are linearly independent with each other.

Now, we first show $\mathcal{V}_{12}$ is linearly independent with $\sum_{s=1,2,3} \mathcal{V}_s$. Suppose there exists a non-trivial column vector $v \in \mathcal{V}_{12}$ and also $\in \sum_{s=1,2,3} \mathcal{V}_s$. Then, there exists a group of all-nonzero coefficients $c_1$ and $c_2$, such that $v = \mathcal{V}_{12} \cdot c_1 = [V_1\ V_2\ V_3] \cdot c_2$, where $c_1 \in \mathbb{C}^{r_{12} \times 1}$ and $c_2 \in \mathbb{C}^{(r_1+r_2+r_3) \times 1}$. The last $r_3$ coefficients in $c_2$ must be all 0, otherwise $H_3 \cdot v$ will not be 0, which contradicts the fact that $v \in \mathcal{V}_{12}$. As a result, we have that $v = \mathcal{V}_{12} \cdot c_1 = [V_1\ V_2] \cdot \hat{c}_2$, which contradicts with the fact that $\mathcal{V}_{12}$ is linearly independent with $\mathcal{V}_1 + \mathcal{V}_2$. Consequently, $\mathcal{V}_{12}$ is linearly independent with $\sum_{s=1,2,3} \mathcal{V}_s$.

Next, we show $\mathcal{V}_{13}$ is linearly independent with $\sum_{s=1,2,3,12} \mathcal{V}_s$. The idea is similar. Suppose there exists a non-zero column vector $v \in \mathcal{V}_{13}$ and also $\in \sum_{s=1,2,3,13} \mathcal{V}_s$. Then, there exists a group of all-nonzero coefficients such that $v = \mathcal{V}_{13} \cdot c_1 = [V_1\ V_2\ V_3\ \mathcal{V}_{12}] \cdot c_2$, where $c_1 \in \mathbb{C}^{r_{12} \times 1}$ and $c_2 \in \mathbb{C}^{(r_1+r_2+r_3+r_{12}) \times 1}$. Since $H_2 \cdot \mathcal{V}_{13} = 0$, we have $H_2[V_1\ V_2\ V_3\ \mathcal{V}_{12}]c_2 = 0$. Since $H_2V_1$ and $H_2V_3$ are both equal to all-zero vector, we have that the rest part $H_2[V_2\ \mathcal{V}_{12}]\hat{c}_2 = 0$, where $\hat{c}_2$ is the column vector containing all the coefficients in $c_2$ that are corresponding to the columns of $V_2$ and $\mathcal{V}_{12}$. Since $\mathcal{V}_2$ is linearly independent with $\mathcal{V}_{12}$, $H_2[V_2\ \mathcal{V}_{12}]\hat{c}_2 = 0$ leads to $\hat{c}_2 = 0$. Hence, the other coefficient in $c_2$, i.e., those corresponding to the columns of $V_1$ and $V_3$, can not be all-zeros. Now, we have $v = \mathcal{V}_{13} \cdot c_1 = [V_1\ V_3] \cdot \hat{c}_2$, where $\hat{c}_2$ contains all the coefficients in $c_2$ that are corresponding to the columns of $V_1$ and $V_3$. This again contradicts with the fact that $\mathcal{V}_{13}$ is linearly independent with $\mathcal{V}_1 + \mathcal{V}_3$. Thus, $\mathcal{V}_{13}$ is linearly independent with $\sum_{s=1,2,3,12} \mathcal{V}_s$.

In sum, $\mathcal{V}_{12}$, $\mathcal{V}_{13}$ and $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ are linearly independent with each other. \hfill \Box

\textbf{Remark 7.} Lemma 10 is also true if we replace $\mathcal{V}_{12}$ or $\mathcal{V}_{13}$ with $\mathcal{V}_{23}$. In other words, any two of $\mathcal{V}_{12}$, $\mathcal{V}_{13}$ and $\mathcal{V}_{23}$ are linearly independent, and also linearly independent to $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$.

\textbf{Lemma 11.} The images of $\mathcal{V}_{13}$ and $\mathcal{V}_{12}$ at Receiver $R_1$, i.e., $\text{Span}(H_1V_{13})$ and $\text{Span}(H_1V_{12})$, will intersect with each other at a subspace with rank equal to $(r_1 + r_{13} + r_{12} - n_1)^+$, and the images of $\mathcal{V}_{23}$ and $\mathcal{V}_{12}$ at Receiver $R_2$, i.e., $\text{Span}(H_2V_{23})$ and $\text{Span}(H_2V_{12})$, will intersect with each other at a subspace with rank equal to $(r_2 + r_{23} + r_{12} - n_2)^+$, and the images of $\mathcal{V}_{13}$ and $\mathcal{V}_{23}$ at Receiver $R_3$,
\( i.e., \text{Span}(H_3V_{13}) \) and \( \text{Span}(H_3V_{23}) \), will intersect with each other at a subspace with rank equal to \( (r_3 + r_{13} + r_{23} - n_3)^+ \).

**Proof.** Note that the three statement are symmetric according to the indexes, we only prove the first case at receiver \( R_1 \), the other two follow in the same manner.

Consider nullspaces \( \mathcal{N}(H_2) \) and \( \mathcal{N}(H_3) \), from which \( \mathcal{V}_{13} \) and \( \mathcal{V}_{12} \) come from. \( \mathcal{N}(H_2) \) is obtained from the null space of \( H_2 \), and \( \mathcal{N}(H_3) \) is obtained from the null space of \( H_3 \). Consequently, \( \mathcal{N}(H_2) \) and \( \mathcal{N}(H_3) \) have nothing to do with each other and also with channel matrix \( H_1 \). The column vectors of \( H_1 \cdot \mathcal{N}(H_2) \) and \( H_1 \cdot \mathcal{N}(H_3) \) will be linear dependent only if they have to be linearly dependent. More specifically, if \( \text{rank}(H_1 \cdot \mathcal{N}(H_2)) + \text{rank}(H_1 \cdot \mathcal{N}(H_3)) \) is greater than the total available signal dimension at receive 1, i.e., \( n_1, \text{Span}(H_1 \cdot \mathcal{N}(H_2)) \) and \( \text{Span}(H_1 \cdot \mathcal{N}(H_3)) \) will overlap together almost surely at a subspace with dimension of \( \text{Rank}(H_1 \cdot \mathcal{N}(H_2)) + \text{Rank}(H_1 \cdot \mathcal{N}(H_3)) - n_1 \), otherwise, they will not. Equivalently, we have that the dimension of the intersection of \( \text{Span}(H_1 \cdot \mathcal{N}(H_2)) \) and \( \text{Span}(H_1 \cdot \mathcal{N}(H_3)) \) is equal to \( \text{Rank}(H_1 \cdot \mathcal{N}(H_2)) + \text{Rank}(H_1 \cdot \mathcal{N}(H_3)) - n_1^+ \) almost surely.

Recall that \( \mathcal{N}(H_2) \) consists of three linearly independent subspaces \( \mathcal{V}_1, \mathcal{V}_3 \) and \( \mathcal{V}_{13} \), i.e.,

\[
\mathcal{N}(H_2) = \mathcal{V}_1 \oplus \mathcal{V}_3 \oplus \mathcal{V}_{13}.
\]

Since \( H_1V_3 = 0 \), we have that \( \text{Span}(H_1 \cdot \mathcal{N}(H_2)) = \text{Span}([H_1V_1 \ H_1V_{13}]) \) and \( \text{Rank}(H_1 \cdot \mathcal{N}(H_2)) = r_1 + r_{13} \). Similarly, we have that \( \text{Span}(H_1 \cdot \mathcal{N}(H_3)) = \text{Span}([H_1V_1 \ H_1V_{12}]) \) and \( \text{Rank}(H_1 \cdot \mathcal{N}(H_3)) = r_1 + r_{12} \). Thus, we obtain that \( \text{Span}(H_1 \cdot \mathcal{N}(H_2)) \cap \text{Span}(H_1 \cdot \mathcal{N}(H_3)) = \text{Span}([H_1V_1]) \oplus (\text{Span}(H_1 \cdot V_{13}) \cap \text{Span}(H_1 \cdot V_{12})). \) Consequently, the rank of \( \text{Span}(H_1 \cdot \mathcal{N}(H_2)) \cap \text{Span}(H_1 \cdot \mathcal{N}(H_3)) \) is equal to the rank of \( \text{Span}(H_1 \cdot \mathcal{N}(H_2)) \cap \text{Span}(H_1 \cdot \mathcal{N}(H_3)) \) minus the rank of \( \text{Span}([H_1V_1]) \), which is equal to \( (r_1 + r_{12} + r_1 + r_{13} - n_1 - r_1)^+ = (r_1 + r_{12} + r_{13} - n_1)^+ \). □

**Remark 8.** From the proof of Lemma 11, we can see the benefit of constructing \( V_{13} \) such that \( H_1V_{13} \perp H_1V_1 \), we could restrict the attention to the intersection between \( \text{Span}(H_1V_{13}) \) and \( \text{Span}(H_1V_{12}) \), rather than getting \( \text{Span}(H_1V_1) \) involved together.

**Remark 9.** Interestingly, one can check that the three ranks in Lemma 11 are equal to each other, i.e.,

\[
(r_1+r_{13}+r_{12} - n_1)^+ = (r_2+r_{23}+r_{12} - n_2)^+ = (r_3+r_{13}+r_{23} - n_3)^+ = (r_1+r_{2}+r_{3}+r_{12}+r_{13}+r_{23} - m)^+.
\]
We denote this value as \((r_X)^+\) and will show later in Section 3.4.3 the hidden reason of such coincidence.

### 3.4.3 \(V_{13X}, V_{12X}\) and \(V_{13R}, V_{23R}, V_{12R}\)

According to Lemma 11, \(\text{Span}(H_1V_{13})\) and \(\text{Span}(H_1V_{12})\) intersect with each other at \((r_X)^+\) dimensions. Define subspace \(U_X\) and let

\[
U_X = \text{Span}(H_1V_{13}) \cap \text{Span}(H_1V_{12}).
\]

\(U_X\) and one of its basis matrix \(U_X\) can be obtained using Algorithm 2 given in Section 3.2.3.2. Then, we could further partition \(V_{13}\) into two linearly independent subspaces, namely \(V_{13X}\) and \(V_{13R}\), such that \(V_{13} = V_{13R} \oplus V_{13X}\) and their images at receiver \(R_1\) are equal to \(U_X\) and the orthogonal complement of \(U_X\) in \(\text{Span}(H_1V_{13})\), respectively. In other words, we would like

\[
H_1V_{13X} = U_X
\]

\(3.41\)

\[
H_1V_{13R} = (H_1V_{13})^\dagger(U_X)^\dagger H_1V_{13}.
\]

\(3.42\)

A feasible solution of basis matrices \(V_{13X}\) and \(V_{13R}\) can be obtained from the following Corollary.

**Corollary 9.** Given \(H_1\) and \(U_X\), one feasible solution for \(V_{13X}\) and \(V_{13R}\) is given by

\[
V_{13X} = V_{13} \left( (H_1V_{13})^\dagger (H_1V_{13}) \right)^{-1} (H_1V_{13})^\dagger U_X.
\]

\(3.43\)

\[
V_{13R} = V_{13} \left( (H_1V_{13})^\dagger (H_1V_{13}) \right)^{-1} (H_1V_{13})^\dagger (U_X)^\dagger H_1V_{13}.
\]

\(3.44\)

**Proof.** Because \(n_1 < m\), \(H_1^\dagger H_1\) is a rank-deficient \(m \times m\) square matrix, which is not invertible. Thus, we could not obtain \(V_{13X}\) simply as \((H_1^\dagger H_1)^{-1} H_1^\dagger U_X\), as what we have done in (3.38) for \(V_{13}\).

Since \(V_{13X}\) is a subspace of \(V_{13}\), each column vector of \(V_{13X}\) can be written as a linear combination of the basis vectors of \(V_{13}\). Thus, we can write \(V_{13X}\) in the following way: \(V_{13X} = V_{13}X\), where \(X\) is a \(r_{13} \times r_X\) matrix with full column rank. Then, the problem is transformed to finding a matrix \(X\) such that \(H_1V_{13}X = U_X\). Here, \(H_1V_{13}\) is a \(n_1 \times r_{13}\) matrix and we have that \(n_1 \geq r_{13}\).
Consequently, \((H_1V_{13})^\dagger(H_1V_{13})\) will be a \(r_{13} \times r_{13}\) full-rank square matrix and thus invertible. Then, we can obtain \(X\) via \(X = ((H_1V_{13})^\dagger(H_1V_{13}))^{-1}(H_1V_{13})^\dagger U_X\). Finally, we have that

\[
V_{13X} = V_{13}X = V_{13}((H_1V_{13})^\dagger(H_1V_{13}))^{-1}(H_1V_{13})^\dagger U_X.
\]

The calculation of \(V_{13R}\) follows in the same way.

Similarly, we can partition subspace \(V_{12}\) into two linearly independent parts: \(V_{12R}\) and \(V_{12X}\), such that \(V_{12} = V_{12R} \oplus V_{12X}\) and their images at receiver \(R_1\) are equal to \(U_X\) and the orthogonal complement of \(U_X\) in \(\text{Span}(H_1V_{12})\), respectively. In other words, we would like

\[
H_1V_{12X} = U_X
\]
\[
H_1V_{12R} = (U_X)^\dagger H_1V_{12}
\]

and one pair of feasible solution for \(V_{12R}\) and \(V_{12X}\) is given by

\[
V_{12X} = V_{12}((H_1V_{12})^\dagger (H_1V_{12}))^{-1} (H_1V_{12})^\dagger U_X
\]
\[
V_{12R} = V_{12}((H_1V_{12})^\dagger (H_1V_{12}))^{-1} (H_1V_{12})^\dagger (U_X)^\dagger H_1V_{12}.
\]

Next, consider \(V_{23}\). We do not partition it into two subspaces as what we have done to \(V_{13}\) and \(V_{12}\). However, we show that from \(V_{13X}\) and \(V_{12X}\), we could obtain a subspace in \(V_{23}\) in the following lemma.

**Lemma 12.** There exists a \(r_X\)-dimensional subspace \(V_{23X}\), such that \(V_{23X} \subseteq V_{23}\) and \(V_{23X} \subseteq V_{13X} \oplus V_{12X}\), where \(r_X = \text{rank}(V_{13X}) = \text{rank}(V_{12X})\).

**Proof.** From the constructions of \(V_{13X}\) and \(V_{12X}\), we have that their ranks are both equal to \(r_X\), and their images at receiver \(R_1\) are both equal to subspace \(U_X\). In other words, they are aligned together at receiver \(R_1\).

Let \(u_1\) be any \(n_1 \times 1\) column vector in \(U_X\). Then, there exists a pair of vectors \(v_{13X}\) and \(v_{12X}\) in \(V_{13X}\) and \(V_{12X}\), respectively, such that \(H_1v_{13X} = H_1v_{12X} = u_1\). Furthermore, we denote
\( H_{123}v_{13X} = \begin{bmatrix} u_1 \\ 0 \\ u_3 \end{bmatrix} \) and \( H_{123}v_{12X} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \), where \( u_2 \) and \( u_3 \) are some non-zero column vector with size \( n_2 \times 1 \) and \( n_3 \times 1 \), respectively. Then, let beamformer \( v_{23X} = v_{12X} - v_{13X} \). We have

\[
H_{123}v_{23X} = \begin{bmatrix} u_1 \\ u_2 \\ -u_3 \end{bmatrix},
\]

which means \( v_{23X} \in N(H_1) = \text{Span}([V_2 \ V_3 \ V_{23}]) \). Since \( H_2v_{12X} \perp H_2V_2 \) and \( H_3v_{12X} \perp H_3V_3 \), we have

\[
\begin{bmatrix} 0 \\ u_2 \\ -u_3 \end{bmatrix} \perp H_{123}V_2 \quad \text{and} \quad \begin{bmatrix} 0 \\ u_2 \\ -u_3 \end{bmatrix} \perp H_{123}V_3.
\]

As a result, \( \begin{bmatrix} 0 \\ u_2 \\ -u_3 \end{bmatrix} \) can only lie in \( \text{Span}(H_{123}V_{23}) \) and \( v_{23X} \in \text{Span}(V_{23}) \).

For each of the \( r_X \) basis vectors of \( U_X \), their exists such a pair of vectors \( v_{13X} \) and \( v_{12X} \), from which we could obtain such a vector \( v_{23X} \). The \( r_X \) vectors together span subspace \( V_{23X} \), which is a subspace of \( V_{23} \), and also a subspace of \( V_{13X} \oplus V_{12X} \).

It can be easily check that the image of \( v_{23X} \) is aligned together with the image of \( v_{13X} \) at receiver \( R_3 \), and aligned together with the image of \( v_{12X} \) at receiver \( R_2 \). Consequently, we have that, if \( \text{Span}(H_1V_{13}) \) intersects with \( \text{Span}(H_1V_{12}) \) for \( r_X \) dimension, \( \text{Span}(H_3V_{13}) \) will also intersect with \( \text{Span}(H_3V_{23}) \) for \( r_X \) dimension and \( \text{Span}(H_2V_{12}) \) will also intersect with \( \text{Span}(H_2V_{23}) \) for \( r_X \) dimension. The symmetry among three receivers guarantee that the dimensions of intersections will be equal with each other. This is also the hidden reason of the equalities in Remark 9.

It is worth noting that, although \( v_{23X} \in \text{Span}(V_{23}) \), it is just a linear combination of \( v_{12X} \) and \( v_{13X} \). In other words, \( V_{23X} \) is implicitly contained in \( V_{12X} \oplus V_{13X} \). Furthermore, we have the following result of alignments:

\[
\begin{align*}
\text{Span}(H_1V_{13X}) &= \text{Span}(H_1V_{12X}) \\
\text{Span}(H_2V_{12X}) &= \text{Span}(H_2V_{23X}) \\
\text{Span}(H_3V_{13X}) &= \text{Span}(H_3V_{23X}).
\end{align*}
\]

For subspace \( V_{23} \), only the part that is independent with \( V_{23X} \) is new here, and we name it
\[ V_{23^R}. \] The dimension of \( V_{23^R} \) is equal to \( r_{23} - (r_X)^+ \). Similarly, we construct \( V_{23^R} \) such that its image at receiver \( R_3 \) is orthogonal to that of \( V_{23^X} \) in \( \text{Span}(H_3V_{23}) \), i.e.,

\[ H_3V_{23^R} = (H_3V_{23^X})_{H_3V_{23}}^\perp. \]

Here, we do not need to calculate any basis vector of \( V_{23^X} \) since \( \text{Span}(H_3V_{23^X}) = \text{Span}(H_3V_{13^X}) \), we have that \( H_3V_{23^R} = (H_3V_{13^X})_{H_3V_{23}}^\perp \). A feasible solution of \( V_{23^R} \) is given by

\[ V_{23^R} = V_{23} \left( (H_3V_{23})^\dagger(H_3V_{23}) \right)^{-1} (H_3V_{23})^\dagger(H_3V_{13^X})_{H_3V_{23}}^\perp. \]

(3.47)

### 3.4.5 \( V_{12^3} \)

So far, we have already partitioned 8 linearly independent subspaces from the \( m \)-dimensional precoding space, i.e., \( V_1, V_2, V_3, V_{13^R}, V_{12^R}, V_{13^X}, V_{12^X} \) and \( V_{23^R} \). Their common characteristic is that beamformers from each of them can be zero-forced at one or two of the three receivers. Thus, we call them zero-forcing beamformers.

The subspace of zero-forcing beamformers have occupied all together \( r_1 + r_2 + r_3 + r_{13} + r_{12} + r_{23} - (r_X)^+ \) of the \( m \) dimensions. The rest dimension of beamformers available at the transmitter is given by

\[ r_{123} = m - (r_1 + r_2 + r_3 + r_{13} + r_{12} + r_{23} - (r_X)^+) \]

\[ = -r_X + (r_X)^+ \]

\[ = (-r_X)^+. \]

and we define this remainder subspace as \( V_{12^3} \). Beamformers chosen from \( V_{12^3} \) will impact all three receivers. For this part, we can randomly generate \( r_{123} \) beamformers, and they will almost surely be linearly independent with previous beamformers and received by all three receivers, i.e., the probability that they are accidentally zero-forced at any receiver is 0.

**Remark 10.** From the value of \( r_{123} \), we can discover that \( V_{12^3} \) and \( [V_{12^X} V_{13^X}] \) won’t exist at the same time, because at most one of \((-r_X)^+ \) and \((r_X)^+ \) can be non-zero. The reason behind is that,
if there are still independent beamformer dimensions that can not be zero-forced at any receiver, the zero-forcing beamformers won’t be forced to intersect with each other.

### 3.4.5 Summary

Now, we have decomposed the \( m \)-dimensional transmitter beamformer vector space into 9 linearly independent subspaces. An illustration of different beamformer subspaces is given previous in Figure 3.7 of Section 3.3.1. As a matter of convenience, we summarize all the calculations of basis matrices of beamformer spaces here together as algorithm 4 below.

### 3.4.6 Algorithm to construct the nine subspaces

Algorithm 4:

Follow the steps below to obtain the basis matrices of the nine independent subspaces.

- **Step 1:** Use algorithm 1 to calculate the null space of matrix \( H_{23} \), \( H_{13} \) and \( H_{12} \), and obtain \( V_1 \), \( V_2 \) and \( V_3 \), respectively.

\[
V_1 = N(H_{23}) \\
V_2 = N(H_{13}) \\
V_3 = N(H_{12})
\]

- **Step 2:** Use algorithm 3 to calculate the orthogonal complement of subspace. Calculate subspace \( V_{23} \), \( V_{13} \) and \( V_{12} \), which is given by equations (3.13), (3.14) and (3.15), respectively.

- **Step 3:** Use algorithm 2 to find the intersection of \( \text{span}(H_1V_{13}) \) and \( \text{span}(H_1V_{12}) \), and obtain a basis matrix of the intersection as \( U_X \).

- **Step 4:** Calculate the basis matrices of subspace \( V_{13X}, V_{13R}, V_{12X}, V_{12R} \) and \( V_{23R} \), which are given by equations (3.43)- (3.47), respectively.

- **Step 5:** Randomly generate a \( m \times r_{123} \) matrix as \( V_{123} \).
3.4.7 Symbol extensions

The matrix decomposition is based merely on the channel matrices $H_{123}$, and the dimension of each of the nine subspaces is determined by the numbers of antennas at four terminals. In the case that symbol extensions (SE) are required to achieve some fractional-valued degrees of freedom corner points, our matrix decomposition approach can be applied to the equivalent channel matrices after symbol extension, i.e., $\hat{H}_i = I_T \otimes H_i$, where $\hat{H}_i$ denotes the equivalent channel matrix from the transmitter to receiver $i$ after symbol extensions, $T$ is the length of symbol extensions, $I_T$ is the $T \times T$ identity matrix, $\otimes$ denotes the Kronecker product of matrices. After doing symbol extensions, the dimension of each nullspace increases proportionally to $T$. The dimension of each of the nine subspaces also increases by $T$ times, and we have that the basis matrices after symbol extensions are $\hat{V}_s = I_T \otimes V_s$, where $s \in \{1, 2, 3, 13R, 12R, 23R, 13X, 12X, 123\}$. According to our construction approach, all dependence and independence results at the receivers stay the same. Thus, nothing surprising happens (as in the SISO $X$ channel etc.), and techniques such as asymmetric complex signaling (ACS) are not relevant in our broadcast channel model. In fact, from the results of [32, 9], careful readers can find that in all current known cases in which ACS is necessary, there are multiple transmitters and at least one of the transmitters has single antennas. However, in our broadcast channel case, there is only one transmitter. And most importantly, in the case when the transmitter has only 1 antenna, the DoF region will trivially be $\text{Sum \ dof} \leq 1$, and all the corner points are integer-valued, and hence neither SE nor ACS is not necessary.

3.5 Optimal linear precoding Scheme

In this section, we describe the linear precoding scheme based on the decomposition of channel matrix introduced in Section 3.4.
3.5.1 Basic principle of beamformer allocation

When designing beamformers for a certain message, we follow the basic idea of minimizing the unnecessary interference it causes to its unintended receivers. For example, when transmitting message $W_1$, which is only desired by receiver 1, we try to minimize its impact to receivers 2 and 3. Thus, it is preferable that we choose beamformers that would be zero-forced at both receivers 2 and 3, i.e., from subspace $\mathcal{V}_1$. Similarly, when transmitting message $W_{23}$, which is desired by both receivers 2 and 3, we should choose beamformers that would be zero-forced at receiver 1, i.e., from subspace $\mathcal{V}_{23R}$. However, if there is not enough appropriate zero-forcing beamformers available, we then choose beamformers that would impact fewer unintended receivers.

Now, let’s have a detailed look at what messages each beamformer subspace can transmit. We use $v_s$ to denote a beamformer chosen from unused dimensions from subspace $\mathcal{V}_s$. In our precoding scheme, all precoding beamformers are chosen from the column vectors of matrix $V$.

Let $i, j \in \{1, 2, 3\}$ and $i < j$, we have that

- $v_i$ can be used to send 1 additional independent symbol of $W_i$.
  
i = 1 for example. This follows by the fact that $v_1$ won’t be received by receivers 2 and 3, i.e., using $v_1$ or not won’t have adverse impact on the decoding of existing messages at receivers 2 and 3. Moreover, the image of $v_1$ at receiver 1, i.e., $H_1v_1$, is linearly independent with that of any other existing beamformers, if any, received at receiver 1. Thus, any other existing symbols desired by receiver 1 and the extra $v_1$ are both decodable at receiver 1.

- $v_{ijR}$ can be used to send 1 additional independent symbol of $W_{ij}$ or $W_i$ or $W_j$.
  
Similar with the analysis of $v_i$, using $v_{ijR}$ to transmit one symbol of $W_{ij}$ or $W_i$ or $W_j$ guarantees this symbol, as well as other existing symbols, can be decoded at their intended receivers. However, if we only use $v_{ijR}$ to transmit a symbol from $W_i$, it will waste one available dimension at receiver $j$. Sometimes, this kind of waste is inevitable, especially when receiver $i$ has relatively higher data rate requirement.

- a pair of $v_i$ and $v_j$ can be used to send 1 additional independent symbol of $W_{ij}$
This can be achieved by simply making two private messages to receivers $i$ and $j$, respectively, both equal to message $W_{ij}$.

- a pair of $v_1$ and $v_{23R}$, can be used to send 1 additional independent symbol of $W_{123}$, so can some other similar combinations.

Similarly, this can be achieved by using $v_1$ to transmit a message to receiver 1 and using $v_{23R}$ to transmit a message to receiver 2 and 3, and making both these two messages equal to $W_{123}$.

**Remark 11.** If a beamformer can be used to send a certain kind of message $W$, it also can be used for another kind of message whose intended receivers are a subset of those of $W$. For example, the beamformer which can be used to send a symbol of $W_{123}$ can also be used to send $W_{12}$ or $W_1$, etc.

Next, let’s consider the subspaces $\mathcal{V}_{13X}$ and $\mathcal{V}_{12X}$, whose ranks are both $x$. Recall that subscript ‘X’ here indicates ‘intersection’, and these two subspaces are obtained as the intersection of $\mathcal{V}_{13}$ and $\mathcal{V}_{12} \oplus \mathcal{V}_{23}$ and the intersection of $\mathcal{V}_{12}$ and $\mathcal{V}_{13} \oplus \mathcal{V}_{23}$, respectively. Furthermore, $\mathcal{V}_{23X}$ is implicitly contained in subspace $\mathcal{V}_{13X} \oplus \mathcal{V}_{12X}$. In the rest of the chapter, we refer to subspace $\mathcal{V}_X = \mathcal{V}_{13X} \oplus \mathcal{V}_{12X}$ as the **intersection part**. For this part, we consider its capability as a whole, and the beamformers from it always appear in pairs, i.e., $(v_{12X}, v_{13X})$. In other words, for each column vector $v_{12X}$ in $\mathcal{V}_{12X}$, there exists a column vector $v_{13X}$ in $\mathcal{V}_{13X}$ such that a linearly combination of $v_{12X}$ and $v_{13X}$ is in $\mathcal{V}_{23X}$. Hence, we do not consider what messages can be transmitted via $v_{12X}$ or via $v_{13X}$ separately. However, we consider the subspace $\text{Span}([v_{12X} \ v_{13X}])$ as an integral component.

Suppose the $x$ independent beamformers in $\mathcal{V}_{13X}$, $v_{13X}^i$ ($i = 1, ..., x$), align with $v_{23X}^i$, respectively, i.e., $v_{12X}^i = v_{13X}^i - v_{23X}^i$.

- $V_X$ can be used to send any combinations of broadcast messages, i.e., $W_{123}$, $W_{12}$, $W_{13}$ and $W_{23}$, as long as the following conditions are satisfied.

\[
\begin{align*}
    d_{123} + d_{12} + d_{13} & \leq x \\
    d_{123} + d_{12} + d_{23} & \leq x
\end{align*}
\]
\[ d_{123} + d_{23} + d_{13} \leq x, \]

In other words, the number of symbols intended for any receiver cannot exceed \( x \).

Obviously, these three conditions are all outer bounds, they simply state the fact that the number of independent symbols decodable at each receiver can not exceed the dimension of its signal space. We now show it is achievable. First, consider the \( d_{123} \) symbols of \( W_{123} \), we simply use the first \( d_{123} \) pairs of \((v^i_{13}, v^i_{12})\) to transmit them, and then the problem is to transmit \( W_{12} \), \( W_{13} \) and \( W_{23} \) using the rest \( x - d_{123} \) pairs of beamformers.

We show the idea of precoding scheme with a specific example. Suppose, we need to send 3 symbols of \( W_{12} \), 2 symbols of \( W_{13} \) and 2 symbols of \( W_{23} \) using the rest 5 pairs of beamformers. First, consider the \( d_{123} \) symbols of \( W_{123} \), we simply use the first \( d_{123} \) pairs of \((v^i_{13}, v^i_{12})\) to transmit them, and then the problem is to transmit \( W_{12} \), \( W_{13} \) and \( W_{23} \) using the rest \( x - d_{123} \) pairs of beamformers.

We show the idea of precoding scheme with a specific example. Suppose, we need to send 3 symbols of \( W_{12} \), 2 symbols of \( W_{13} \) and 2 symbols of \( W_{23} \) using the rest 5 pairs of beamformers. First, use beamformers \( v^{13}_{12} \) to transmit three symbols of message \( W_{12} \), and use beamformers \( v^{45}_{13} \) to transmit the two symbols of message \( W_{13} \). Then, use beamformers \( v^{23}_{23} + v^{45}_{23} \) to transmit the two symbols of message \( W_{23} \). In this way, the signal received by three receivers will be \( H_1[v^{13}_{12} v^{45}_{13}], H_2[v^{13}_{12} v^{23}_{23} + v^{45}_{23}] \) and \( H_3[v^{45}_{13} v^{23}_{23} + v^{45}_{23}] \), respectively. The five symbols received by receiver 1 are linearly independent with each other. Thus, receiver 1 can decode all five symbols of message \( W_{12} \) and \( W_{13} \). At receiver 2, signals \( H_2 v^{23}_{12} \) and \( H_2 v^{45}_{23} \) are aligned together and are thus not directly decodable. However, the other three beamformers \( H_2 v^{13}_{12} \) and \( H_2 v^{45}_{23} \) each occupies an independent dimension and can be decoded first. after decoding \( H_2 v^{45}_{23} \), receiver 2 also knows the coefficients of \( H_2 [v^{23}_{23} + v^{45}_{23}] \), and thus it can remove \( H_2 v^{23}_{23} \) from its received signal, and finally be able to decode \( v^{23}_{23} \). Similarly, for receiver 3, it can first decode \( v^{23}_{23} + v^{45}_{23} \), and then it can remove \( H_3 v^{45}_{23} \) for its received signal, and thus be able to decode \( v^{45}_{13} \). In sum, all 7 symbols are decodable at their respectively intended receivers.

For general \( d_{12}, d_{13} \) and \( d_{23} \) values, since \( d_{12} + d_{13} \leq x - d_{123} \), we can use \( v^{d_{12}}_{12} \) to transmit \( W_{12} \), and use \( v^{d_{12}+1:d_{12}+d_{13}}_{13} \) to transmit \( W_{13} \), and use \( v^{d_{12}+d_{13}+1:x-d_{123}}_{23} \) and \( v^{d_{12}+k+1:d_{12}+d_{23}+1:k+1} \) to transmit \( W_{23} \), where \( k = (d_{23} + d_{123} + d_{12} + d_{13} - x)^+ \). All symbols will be...\(^4\)

\(^4\) Here we use \( v^{13}_{12} \) to denote three column vectors \( v^1_{12}, v^2_{12} \) and \( v^3_{12} \). Later similarly, \( v^{23}_{23} + v^{45}_{23} \) denotes two column vectors \( v^{23}_{23} + v^{45}_{23} \) and \( v^{23}_{23} + v^{45}_{23} \).
decodable at their intended receivers.

- $V_X$ can be used to send any combinations of private messages, i.e., $W_1$, $W_2$ and $W_3$, as long as the following conditions are satisfied.

\[
d_1, d_2, d_3 \leq x \\
d_1 + d_2 + d_3 \leq 2x
\]

This is followed by the fact that each pair of $v_{i3X}^i$ and $v_{i2X}^i$ can be used to send 1 symbol of $W_i$ together with 1 symbol of $W_j$, where $i, j = 1, 2, 3$ and $i \neq j$. Because we do not need the symbols be decodable at receiver $k$ ($k \neq i, j$), so the beamformers are allowed to be aligned at receiver $k$. This works as if the two interference symbols are aligned.

- When using $V_X$ to transmit the mixture of broadcast messages and private messages, one should keep in mind that for each symbol of private message, it will cause interference to one of the other two receivers. For example, if using $v_{13}$ to sending a symbol for message $W_1$, then it will cause interference at receiver 3.

So far, we have shown what a beamformer of a certain beamformer subspace from non-intersection part or a combination of several different types of beamformers from non-intersection part can do. We have also shown what symbols can be sent using the intersection part. Now, let’s look at a more complicated case, the combination of beamformers from non-intersection part and intersection part.

For example, suppose we need to send some symbols of $W_{12}$ and 1 symbol of $W_{23}$, and we have one $v_2$ and 3 pairs of intersection beamformers($v_{12}^{13}, v_{13}^{12}$) available. Using $v_2$, by its own, we can not transmit $W_{12}$, and using the intersection, we can send 2 symbols of $W_{12}$ and 1 symbols of $W_{23}$. However, we show that with $v_2$ and the intersection, we can transmit 3 symbols of $W_{12}$ and 1 symbols of $W_{23}$.

When only the intersection part is available, if using beamformer $v_{23X}$ to transmit $W_{23}$, the paired $v_{12X}$ is wasted and can not be used to send message $W_{12}$ because it will be aligned with $v_{23X}$ at receiver 2, such that receiver 2 cannot distinguish $W_{12}$ and $W_{23}$. However, if we have an
extra $v_2$ available, we can send one symbol of $W_{12}$ using beamformer $v_{12X} + v_2$. Since receiver 2 can decode $v_2$, it also knows the coefficient of $v_{12X} + v_2$, and thus it could subtract $H_2 v_{12X}$ from $H_2[v_{23X} v_{12X}]$ and then decode $v_{23X}$. The extra beamformer $v_2$ helps receiver 2 to solve the problem of the alignment of signal caused by $v_{12X}$ and $v_{23X}$.

So far, we have shown the ability of each beamformer and different combinations of them, one should be able to get a general idea of how to allocate all the beamformers. Here we provide an illustrative precoding steps for reference. However, it is not a strict precoding algorithm.

### 3.5.1.1 Steps of allocating the beamformers

1. Assign beamformers from subspace $V_s$ to message $W_s$, where $s \in \{1, 2, 3\}$.

2. Assign beamformers from subspace $V_{sR}$ to message $W_s$, where $s \in \{12, 13, 23\}$.

3. Assign beamformers from subspace $V_{123}$ to message $W_{123}$.

4. Assign beamformers from subspace $(V_{13X}, V_{12X})$ to message $W_s$, where $s \in \{12, 13, 23\}$. If possible, use extra beamformer from $e V_1$ or $V_2$ or $V_3$ to assist the transmission.

5. If there are unassigned message symbols, assign beamformers from subspace $(V_{13X}, V_{12X})$ to messages $W_s$, where $s \in \{1, 2, 3\}$.

6. If there are unassigned message symbols remaining, assign beamformers from subspace $V_{s1}$ to message $W_{s2}$, where $s_2 \subset s_1$, e.g., use 1 beamformer from $V_{12R}$ or $V_{12X}$ to transmit a symbol of message $W_1$.

7. If there are more unassigned message symbols, assign multiple beamformers from several subspaces to a multicast message, e.g., using 1 beamformer from $V_1$ and 1 beamformer from $V_2$ to send a symbol of message $W_{12}$, and etc.

One can observe that, in the first 3 steps, the beamformer resources are perfectly consumed and there is no unexpected interference or waste of dimensions at the receivers. In Steps 4 and 5,
the intersection part is used. However, as was mentioned previously, due to the fact that it may introduce interference at the unintended receivers, we need to implement interference alignment if possible to best utilize the beamformer resources. Steps 6 and 7 are applicable when there are more demands for certain kinds of messages.

### 3.5.2 Achievable LDoF region of each class

Due to the forbidding number of possible antenna configurations, it is difficult to analytically establish the achievable linear degrees of freedom region under arbitrary antenna setting. However, a closed form solution is possible in the symmetric case, i.e., \( n_1 = n_2 = n_3 = n \).

Before we construct the achievable LDoF region using the entire \( m \)-dimensional beamformer vector space, let us look at the achievable region using beamformers chosen from some classes of subspaces. In the following content, we refer to \( V_1 \), \( V_2 \) and \( V_3 \) as class A subspaces, \( V_{13R} \), \( V_{12R} \) and \( V_{23R} \) as class B subspaces, \( V_{13X} \) and \( V_{12X} \) as class C subspaces, and \( V_{123} \) as class D subspace.

In this section, we do not consider fractional-valued DoF tuples. As was mentioned previously in Section 3.4.7, by doing symbol extensions, all fractional points can be turned into integer-valued, and the dimensions of subspaces are all amplified in equal proportion. In other words, any fractional-valued DoF tuple problem can be transferred to integer-valued DoF tuple accordingly by doing symbol extensions.

**Lemma 13.** The achievable LDoF region of transmitting in the union of \( x \) dimensional \( V_1 \), \( x \) dimensional \( V_2 \) and \( x \) dimensional \( V_3 \) is given by

\[
\mathbb{D}_A(x) = \left\{ (d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) \in \mathbb{R}^E_+ : \\
\quad d_1 + d_{12} + d_{13} + d_{123} \leq x \\
\quad d_2 + d_{12} + d_{23} + d_{123} \leq x \\
\quad d_3 + d_{13} + d_{23} + d_{123} \leq x \right\}.
\]

Here subscript ‘A’ indicates class A beamformers. The same goes for subscript ‘B’, ‘C’ and ‘D’, which will appear later.
Proof. The allocation of beamformers for this case is straightforward. For each symbol of message $W_S (S \in E)$, it requires one dimension from each of $V_i$, where $i \in S$. The total number of independent beamformers required from subspace $V_i$ ($i \in \{1, 2, 3\}$) is equal to the total number of streams associated with this subspace, i.e., $\sum_{i \in S} d_S$. For any DoF tuple $\bar{d} \in \mathbb{D}_A(x)$, the required dimension for each one of $V_1$, $V_2$ and $V_3$ is no greater than $x$. Thus, $\mathbb{D}_A(x)$ is achievable.

Lemma 14. The achievable LDoF region of transmitting in the union of $x$ dimensional $V_{12R}$, $x$ dimensional $V_{13R}$ and $x$ dimensional $V_{23R}$ is given by

$$
\mathbb{D}_B(x) = \{(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) \in \mathbb{R}_+^E : \\
\begin{aligned}
&d_1 + d_{12} + d_{13} + d_{123} \leq 2x \\
&d_2 + d_{12} + d_{23} + d_{123} \leq 2x \\
&d_3 + d_{13} + d_{23} + d_{123} \leq 2x \\
&d_1 + d_2 + d_3 + 2d_{12} + d_{13} + d_{23} + 2d_{123} \leq 4x \\
&d_1 + d_2 + d_3 + d_{12} + 2d_{13} + d_{23} + 2d_{123} \leq 4x \\
&d_1 + d_2 + d_3 + d_{12} + d_{13} + 2d_{23} + 2d_{123} \leq 4x \\
&2d_1 + 2d_2 + 2d_3 + 2d_{12} + 2d_{13} + 2d_{23} + 3d_{123} \leq 6x
\end{aligned}
\}.$$

Proof. To begin with, consider message $W_{12}$, $W_{13}$ and $W_{23}$. We first show that if there only exist these three messages, the achievable region is

$$
\begin{cases}
\begin{aligned}
d_{12} + d_{13} & \leq 2x \\
d_{12} + d_{23} & \leq 2x \\
d_{13} + d_{23} & \leq 2x.
\end{aligned}
\end{cases}
$$

(3.48)

Since the region is isotropic, we can assume $d_{12} \geq d_{13} \geq d_{23}$ without loss of generality.

If $d_{12} \leq x$, we also have that $d_{13} \leq x$ and $d_{23} \leq x$. Thus, we can transmit all symbols of $W_S (S \in \{12, 13, 23\})$ in subspace $V_{SR}$. The decodability follows directly from the decomposition of beamformer space and the independence properties of each subspace. If $d_{12} > x$, we transmit $x$
symbols of $W_{12}$ using entire $V_{12R}$ and then transmit the remaining $d_{12} - x$ symbols of $W_{12}$ using beamformer $v_{13R}^i + v_{23R}^i$ ($i = 1, 2, \ldots, d_{12} - x$). The remaining dimensions left in subspaces $V_{13R}$ and $V_{23R}$ are both equal to $x - (d_{12} - x) = 2x - d_{12}$. Since $d_{13} \leq 2x - d_{12}$ and $d_{23} \leq 2x - d_{12}$, there are enough available dimensions left in subspaces $V_{13R}$ and $V_{23R}$ for all $d_{13}$ symbols of message $W_{13}$ and $d_{23}$ symbols of message $W_{23}$. Thus, any DoF tuple satisfies inequality (3.48) are achievable.

Next, consider message $W_{123}$. We show that a combination of one beamformer from each of $V_{12R}$, $V_{13R}$ and $V_{23R}$ can transmit 2 linearly independent symbols of $W_{123}$.

Suppose we have beamformers $v_{12R}$, $v_{13R}$ and $v_{23R}$ available for transmitting message $W_{123}$. Let $\tilde{v}_{123}^1 = v_{12R} + v_{13R}$, and $\tilde{v}_{123}^2 = v_{12R} + v_{23R}$. First, we have that $\tilde{v}_{123}^1$ is decodable at receiver 1 and 3, since its component $v_{13R}$ can be uniquely decoded at receiver 1 and 3. Similarly, $\tilde{v}_{123}^2$ is decodable at receiver 2 and 3, since its component $v_{23R}$ can be uniquely decoded at receiver 2 and 3. Next, consider $\tilde{v}_{123}^1$ at receiver 2. Because $\tilde{v}_{123}^2$ is decodable at receiver 2, receiver 2 can decode $\tilde{v}_{123}^2$ and then eliminate the portion of $H_2 \cdot v_{12R}$ contributed by $H_2 \cdot \tilde{v}_{123}^2$. By decoding the remainder of $H_2 \cdot v_{12R}$, receiver 2 can decode $\tilde{v}_{123}^1$. Similarly, by decoding $\tilde{v}_{123}^1$ and eliminating the portion of $H_1 \cdot v_{12R}$ contributed by $H_1 \cdot \tilde{v}_{123}^1$, and then decoding the remainder of $H_1 \cdot v_{12R}$ contributed by $H_1 \cdot \tilde{v}_{123}^2$, receiver 1 can decode $\tilde{v}_{123}^2$. In sum, both $\tilde{v}_{123}^1$ and $\tilde{v}_{123}^2$ can be decoded by all three receivers.

To transmit $d_{123}$ symbols of message $W_{123}$, we need to consume $\frac{d_{123}}{2}$ dimensions from each of $V_{12R}$, $V_{13R}$ and $V_{23R}$. The remaining dimension would all be $x - \frac{d_{123}}{2}$, where $d_{123} \leq 2x$. Consequently, the achievable DoF for $W_{12}$, $W_{13}$ and $W_{23}$ is changed to

$$
\begin{align*}
  d_{12} + d_{13} &\leq 2 \left(x - \frac{d_{123}}{2}\right) \\
  d_{12} + d_{23} &\leq 2 \left(x - \frac{d_{123}}{2}\right) \\
  d_{13} + d_{23} &\leq 2 \left(x - \frac{d_{123}}{2}\right) .
\end{align*}
$$

Finally, consider private messages $W_1$, $W_2$ and $W_3$. As is mentioned previously, each independent beamformer of $v_{ijR}$ can also be used to transmit a symbol of $W_i$ or $W_j$. So $d_{12}$ can be
partitioned into three parts:

\[ d_{12} = d_{12,1} + d_{12,2} + d_{12,12}, \quad (3.50) \]

while \( d_{12,s} \) \((s = 1, 2, 12)\) means that \( d_{12,s} \) beamformers originally used to transmit message \( W_{12} \) are used to transmit message \( W_S \) now. Similarly, we have partitions that

\[ d_{13} = d_{13,1} + d_{13,3} + d_{13,13} \quad (3.51) \]
\[ d_{23} = d_{23,2} + d_{23,3} + d_{23,23} \quad (3.52) \]

Since each private message is transmitted via beamformers originally for two broadcast messages, we have that

\[ d_1 = d_{12,1} + d_{13,1} \quad (3.53) \]
\[ d_2 = d_{12,2} + d_{23,2} \quad (3.54) \]
\[ d_3 = d_{13,3} + d_{23,3} \quad (3.55) \]

Now, the actual numbers of symbols of message \( W_{12}, W_{13} \) and \( W_{23} \) are \( d_{12,12}, d_{13,13} \) and \( d_{23,23} \), respectively.

So far we obtain an achievable region for all 7 messages as

\[
\begin{align*}
(d_{12,1} + d_{12,2} + d_{12,12}) + (d_{13,1} + d_{13,3} + d_{13,13}) & \leq 2x - d_{123} \\
(d_{12,1} + d_{12,2} + d_{12,12}) + (d_{23,2} + d_{23,3} + d_{23,23}) & \leq 2x - d_{123} \\
(d_{13,1} + d_{13,3} + d_{13,13}) + (d_{23,2} + d_{23,3} + d_{23,23}) & \leq 2x - d_{123}
\end{align*}
\]

as well as all implicit constraints, i.e., each element is non-negative and \( x - \frac{d_{123}}{2} \geq 0 \).

It’s worth noting that we introduced 6 extra auxiliary variables here, i.e., \( d_{12,1}, d_{12,2}, d_{13,1}, d_{13,3}, d_{23,2} \) and \( d_{23,3} \). However, there are only 3 independent ones according to equalities (3.53)-(3.55). Here, we keep \( d_{12,2}, d_{23,3} \) and \( d_{13,1} \). Plugging the equalities and rewriting the region, we have that an achievable region is

\[
\{ (d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}, d_{12,2}, d_{23,3}, d_{13,1}) \in \mathbb{R}^{10}_+ : \}
\]
\[ d_1 + d_3 + d_{12} + d_{13} + d_{12,2} - d_{23,3} \leq 2x - d_{123} \]  
(3.57)

\[ d_1 + d_2 + d_{12} + d_{23} + d_{23,3} - d_{13,1} \leq 2x - d_{123} \]  
(3.58)

\[ d_2 + d_3 + d_{13} + d_{23} + d_{13,1} - d_{12,2} \leq 2x - d_{123} \]  
(3.59)

\[ 2x - d_{123} \geq 0 \]  
(3.60)

\[ d_1 - d_{13,1} \geq 0 \]  
(3.61)

\[ d_2 - d_{12,2} \geq 0 \]  
(3.62)

\[ d_3 - d_{23,3} \geq 0 \} \]  
(3.63)

where inequalities (3.61)-(3.63) come from the implicit constraints that \( d_{12,1}, d_{23,2}, d_{13,3} \geq 0 \).

This region is a 10-dimensional polyhedron defined via 17 inequalities which include 3 auxiliary variables. Now, we need to project this polyhedron onto the 7-dimensional positive orthant \( \mathbb{R}^E_+ \), which can be performed via the Fourier-Motzkin Elimination (FME) [3]. FME can be run with the assistance of various computer softwares, such as “FourierMotzkinGUI” [40]. After doing FME, the 7-dimensional projection is shown to be exactly the same as \( \mathbb{D}_B(x) \).

\[ \text{Lemma 15. The achievable LDoF region of transmitting in the union of } x \text{ dimensional intersection part, i.e., } x \text{ dimension } \mathcal{V}_{12X} \text{ and } x \text{ dimensional } \mathcal{V}_{13X}, \text{ is given by} \]

\[ \mathbb{D}_C(x) = \{ (d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) \in \mathbb{R}^E_+ : \]

\[ d_1 + d_{12} + d_1 + d_{13} + d_{123} \leq x \]

\[ d_2 + d_{12} + d_{23} + d_{123} \leq x \]

\[ d_3 + d_{13} + d_{23} + d_{123} \leq x \]

\[ d_1 + d_2 + d_3 + 2d_{12} + d_{13} + d_{23} + 2d_{123} \leq 2x \]

\[ d_1 + d_2 + d_3 + d_{12} + 2d_{13} + d_{23} + 2d_{123} \leq 2x \]

\[ d_1 + d_2 + d_3 + d_{12} + d_{13} + 2d_{23} + 2d_{123} \leq 2x \}

\[ \text{Proof. As is analyzed previously, the dimension of subspace } \mathcal{V}_{12X} \text{ is always equal to that of subspace } \mathcal{V}_{13X}, \text{ and they implicitly contain the subspace } \mathcal{V}_{23X}. \text{ The images of } \mathcal{V}_{12X} \text{ and } \mathcal{V}_{13X} \text{ at receiver 1} \]
span the same subspace. Let $v^{i}_{12}$ ($i = 1, ..., x$) be aligned with $v^{i}_{13}$ at receiver 1, i.e., $H_{1}v^{i}_{12} = H_{1}v^{i}_{13}$, then beamformer $v^{i}_{12} - v^{i}_{13}$ will only be received by receiver 2 and 3. In other words, $v^{i}_{12} - v^{i}_{13}$ works equivalently as a $v^{i}_{23}$. Although we express this subspace as two components $V_{12}$ and $V_{13}$, it can be also expressed as $V_{12}$ and $V_{23}$, or $V_{13}$ and $V_{23}$.

To show that $D_{C}(x)$ is achievable, we, again, consider only message $W_{12}$, $W_{13}$, $W_{23}$ and $W_{123}$. It is already shown in Section 3.5.1 that the achievable region is

$$
\begin{align*}
&d_{12} + d_{13} \leq x - d_{123} \\
&d_{12} + d_{23} \leq x - d_{123} \\
&d_{13} + d_{23} \leq x - d_{123}
\end{align*}
$$

(3.64)

Next, consider private messages $W_{1}$, $W_{2}$ and $W_{3}$. Similar as the analysis in the proof of Lemma 14, each beamformer originally assigned to send message $W_{ij}$ can also be used to transmit message $W_{i}$ or $W_{j}$. Thus, the message splitting method is also applicable in this case, and we obtain following achievable region for 7 messages

$$
\begin{align*}
&(d_{12,1} + d_{12,2} + d_{12,12}) + (d_{13,1} + d_{13,3} + d_{13,13}) \leq x - d_{123} \\
&(d_{12,1} + d_{12,2} + d_{12,12}) + (d_{23,2} + d_{23,3} + d_{23,23}) \leq x - d_{123} \\
&(d_{13,1} + d_{13,3} + d_{13,13}) + (d_{23,2} + d_{23,3} + d_{23,23}) \leq x - d_{123}
\end{align*}
$$

(3.65)

which can be rewritten as

$$
\begin{align*}
&d_{1} + d_{12} + d_{13} + d_{12,2} + d_{13,3} \leq x - d_{123} \\
&d_{2} + d_{12} + d_{23} + d_{23,3} + d_{12,1} \leq x - d_{123} \\
&d_{3} + d_{13} + d_{23} + d_{13,1} + d_{23,2} \leq x - d_{123}
\end{align*}
$$

(3.66)

Recall that in the proof of Lemma 14, if beamformer $v_{12}$ is used to transmit message $W_{1}$, the subspace spanned by $H_{2}v_{12}$ at receiver 2 is just wasted, since no other beamformers are mapped to the same signal space at receiver 2. But this is not true for beamformer $v_{12}$. If merely using beamformer pair $(v^{i}_{12}, v^{i}_{13})$, one can at most transmit 1 symbol of broadcast message $W_{12}$ or $W_{13}$.
or $W_{23}$. If, for example, $v^{i}_{12X}$ is used to transmit $W_{12}$, then $v^{i}_{13X}$ will be wasted unless cooperatively working together with some extra independent beamformers. However, if we use $v^{i}_{12X}$ to transmit message $W_1$, one can observe that it is possible to use this pair of beamformers to transmit another symbol of $W_2$ or $W_3$. For example, one can use $v^{i}_{23X}$ to transmit a symbol of $W_3$. Although $H_2v^{i}_{12X}$ (message $W_1$) will be aligned with $H_2v^{i}_{23X}$ (message $W_3$) at receiver 2, it does not matter since receiver 2 does not need to decode message $W_1$ or $W_3$. This actually works the same as interference alignment. Each pair of $(v_{12X}, v_{13X})$ can be used to transmit 2 private messages with different targets.

There are three possibilities of doing interference alignment here, i.e., aligning $v_{12,2}$ and $v_{13,3}$ at receiver 1, $v_{23,3}$ and $v_{12,1}$ at receiver 2, $v_{13,1}$ and $v_{23,2}$ at receiver 3. The total number of alignments can not exceed the number of pairs, i.e., $x$, since each pair of $(v_{12X}, v_{13X})$ can only be used for alignment once. Let $a_i$ ($i = 1, 2, 3$) be the number of alignments happened at receiver $i$, and they need satisfy the following constraints:

$$a_1 + a_2 + a_3 \leq x$$  \hspace{1cm} (3.67)

$$0 \leq a_1 \leq \min(d_{12,2}, d_{13,3})$$  \hspace{1cm} (3.68)

$$0 \leq a_2 \leq \min(d_{12,1}, d_{23,3})$$  \hspace{1cm} (3.69)

$$0 \leq a_3 \leq \min(d_{13,1}, d_{23,2}).$$  \hspace{1cm} (3.70)

The inequalities (3.68)-(3.70) come from the fact that the number of symbols participating in alignment can not exceed the total number of symbols. Applying interference alignment, the achievable region is modified to

$$\begin{align*}
    d_1 + d_{12} + d_{13} + d_{12,2} + d_{13,3} - a_1 &\leq x - d_{123} \\
    d_2 + d_{12} + d_{23} + d_{23,3} + d_{12,1} - a_2 &\leq x - d_{123} \\
    d_3 + d_{13} + d_{23} + d_{13,1} + d_{23,2} - a_3 &\leq x - d_{123}
\end{align*}$$  \hspace{1cm} (3.71)

The region given in (3.71), combined with the constraints (3.67)-(3.70), can be equivalently trans-
formed to
\[
\begin{align*}
\begin{cases}
    d_1 + d_{12} + d_{13} + \max(d_{12,2}, d_{13,2}) &\leq x - d_{123} \\
    d_2 + d_{12} + d_{23} + \max(d_{23,3}, d_{12,1}) &\leq x - d_{123} \\
    d_3 + d_{13} + d_{23} + \max(d_{13,1}, d_{23,2}) &\leq x - d_{123}
\end{cases}
\end{align*}
\]  
(3.72)

Now again, by eliminating three of the six auxiliary variables and only keeping \(d_{12,2}, d_{23,3}, d_{13,1}\), we can obtain following achievable region
\[
\{(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}, d_{12,2}, d_{13,1}) \in \mathbb{R}_+^{10} : \}
\begin{align*}
    &d_1 + d_{12} + d_{13} + \max(d_{12,2}, d_{3} - d_{23,3}) \leq x - d_{123} \\
    &d_2 + d_{12} + d_{23} + \max(d_{23,3}, d_{1} - d_{13,1}) \leq x - d_{123} \\
    &d_3 + d_{13} + d_{23} + \max(d_{13,1}, d_{2} - d_{12,2}) \leq x - d_{123} \\
    &x - d_{123} \geq 0 \\
    &d_1 - d_{13,1} \geq 0 \\
    &d_2 - d_{12,2} \geq 0 \\
    &d_3 - d_{23,3} \geq 0
\end{align*}
(3.73)\(\)\(\)\(\)\(\)
(3.74)\(\)\(\)\(\)\(\)
(3.75)\(\)\(\)\(\)\(\)
(3.76)\(\)\(\)\(\)\(\)
(3.77)\(\)\(\)\(\)\(\)
(3.78)\(\)\(\)\(\)\(\)
(3.79)\(\)

By doing Fourier-Motzkin Elimination and projecting this 10-dimensional polyhedron onto the 7-dimension positive orthant \(\mathbb{R}_+^E\), we obtain that an achievable DoF for this case is equal to \(\mathcal{D}_C(x)\)

Lemma 16. The achievable LDoF region of transmitting in the \(x\) dimensional subspace \(\mathcal{V}_{123}\) is given by
\[
\mathcal{D}_D(x) = \{(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) \in \mathbb{R}_+^E : \\
d_1 + d_2 + d_3 + d_{12} + d_{13} + d_{23} + d_{123} \leq x\}.
\]

Proof. There are all together \(x\) linearly independent beamformers in this subspace, and each of them can be used to transmit one symbol of message of any kind. Any DoF tuple is achievable if its sum DoF does not exceed \(x\).
So far, we have established the achievable LDoF region using beamformers chosen from selected subspace groups. In Section 3.6, we show that the overall LDoF region $D$ given in Theorem 4 is achievable using the entire $m$-dimensional beamformer vector space.

### 3.6 Achievability of $D$

In this section, we prove that $D$ in Theorem 4 is achievable by using the linear precoding scheme introduced in Section 3.5, thus it is an inner bound of the LDoF region.

According to Equation (3.16-3.23), we have that, in the case that $n_1 = n_2 = n_3 = n$, we have

$$
\begin{align*}
a &= b = c = (m - 2n)^+ \\
d &= e = f = (m - n)^+ - 2(m - 2n)^+ \\
x &= -r = (3(m - n)^+ - 3(m - 2n)^+ - m)^+.
\end{align*}
$$

To make the analysis easier, we divide the proof of achievability into three cases each corresponding to the five distinct ranges of the ratio between $m$ and $n$, such that all the $(\cdot)^+$ operations can be removed. Recall that we assume $n \leq m \leq 3n$ in our problem, otherwise, the problem is trivial.

#### 3.6.1 Case 1: $n < m \leq \frac{3}{2}n$

![Figure 3.9: Beamformer space illustration of Case 2](image)
Removing the ()\(^+\) operations, in this case, the DoF region \(\mathcal{D}\) is equal to

\[
\mathcal{D}_1 = \{(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) \in \mathbb{R}^E_+: \\
\begin{align*}
d_1 + d_{12} + d_{13} + d_{123} &\leq n \\
d_2 + d_{12} + d_{23} + d_{123} &\leq n \\
d_3 + d_{13} + d_{23} + d_{123} &\leq n \\
d_1 + d_2 + d_3 + d_{12} + d_{13} + d_{23} + d_{123} &\leq m \\
d_1 + d_2 + d_3 + 2d_{12} + d_{13} + d_{23} + 2d_{123} &\leq 2n \\
d_1 + d_2 + d_3 + d_{12} + 2d_{13} + d_{23} + 2d_{123} &\leq 2n \\
d_1 + d_2 + d_3 + d_{12} + d_{13} + 2d_{23} + 2d_{123} &\leq 2n \\
2d_1 + 2d_2 + 2d_3 + 2d_{12} + 2d_{13} + 2d_{23} + 3d_{123} &\leq 3n \}
\]

where the subscript 1 in \(\mathcal{D}_1\) denotes case 1 (the \(\mathcal{D}_2\) and \(\mathcal{D}_3\), which will appear later, follow the same way).

Consider the decomposition of the precoding matrix. When \(n < m \leq \frac{3}{2}n\), we have that \(a = b = c = x = 0, d = e = f = m - n, r = 3n - 2m\), and the dimensions of the nine beamformer subspaces are \(r_1 = r_2 = r_3 = 0, r_{13} = r_{12} = r_{23} = m - n, r_{13X} = r_{12X} = 0, r_{123} = 3n - 2m\). Deleting the columns in Figure 3.7 with rank 0, the illustration of the \(m\)-dimensional beamformer space is simplified as Figure 3.9, and it is shown that only class B and class D subspace exist. Thus, we can partition the entire beamformer space into two parts: part 1 contains all class B subspaces, i.e., \(\mathcal{V}_{13}, \mathcal{V}_{12} \) and \(\mathcal{V}_{23}\), and part 2 contains the class D subspace, i.e., \(\mathcal{V}_{123}\). The subspace of part 1 is linearly independent of the subspace of part 2. Also, the image of subspaces of part 1 at any receiver is linearly independent with the image of subspace of part 2 at the same receiver. Consequently, transmitting in the subspaces of part 1 is independent of transmitting in the subspace of part 2. There is no mutual interference between them.

From Lemma 14, we can obtain the achievable DoF region of only using beamformer from the subspace of part 1 as \(\mathcal{D}_B(m - n)\). From Lemma 16, we obtain the achievable DoF region of
only using beamformer from the subspace of part 2 as $D_D(3n - 2m)$. Thus, we have an achievable region of the entire system as the Minkowski-Sum of $D_B(m - n)$ and $D_D(3n - 2m)$, which is exactly the same with region $D_1$. In other words, this Minkowski-Sum achievable region is optimal in the case of \( n < m \leq \frac{3}{2}n \), when only class B and class D subspaces exist.

3.6.2 Case 2: \( \frac{3}{2}n < m \leq 2n \)

Figure 3.10: Beamformer space illustration of Case 3

Removing the (\()^+\) operations, in this case, the DoF region $D$ is equal to

\[
D_2 = \{(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) \in \mathbb{R}^E_+: \\
d_1 + d_{12} + d_{13} + d_{123} \leq n \\
d_2 + d_{12} + d_{23} + d_{123} \leq n \\
d_3 + d_{13} + d_{23} + d_{123} \leq n \\
d_1 + d_2 + d_3 + 2d_{12} + d_{13} + d_{23} + 2d_{123} \leq 2n \\
d_1 + d_2 + d_3 + d_{12} + 2d_{13} + d_{23} + 2d_{123} \leq 2n \\
d_1 + d_2 + d_3 + d_{12} + d_{13} + 2d_{23} + 2d_{123} \leq 2n \\
2d_1 + 2d_2 + 2d_3 + 2d_{12} + 2d_{13} + 2d_{23} + 3d_{123} \leq 2m \}.
\]

\footnote{The operation of calculating the Minkowski-Sum of two polytopes can be easily accomplished with the assistance of various computer softwares, such as “polymake” \cite{41}.}
and the constraint (3.27) is redundant. The checking of redundant conditions of a polytope can be done with the assistance of “ploymake” as well.

Consider the decomposition of the precoding matrix. When $\frac{3}{2}n < m \leq 2n$, we have that $a = b = c = r = 0$, $d = e = f = m - n$, $x = 2m - 3n$, and the dimensions of nine beamformer subspaces are $r_1 = r_2 = r_3 = r_{123} = 0$, $r_{13R} = r_{12R} = r_{23R} = 2n - m$, $r_{13X} = r_{12X} = 2m - 3n$. Deleting the columns in Figure 3.7 with rank 0, the illustration of the $m$-dimensional beamformer space is simplified as Figure 3.10, and it is shown that only class B and class C subspaces exist. Again, we partition the entire beamformer space into two parts: part 1 contains all class B subspaces, i.e., $V_{13R}$, $V_{12R}$ and $V_{23R}$, and part 2 contains all class C subspaces, i.e., $V_{13X}$ and $V_{12X}$. Again, beamformers chosen from part 1 and from part 2 are linearly independent with each other, and also there is no mutual interference between the transmission of them.

From Lemma 14, we can obtain the achievable DoF region of only using beamformer from the subspace of part 1 as $\mathbb{D}_B(2n - m)$. From Lemma 15, we obtain the achievable DoF region of only using beamformer from the subspace of part 2 as $\mathbb{D}_C(2m - 3n)$. Thus, we have an achievable region of the entire system as the Minkowski-Sum of $\mathbb{D}_B(2n - m)$ and $\mathbb{D}_C(2m - 3n)$, which is exactly the same with region $\mathbb{D}_2$. In other words, this Minkowski-Sum achievable region is also optimal in the case of $\frac{3}{2}n < m \leq 2n$, when only class B and class C subspaces exist.
3.6.3 Case 3: $2n < m \leq 3n$

Removing the $(\cdot)^+$ operations, in this case, the DoF region $\mathcal{D}$ is equal to

$$\mathcal{D}_3 = \{(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) \in \mathbb{R}_+^E :$$

$$d_1 + d_{12} + d_{13} + d_{123} \leq n$$
$$d_2 + d_{12} + d_{23} + d_{123} \leq n$$
$$d_3 + d_{13} + d_{23} + d_{123} \leq n$$
$$d_1 + d_2 + d_3 + 2d_{12} + d_{13} + d_{23} + 2d_{123} \leq m$$
$$d_1 + d_2 + d_3 + d_{12} + 2d_{13} + d_{23} + 2d_{123} \leq m$$
$$d_1 + d_2 + d_3 + d_{12} + d_{13} + 2d_{23} + 2d_{123} \leq m\}$$

and the constraint (3.27) and (3.31) are redundant.

Consider the decomposition of the precoding matrix. When $2n < m \leq 3n$, we have that $a = b = c = m - 2n, d = e = f = 3n - m, x = 3n - m, r = 0$, and the dimensions of nine beamformer subspaces are $r_1 = r_2 = r_3 = m - 2n, r_{13} = r_{12} = r_{23} = r_{123} = 0, r_{13X} = r_{12X} = 3n - m$.

Deleting the columns in Figure 3.7 with rank 0, the illustration of the $m$-dimensional beamformer space is simplified as Figure 3.11, and it is shown that only class A and class C subspace exist. Again, we partition the entire beamformer space into two parts: part 1 contains all class A subspaces, i.e.,
\(V_1, V_2\) and \(V_3\), and part 2 contains all class C subspaces, i.e., \(V_{13X}\) and \(V_{12X}\). From Lemma 14, we can obtain the achievable DoF region of only using beamformer from the subspace of part 1 as \(D_A(m - 2n)\). From Lemma 15, we obtain the achievable DoF region of only using beamformer from the subspace of part 2 as \(D_C(3n - m)\). Thus, we have an achievable region of the entire system as the Minkowski-Sum of \(D_B(2n - m)\) and \(D_C(2m - 3n)\), which is equal to

\[
\{(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) \in \mathbb{R}_+^E : \\
d_1 + d_{12} + d_{13} + d_{123} \leq n \\
d_2 + d_{12} + d_{23} + d_{123} \leq n \\
d_3 + d_{13} + d_{23} + d_{123} \leq n \\
d_1 + d_2 + d_3 + 2d_{12} + d_{13} + d_{23} + 2d_{123} \leq m \\
d_1 + d_2 + d_3 + d_{12} + 2d_{13} + d_{23} + 2d_{123} \leq m \\
d_1 + d_2 + d_3 + d_{12} + d_{13} + 2d_{23} + 2d_{123} \leq m \\
2d_1 + 2d_2 + d_3 + 2d_{12} + 3d_{13} + 3d_{23} + 4d_{123} \leq m + 2n \\
2d_1 + 2d_2 + 2d_3 + 3d_{12} + 2d_{13} + 3d_{23} + 4d_{123} \leq m + 2n \\
2d_1 + 2d_2 + d_3 + d_{12} + 3d_{13} + 2d_{23} + 4d_{123} \leq m + 2n \}
\]

Comparing this result with \(D_3\), we can find that the Minkowski-Sum region has three additional constraints, which means the Minkowski-Sum region is just a subset of the region \(D_3\). Thus, the Minkowski-Sum region is sub-optimal for the case of \(2n < m \leq 3n\). We could not just transmit independently in two parts and expect their sum to be able to cover the entire LDoF region. In other words, a cooperation between subspace part 1 and part 2 is expected.

We show how beamformers from class A can help the transmission using beamformers from class C in an example. Suppose 1 \(v_1\) and 1 pair of \((v_{12X}, v_{13X})\) is available. We have already known that, using \((v_{12X}, v_{13X})\), 1 symbol of \(W_{12}\) or \(W_{13}\) or \(W_{23}\) can be transmitted, and using \(v_1\) we can only send a symbol of message \(W_1\). Here, we show that using \(v_1\) together with a pair of \((v_{12X}, v_{13X})\), we can transmit 1 symbol of \(W_{12}\) and 1 symbol of \(W_{13}\) at the same time. Let
\( \tilde{v}_{12} = v_{12}X \) and \( \tilde{v}_{13} = v_{13}X + v_1 \). First, we have that \( \tilde{v}_{12} \) can be decoded at receiver 2 and \( \tilde{v}_{13} \) can be decoded at receiver 3. By decoding \( v_1 \), the receiver 1 can decode \( \tilde{v}_{13} \). Then, receiver 1 can eliminate the \( H_{1}v_{13}X \) from its received signal, and thus be able to decode \( v_{12}X \), i.e., \( \tilde{v}_{12} \). The extra beamformer \( v_1 \) gives an additional dimension at receiver 1, such that the cut-set bound is enlarged by 1 at receiver 1. Similarly, if we additionally have \( a \) linearly independent beamformers from \( V_1 \), \( b \) linearly independent beamformers from \( V_2 \), \( c \) linearly independent beamformers from \( V_3 \) and \( x \) pairs of \( (v_{12}X, v_{13}X) \), the achievable DoF region for message \( W_{12}, W_{13} \) and \( W_{23} \) will be increased from

\[
\begin{align*}
\begin{cases}
d_{12} + d_{13} & \leq x \\
d_{12} + d_{23} & \leq x \\
d_{13} + d_{23} & \leq x
\end{cases}
\end{align*}
\tag{3.80}
\]

to

\[
\begin{align*}
\begin{cases}
d_{12} + d_{13} & \leq x + a \\
d_{12} + d_{23} & \leq x + b \\
d_{13} + d_{23} & \leq x + c
\end{cases}
\end{align*}
\tag{3.81}
\]

Now, we are ready to describe the optimal precoding scheme for the case of \( 2n < m \leq 3n \).

First, transmit \( c_s \leq m - 2n \) \( (s \in \{1, 2, 3\}) \) symbols of message \( W_s \) using \( c_s \) dimensions of subspace \( V_s \). The remaining dimension available in subspace \( V_s \) will be \( m - 2n - c_s \). Then, consider the \( x \) dimensional class C subspaces \( V_{12X} \) and \( V_{13X} \). With the assistance of the remaining dimensions from \( V_s \) \( (s \in \{1, 2, 3\}) \), the achievable DoF region DoF for message \( W_{12}, W_{13}, W_{23} \) and \( W_{123} \) will be

\[
\begin{align*}
\begin{cases}
d_{12} + d_{13} & \leq (3n - m) + (m - 2n - c_1) - d_{123} \\
d_{12} + d_{23} & \leq (3n - m) + (m - 2n - c_2) - d_{123} \\
d_{13} + d_{23} & \leq (3n - m) + (m - 2n - c_3) - d_{123}
\end{cases}
\end{align*}
\tag{3.82}
\]
Again, by doing message splitting as was done in Lemma 14, we have the following achievable region

\[
\begin{align*}
    d_{13,1} + d_{12,1} + d_{12} + d_{13} + \max(d_{12,2}, d_{13,3}) & \leq n - c_1 - d_{123} \\
    d_{12,2} + d_{23,2} + d_{12} + d_{23} + \max(d_{23,3}, d_{12,1}) & \leq n - c_2 - d_{123} \\
    d_{13,3} + d_{23,3} + d_{13} + d_{23} + \max(d_{13,1}, d_{23,2}) & \leq n - c_3 - d_{123}
\end{align*}
\] (3.83)

In this scheme, we have that each private message contains three parts. Taking message $W_1$ for example, $c_1$ symbols are send using beamformers from $V_1$, $d_{12,1}$ symbols are send using beamformers split from original beamformers for message $W_{12}$, and $d_{13,1}$ symbols are transmitted using beamformers split from original beamformers for message $W_{13}$. Thus, we have the following equalities for $d_1$, $d_2$ and $d_3$:

\[
\begin{align*}
    d_1 &= d_{12,1} + d_{13,1} + c_1 \\
    d_2 &= d_{12,2} + d_{23,2} + c_2 \\
    d_3 &= d_{13,3} + d_{23,3} + c_3.
\end{align*}
\]

Consequently, only 6 out of the total 9 auxiliary variables are independent ones. Eliminating $c_1$, $c_2$ and $c_3$, we have that

\[
\begin{align*}
    d_1 + d_{12} + d_{13} + d_{123} + \max(d_{12,2}, d_{13,3}) & \leq n \\
    d_2 + d_{12} + d_{23} + d_{123} + \max(d_{23,3}, d_{12,1}) & \leq n \\
    d_3 + d_{13} + d_{23} + d_{123} + \max(d_{13,1}, d_{23,2}) & \leq n
\end{align*}
\] (3.84)

Together with all other implicit constraints, we have that an achievable region for the entire systems is given as

\[
\{(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}, d_{12,1}, d_{12,2}, d_{23,2}, d_{23,3}, d_{13,1}, d_{13,3}) \in \mathbb{R}_+^{13} : \\
    d_1 + d_{12} + d_{13} + d_{123} + \max(d_{12,2}, d_{13,3}) \leq n \} \quad (3.85) \\
    d_2 + d_{12} + d_{23} + d_{123} + \max(d_{23,3}, d_{12,1}) \leq n \} \quad (3.86) \\
    d_3 + d_{13} + d_{23} + d_{123} + \max(d_{13,1}, d_{23,2}) \leq n \} \quad (3.87)
\]
\begin{align}
0 & \leq d_1 - d_{12,1} - d_{13,1} \leq m - 2n \quad \text{(3.88)} \\
0 & \leq d_2 - d_{12,2} - d_{23,2} \leq m - 2n \quad \text{(3.89)} \\
0 & \leq d_3 - d_{13,3} - d_{23,3} \leq m - 2n \} \quad \text{(3.90)}
\end{align}

where inequalities (3.88)-(3.90) come from the constraints that

\[
\begin{cases}
0 \leq c_1 \leq m - 2n \\
0 \leq c_2 \leq m - 2n \\
0 \leq c_3 \leq m - 2n
\end{cases}
\]

By doing Fourier-Motzkin Elimination and projecting this 13-dimensional polyhedron onto the 7-dimension positive orthant $\mathbb{R}_+^E$, we obtain that an achievable DoF for this case is exactly equal to $\mathbb{D}_3$.

Remark 12. In each of the 3 cases, the entire $M$-dimensional beamformer space can be partitioned into two linearly independent parts, where each part contains one of the four classes of beamformer subspaces defined in the precoding matrix decomposition. For case 1 and 2, the Minkowski-Sum of the achievable regions of the two parts are equal to the final LDoF region of the entire system, however, the Minkowski-Sum region is sub-optimal for case 3. The reason behind is that in the optimal precoding scheme of case 3, cooperations between two parts are required, but not necessary in case 1 and 2. We further give an illustration of the difference here. In case 3, one remaining $v_1$ in part 1 alone can only be used to transmit message $W_1$, however, it has the potential to help part 2 transmitting one more $W_{12}$ or $W_{13}$. As a result, the DoF region is extended in the direction of $W_{12}$ and $W_{13}$. However, in case 2, instead of using beamformer $v_{12R}$ to help part 2 transmitting one more $W_{12}$ or $W_{13}$ or $W_{23}$, we can directly use it to transmit a symbol of message $W_{12}$, and then use the pair in part 2 to transmit the other $W_{13}$ or $W_{23}$. There is no extra benefit of cooperation. Similarly, in case 1, such a cooperation is not meaningful in term of overall performance.
3.7 Examples of the precoding scheme

In this section, we give a specific example to illustrate our precoding scheme introduced in Section 3.5 and show how it works.

Consider the system with antenna setting of $M = 12$, $N_1 = N_2 = N_3 = 5$. According to Theorem 4, the LDoF region of this system is given by

\[
\begin{align*}
&d_1 + d_{12} + d_{13} + d_{123} \leq 5 \\
&d_2 + d_{12} + d_{23} + d_{123} \leq 5 \\
&d_3 + d_{13} + d_{23} + d_{123} \leq 5 \\
&d_1 + d_2 + d_3 + 2d_{12} + d_{13} + d_{23} + 2d_{123} \leq 12 \\
&d_1 + d_2 + d_3 + d_{12} + 2d_{13} + d_{23} + 2d_{123} \leq 12 \\
&d_1 + d_2 + d_3 + d_{12} + d_{13} + 2d_{23} + 2d_{123} \leq 12
\end{align*}
\]

Checking numerically, this 7 dimensional polytope has in total 56 non-trivial corner points. We show how to achieve some of them for examples.

First let's look at the decomposition of precoding matrix. Calculating the values of the 8 auxiliary variables in Equations (3.16)-(3.23), we have that $a = b = c = 3$, $d = e = f = 0$, $x = 2$ and $r = 0$. From Figure 3.7, we have that the dimensions of subspaces $V_{12R}$, $V_{13R}$, $V_{23R}$ and $V_{123}$ are all equal to zero. Deleting them from the concatenated matrix $V$, we have $V = [V_1 V_2 V_3 V_{13X} V_{12X}]$ and $\text{Rank}(V_1) = \text{Rank}(V_2) = \text{Rank}(V_3) = 2$ and $\text{Rank}(V_{13X}) = \text{Rank}(V_{12X}) = 3$. The corresponding image at receivers is given by

\[
U = \begin{bmatrix}
A_1 & 0 & 0 & A_{13X} & A_{12X} \\
0 & B_2 & 0 & 0 & B_{12X} \\
0 & 0 & C_3 & C_{13X} & 0
\end{bmatrix}.
\]

By construction, the rank of each sub-block is given as follows:

$\text{Rank}(A_1) = \text{Rank}(B_2) = \text{Rank}(C_3) = 2$
\[
\text{Rank}(A_{13X}) = \text{Rank}(A_{12X}) = \text{Rank}(C_{13X}) = \text{Rank}(B_{12X}) = 3.
\]

If we replace the sub-blocks in matrix \(U\) with its ranks, we have the following dimension matrix

\[
\begin{bmatrix}
2 & 0 & 0 & 3 & 3 \\
0 & 2 & 0 & 0 & 3 \\
0 & 0 & 2 & 3 & 0
\end{bmatrix}.
\]

Furthermore, from the results of Section 3.4, we have that the subspaces spanned by \(A_{13X}\) and \(A_{12X}\) are aligned together at receiver 1, i.e., \(\text{Span}(A_{13X}) = \text{Span}(A_{12X})\), and \(\text{Span}([A_{13X} A_{12X}])\) is linearly independent with \(\text{Span}(A_1)\), \(\text{Span}(B_{12X})\) is linearly independent with \(\text{Span}(B_2)\), and \(\text{Span}(C_{12X})\) is linearly independent with \(\text{Span}(C_3)\).

Now let’s look at the achievable scheme of several corner points of the LDoF region.

**3.7.1** \((d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) = (2, 2, 2, 0, 0, 0, 3)\)

This is an easy case. First assign both 2 dimensions of \(V_i\) \((i = 1, 2, 3)\) to message \(W_i\), i.e., use \(v_i^{1,2}\) for message \(W_i\). Then, use the intersection part to transmit \(W_{123}\), i.e., use \(v_{13x}^{1,3} + v_{12x}^{1,3}\) for the three symbols of \(W_{123}\), respectively.

**3.7.2** \((d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) = (5, 5, 0, 0, 0, 0, 0)\)

First, assign both 2 dimensions of \(V_i\) \((i = 1, 2)\) to message \(W_i\), i.e., use \(v_i^{1,2}\) for message \(W_i\). Then, use the 3 pairs of \((v_{13x}, v_{12x})\) pairs to send the rest \(W_1\) and \(W_2\) symbols. More specifically, use \(v_{13x}^{1,3}\) to transmit 3 symbols of \(W_1\), respectively, and use the implicitly contained \(v_{23x}^{1,3}\) to transmit 3 symbols of \(W_2\), respectively. Note that \(v_{13x}^{1,3}\) and \(v_{23x}^{1,3}\) will be overlap together at receiver 3. However, it does not matter, since receiver 3 does not need to decode either of them.

**3.7.3** \((d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) = (0, 0, 2, \frac{7}{2}, \frac{3}{2}, \frac{3}{2}, 0)\)

Note that in this case, the corner point is not all integer-valued. We can simply use symbol extensions of length 2. The dimensions of different beamformers are all doubled accordingly.
The independence among them are unchanged. The problem is equivalent to that to achieve
\((d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) = (0, 0, 4, 7, 3, 3, 0)\) using beamformers with equivalent dimension matrix equal to
\[
\begin{bmatrix}
4 & 0 & 0 & 6 & 6 \\
0 & 4 & 0 & 0 & 6 \\
0 & 0 & 4 & 6 & 0
\end{bmatrix}
\]

First, for the 4 symbols of message \(W_3\), we use \(v_3^{1:4}\) to transmit them, respectively. Then, we use \(v_{13X}^{1:3}\) to send the 3 symbols of \(W_{13}\), and use \(v_{23X}^{4:6}\) to send the 3 symbols of \(W_{23}\). For \(W_{12}\), we use beamformers \(v_{12X}^{1:3} + v_1^{1:3}, v_{12X}^{1:3} + v_2^{1:3}\) and \(v_1^4 + v_2^4\).

Now, let’s prove the decodability of all symbols at their intended beamformers. Apparently, all 4 \(W_3\) symbols are decodable at receiver 3, since each of them occupies an independent dimension at receiver 3. Next, consider the case that there is no \(W_{12}\) symbols yet. It is easy to see that \(v_{13X}^{1:3}\) are decodable at both receivers 1 and 3, and \(v_{23X}^{4:6}\) are decodable at both receivers 2 and 3, since they also occupy independent dimensions at their intended receivers. Finally, consider the 7 beamformers for message \(W_{12}\). Since receiver 1 can decode beamformer \(v_1^{1:4}\) and receiver 2 can decode beamformer \(v_2^{1:4}\), beamformer \(v_1^4 + v_2^4\) for \(W_{12}\) can thus be received and recovered by both receiver 1 and receiver 2, and receiver 1 can recover \(v_{12X}^{1:3} + v_1^{1:3}\), receiver 2 can recover \(v_{12X}^{1:3} + v_2^{1:3}\). So we only need to show receiver 1 is also able to recover \(v_{12X}^{1:3} + v_1^{1:3}\), and receiver 2 is able to recover \(v_{12X}^{1:3} + v_1^{1:3}\). We also need to show the transmission of \(W_{12}\) doesn’t interfere with existing symbols of \(W_{23}\) and \(W_{13}\).

Here we prove for \(v_{12X}^{4:6} + v_2^{1:3}\) as an example.

Note that \(v_{23X}^{4:6}\) has been used for \(W_{23}\), when we use \(v_{12X}^{4:6} + v_2^{1:3}\) to send \(W_{12}\), the signal received by receiver 1 is in new independent dimensions, i.e., \(H_1 v_{12X}^{4:6}\), so it’s decodable by receiver 1. However, \(H_2 v_{12X}^{4:6}\) and \(H_2 v_{23X}^{4:6}\) lie in the same subspace, which means \(v_{12X}^{4:6} + v_2^{1:3}\) would bring problem for receiver 2 to decode \(W_{23}\). As we have mentioned before, this interference can be removed because receiver 2 is able to decode \(v_2^{1:3}\) and consequently it can subtract \(H_2 v_{12X}^{4:6}\) from its signal and get pure \(H_2 v_{23X}^{4:6}\). So \(v_{12X}^{4:6} + v_2^{1:3}\) won’t impact the decoding of \(W_{23}\) at receiver 2.
Since $W_{13}$ is using the first 3 intersection vector ($\mathbf{v}^{1\times 3}_{13X}$), $\mathbf{v}^{4\times 6}_{12X} + \mathbf{v}^{1\times 3}_{2}$ won’t impact $W_{13}$ either.

Similarly, we have that $\mathbf{v}^{1\times 3}_{12X} + \mathbf{v}^{1\times 3}_{1}$ is decodable at receiver 2 and it doesn’t interfere with $W_{13}$.

3.7.4 $(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) = (2, 2, 0, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}, 0)$

Similarly, in this case, it is equivalent to that to achieve $(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) = (4, 4, 0, 1, 5, 5, 0)$ using beamformers with equivalent dimension matrix equal to

$$
\begin{bmatrix}
4 & 0 & 0 & 6 & 6 \\
0 & 4 & 0 & 0 & 6 \\
0 & 0 & 4 & 6 & 0
\end{bmatrix}.
$$

First, allocate entire 4 dimensions of $V_1$, i.e., $\mathbf{v}^{1\times 4}_1$, to 4 symbols of message $W_1$, and allocate entire 4 dimensions of $V_2$, i.e., $\mathbf{v}^{1\times 4}_2$, to 4 symbols of message $W_2$. Then, we use $\mathbf{v}^{1\times 5}_{23}$ to transmit 5 symbols of message $W_{23}$, use $\mathbf{v}^{1\times 4}_{13} + \mathbf{v}^{1\times 4}_3$ and $\mathbf{v}^{6}_{13}$ to transmit 5 symbols of message $W_{13}$, and use $\mathbf{v}^{5}_{12} + \mathbf{v}^{6}_{12}$ to transmit 1 symbol of $W_{12}$.

The $W_1$ and $W_2$ is guaranteed to be able to be recovered at receiver 1 and receiver 2, respectively. Receiver 3 can decode $\mathbf{v}^{1\times 4}_{13} + \mathbf{v}^{1\times 4}_3$ and $\mathbf{v}^{6}_{13}$, and then by removing $H_3 \cdot \mathbf{v}^{1\times 4}_{13}$ from its received signal, it can decode all 5 symbols of $W_{23}$. Receiver 2 can first decode $\mathbf{v}^{5}_{12} + \mathbf{v}^{6}_{12}$, and then by removing $H_2 \cdot \mathbf{v}^{5}_{12}$, it can recover all 5 symbols of $W_{23}$. Receiver 1 is able to decode $\mathbf{v}^{1\times 4}_{13} + \mathbf{v}^{1\times 4}_3$ and $\mathbf{v}^{5}_{12} + \mathbf{v}^{6}_{12}$, and then by removing $H_1 \cdot \mathbf{v}^{6}_{12}$, it is able to decode $\mathbf{v}^{6}_{13}$.

In sum, all messages can be recovered at their intended receivers, respectively.

3.8 The LDoF region with general antenna setting

In this section, we present the 9-dimensional linear degrees of freedom region of the 3-user MIMO broadcast channel with general message sets under arbitrary antenna setting. The proofs of converse and achievability follow in the same manner as that in the proofs under symmetric antenna setting. However, due to the extensive possible relationships among the values of $m, n_1,$
$n_2$ and $n_3$, the detail is rather tedious and complicated. Thus, we present our current result as a conjecture here.

Again, we assume that $m \leq n_1 + n_2 + n_3$, otherwise, the problem is trivial. Also, we assume that $m \geq \max(n_1, n_2, n_3)$. If any receiver has more antenna than the transmitter does, it is easy to show that this receiver has redundant antennas, and the DoF region of the system will be the same if we reduce the number of antenna of that receiver to $m$.

**Conjecture 1.** Assuming that $m \leq n_1 + n_2 + n_3$ and $m \geq \max(n_1, n_2, n_3)$, the linear DoF region of the three-user broadcast channel with general message sets is given by

$$\{(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123}) \in \mathbb{R}_+^E :$$

\begin{align*}
&d_1 + d_{12} + d_{13} + d_{123} \leq n_1 \\
&d_2 + d_{12} + d_{23} + d_{123} \leq n_2 \\
&d_3 + d_{13} + d_{23} + d_{123} \leq n_3 \\
&d_1 + d_2 + d_3 + d_{12} + d_{13} + d_{23} + d_{123} \leq m \\
&d_1 + d_2 + d_3 + 2d_{12} + d_{13} + d_{23} + 2d_{123} \\
&\leq n_1 + n_2 + (m - n_1 - n_2)^+ \\
&d_1 + d_2 + d_3 + d_{12} + 2d_{13} + d_{23} + 2d_{123} \\
&\leq n_1 + n_3 + (m - n_1 - n_3)^+ \\
&d_1 + d_2 + d_3 + d_{12} + d_{13} + 2d_{23} + 2d_{123} \\
&\leq n_2 + n_3 + (m - n_2 - n_3)^+ \\
&2d_1 + 2d_2 + 2d_3 + 2d_{12} + 2d_{13} + 2d_{23} + 3d_{123} \\
&\leq 2m + (n_1 + n_2 + n_3 - 2m)^+ \}.$$

### 3.9 Conclusion

The optimal linear coding scheme of the 3-user MIMO broadcast channel with general message sets is studied in this chapter. We propose a novel perspective of analyzing the channels via subspace
resource identification and decomposing the overall $m$-dimensional beamformer vector space into a series of linearly independent subspaces. By appropriately allocating beamformers from each subspace to their best matching messages, the interference reaching at each receiver is minimized. An outer bound on the 7-dimensional linear degrees of freedom region is given under the symmetric antenna setting, i.e., $(m, n, n, n)$ case. The outer bound is further shown to be achievable using our proposed precoding scheme. In other words, it is the LDoF region of the system, and the precoding scheme is optimal under the consideration of LDoF. Finally, the result is extended to the general asymmetric antenna configurations.
Chapter 4

Degrees of Freedom of the Two-User MIMO Broadcast Channel with Private and Common Messages Under Hybrid CSIT Models

4.1 Introduction

Multiple-input multiple-output (MIMO) systems can provide a multiplicative gain in capacity compared to their single-input single-output (SISO) counterparts, with the multiplicative factor variously referred to as capacity pre-log, spatial multiplexing gain, or degrees of freedom. For example, the point-to-point (PTP) MIMO system with $M$ transmit antennas and $N$ receive antennas has $\min(M,N)$ degrees of freedom, i.e., its capacity grows linearly with $\min(M,N)$ in the high signal-to-noise ratio (SNR) regime [42]. Moreover, in order to achieve this rate of growth of the capacity, channel state information at the transmitter (CSIT) is not needed.

However, CSIT plays a vital role in multi-user channels. For example, in the two-user MIMO broadcast channel (BC), CSIT can be used to send information along different zero-forcing beams to the two receivers simultaneously so as to not create interference at unintended receivers [3]. The sum-DoF of $\min(M,N_1 + N_2)$ can be achieved in this way, where $M, N_1, N_2$ are the numbers of antennas at the transmitter and at Receivers 1 and 2, respectively; in effect, from the DoF perspective, the availability of CSIT is the antidote that exactly neutralizes the fact that the receivers are distributed and non-cooperating. Another example is the two-user MIMO X channel, a two-transmit, two-receive interference network where each transmitter has two independent messages, one for each receiver, in which CSIT can be used for zero-forcing beamforming as well as to align interference from the two unintended messages into the same subspace (to the extent possible).
at the receiver where they are not desired [8, 43]. The implementation of both transmitter zero-forcing and interference alignment requires CSIT. Without CSIT, the DoF collapse to the extent that time-division alone is DoF-optimal [44, 19].

Henceforth, the term “two-user MIMO BC” refers to the BC with general antenna configuration as defined above, and will also be referred to as the \((M, N_1, N_2)\) BC. Without loss of generality, we assume that \(N_1 \geq N_2\) throughout.

Since the receivers are able to save and post-process the data, we will assume, as is commonly done in the literature, that there is perfect channel state information at the receivers (CSIR). However, the benefits of perfect and instantaneous CSIT notwithstanding, practical settings in which such CSIT can be acquired are more of an exception than the rule. The typical approach to obtain CSIT is to transmit pilot signals, have the receivers measure the channel state and send this measured channel state back to the transmitter via feedback links [45]. In constant or slowly time-varying networks, it may be reasonable to assume that the channel state information at the transmitter(s) acquired in this manner remains unchanged and valid when it is used for the subsequent transmission.

But if the delay between the time when the channel state information is measured and the time when it is used at the transmitters is non-negligible compared to the rate of channel variation, the transmitter cannot use the outdated channel state information as if it were current. A natural way to deal with this delay is to predict the current channel state using previous information and the channel time-correlation model, and then use the predicted channel state as if it were the true channel state in a scheme designed for the prefect CSIT case [46]. In this scheme, the accuracy of prediction plays a significant role on the effective (finite SNR) multiplexing gain achieved.

When the delay is significant compared to the rate of channel variation however, even this prediction-based approach may fail in that the predicted values are poor estimates of the current channel state. In such cases, one may be better-off relying only on past channel states, even if they are independent of the current state, i.e., even if they are completely outdated. Such an approach was proposed in [47]. It was shown that even when channel fading states across symbols
are independent and identically distributed (i.i.d.), in which predicting the channel state based on past information is impossible, finitely delayed channel state information is still useful in many cases (an advantage of allowing an arbitrary finite delay is that accurate estimation of the channel state becomes possible). For example, the multi-input, single-output (MISO) broadcast channel with \( K \) transmit antennas and \( K \) single antenna receivers can achieve a sum-DoF of \( \frac{K}{\log K} \) with delayed (and accurate) CSIT, while only 1 degree of freedom is achievable when there is no CSIT [19]. Although \( \frac{\log K}{1+1/2+...+1/K} \) is much smaller than \( K \), which is the sum DoF of the same system under the assumption of perfect CSIT, the scaling of sum DoF as \( O(K/\log K) \) is still significant and inspiring compared to the no CSIT result. In [48], the authors extend the MISO BC results to the MIMO BC with an arbitrary number of antennas at each terminal. An outer bound of the DoF region is provided, which is further shown to be tight for the two-user case in [48] and for certain symmetric three-user cases (with equal numbers of antennas at all receivers) in [49] by providing the respective DoF-region-optimal achievability schemes. The key idea of using delayed CSIT in [47, 48] is that, the interference experienced by a certain receiver at a previous time is useful in the future for another receiver where that interference is a desired signal. If the transmitter re-sends a copy of that previous interference, which it can obtain using delayed CSIT feedback, not only does it benefit the other receiver where that interference is desired but it would also not cause interference at the same user again, since this user is able to cancel its influence using the saved version of the past received signal containing that interference. Thus, in this phase, transmission could be more efficient than under the no CSIT assumption.

Besides these symmetric or homogeneous CSIT assumptions in which the availability of CSI from all receivers are at the same level (i.e., perfect (P), delayed (D) or no (N) CSIT), there are more general, and perhaps more commonly occurring, scenarios in which one can expect different types of CSI from different receivers due to the heterogeneity of channel variations. In [50], the DoF region of the two-user MIMO BC is studied in which the one receiver’s channel is known instantaneously and perfectly at the transmitter, whereas the other receiver’s channel is known to
it in a delayed manner. The DoF region in this hybrid setting, henceforth called the ‘PD’ case\(^1\)
is, in general, larger than that in the symmetric delayed ‘DD’ CSIT case, and smaller than that in
the symmetric perfect ‘PP’ CSIT case. Such a phenomenon is also observed in the two-user MIMO
interference channel in [51]. Such results on the sensitivity of even the DoF of wireless networks to
the extent of availability of CSIT underscore the importance of hybrid CSIT models.

The ‘PN’ case, in which perfect CSI is available from one receiver and no CSI is available
from the other, is more challenging. The authors of [52] introduce the idea of “aligned image sets”
and prove that the special case of the two-user MISO BC with perfect CSI from one receiver and
finite-precision CSIT from the other (hence including the ‘PN’ case), has a maximum sum DoF of
just 1. In particular, the perfect channel knowledge for one user at the transmitter does not help in
improving the DoF beyond that of the ‘NN’ case. The works of [53, 54, 55] investigate the MISO
BC for more than two users under hybrid CSIT. In particular, [55] makes significant progress that
includes the exact DoF under the constraint of linear encoding strategies (known as the linear DoF,
denoted LDoF) for the three-user MISO BC for all possible \(3^3\) hybrid CSIT models. In spite of
these advances on the MISO BC however, the generalization of the result of [52] on the two-user
MISO BC to the \((M,N_1,N_2)\) BC is, to the best of the authors’ knowledge, an open problem.

Because of the special difficulty that hybrid CSIT models pose even in the two-user MIMO
BC when exactly one of the two channels is not known at the transmitter, we classify the nine
hybrid CSIT models as belonging to one of two types throughout this paper. Type I contains the
five hybrid CSIT models \{‘NN’, ‘DD’, ‘DP’, ‘PD’, ‘PP’\}\(^2\) in which either both channels are not
known or each of the two channels is known perfectly or with delay. Type II contains the other
four hybrid CSIT models \{‘ND’, ‘DN’, ‘NP’, ‘PN’\}, in which exactly one of the two channels is not
known at the transmitter.

For Type I models, the exact DoF region of the two-user MIMO BC with private messages

\(^1\) For the nine possible hybrid CSIT cases, we use a concatenation of two letters each drawn from the alphabet \(\{P, D, N\}\) to denote the status of CSI from Receivers 1 and 2, in that order. For example, ‘PD’ means that the transmitter has perfect knowledge of the first receiver’s channel state and delayed knowledge of the second receiver’s channel state.

\(^2\) Because we assume throughout that \(N_1 \geq N_2\), symmetric hybrid CSIT models, such as ‘PD’ and ‘DP’ or ‘PN’ and ‘NP’ must be considered as two distinct models.
only (henceforth referred to as the BC-PM) have been found in the literature [48, 50, 44, 19]. One contribution of this paper is the complete characterization of the LDoF regions of the $(M, N_1, N_2)$ BC-PM for the Type II hybrid CSIT models. A key result we obtain in this regard is a tight outer bound on the LDoF region for the ‘PN’ (and ‘DN’) hybrid CSIT models.

While much work has been devoted to the study of transmitting private messages (i.e., multiple unicasting) over the broadcast channel (i.e., the BC-PM), the more general as well as the more interesting problem of simultaneous groupcasting has received much less attention. In simultaneous groupcasting, there may be exponentially many (in number of receivers) independent messages, one message desired by each distinct subset or group of receivers.

In this paper, we study the two-user fast fading Gaussian MIMO $(M, N_1, N_2)$ BC with simultaneous two-unicasting and multicasting, i.e., the transmitter has two independent private messages intended for each of the two users, respectively, and one common multicast message which is desired at both users. Henceforth, we will refer to this broadcast channel with the three messages simply as the MIMO BC-CM, or as the $(M, N_1, N_2)$ BC-CM. The BC-PM is evidently a special case of the BC-CM, as is the BC with degraded messages (i.e., with a private message intended for one receiver and a common message for both receivers), denoted henceforth as the BC-DM.

The fixed two-user Gaussian MIMO BC-CM (without fading) under perfect CSIT has been extensively studied previously. An achievable scheme consisting of a linear superposition of Gaussian codewords for the common message with a dirty-paper coding (DPC) scheme for the private messages was proposed in [56]. The resulting inner bound (the DPC region) on the capacity region, was shown to be tight in certain sub-regions in [57]. Meanwhile, the DoF region of the two-user MIMO BC-CM, also under the perfect CSIT assumption, was obtained in [24]. In [24], the generalized singular value decomposition (GSVD) was used to construct a parallel Gaussian broadcast channel so as to obtain an outer bound on the DoF region, and it was shown that that bound can be attained by an achievable scheme also based on the GSVD. As a special case of a more general result on the interference channel with general message sets, [31, 43] also obtain the DoF region but
with a scheme based just on the singular value decomposition (SVD)\(^3\). An outer bound based on the GVSD and relaxation of the input power in [24] (a refinement of that in [12]) is shown therein to be within an SNR-independent (but channel-dependent) constant of the DPC region of [56], thereby providing an approximation of the capacity region within an SNR-independent additive gap. Finally, the authors of [13] prove the optimality of Gaussian inputs in Marton’s inner bound to establish that the DPC region of [56] is indeed the capacity region of the two-user Gaussian MIMO BC-CM.

In what are the main results of this work, we establish the DoF regions of the two-user fast fading \((M, N_1, N_2)\) BC-CM under the Type I hybrid CSIT models and the LDoF regions for the Type II hybrid CSIT models. These results represent significant progress on the understanding of the BC-CM beyond the perfect CSIT (or ‘PP’) setting in practically relevant scenarios, where we associate practical relevance to fast fading and the extent of availability of CSIT. It is further conjectured that, for the Type II cases, the obtained LDoF regions are also the respective DoF regions.

In obtaining the outer bounds for the DoF/LDoF regions for the \((M, N_1, N_2)\) BC-CM, we demonstrate the relationship between the two-user MIMO BC-CM and the two-user MIMO BC-PM. The key idea for obtaining the outer bound on the DoF (or LDoF) region of the BC-CM is via the approach of loosening the decoding requirement of the common message so that it is decoded only at one receiver. In other words, the common message is devolved into either one or the other of the two private messages, and the outer bounds for the resulting MIMO BC-PM are then used to obtain outer bounds for the MIMO BC-CM. Remarkably, this approach works for all the nine hybrid CSIT cases, in the sense that it produces for the MIMO BC-CM tight outer bounds for the DoF regions under hybrid CSIT cases of Type I and tight outer bounds for the LDoF regions under hybrid CSIT cases of Type II.

Then, it is shown that all the corner points of the three-dimensional DoF (respectively, LDoF)

\(^3\) Indeed, the DoF region for the \(2 \times 2\) network seen as two interfering BC-CMs (i.e., with two different transmitters but with common receivers) with six messages altogether is also fully established as a special case of an even more general result in [31, 43].
outer bound regions of MIMO BC-CM thus obtained under all hybrid CSIT assumptions except
the (previously known) ‘PP’ case have at least one zero element. The achievability proof in each
case thus consists of solving one of two sub-problems: the achievability of the DoF (LDoF) region of
MIMO BC-PM and the achievability of the DoF/LDoF region of MIMO BC-DM, both using linear
encoding strategies. We obtain the achievability schemes for the private message MIMO BC for the
Type II hybrid CSIT models (with those for Type I known in the literature) corresponding to all
relevant corner points of the outer bound regions of the BC-CM. We also obtain linear achievability
schemes for the MIMO BC-DM for both Type I and Type II hybrid CSIT models corresponding to
all relevant corner points of the outer bound regions of the BC-CM. Any DoF-tuple in the DoF (or
LDoF respectively) region of the BC-CM is then achieved using these strategies via time-sharing.
Remarkably again, this high-level description of the overall strategy for obtaining the DoF/LDoF
region for the MIMO BC-CM applies to each one of the hybrid CSIT models (except the previously
known ‘PP’ case). In other words, in each of the eight non-‘PP’ cases, it is sufficient to time-share
between schemes designed for the BC-PM and the BC-DM.

Notation: $\mathbb{R}_n^+$ and $\mathbb{Z}_n^+$ denote the set of $n$-tuples nonnegative real numbers and integers,
respectively. $(x)^+$ means $\max(x, 0)$. $\text{null}(A)$ denotes the nullspace of the linear transformation $A$.

### 4.2 System Model, DoF and LDoF

![Figure 4.1: The 2-user MIMO broadcast channel with common message](image-url)
In this section, we define the system model of the two-user MIMO BC-CM under hybrid CSIT and the DOF and LDoF metrics.

Consider the MIMO \((M, N_1, N_2)\) Gaussian broadcast channel with arbitrary antennas setting, i.e., the transmitter has \(M\) antennas and the two users have \(N_1, N_2\) receive antennas, respectively. We will assume without loss of generality that \(N_1 \geq N_2\), because if \(N_1 < N_2\), we could exchange the indexes of the two users. As is shown in Figure 4.1, the transmitter has two private messages \(W_1\) and \(W_2\) intended for two receivers, respectively, and one common message \(W_0\), which is desired by both receivers. The channel matrices \(H_1(t) \in \mathbb{C}^{N_1 \times M}\) and \(H_2(t) \in \mathbb{C}^{N_2 \times M}\) are i.i.d. across time and receiver indexes, and their entries are i.i.d. standard complex normal \(\mathcal{CN}(0,1)\) random variables. The transmitter can have either perfect/instantaneous (P), delayed (D) or no (N) channel state information (CSI) available from each receiver. When considering the delayed CSIT, without loss of generality, the delay can be taken to be one time unit. Hence, the transmitter with delayed knowledge of receiver \(r\)’s channel knows \(H_r(t-1)\) at time \(t\).

As previously stated in Footnote 2, for the nine possible hybrid CSIT cases, we use a concatenation of two letters each drawn from the alphabet \(\{P, D, N\}\) to denote the status of CSI from Receivers 1 and 2, in that order. For example, ‘PD’ means that the transmitter has perfect knowledge of the first receiver’s channel state and delayed knowledge of the second receiver’s channel state.

The signals received at receiver \(r\) \((r = 1, 2)\) at time \(t\) is given by

\[
Y_r(t) = H_r(t)X(t) + Z_r(t),
\]  

(4.1)

where \(X(t) \in \mathbb{C}^{M \times 1}\) is the transmitted signal at time \(t\), \(Z_r(t) \in \mathbb{C}^{N_r \times 1}\) is the additive white Gaussian noise (AWGN) vector at receiver \(r\). The channel input is subject to an average power constraint, which is taken to be \(E(X^\dagger(t)X(t)) \leq P\) for all \(t\) (the superscript \(\dagger\) denotes complex conjugate transpose). For codewords occupying \(t_0\) channel uses, we say that the rate-tuple \((R_1, R_2, R_0)\) is achievable if the probabilities of decoding error for all three messages can be made arbitrarily small simultaneously by choosing appropriately large \(t_0\). The capacity region \(\mathcal{C}(P)\) is then defined as the
set of all achievable rate tuples \((R_1, R_2, R_0)\), while the DoF region \(\mathcal{D}\) is defined as

\[
\mathcal{D} \triangleq \left\{ (d_1, d_2, d_0) \in \mathbb{R}^3_+ : \forall (\omega_1, \omega_2, \omega_0) \in \mathbb{R}^3_+ \sum_{x=0,1,2} \omega_x d_x \leq \limsup_{\rho \to \infty} \left[ \sup_{R(\rho) \in C(\rho)} \frac{\sum_{x=0,1,2} \omega_x R_x(\rho)}{\log(\rho)} \right] \right\}.
\]

If we restrict ourselves to linear coding strategies as defined in [38, 39], in which the degrees of freedom simply indicates the dimension of the linear subspace of transmitted signal, we obtain the linear DoF (denoted LDoF) of the system. More specifically, consider a linear coding scheme with block length \(T\). At time \(t\), \((t = 1, ..., T)\), the three messages are modulated with precoding matrix \(V_i(t)\) \((i = 1, 2, 0)\), respectively. The column size of matrix \(V_i(t)\) is equal to the number of independent information symbols of message \(W_i\) that will be transmitted in the entire \(T\) time slots. The signal transmitted by the transmitter at time \(t\) can be written as

\[
S(t) = \sum_{i=0}^{2} V_i(t)x_i^{(T)}
\]

where \(x_i^{(T)} \in \mathbb{C}^{m_i^{(T)} \times 1}\) contains the entire \(m_i^{(T)}\) information symbols. Ignoring noise, the signal received by receiver \(r\) \((r = 1, 2)\) is equal to \(H_r(t)S(t)\). Letting \(V_i^{(T)}\) be the overall precoding matrix of message \(W_i\) of the entire block, and \(V_i(t) \in \mathbb{C}^{M \times m_i^{(T)}}\) be its \(t^{th}\) block row (that determines the transmitted signal at time \(t\)), we have that

\[
V_i^{(T)} = \begin{bmatrix}
V_i(1) \\
V_i(2) \\
\vdots \\
V_i(T)
\end{bmatrix}.
\]

The equivalent overall channel matrix will be the block diagonal matrix given by

\[
H_r^{(T)} = \begin{bmatrix}
H_r(1) & 0 & \cdots & 0 \\
0 & H_r(2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_r(T)
\end{bmatrix}.
\]
At receiver $r$, the corresponding signal subspace is $\text{Span}(H_r^{(T)}[V_r^{(T)} V_0^{(T)}])$, the interference subspace is $\text{Span}(H_r^{(T)} V_{r'}^{(T)})$, where $r = 1, 2$ and $r' = 3 - r$. In order to decode the information symbols correctly, the signal subspace and interference subspace must be linearly independent with each other and the signal subspace must reserve the full column rank. In other words, the following two constraints need to be satisfied for both $r = 1$ and $r = 2$:

$$\text{rank}(H_r^{(T)}[V_r^{(T)} V_0^{(T)} V_r^{(T)}]) = \text{rank}(H_r^{(T)}[V_r^{(T)} V_0^{(T)}]) + \text{rank}(H_r^{(T)} V_{r'}^{(T)})$$ (4.2)

$$\text{rank}(H_r^{(T)}[V_r^{(T)} V_0^{(T)}]) = m_i^{(T)} + m_0^{(T)}$$ (4.3)

Based on this setting, we now define the LDoF of MIMO 2-user BC-CM. The DoF tuple $(d_1, d_2, d_0)$ is linearly achievable if there exists a sequence of linear encoding strategies with block length of $T$, such that for each $T$ and the choice of $m_i^{(T)} (i = 1, 2, 0)$, $V_i^{(T)}$ satisfy the decodability conditions (4.2) and (4.3) with probability 1, and

$$d_i = \lim_{T \to \infty} \frac{m_i^{(T)}}{T}$$

holds for each $i = 1, 2, 0$. We also define the LDoF region, denoted $\mathbb{D}_L$, as the closure of the set of all achievable 3-tuple $(d_1, d_2, d_0)$.

### 4.3 Main Results

As stated previously, in this paper, we establish the DoF regions of the general 3-message, 2-user MIMO $(M, N_1, N_2)$ BC-CM under the five hybrid CSIT models of Type I, i.e., ‘NN’, ‘DD’, ‘DP’, ‘PD’ and ‘PP’ cases. For the hybrid CSIT models of Type II, i.e., ‘PN’, ‘DN’, ‘NP’ and ‘ND’ cases, LDoF regions are established. These LDoF regions are also conjectured to be the DoF regions for the respective hybrid CSIT models.

The associated specializations to the degraded messages MIMO BC-DM under all hybrid CSIT models are also new. Furthermore, the specializations of the LDoF regions of the MIMO BC-CM to the private-message MIMO BC-PM under hybrid CSIT models of Type II are new as well.
The following two theorems present the main results on the MIMO BC-PM and the more general MIMO BC-CM, respectively. In the first case, we also include the previously known DoF regions (for Type I models) for the sake of completeness.

Table 4.1: DoF region of the \((M, N_1, N_2)\) BC-PM under different CSIT assumptions

<table>
<thead>
<tr>
<th>Perfect CSIT (PP)</th>
<th>Hybrid CSIT (PD)</th>
<th>Hybrid CSIT (PN)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_1 \leq N_1)</td>
<td>(\frac{d_1}{\min(M,N_1)} \leq 1)</td>
<td>(\frac{d_1}{\min(M,N_1)} + \frac{d_2}{\min(M,N_2)} \leq 1)</td>
</tr>
<tr>
<td>(d_2 \leq N_2)</td>
<td>(\frac{d_2}{\min(N_1+N_2)} \leq 1)</td>
<td>(\frac{d_1}{\min(M,N_1)} + \frac{d_2}{\min(M,N_2)} \leq 1)</td>
</tr>
<tr>
<td>(d_1 + d_2 \leq M)</td>
<td>(\frac{d_1}{\min(M,N_1)} + \frac{d_2}{\min(M,N_2)} \leq 1)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hybrid CSIT (DP)</th>
<th>Delayed CSIT (DD)</th>
<th>Hybrid CSIT (DN)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{d_1}{\min(M,N_1)} + \frac{d_2}{\min(M,N_1+N_2)} \leq 1)</td>
<td>(\frac{d_1}{\min(M,N_1+N_2)} + \frac{d_2}{\min(M,N_2)} \leq 1)</td>
<td>(\frac{d_1}{\min(M,N_1)} + \frac{d_2}{\min(M,N_2)} \leq 1)</td>
</tr>
<tr>
<td>(\frac{d_2}{\min(M,N_2)} \leq 1)</td>
<td>(\frac{d_1}{\min(M,N_1)} + \frac{d_2}{\min(M,N_1+N_2)} \leq 1)</td>
<td></td>
</tr>
<tr>
<td>(\frac{d_1}{\min(M,N_1)} \leq 1)</td>
<td>(\frac{d_1}{\min(M,N_1+N_2)} \leq 1)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hybrid CSIT (NP)*</th>
<th>Hybrid CSIT (ND)*</th>
<th>No CSIT (NN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{d_1+d_2}{\min(M,N_1)} \leq 1)</td>
<td>(\frac{d_1}{\min(M,N_1+N_2)} \leq 1)</td>
<td>(\frac{d_1}{\min(M,N_1)} + \frac{d_2}{\min(M,N_2)} \leq 1)</td>
</tr>
<tr>
<td>(\frac{d_2}{\min(M,N_2)} \leq 1)</td>
<td>(\frac{d_2}{\min(M,N_2)} \leq 1)</td>
<td></td>
</tr>
</tbody>
</table>

* means the region is LDoF region.

The results in black are already known in the literature, while results in red are new results obtained in this paper.

**Theorem 5.** Let \(N_1 \geq N_2\). The DoF regions of the two-user MIMO BC-PM under the CSIT assumptions of Type I are given in Table 4.1, and the LDoF regions are provided in the same table for Type II hybrid CSIT models. We name the region of case ‘\(X_1X_2\)’ where \(X_1, X_2 \in \{P, D, N\}\),
as $D_{BC}^{X_1X_2}$. The label $(X_1X_2)$ denote the cases for which the corresponding region is the DoF region, whereas $(X_1X_2)^*$ is used to denote the LDoF cases in Table 4.1.

Proof. The DoF regions for Type I cases ‘PP’, ‘DD’ and ‘NN’ are solved in [24], [48] and [44] and [19], respectively. The DoF regions for the two remaining Type I cases ‘PD’ and ‘DP’ were solved in [50]. We give proofs for the LDoF regions shown in Table 4.1 for the Type II cases, i.e., under ‘PN’, ‘DN’, ‘NP’ and ‘ND’ CSIT. Among these, we show that the ‘PN’ case is key. In other words, given the solution to the ‘PN’ case, the other three cases can be easily resolved.

Because the proof of the ‘PN’ case is rather involved, we defer it to Section 4.4 for the sake of readability of this proof. Based on the result of ‘PN’ case however, the LDoF region of ‘DN’ case is immediately obtained since it is outer bounded by the LDoF region of the ‘PN’ case and inner bounded by the LDoF region of the ‘NN’ case, the two of which are seen (from Table 4.1) to be the same. The proof for the two remaining cases, ‘NP’ and ‘ND’, are given in what follows.

Consider the ‘NP’ case. First, $\frac{d_2}{\min(M,N_2)} \leq 1$ is a trivial outer bound. Then, by adding $N_1 - N_2$ extra antennas to Receiver 2, we have a new system with $N_1$ antennas at both receivers. Since adding extra antennas does not shrink the LDoF region, the LDoF region of this new system is an outer bound on that of the original system. From the result for the ‘PN’ case, we have that an outer bound on the LDoF region of the new system is $\frac{d_1 + d_2}{\min(M,N_1)} \leq 1$. Thus, it is also an outer bound for the original system under the ‘NP’ assumption. Next, consider achievability. We only consider the case that $M > N_2$, since otherwise, the achievable scheme is trivial since random beamforming suffices. Since the transmitter has perfect channel state information from Receiver 2, it is possible that it sends some symbols of message $W_1$ in the null space of $H_2$, such that this part can be zero-forced at Receiver 2. The maximum number of such streams that can be zero-forced is $M - N_2$. To achieve any integer-valued DoF pair $(d_1,d_2)$ within the outer bound, we use the following precoding scheme. For the entire $d_1$ streams of message $W_1$, $d_1^Z = \min(d_1, M - N_2)$ of them are transmitted using zero-forcing and thus will not be received at Receiver 2. The rest of the $d_1 - d_1^Z$ streams will be transmitted using random beamforming. For message $W_2$, we transmit
all its symbols using random beamforming. Now, consider the signal received by Receiver 1. It consists of $d_1 + d_2$ independent messages. Since $d_1 + d_2 \leq N_1$, Receiver 1 will be able to decode all these symbols. Receiver 2 would receive $d_1 - d_1^Z + d_2$ independent symbols. If $d_1 \leq M - N_2$, then $d_1 - d_1^Z + d_2 = d_2 \leq N_2$. If $d_1 > M - N_2$, then

$$\begin{align*}
d_1 - d_1^Z + d_2 &= d_1 - (M - N_2) + d_2 \\
&= (d_1 + d_2 - M) + N_2 \\
&\leq N_2.
\end{align*}$$

In summary, the number of independent symbols received by Receiver 2 is also no greater than its antenna numbers. Thus, Receiver 2 will be able to recover all these symbols. As a result, the DoF tuple $(d_1, d_2)$ is achieved. Since all the corner points of the outer bound are integer-valued and thus achievable, the entire region is achievable using time sharing.

Finally, consider the ‘ND’ case. The two outer bounds come from the fact that the LDoF region of ‘ND’ case is a subset of that of the LDoF region of the ‘NP’ case and also a subset of that of the DoF region of the ‘DD’ case. The achievability of ‘ND’ case is somewhat more involved and is given later in Section 4.6.1.

Next, we state and prove our more general results for the three-message BC-CM. For the BC-CM model, only the perfect CSIT (PP) was studied previously in the literature [50].
Table 4.2: DoF region of two-user BCCM under different hybrid CSIT assumptions

<table>
<thead>
<tr>
<th>Perfect CSIT (PP)</th>
<th>Hybrid CSIT (PD)</th>
<th>Hybrid CSIT (PN)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1 + d_0 \leq N_1$</td>
<td>$\frac{d_1 + d_0}{\min(M,N_1)} \leq 1$</td>
<td>$\frac{d_1 + d_0}{\min(M,N_1)} + \frac{d_2 + d_0}{\min(M,N_2)} \leq 1$</td>
</tr>
<tr>
<td>$d_2 + d_0 \leq N_2$</td>
<td>$\frac{d_1}{\min(M,N_1+N_2)} + \frac{d_2 + d_0}{\min(M,N_2)} \leq 1$</td>
<td></td>
</tr>
<tr>
<td>$d_1 + d_2 + d_0 \leq M$</td>
<td>$\frac{d_1}{\min(M,N_1+N_2)} + \frac{d_2 + d_0}{\min(M,N_2)} \leq 1$</td>
<td></td>
</tr>
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</table>

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<th>Hybrid CSIT (DP)</th>
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<th>Hybrid CSIT (DN)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d_1 + d_0}{\min(M,N_1)} + \frac{d_2}{\min(M,N_1+N_2)} \leq 1$</td>
<td>$\frac{d_1}{\min(M,N_1+N_2)} + \frac{d_2 + d_0}{\min(M,N_2)} \leq 1$</td>
<td>$\frac{d_1}{\min(M,N_1)} + \frac{d_2 + d_0}{\min(M,N_2)} \leq 1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hybrid CSIT (NP)*</th>
<th>Hybrid CSIT (ND)*</th>
<th>No CSIT (NN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d_1 + d_2 + d_0}{\min(M,N_1)} \leq 1$</td>
<td>$\frac{d_1 + d_2 + d_0}{\min(M,N_1+N_2)} \leq 1$</td>
<td>$\frac{d_1 + d_2 + d_0}{\min(M,N_1)} \leq 1$</td>
</tr>
</tbody>
</table>

* means the region is LDoF region.

The results in black are already known in the literature, while results in red are new results obtained in this paper.

**Theorem 6.** Let $N_1 \geq N_2$. The DoF regions of the two-user BC-CM under the hybrid CSIT assumptions of Type I and the LDoF regions for the hybrid CSIT assumptions of Type II are given in Table 4.2. We name the region of case ‘$X_1X_2$’ as $D_{BC-CM}^{X_1,X_2}$, where $X_1, X_2 \in \{P, D, N\}$. As in Table 4.1, the label $(X_1X_2)$ denote the cases for which the corresponding region is the DoF region whereas $(X_1X_2)^*$ is used to denote the LDoF cases in Table 4.2.

**Proof.** We give the converse proof for case ‘DD’. The proofs for all the other cases follow in the same manner.
First, let us loosen the decoding requirement of the common message $W_0$ and only require the first user to be able to decode it, such that $W_0$ degenerates into $W_1$. Since loosening decoding requirement won’t hurt, the DoF region of this new system is an outer bound of that of the original system. The new system is a MIMO BC-PM, whose DoF region is given in Theorem 5. Thus, we obtain the following two outer bounds for the BC-CM system under delayed CSIT as

$$\frac{(d_1 + d_0)}{\min(M, N_1 + N_2)} + \frac{d_2}{\min(M, N_2)} \leq 1 \quad (4.4)$$

$$\frac{(d_1 + d_0)}{\min(M, N_1)} + \frac{d_2}{\min(M, N_1 + N_2)} \leq 1 \quad (4.5)$$

Similarly, we can also require only the second user to be able to decode the common message $W_0$ and obtain another two outer bounds as

$$\frac{d_1}{\min(M, N_1 + N_2)} + \frac{(d_2 + d_0)}{\min(M, N_2)} \leq 1 \quad (4.6)$$

$$\frac{d_1}{\min(M, N_1)} + \frac{(d_2 + d_0)}{\min(M, N_1 + N_2)} \leq 1 \quad (4.7)$$

Combining outer bounds (4.4), (4.5), (4.6) and (4.7) together, it can be verified that the constraints (4.4) and (4.7) are redundant. After deleting these two redundant constraints, we obtain our final outer bound (4.5) and (4.6), which is the same with the region $D^{DD}_{BC-CM}$ shown in Table 4.2.

The approach of relaxing the decoding requirement at one receiver or the other to get two groups of outer bounds (on DoF or LDoF, as appropriate) can be used for each of the nine different hybrid CSIT cases. It is left to the reader to verify that the DoF/LDoF region outer bounds thus obtained are exactly as in Table 4.2.

Remarkably, the outer bounds obtained via this approach are tight in every case for the MIMO BC-CM under the corresponding hybrid CSIT assumption. The achievability proofs are provided later in Section 4.6.2 (including the ‘PP’ case solved more simply than in [24], without requiring the GSVD as in [24]) to 4.6.8.

**Conjecture 2.** The LDoF regions given in Theorem 5 and Theorem 6 for the hybrid CSIT models of Type II are also the DoF regions for the respective settings.
Note that the proof of this conjecture reduces to demonstrating that the LDoF region given later in Theorem 7 for the MIMO BC-PM is also the DoF region for the ‘PN’ (and hence ‘DN’) setting since all the outer bound arguments of Theorems 5 and 6 are then valid with statements about LDoF regions replaced by the corresponding ones for DoF regions, and moreover, all the achievability schemes used to prove Theorems 5 and 6 are linear as well.

The above conjecture is true for the MISO BC-PM and BC-CM (when \( N_1 = N_2 = 1 \)), since the corresponding result was recently established for the MISO BC-PM in [52] under the ‘PN’ setting.

4.4 The MIMO BC-PM under hybrid CSIT of type ‘PN’ (and ‘DN’)  

In this section, we present one of the key contributions of our paper, i.e., the converse proof of the LDoF region of the two-user MIMO BC-PM with no CSIT available from the receiver which has fewer antennas. This is also the key result for solving the remaining four open cases, i.e., ‘PN’, ‘DN’, ‘NP’ and ‘ND’.

The LDoF region for the ‘PN’ hybrid CSIT model (which is identical to that of the ‘DN’ model) for the MIMO BC-PM is given in Theorem 7. Before proving that theorem, we first give an intermediate result below, with its proof provided in Appendix C.1.

**Lemma 17.** For the 2-user MIMO BC-PM with hybrid CSIT of type ‘PN’, if \( N_2 \leq \min(M, N_1) \), considering any linear coding scheme as described as in Section 4.2, if \( V_1^{(T)} \) is decodable (i.e., the symbols of message \( W_1 \) are all decodable) at receiver 1, we have that

\[
\frac{\text{rank}(H_1^{(T)}V_1^{(T)})}{\text{rank}(H_2^{(T)}V_1^{(T)})} \leq \frac{\min(M, N_1)}{N_2}
\]

for arbitrary \( T \) and \( V_1^{(T)} \).

Based on this result, we give the converse proof of the LDoF region.

**Theorem 7.** For the 2-user MIMO BC-PM, if no channel state information is available from the receiver which has fewer antennas, the availability of channel state information (delayed or
instantaneous), or lack thereof, from the other receiver will not impact the degrees of freedom region of the system when only considering linear coding strategies. In other words, if \( N_1 \geq N_2 \), the LDoF regions of the system are the same under the CSIT assumption of type ‘PN’, ‘DN’ and ‘NN’, and is given by

\[
\frac{d_1}{\min(M, N_1)} + \frac{d_2}{\min(M, N_2)} \leq 1. \tag{4.8}
\]

**Proof.** This region can be achieved by random beamforming and the simple time-division scheme even with no CSIT. Thus, we only need to prove that (4.8) is an outer bound on the LDoF region of the MIMO BC-PM if no CSI is available from Receiver 2, which has fewer antennas.

In the case that \( M \leq N_2 \), inequality (4.8) becomes to \( d_1 + d_2 \leq M \), which is a trivial outer bound. Thus, we only need to consider the case that \( M > N_2 \).

Consider any linear coding strategy as described as in Section 4.2. In this problem, since the common message \( W_0 \) is not relevant, we remove it from all conditions. Since the total dimension of signal space at Receiver 2 is equal to \( T \cdot N_2 \) in the entire transmission block of length \( T \), we have that

\[
\text{rank}(H_2^{(T)}[V_1^{(T)} V_2^{(T)}]) \leq T \cdot N_2 \tag{4.9}
\]

From constraints (4.2) and (4.3), we have that

\[
\text{rank}(H_2^{(T)} V_1^{(T)}) + \text{rank}(H_2^{(T)} V_2^{(T)}) \leq T \cdot N_2 \tag{4.10}
\]

\[
\text{rank}(H_1^{(T)} V_1^{(T)}) = m_1^{(T)} \tag{4.11}
\]

\[
\text{rank}(H_2^{(T)} V_2^{(T)}) = m_2^{(T)} \tag{4.12}
\]

According to Lemma 17, we have that

\[
\frac{N_2}{\min(M, N_1)} \text{rank}(H_1^{(T)} V_1^{(T)}) \leq \text{rank}(H_2^{(T)} V_1^{(T)}). \tag{4.13}
\]

Together with (4.10), we have that

\[
\frac{N_2}{\min(M, N_1)} \text{rank}(H_1^{(T)} V_1^{(T)}) + \text{rank}(H_2^{(T)} V_2^{(T)}) \leq T \cdot N_2
\]
which can be rewritten as
\[
\frac{\text{rank}(H_1^{(T)}V_1^{(T)})}{\min(M, N_1)} + \frac{\text{rank}(H_2^{(T)}V_2^{(T)})}{N_2} \leq T.
\]
From Definition 1, we have that
\[
\frac{d_1}{\min(M, N_1)} + \frac{d_2}{N_2} = \lim_{T \to \infty} \frac{1}{T} \left( \frac{m_1^{(T)}}{\min(M, N_1)} + \frac{m_2^{(T)}}{N_2} \right)
\]
\[
= \lim_{T \to \infty} \frac{1}{T} \left( \frac{\text{rank}(H_1^{(T)}V_1^{(T)})}{\min(M, N_1)} + \frac{\text{rank}(H_2^{(T)}V_2^{(T)})}{N_2} \right)
\]
\[
\leq 1.
\]
Thus, inequality (4.8) is an outer bound on the LDoF region of the two-user MIMO BC-PM if no CSI is available from Receiver 2, which has fewer antennas.

Remark 13. The condition that $N_1 \geq N_2$ is important in Theorem 7. The perfect CSI from Receiver 1 can in general help in reducing interference received by Receiver 1. However, since Receiver 1 has more antennas than Receiver 2 does, it can handle more information than Receiver 2, which in turn must be able to recover all messages if $d_2 \neq 0$. Consequently, linear techniques such as zero-forcing message $W_2$ at Receiver 1 are not necessary, and hence the CSI from Receiver 1 is not useful when considering the LDoF region result.

### 4.5 Broadcast Channel with the Degraded Message Set ($W_1$, $W_0$)

![Figure 4.2: Broadcast channel with degraded message set under hybrid CSIT of type ‘ND’](image-url)
Notably, as stated in the introduction, for all but the ‘PP’ CSIT model, the corner points of the 3-dimensional DoF/LDoF regions of Table 4.2 for the BC-CM all have at least one zero element. This means that achievability results for these models for the BC-CM can be proved by establishing achievability of those corner points which involve achievability for the 2-message BC-PM and BC-DM problems. As noted previously, achievability results for the BC-PM under Type I models are known from the literature and those for Type II models were given in the proof of Theorem 5, with the exception of the ‘ND’ model, which, as stated therein, will be given in Section 4.6.1.

Therefore, in this section, our focus is on achievability results for the MIMO BC-DM (shown in Figure 4.2) because these results are a stepping stone to the (remaining) achievability proofs of Theorems 5 and 6. The reason the proof of achievability for the BC-PM under the ‘ND’ model is deferred to Section 4.6.1 is that it also can be deduced from the achievability under the same model for the BC-DM which is provided in this section.

We have the same physical structure as BC-CM (Figure 4.1) but now with just two messages, \( W_1 \) and \( W_0 \). The first user requires both messages, and the second user needs to only decode the common message \( W_0 \). If Receiver 2 has more antennas than Receiver 1 does, Receiver 2 will be able to recover all the messages that Receiver 1 can recover. In this case, random beamforming is optimal no matter what types of CSIT is available, and the DoF region would simply be \( d_1 + d_0 \leq \min(M, N_1) \). Thus, the \( N_1 < N_2 \) case is trivial.

Let us consider the case of \( N_1 \geq N_2 \). The CSIT assumption of type ‘ND’ is of particular interest, since it is the key to solving the problem under many other hybrid CSIT assumptions, as will be shown later.

**Theorem 8.** If \( N_1 \geq N_2 \), in the case that the transmitter has no CSI from Receiver 1 but has delayed CSI from Receiver 2, i.e., hybrid CSIT of type ‘ND’, the DoF region of the 2-user MIMO BC-DM is given by

\[
\mathcal{D}_{\text{BC-DM}}^{ND} = \left\{ (d_1, d_0) \middle| d_1, d_0 \geq 0, \right\}
\]
\[
\frac{d_1}{\min(M, N_1 + N_2)} + \frac{d_0}{\min(M, N_2)} \leq 1 \tag{4.14}
\]
\[
\frac{d_1 + d_0}{\min(M, N_1)} \leq 1 \tag{4.15}
\]

**Proof.** By setting the value of \(d_2\) in the DoF region of \(\mathbb{D}_{BC-CM}^{DD}\) in Table 4.2 to 0, we have an outer bound on the DoF region of the BC-DM under the ‘DD’ hybrid CSIT assumption given by the inequality (4.14). This is therefore also an outer bound on the LDoF region under the ‘ND’ assumption when no CSIT is available from Receiver 1. The outer bound (4.15) is a simple cut-set bound. Thus, we only need to prove the achievability of \(\mathbb{D}_{BC-DM}^{ND}\).

In Figure 4.3: The typical shape of \(\mathbb{D}_{BC-DM}^{ND}\). There are two line boundaries (red), and the achievable region is the quadrangle (yellow).

The typical shape of \(\mathbb{D}_{BC-DM}^{ND}\) is shown in Figure 4.3. The two corner points \((d_1, d_0) = (\min(M, N_1), 0)\) and \((0, \min(M, N_2))\) are trivially achievable. So to prove the achievability of \(\mathbb{D}_{BC-DM}^{ND}\), it is sufficient to prove that point P, the intersection of the two edges, is achievable. The entire region can then be achieved by time-sharing.

We divide the proof of achievability of corner point P into 4 cases.

**Case 1:** \(M \leq N_2\)

In this case, the two constraints (4.14) and (4.15) are identical to \(d_1 + d_0 \leq M\). This region is
achievable with random beamforming even with no CSIT, so it is trivially achieved with CSIT of type ‘ND’.

Case 2: \(N_2 < M < N_1\)

In this case, the two constraints (4.14) and (4.15) become \(\frac{d_1}{M} + \frac{d_0}{N_2} \leq 1\) and \(\frac{d_1 + d_0}{M} \leq 1\). Since \(N_2 < M\), the second inequality \(\frac{d_1 + d_0}{M} \leq 1\) is redundant. Since \((d_1, d_0) = (M, 0)\) and \((0, N_2)\) are achievable with random beamforming even with no CSIT, using time-sharing, all points in \(\{(d_1, d_0) \mid d_1, d_0 \geq 0, \frac{d_1}{M} + \frac{d_0}{N_2} \leq 1\}\) are achievable even with no CSIT. Thus, the outer bound is also trivially achieved with CSIT of type ‘ND’.

In Cases 1 and 2, it is easy to see that the regions are actually equal to the corresponding DoF regions under no CSIT assumption, so the achievability proof is trivial. For the following two cases, a particular achievability scheme is needed to achieve corner point P. In this scheme, the entire transmission is divided into several phases. The operations in specific phases are completely different for different systems. Since the coding scheme here is almost identical in the remaining 2 cases, we describe it with an example for Case 3, and then derive it in general in Case 4.

Case 3: \(N_1 \leq M < N_1 + N_2\)

In this case, the two constraints (4.14) and (4.15) become \(\frac{d_1}{M} + \frac{d_0}{N_2} \leq 1\) and \(\frac{d_1 + d_0}{N_1} \leq 1\), and the intersection P is given by

\[
P = \left( \frac{M(N_1 - N_2)}{M - N_2}, \frac{(M - N_1)N_2}{M - N_2} \right).
\]

Consider an example wherein \(M = 5\), \(N_1 = 4\), and \(N_2 = 2\), then \(P = (\frac{10}{3}, \frac{2}{3})\). To achieve this DoF pair, we need to transmit, in 3 time slots, 10 independent symbols of private messages \(W_1\) to receiver one and 2 independent symbols of common message \(W_0\) to both receivers. Let us divide the transmission into two phases.
Phase one consists of $N_1 - N_2 = 2$ time slots. At each time slot, the transmitter sends $M = 5$ independent $W_1$ symbols intended for the first user through the $M = 5$ transmit antennas. Let the 5 symbols at time slot $t$ be $\{u_{t,i}\}$, where $t \in \{1, 2\}$ and $i \in \{1, \ldots, 5\}$. Consider the signal received by the first user. As is shown in Figure 4.4, at time slot $t$, user one will receive $N_1 = 4$ independent linear combinations of symbols $u_{t,1}, u_{t,2}, u_{t,3}, u_{t,4}$ and $u_{t,5}$, which are named $L_{t1}(u_{t,1}^{t,5}), L_{t2}(u_{t,1}^{t,5}), L_{t3}(u_{t,1}^{t,5})$ and $L_{t4}(u_{t,1}^{t,5})$, respectively. We have that

$$
\begin{bmatrix}
L_{t1}(u_{t,1}^{t,5}) \\
L_{t2}(u_{t,1}^{t,5}) \\
L_{t3}(u_{t,1}^{t,5}) \\
L_{t4}(u_{t,1}^{t,5})
\end{bmatrix} = H_1(t) \begin{bmatrix}
u_{t,1}^* \\
u_{t,2}^* \\
u_{t,3}^* \\
u_{t,4}^* \\
u_{t,5}^*
\end{bmatrix}.
$$

Similarly, user two will receive $N_2 = 2$ independent linear combinations of symbols $u_{t,1}^{t,5}$, which are named $I_{t1}(u_{t,1}^{t,5})$ and $I_{t2}(u_{t,1}^{t,5})$. These messages are intended only for user one, thus they are interference at user two. However, they are still useful as explained later in phase two. We have
that

\[
\begin{bmatrix}
I_{t1}(u_{t1}^{t,5}) \\
I_{t2}(u_{t1}^{t,5})
\end{bmatrix} = H_2(t) \begin{bmatrix} u_{t1,1}^* & u_{t2,2}^* & u_{t3,3}^* & u_{t4,4}^* & u_{t5,5}^* \end{bmatrix}^* .
\]

We can observe that at each time slot, the transmitter sends 5 symbols for user one, and
user one has already obtained 4 independent equations/combinations of them. Thus, user one
only needs one more independent equation of these 5 symbols to be able to decode them successfully.

Phase two consists of \(M - N_1 = 1\) time slot. In this phase, the transmitter will send
\((M - N_1)N_2 = 2\) independent symbols \(v_{1,1}\) and \(v_{1,2}\) of common message \(W_0\). Note that the
channel matrices \(H_2(t)\) during phase one are known to the transmitter due to the delayed CSIT
assumption. As a result, the transmitter knows \(I_{t1}(u_{t1,1}^{t,5})\) and \(I_{t2}(u_{t1,1}^{t,5})\), where \(t = 1, 2\). Since \(H_1\)
and \(H_2\) are generic matrices and i.i.d. cross time and receiver indexes, \(I_{t1}(u_{t1,1}^{t,5})\) will be linearly
independent with \(L_{t1}(u_{t1,1}^{t,5}), L_{t2}(u_{t2,1}^{t,5}), L_{t3}(u_{t3,1}^{t,5})\) and \(L_{t4}(u_{t4,1}^{t,5})\) almost surely, because the number of
combinations is no greater than the number of independent symbols. If the transmitter could send
\(I_{t1}(u_{t1,1}^{t,5})\) to user one, then user one will be able to decode all 5 symbols, i.e., \(u_{t1,1}^{t,5}\).

As is shown in Figure 4.4, in the third time slot, the transmitter will send 4 symbols, i.e.,
\(I_{11}(u_{1,1,1}^{1,5}), I_{21}(u_{2,1,1}^{2,5}), v_{1,1}\) and \(v_{1,2}\), through four of its antennas. Since user one has four receive
antennas, it will be able to decode all of messages \(I_{11}(u_{1,1}^{1,5}), I_{21}(u_{2,1}^{2,5}), v_{1,1}\) and \(v_{1,2}\). Then, using
\(I_{t1}(u_{t1,1}^{t,5})\) as well as \(L_{t1}(u_{t1,1}^{t,5}), L_{t2}(u_{t2,1}^{t,5}), L_{t3}(u_{t3,1}^{t,5})\) and \(L_{t4}(u_{t4,1}^{t,5})\), user one can decode message \(u_{t1,1}^{t,5}\) (for
\(t = 1, 2\)). In others words, user one can decode both the 2 symbols of common message \(W_0\) and
the 10 symbols of private messages \(W_1\).

Next, consider user two. In the third time slot, it will receive two independent linear combina-
tions of messages \(I_{11}(u_{1,1}^{1,5}), I_{21}(u_{2,1}^{2,5}), v_{1,1}\) and \(v_{1,2}\). However, since user two has already known\(^4\)
\(I_{11}(u_{1,1}^{1,5})\) and \(I_{21}(u_{2,1}^{2,5})\), it can subtract them from the signal it receives. After removing the \(I_{11}(u_{1,1}^{1,5})\)
and \(I_{21}(u_{2,1}^{2,5})\), it is as if user two has 2 independent linear combinations of only \(v_{1,1}\) and \(v_{1,2}\). As a
result, user two can decode the 2 symbols of common message \(W_0\).

\(^4\) In fact, what user two knows are noisy versions of \(I_{11}(u_{1,1}^{1,5})\) and \(I_{21}(u_{2,1}^{2,5})\). However, noise can be neglected when
considering a DoF analysis.
In summary, the transmitter successfully send 10 symbols of $W_1$ to user one and 2 symbols of $W_0$ to both users in three time slots, i.e., the DoF $(\frac{10}{3}, \frac{2}{3})$ is achievable.

**Case 4: $M \geq N_1 + N_2$**

In this case, the two constraints (4.14) and (4.15) become

$$\frac{d_1}{N_1 + N_2} + \frac{d_0}{N_2} \leq 1 \quad \text{and} \quad \frac{d_1 + d_0}{N_1} \leq 1,$$

and the intersection $P$ is given by

$$P = \left( \frac{N_1^2 - N_2^2}{N_1}, \frac{N_2^2}{N_1} \right).$$

The achievability scheme is almost identical with that in Case 3. We derive it in general here for Case 4. Note that it is sufficient to use only $N_1 + N_2$ transmit antennas to achieve the corner point, i.e., there are redundant antennas at the transmitter. Hence, without loss of generality, we assume $M = N_1 + N_2$ in the following analysis.

Phase one consists of $N_1 - N_2$ time slots. At each time slot, the transmitter sends $M = N_1 + N_2$ independent $W_1$ symbols intended for the user one through the $M = N_1 + N_2$ transmit antennas. Let the $N_1 + N_2$ symbols at time slot $t$ be $\{u_{t,i}\}$, where $t \in \{1, ..., N_1 - N_2\}$ and $i \in \{1, ..., N_1 + N_2\}$.

In phase one, transmitter sends out altogether $(N_1 - N_2) \cdot (N_1 + N_2) = N_1^2 - N_2^2$ symbols of $W_1$.

Consider the signal received by Receiver 1. At time slot $t$, user one will receive $N_1$ independent linear combinations of symbols $u_{t,1}^{t,N_1+N_2}$, which are denoted as $L_{tk}(u_{t,1}^{t,N_1+N_2})$, where $k \in [1 : N_1]$.

Similarly, user two will receive $N_2$ independent linear combinations of symbols $u_{t,1}^{t,N_1+N_2}$, which are denoted as $I_{tj}(u_{t,1}^{t,N_1+N_2})$, where $j \in \{1, ..., N_2\}$.

We can observe that at each time slot, the transmitter sends $N_1 + N_2$ symbols of $W_1$ for Receiver 1, and Receiver 1 obtains $N_1$ independent equations/combinations of them. Thus, Receiver 1 only needs $N_2$ more independent equation of these $N_1 + N_2$ symbols so as to be able to decode them successfully.

Phase two consists of $N_2$ time slots. Note that the channel matrices $H_2(t)$ during phase
one are known to the transmitter due to the delayed CSIT. As a result, the transmitter knows
$I_{tj}(u_{t,1}^{t, N_1+N_2})$, where $t \in \{1, ..., N_1 - N_2\}$ and $j \in \{1, ..., N_2\}$. Since $H_1$ and $H_2$ are generic matrix
and i.i.d. cross time and receiver indexes, $I_{tj}(u_{t,1}^{t, N_1+N_2})$ will be linearly independent with each
other and also with $L_{tk}(u_{t,1}^{t, N_1+N_2})$ almost surely, because the number of linear combinations is no
greater than the number of independent symbols. There are in sum $(N_1 - N_2) \cdot N_2$ messages as
$I_{tj}(u_{t,1}^{t, N_1+N_2})$, and we equally divide them into $N_2$ groups, where each group contains $N_1 - N_2$ of them.

At each time slot of phase two, the transmitter sends $N_2$ independent symbols of common
message $W_0$ and one of the $N_2$ groups of $I_{tj}(u_{t,1}^{t, N_1+N_2})$, so that there are $N_2 + (N_1 - N_2) = N_1$
symbols in total. Since user one has $N_1$ antennas, it is able to decode all these $W_0$ symbols and
$I_{tj}(u_{t,1}^{t, N_1+N_2})$. Then, together with the previously saved $L_{tk}(u_{t,1}^{t, N_1+N_2})$ symbols, Receiver 1 is able
to decode all $u_{t,1}^{t, N_1+N_2}$.

Next, consider Receiver 2. At each time slot, it will receive $N_2$ independent linear com-
binations of $N_2$ symbols of $W_0$ messages and $N_1 - N_2$ symbols of $I_{tj}(u_{t,1}^{t, N_1+N_2})$. However, since
Receiver 2 has already known all $I_{tj}(u_{t,1}^{t, N_1+N_2})$, it can subtract them from the signal it receives.
After removing the contributions of $I_{tj}(u_{t,1}^{t, N_1+N_2})$ on the received signal, it is as if Receiver 2 has
$N_2$ independent linear combinations of $N_2$ symbols of $W_0$. As a result, Receiver 2 can decode all
the symbols of common message $W_0$.

In summary, the transmitter successfully send $N_2^2 - N_2^2$ symbols of $W_1$ to Receiver 1 and $N_2^2$
symbols of $W_0$ to both receivers in $(N_1 - N_2) + N_2 = N_1$ time slots, i.e., the DoF $\left(\frac{N_2^2 - N_2^2}{N_1}, \frac{N_2^2}{N_1}\right)$ is
achieved.

All the cases together prove corner point P in Figure 4.3 is achievable, and hence the DoF
region described in Theorem 8 is achievable.

4.6 Achievability proof of Theorem 5 and 6

In this section, we give all the remaining achievability proofs of Theorems 5 and 6.
4.6.1 ‘ND’ case of Theorem 5

By comparing the region $D_{BC-PM}^{ND}$ in Table 4.1 with $D_{BC-DM}^{ND}$ given in Theorem 8, we find that the shapes of these two regions are exactly the same except that one of them contains $d_2$ and the other one contains $d_0$. Since the decoding requirement of message $W_0$ is higher than that of message $W_2$, each symbol of message $W_0$ can be thought of as a symbol of message $W_2$ by not requiring Receiver 1 to be able to decode it. If a DoF tuple $(d_1, d_0)$ is achievable in the BC-DM, the DoF tuple $(d_1, d_2)$, where $d_2 = d_0$ is also achievable in the same physical channel. In other words, the DoF region $D_{BC-DM}^{ND}$ is achievable for the MIMO BC-PM. The achievable scheme is the same as the scheme given in Section 4.5.

4.6.2 ‘PN’, ‘DN’ and ‘NN’ cases of Theorem 6

The DoF regions for these three cases are achievable by the simple random beamforming and time-division scheme.

4.6.3 ‘PP’ case of Theorem 6

To start, we propose a precoding scheme and show that it can achieve all the integer-valued DoF tuples within the region $D_{BC-CM}^{PP}$. This scheme is also a special case/simplification of the precoding scheme for the more general 2×2 interference network with general message sets proposed in [31, 43]. Since it does not involve the GSVD, it is also simpler than the scheme given in [24].

In the case that $M > N_1$, the null space of channel $H_1$ is not empty. By transmitting symbols of $W_2$ using beamformers picked from the null space of $H_1$, i.e., $\text{null}(H_1)$, we can zero-force these symbols at Receiver 1 and thus reduce the interference message $W_2$ brings to Receiver 1. The maximum number of such independent symbols is equal to $(M - N_1)^+$. Similarly, if $M > N_2$, we can zero-force, maximally, $(M - N_2)^+$ independent symbols of message $W_1$ at Receiver 2. So, the basic idea of the precoding scheme is that to first transmit as many symbols of private message $W_i$ ($i = 1, 2$) as possible in the nullspace $\text{null}(H_j)$ ($j = 3 - i$) and then send the rest of the symbols of $W_1$ and $W_2$ and all the symbols of message $W_0$ using random beamforming. To obtain a basis
of \text{null}(H_i), we can do a singular value decomposition (SVD) of matrix \( H_i \) while arranging the singular values in non-increasing order. Then, the last \((M - N_i)^+\) right-singular column vectors, which are corresponding to singular value 0, will form a basis of \text{null}(H_i).

Suppose \( \vec{d} = (d_1, d_2, d_0) \in \mathbb{Z}_+^3 \) and \( \vec{d} \in \mathbb{D}_{BC-CM}^P \). Define \( d_i^Z = \min(d_i, (M - N_{3-i})^+) \) and \( d_i^R = d_i - d_i^Z \), where \( i = 1, 2 \). Here \( d_i^Z \) is the number of \( W_i \) symbols that will be zero-forced at Receiver \( 3 - i \), and \( d_i^R \) is the number of \( W_i \) symbols that will be transmitted using random beamforming. Construct matrix \( V_i^Z \) and \( V_i^R \) such that their column vectors are the zero-forcing beamformers and random beamformers for message \( W_i \), respectively. Construct matrix \( V_0 \) such that its column vectors are random beamformers for message \( W_0 \).

Now, consider the signal received at Receiver 1. Dropping the time index, we have \( Y_1 = H_1 \cdot (V_1^Z S_1^Z + V_1^R S_1^R + V_2^R S_2^R + V_0 S_0) \), where the \( S \)'s are the corresponding messages. According to the above precoding scheme, \( V_1^Z \) is generated from the nullspace \text{null}(H_2), so it is independent with channel \( H_1 \). Meanwhile, \( V_1^R, V_2^R \) and \( V_0 \) are all generated randomly, and thus they are also independent with channel \( H_1 \). Since channel \( H_1 \) is a full matrix with generic elements, the columns of \([H_1 \cdot V_1^Z \ H_1 \cdot V_1^R \ H_1 \cdot V_2^R \ H_1 \cdot V_0]\) will be linearly dependent only if they have to be linearly dependent.

Since \( d \in \mathbb{D}_{BC-CM}^P \), we have that

\[
\begin{align*}
d_0 + d_1 & = d_0 + d_1^R + d_1^Z \leq N_1 \\
d_0 + d_1 + d_2 & = d_0 + d_1^R + d_1^Z + d_2^R + d_2^Z \leq M.
\end{align*}
\]

Next, consider the sum of \( d_0 + d_1^R + d_1^Z + d_2^R \). In the case that \( M > N_1 \): if \( d_2 > M - N_1 \), we have that \( d_2^Z = M - N_1 \) and

\[
\begin{align*}
d_0 + d_1^R + d_1^Z + d_2^R + (M - N_1) & \leq M,
\end{align*}
\]

which leads to \( d_0 + d_1^R + d_1^Z + d_2^R \leq N_1 \); if \( d_2 \leq M - N_1 \), we have \( d_2^Z = d_2 \) and \( d_2^R = 0 \), such that

\[
d_0 + d_1^R + d_1^Z + d_2^R = d_0 + d_1^R + d_1^Z \leq N_1.
\]
In the case that $M \leq N_1$, we have $d^Z_2 = 0$ and $d^R_2 = d_2$, such that

$$d_0 + d^R_1 + d^Z_1 + d^R_2 = d_0 + d^R_1 + d^Z_1 + d^R_2 \leq M \leq N_1.$$ 

Thus, we have that $d^R_0 + d^R_1 + d^Z_1 + d^R_2 \leq \min(M, N_1) = \text{rank}(H_1)$ in all cases. As a result, the column vectors of $[H_1V^Z_1 H_1V^R_1 H_1V^R_2 H_1V^R_0]$ will be almost surely linearly independent with each other, since the number of vectors is no greater than the rank of $H_1$. Consequently, Receiver 1 can recover all symbols of message $W_1$ and $W_0$ via linear decoding.

Following the same argument, we have that all symbols of message $W_2$ and $W_0$ are also distinguishable at Receiver 2. In other words, the degrees of freedom $(d_1, d_2, d_0)$ is achieved. It is worth noting that only zero-forcing, which needs singular value decomposition (SVD), and random beamforming are required in the optimal precoding scheme.

So far, we have proved that all integer-valued degrees of freedom tuples in $\mathbb{D}^P_{BC-CM}$ are achievable. It is easy to verify that, no matter what values $M$, $N_1$ and $N_2$ are, all the corner points of the 3-D region $\mathbb{D}^P_{BC-CM}$ are integer-valued and thus achievable. Consequently, the entire region of $\mathbb{D}^P_{BC-CM}$ is achievable using time sharing, and we have proved that $\mathbb{D}^P_{BC-CM}$ is the DoF region.

Note that [24] also studies the DoF region of a 2-user BC-CM system with perfect CSIT, but with fixed channels and arbitrary channel matrices (not necessarily generic), and here we are dealing with fast fading channel where $H$’s are i.i.d. across time. The converse proof is made much simpler than in [24] by using the basic strategy of loosening decoding requirement. Also, in the proof of achievability, we propose a relatively simpler scheme, which needs singular value decomposition (SVD) instead of generalized singular value decomposition (GSVD).
4.6.4 ‘DD’ case of Theorem 6

Figure 4.5: The typical shape of $\mathbb{D}^{DD}_{BC-CM}$. The two planes/constraints, spanned by points $(A_1, A_2, A_0)$ and by $(B_1, B_2, B_0)$, intersect at line $L$. The final DoF region is a pentahedron, whose vertices are 0, $A_1$, $B_0$, $B_2$, $P_1$ and $P_2$.

Observe that $\mathbb{D}^{DD}_{BC-CM}$ is a three-dimensional pentahedron. The typical shape of $\mathbb{D}^{DD}_{BC-CM}$ is shown in Figure 4.5 above. There are five non-trivial corner points on the pentahedron’s boundary, and it is sufficient to prove these corner points are achievable because the entire region can then be achieved using time-sharing.

The three corner points $(d_1, d_2, d_0)$ on the axes, i.e., $A_1 = (\min(M,N_1),0,0)$, $B_2 = (0,\min(M,N_2),0)$ and $B_0 = (0,0,\min(M,N_2))$, can be achieved even with no CSIT. Hence, they are trivially achieved with delayed CSIT. The corner point $P_1$ lies in the plane $\{(d_1,d_2,d_0)|d_0 = 0\}$. It is actually the exact same corner point as that in the MIMO BC-PM with delayed CSIT. Thus, it is achievable using the transmission scheme proposed in [48]. The corner point $P_2$ lies in the plane $\{(d_1,d_2,d_0)|d_2 = 0\}$. This point is exactly the same corner point as the one we considered in Section 4.5, so it is achievable using the transmission scheme described there. Since $P_2$ can be achievable under ‘ND’ CSIT assumption, it is also achievable under the ‘DD’ CSIT assumption using the same coding scheme.
Hence, all the corner points are shown to be achievable, and thus the entire region $D_{BC-CM}^{DD}$ is achievable using time-sharing.

### 4.6.5 The ‘PD’ case of Theorem 6

Again, in the ‘PD’ case, there are two non-trivial corner points which are not on the axes. One of them lies in the plane $\{(d_1, d_2, d_0)|d_0 = 0\}$. It is actually the exact same corner point as that in the MIMO BC-PM with ‘PD’ CSIT. Thus, it is achievable using the transmission scheme introduced for the BC-PM in [50]. The other corner point lies in the plane $\{(d_1, d_2, d_0)|d_2 = 0\}$. It is actually the exact same corner point as the one we considered in Section 4.5, and it is achievable even under ‘ND’ CSIT assumption, so it is also achievable under the ‘PD’ CSIT assumption using the same coding scheme.

Hence, all the corner points are shown to be achievable, and thus the entire region $D_{BC-CM}^{PD}$ is achievable using time-sharing.

### 4.6.6 ‘DP’ case of Theorem 6

For the case of ‘DP’, although the shape of region seems to be symmetric with that of case ‘PD’, the two non-trivial corner points are still in the plane $d_0 = 0$ and $d_2 = 0$, since $N_1 \geq N_2$. The corner point in the plane $d_0 = 0$ is again achievable using the scheme introduced for the BC-PM in [50] for ‘PD’ case. The corner point in the plane $d_2 = 0$ is equal to $(\min(M, N_1) - \min(M, N_2), 0, \min(M, N_2))$. To achieve this point, we use following scheme. First, transmit the common message $W_0$ using random beamforming. Hence it will occupy $\min(M, N_2)$ dimensions each at the two receivers. Then, since the transmitter has perfect knowledge of channel $H_2$, zero-forcing some or all of private message $W_1$ at Receiver 2 is possible. Because we have $\min(M, N_1) - \min(M, N_2) \leq (M - N_2)^+$, which is the rank of the null-space of $H_2$, we can actually zero-force all $\min(M, N_1) - \min(M, N_2)$ streams of private message $W_1$ at Receiver 2. Consequently, Receiver 2 is able to recover the $\min(M, N_2)$ streams of common message $W_0$, and Receiver 1 is able to recover the altogether $\min(M, N_1)$ streams of message $W_1$ and $W_0$, since the number of
independent streams at neither receiver is greater than its number of antennas.

It is worth noting that it requires 'DP' CSIT to achieve the corner point in the plane $d_0 = 0$, however, ‘NP’ CSIT is enough to achieve the corner point in the plane $d_2 = 0$.

4.6.7 ‘NP’ case of Theorem 6

From Section 4.6.6, we can obtain that, under ‘NP’ CSIT assumption, the region $d_1 + d_0 \leq \min(M,N_1)$, $d_0 \leq \min(M,N_2)$ is achievable. By loosening the decoding requirement of part of message $W_0$ and only require Receiver 2 to be able to decode them, this part of $W_0$ will degenerate into message $W_2$. Since loosening the decoding requirement won’t hurt, we have that $d_1 + (d_2 + d_0) \leq \min(M,N_1)$, $(d_2 + d_0) \leq \min(M,N_2)$ is also achievable, which is the same as region $\mathbb{D}_B^{NP}_{BC-CM}$.

4.6.8 ‘ND’ case of Theorem 6

Again, the two non-trivial corner points are in the plane $d_0 = 0$ and $d_2 = 0$. The one in the plane $d_0 = 0$ is the same as the corner point for the BC-PM given in Section 4.6.1 and is thus achievable. The other one in the plane $d_2 = 0$ is the same as the corner point given in Section 4.5 for the BC-DM and is thus achievable.

4.7 Conclusion

In this paper, we study the DoF of MIMO BC with private and common messages (BC-CM) under all possible hybrid CSIT assumptions. For the five Type I hybrid CSIT assumptions, we obtained the DoF regions and for the remaining four Type II CSIT assumptions we obtain the LDoF regions. The outer bounds on the DoF region for the Type I CSIT assumptions are obtained as extensions of the respective DoF regions for the MIMO BC with private messages (BC-PM), which are known from previous literature. The outer bounds on the LDoF region for the Type II CSIT assumptions are obtained from the respective outer bounds on the LDoF region for the MIMO BC with private messages (BC-PM), which in turn are also obtained in this paper.

As the most important converse proof of this paper, we show in Theorem 7 that if no channel
information is available from the receiver which has fewer antennas, the availability of channel state information from the other receiver will not impact the DoF region of the 2-user MIMO BC-PM when only considering linear encoding strategies. In other words, channel state information from the receiver with more antennas does not help if no channel state information is available from the receiver with fewer antenna. The converse proof of the LDoF region for the MIMO BC-PM and BC-CM under Type II hybrid CSIT assumptions all follow from this theorem. For the achievability proof, it is shown that every corner point of the MIMO BC-CM DoF in the non-‘PP’ problem or LDoF regions is either a corner point of the BC-PM or a corner point of the BC with degraded messages (BC-DM). Thus, the achievability of the BC-CM DoF/LDoF region is decomposed into a series of sub-problems. An important such sub-problem is the MIMO BC-DM with private message to Receiver 1 (with greater number of receive antennas than Receiver 2) and a common message under hybrid CSIT assumption in which Receiver 1’s channel is unknown at the transmitter and Receiver 2’s channel is known with delay. For this setting, we propose a two-phase coding scheme to show that the outer bound on its LDoF region is tight. This sub-problem is shown to be the foundation of the achievability proof for the DoF/LDoF region of the MIMO BC-CM under multiple CSIT assumptions.

The results of this work give rise to several interesting future research directions. One such direction is to prove our conjecture that the LDoF regions obtained in this paper are indeed the DoF regions in each of the four hybrid CSIT models in the two-user MIMO BC-PM setting, as well as in the more general two-user MIMO BC-CM. In fact, it is sufficient to prove that Theorem 7 holds despite removing the restriction of linear encoding strategies, since all the other converses follow that case of MIMO BC-PM as they do in this paper but with that restriction in place. Another direction for future research is generalizing the results of this paper for the private messages only setting to the three-user MIMO BC with a general antenna configuration. Furthermore, the DoF or even the LDoF region of the MIMO broadcast channel with a general message set, consisting of seven different messages (one for each subset of receivers where it is desired) even in the perfect CSIT is an intriguing open problem.
In this thesis, we study the fundamental limits of selected wireless networks.

First, the degrees of freedom region is established for the MIMO $2 \times 2$ interference network, in which there can be at most 9 independent messages. In the achievable scheme, the precise roles played by transmit zero-forcing, interference alignment, random beamforming, symbol extensions and asymmetric complex signaling (ACS) are delineated. We also identify a class of antenna settings in which ACS is required to achieve the fractional-valued corner points. To construct the achievable DoF region of the system, we utilize the approach of rate splitting. More specifically, each of the four private messages, i.e., $W_{11}$, $W_{12}$, $W_{21}$ and $W_{22}$, is split into three linearly independent parts according to the interference eliminating techniques applied to them, and similarly, each of the two cognitive and common messages, i.e., $W_{01}$ and $W_{02}$, is split into two linearly independent parts. Then, based on the independence requirements of the decodability, we construct the 17-dimensional achievable DoF region of the split message sets. By projecting this higher-dimensional polytope to the desired 9 dimensions, we obtain the achievable DoF region of the system.

Then, we study the degrees of freedom region for the MIMO 3-user BC with general message sets under the restriction of linear encoding strategies. We introduce the novel idea of channel decomposition. The entire $M$-dimensional transmitter beamformer space is partitioned into nine linearly independent subspaces according to their diverse impacts on the three receivers. Then, these nine subspaces are further divided into four groups. Using the idea of rate splitting, we construct an achievable LDoF region for each group of subspaces. It is shown that under certain
antenna settings, the Minkowski-sum of the obtained achievable LDoF region of existing groups is identical to the outer bound on the LDoF region of the entire system. In other words, transmitting parallelly through each group is LDoF optimal. In the other antenna settings, the idea of parallel transmitting is LDoF-sub-optimal, and the cooperation among transmitting via beamformers chosen from different groups of subspaces is desired to achieve the outer bound.

Finally, we study the degrees of freedom regions of the two-user MIMO BC-CM that includes private and common messages under fast fading. The hybrid CSIT models are considered wherein the transmitter has either perfect/instantaneous, delayed or no channel state information from each of the two receivers. The DoF regions are established for the five hybrid CSIT models in which either both channels are unknown at the transmitter or each of the two channels is known perfectly or with delay. In the four remaining cases in which exactly one of the two channels is unknown at the transmitter, the LDoF regions are established and conjectured to also be the DoF region. As the key to the converse proofs of the LDoF region of the MIMO BC-CM under such hybrid CSIT assumptions, we show that, when only considering linear encoding strategies, the channel state information from the receiver with more antennas does not help if there is no channel state information available from the receiver with fewer antennas.

The work of the thesis makes significant progress in the study of general message sets problem. We show that the ideas of channel decomposition, rate splitting or combining them together are indeed useful in exploring the achievable DoF/LDoF regions of various wireless networks. The works of general message sets sheds light on how the same physical network could be used by a suitable choice of message sets to take most advantage of the channel resource in a flexible and efficient manner.
Bibliography


Appendix A

MIMO 2 × 2 Interference Network

A.1 Equivalence of $\mathbb{D}$ and $\mathbb{D}_{eq}$

Proof. To make the expressions more concise, we define

\[ d_{\text{sum},1} = d_1 + d_2 + d_0 + d_{01} + d_{11} + d_{12} \]
\[ d_{\text{sum},2} = d_1 + d_2 + d_0 + d_{02} + d_{21} + d_{22}. \]

Let $\mathbf{d} = (d_{11}, d_{21}, d_{12}, d_{22}, d_1, d_2, d_{01}, d_{02}, d_0)$. First, prove if $\mathbf{d} \in \mathbb{D}_{eq}$, then $\mathbf{d} \in \mathbb{D}$.

Since $\mathbf{d} \in \mathbb{D}_{eq}$, there exists at least a tuple $(Z_{11}, Z_{12}, Z_{21}, Z_{22}, A_1, A_2, Z_{01}, Z_{02}) \in \mathbb{R}^A_+$ which satisfies the conditions in (2.53)-(2.64), such that inequalities (2.48)-(2.52) are all satisfied. Then, from inequalities (2.48), (2.54), (2.57) and (2.58), we get

\[ d_{\text{sum},1} + d_{21} + d_{22} + d_{02} \leq N_1 + (M_1 - N_1)^+ + d_{22} + d_{02}. \]

Hence,

\[ d_{\text{sum},1} + d_{21} \leq N_1 + (M_1 - N_1)^+ = \max(M_1, N_1), \]

which is inequality (2.8) in the definition of $\mathbb{D}$. Similarly, it can be shown that

\[ d_{\text{sum},1} + d_{22} \leq N_1 + (M_2 - N_1)^+ = \max(M_2, N_1) \]
\[ d_{\text{sum},2} + d_{11} \leq N_2 + (M_1 - N_2)^+ = \max(M_1, N_2) \]
\[ d_{\text{sum},2} + d_{12} \leq N_2 + (M_2 - N_2)^+ = \max(M_2, N_2). \]
which are inequalities (2.9)-(2.11) in the definition of $\mathbb{D}$.

Again, from inequalities (2.48), (2.56), (2.57) and (2.58), we get

$$d_{\text{sum},1} + d_{21} + d_{22} + d_{02} \leq N_1 + d_{21} + d_{22} + d_{02} - A_1,$$

hence,

$$d_{\text{sum},1} \leq N_1 - A_1 \leq N_1,$$

which is inequality (2.12) in the definition of $\mathbb{D}$. Similarly, we have

$$d_{\text{sum},2} \leq N_2 - A_2 \leq N_2,$$

which is inequality (2.13) in the definition of $\mathbb{D}$.

Furthermore, inequalities (2.14)-(2.16) hold for $\overrightarrow{d}$ since they are also contained in the definition of $\mathbb{D}_{eq}$. Consequently, all inequalities in the definition of $\mathbb{D}$ are satisfied and we have that $\overrightarrow{d}$ also belongs to $\mathbb{D}$. Thus,

$$\mathbb{D}_{eq} \subseteq \mathbb{D}. \quad (A.1)$$

Next, we prove that if $\overrightarrow{d} \in \mathbb{D}$, then $\overrightarrow{d} \in \mathbb{D}_{eq}$.

For each $\overrightarrow{d} \in \mathbb{D}$, we choose the value for $(Z_{11}, Z_{12}, Z_{21}, Z_{22}, A_1, A_2, Z_{01}, Z_{02})$ according to equations (2.17)-(2.25). It is straightforward to verify the above choices satisfy the constraints (2.53)-(2.64). Also, by exhaustively enumerating all possible relations among $M_1, M_2, N_1, N_2, d$ and removing the $(\cdot)^+$ and $\min(\cdot, \cdot)$ operators, and substituting the values of the 8 auxiliary variables, we can verify that if inequalities (2.8)-(2.13) hold, then inequalities (2.48) and (2.49) also hold. Inequalities (2.50)-(2.52) automatically hold since they are contained in the definition of $\mathbb{D}$, and hence $\overrightarrow{d}$ also belongs to $\mathbb{D}_{eq}$. Thus

$$\mathbb{D} \subseteq \mathbb{D}_{eq}. \quad (A.2)$$

Together with (A.1), we have $\mathbb{D} = \mathbb{D}_{eq}$. \qed
Appendix B

MIMO 3-user BC with general message sets

B.1 Two useful outer bounds

In this section, we give the proofs for Lemma 7 and Lemma 8.

Lemma 7. An outer bound on the linear DoF region of the three-user broadcast channel with message sets \((W_1, W_2, W_3, W_{12})\) is given by

\[ d_1 + d_2 + d_3 + 2d_{12} \leq \max(m, 2n). \] (B.1)

Proof. If \(m \leq n\), from the cut-set bound \(d_1 + d_2 + d_3 + d_{12} \leq m\), we have that \(d_1 + d_2 + d_3 + 2d_{12} \leq 2m \leq 2n\).

Next, consider the case that \(m > n\). Consider any linear coding scheme as described in Section 3.2.4. From constraint (3.1) - (3.4), we have that

\[ \text{Rank}(H_1^T[U_1^{(T)} U_{12}^{(T)}]) + \text{Rank}(H_2^T[U_2^{(T)} U_{12}^{(T)}]) \leq T \cdot n \] (B.2)

\[ \text{Rank}(H_2^T[U_2^{(T)} U_{12}^{(T)}]) + \text{Rank}(H_1^T[U_1^{(T)} U_{12}^{(T)}]) \leq T \cdot n \] (B.3)

\[ \text{Rank}(H_1^T[U_1^{(T)} U_{12}^{(T)}]) = m_1^{(T)} + m_{12}^{(T)} \] (B.4)

\[ \text{Rank}(H_2^T[U_2^{(T)} U_{12}^{(T)}]) = m_2^{(T)} + m_{12}^{(T)} \] (B.5)

Meanwhile, we have that

\[ \text{Rank}(H_1^T U_3^{(T)}) + \text{Rank}(H_2^T U_3^{(T)}) \leq \text{Rank}(H_1^T[U_2^{(T)} U_{12}^{(T)}]) + \text{Rank}(H_2^T[U_1^{(T)} U_{12}^{(T)}]) \] (B.6)
and
\[
\text{Rank} \left( \begin{bmatrix} H_1^{(T)} \\ H_2^{(T)} \end{bmatrix} U_3^{(T)} \right) \leq \text{Rank}(H_1^{(T)} U_3^{(T)}) + \text{Rank}(H_2^{(T)} U_3^{(T)}). \quad (B.7)
\]

Since the dimension of null-space \( N \left( \begin{bmatrix} H_1^{(T)} \\ H_2^{(T)} \end{bmatrix} \right) \) is equal to \( T \cdot (m - 2n)^+ \), there can be at most a rank lost of \( T \cdot (m - 2n)^+ \). Consequently, we have that
\[
\text{Rank} \left( \begin{bmatrix} H_1^{(T)} \\ H_2^{(T)} \end{bmatrix} U_3^{(T)} \right) \geq m_3^{(T)} - T \cdot (m - 2n)^+. \quad (B.8)
\]

Together from (B.2) - (B.8), we have that
\[
m_1^{(T)} + m_2^{(T)} + m_3^{(T)} + 2m_{12}^{(T)} \leq T \cdot (2n + (m - 2n)^+)
\]
\[
= T \cdot \max(m, 2n).
\]

From definition 1, we have that
\[
d_1 + d_2 + d_3 + 2d_{12} = \lim_{T \to \infty} \frac{1}{T} \left( m_1^{(T)} + m_2^{(T)} + m_3^{(T)} + 2m_{12}^{(T)} \right)
\]
\[
\leq \max(m, 2n)
\]

\[\square\]

**Lemma 8.** An outer bound on the linear DoF region of the three-user broadcast channel with message sets \((W_1, W_2, W_3, W_{123})\) is given by
\[
2d_1 + 2d_2 + 2d_3 + 3d_{123} \leq \max(2m, 3n). \quad (B.9)
\]

**Proof.** We divide the proof of theorem 8 into 5 cases. For simplicity, we omit the superscript \((t)\) in the discuss.

Case 1: \( m \leq n \).
From the cut-set bound \(d_1 + d_2 + d_3 + d_{12} \leq m\), we have that \(2d_1 + 2d_2 + 2d_3 + 3d_{12} \leq 3m \leq 3n\). Bound (B.9) is satisfied.

Case 2: \(n < m \leq \frac{3}{2}n\).

In this case, we first show that for any precoding matrices \(U_1, U_2, U_3\) and \(U_{123}\), there exist another set of precoding matrices \(\hat{U}_1, \hat{U}_2, \hat{U}_3\) and \(\hat{U}_{123}\), which has the same dimension with \(U_1, U_2, U_3\) and \(U_{123}\), respectively, at each time slot and can achieve the same linear degrees of freedom. Meanwhile, \(\hat{U}_1, \hat{U}_2, \hat{U}_3\) and \(\hat{U}_{123}\) satisfy the following additional condition: \(\text{Span}(H_j\hat{U}_i)\) is linearly independent with \(\text{Span}(H_j\hat{U}_k)\) for any mutually different \(i, j, k \in \{1, 2, 3\}\).

Take \(\text{Span}(H_1\hat{U}_2)\) and \(\text{Span}(H_1\hat{U}_3)\) for example. If there exists a pair of \(u_2\) and \(u_3\) which will align at receiver 1, i.e., \(H_1u_2 = H_1u_3 \neq 0\), then we have that \(u_2 - u_3\) lies in the null space of channel \(H_1\). The dimension of \(\mathcal{N}(H_1), \mathcal{N}(H_2)\) and \(\mathcal{N}(H_3)\) are all equal to \(m - n\). Since \(m < \frac{3}{2}n\), we have that \(3(m - n) < m\). Thus, the \(\mathcal{N}(H_1), \mathcal{N}(H_2)\) and \(\mathcal{N}(H_3)\) will be linearly independent of each other almost surely. Consequently, we have that \(H_3u_2 = 0\) and \(H_2u_3 = 0\) can not be sure at the same time. Now, we modify the precoding matrix according to the following rule: if \(H_3u_2 \neq 0\), we change the basis vector \(u_2\) in \(U_2\) to \(u_2 - u_3\) and keep the other basis vector the same, and we name the modified matrix as \(\hat{U}_2\); if \(H_3u_2 = 0\), we change the basis vector \(u_3\) in \(U_3\) to \(u_2 - u_3\) and keep the other basis vector the same, and we name the modified matrix as \(\hat{U}_3\). By doing this kind of modification, we transfer the original interference alignment of one symbol of \(W_2\) with one symbol of \(W_3\) at receiver 1 to zero-forcing one of them at receiver 1 and letting the other one occupy the same subspace at receiver 1 by itself.

Next, we show that the decodability requirement at receiver 2 and 3 are also satisfied. In other words, if \(U_2\) is the one which is modified, the following equalities still hold.

\[
\text{Rank}(H_1[U_1 \hat{U}_2 U_3 U_{123}]) = \text{Rank}(H_1[U_1 U_{123}]) + \text{Rank}(H_1[\hat{U}_2 U_3]) \quad (B.10)
\]
\[
\text{Rank}(H_2[U_1 \hat{U}_2 U_3 U_{123}]) = \text{Rank}(H_2[\hat{U}_2 U_{123}]) + \text{Rank}(H_2[U_1 U_3]) \quad (B.11)
\]
\[
\text{Rank}(H_3[U_1 \hat{U}_2 U_3 U_{123}]) = \text{Rank}(H_3[U_3 U_{123}]) + \text{Rank}(H_3[U_1 \hat{U}_2]) \quad (B.12)
\]
\[ \text{Rank}(H_1[U_1 U_{123}]) = m_1 + m_{123} \] (B.13)
\[ \text{Rank}(H_2[\hat{U}_2 U_{123}]) = m_2 + m_{123} \] (B.14)
\[ \text{Rank}(H_3[U_3 U_{123}]) = m_3 + m_{123}. \] (B.15)

(B.10) and (B.13) follow from the fact that interference alignment and zero-forcing both consume one available dimension from the nullspace \( \mathcal{N}(H_1) \) and have equivalent affect at receiver 1. (B.15) holds since there is no change on either side of the equation. (B.14) holds since \( m < 2n \) such that \( u_2 - u_3 \) can not be zero-forced at receiver 2 at the same time. For equality (B.11), since \( H_2u_2 \) can not be zero, if there exists a vector \( u \) such that \( H_2u \in \text{Span}(H_2[\hat{U}_2 U_{123}]) \) and also \( \in \text{Span}(H_2[U_1 U_3]) \), let \( \alpha \) be the coefficient of basis vector \( u_2 - u_3 \) to construct \( u \), then it is easy to obtain that \( H_2(u + \alpha u_3) \) will belong both to \( \text{Span}(H_2[U_2 U_{123}]) \) and \( \text{Span}(H_2[U_1 U_3]) \). This contradicts with the original decodability requirement and can not be true. Consequently, no vector can belong to \( \text{Span}(H_2[\hat{U}_2 U_{123}]) \) and \( \text{Span}(H_2[U_1 U_3]) \) at the same time, and thus we have equality (B.11). Similarly, since \( H_3u_2 \neq 0 \), we have equality (B.12).

So far, we have shown that \([U_1 \hat{U}_2 U_3 U_{123}]\) (or \([U_1 U_2 \hat{U}_3 U_{123}]\) if \( u_3 \) is the one which is modified) achieves the same DoF tuple with \([U_1 U_2 U_3 U_{123}]\). By iteratively transfer all interference alignment at all three receivers, we continuously update matrix \( U_1, U_2 \) and \( U_3 \) and finally obtain \( \hat{U}_1, \hat{U}_2 \) and \( \hat{U}_3 \), such that only zero-forcing exists and no alignment of interference happens. In other words, the interference subspace, \( \text{Span}(H_j\hat{U}_i) \) is linearly independent with \( \text{Span}(H_j\hat{U}_k) \) for any \( i, j, k \in \{1, 2, 3\} \) and mutually different from each other. Finally, we choose \( \hat{U}_{123} = U_{123} \).

Now let \( m_{i,j} \) denote the number of streams of \( W_i \) that will be zero-forcing at receiver \( j \), and \( U_{i,j} \) denote corresponding beamformer sub-matrices. Since \( m \leq 2n \), it is not possible that a beamformer can be zero-forced at two receivers at the same time. In other words, the \( m_{i,j} \) streams of message \( W_i \) be zero-forced at receiver \( j \) will be linearly independent with the \( m_{i,k} \) streams be zero-forced at receiver \( k \). Thus, \( m_i - m_{i,j} - m_{i,k} \geq 0 \), where \( i, j, k \in \{1, 2, 3\} \) and are mutually different from each other. At receiver 1, we have that
\[ \text{Rank}(H_1[U_2 U_3]) = \text{Rank}(H_1U_2) + \text{Rank}(H_1U_3) \]
Together with (B.10) and (B.13), we have following bound

\[ m_1 + m_{123} + m_2 - m_{2,1} + m_3 - m_{3,1} \leq T \cdot n \]  

(B.16)

similarly, we have

\[ m_2 + m_{123} + m_1 - m_{1,2} + m_3 - m_{3,2} \leq T \cdot n \]  

(B.17)

\[ m_3 + m_{123} + m_1 - m_{1,3} + m_2 - m_{2,3} \leq T \cdot n. \]  

(B.18)

Adding (B.16), (B.17) and (B.18) together, and together with \( m_i - m_{i,j} - m_{i,k} \geq 0 \), we have that

\[ 2m_1 + 2m_2 + 2m_3 + 3m_{123} \leq 3T \cdot n. \]

Thus, we have that

\[
2d_1 + 2d_2 + 2d_3 + 3d_{123} = \lim_{T \to \infty} \frac{1}{T} \cdot (2m_1 + 2m_2 + 2m_3 + m_{123})
\leq 3n.
\]

Case 3: \( \frac{3}{2} n < m \leq 2n \).

In this case, we use the same strategy as in case 2 and transfer the interference alignment to equivalent zero-forcing. However, since \( m > \frac{3}{2} n \), we have that \( 3(m - n) > m \). Thus, the nullspaces \( \mathcal{N}(H_1), \mathcal{N}(H_2) \) and \( \mathcal{N}(H_3) \) will not be linearly independent almost surely. According to the analysis in Section 3.4, the dimension of the intersection, i.e., \( x \) in equation (3.22), is equal to \( 2m - 3n \).

Consider again that if there exists a pair of \( u_2 \) and \( u_3 \) which will align at receiver 1, i.e., \( H_1u_2 = H_1u_3 \neq 0 \), then we have that \( u_2 - u_3 \) lies in the null space of channel \( H_1 \). However, in case 3, it is possible that we have that \( H_3u_2 = 0 \) and \( H_2u_3 = 0 \) at the same time due to the intersection of \( \mathcal{N}(H_1), \mathcal{N}(H_2) \) and \( \mathcal{N}(H_3) \). For this kind of \((u_2, u_3)\) pair, it is not possible to transfer the alignment to zero-forcing one of them at receiver 1. For example, if we replace \( u_2 \) by \( u_2 - u_3 \) and zero-force
it at receiver 1. \( H_3(u_2 - u_3) = H_3u_3 \in \text{span}(H_3U_3) \) will break the decodability requirement at receiver 3. Let the number of such \((u_2, u_3)\) pairs be \( A_{23} \), and the numbers of similar \((u_1, u_3)\) and \((u_1, u_2)\) pairs be \( A_{13} \) and \( A_{12} \), respectively. We have that \( A_{12} + A_{13} + A_{23} \leq x = 2m - 3n \), since each pair consumes one dimension in the intersection subspace.

We still iterative update matrix \( U_1, U_2 \) and \( U_3 \) to obtain \( \hat{U}_1, \hat{U}_2 \) and \( \hat{U}_3 \), and leave these exceptional pairs unchanged. In other words, we only reserve the aforementioned interference alignments which happen in the intersection subspace. Then, we have that

\[
\text{Rank}(H_1[U_2 U_3]) = \text{Rank}(H_1U_2) + \text{Rank}(H_1U_3) - A_{23} = m_2 - m_{2,1} + m_3 - m_{3,1} - A_{23}. \tag{B.19}
\]

Similarly, we have that

\[
\text{Rank}(H_2[U_1 U_3]) = m_1 - m_{1,2} + m_3 - m_{3,2} - A_{13}. \tag{B.20}
\]

\[
\text{Rank}(H_3[U_1 U_2]) = m_1 - m_{1,3} + m_2 - m_{3,3} - A_{12}. \tag{B.21}
\]

Since \( m \leq 2n \), it is not possible that a beamformer can be zero-forced at two receivers at the same time. In other words, the \( m_{i,j} \) streams of message \( W_i \) be zero-forced at receiver \( j \) will be linearly independent with the \( m_{i,k} \) streams be zero-forced at receiver \( k \). Thus, \( m_i - m_{i,j} - m_{i,k} \geq 0 \).

Together with (B.10) - (B.15) and sum the inequalities together, we have that

\[
2m_1 + 2m_2 + 2m_3 + 3m_{123} \leq 3n + A_{23} + A_{12} + A_{13} \leq 3n + 2m - 3n = 2m
\]

Consequently, we obtain the outer bound on the linear DoF region as

\[
2d_1 + 2d_2 + 2d_3 + 3d_{123} \leq 2m.
\]
Case 4: $2n < m \leq 3n$.

In this case, the intersection of $\mathcal{N}(H_1)$, $\mathcal{N}(H_2)$ and $\mathcal{N}(H_3)$ also exists and its dimension is equal to $3n - m$, according to equation (3.22). Furthermore, since $m > 2n$, nullspaces $\mathcal{N}(H_{12})$, $\mathcal{N}(H_{13})$ and $\mathcal{N}(H_{23})$ all exist, and their dimensions are all equal to $m - 2n$. Thus, it is possible that one beamformer can be zero-forced at two receivers. Consequently, it is possible that $m_i - m_{i,j} - m_{i,k}$ is less than zero. The minimal value of $m_i - m_{i,j} - m_{i,k}$ is equal to $-(m - 2n)$, which is achieved when all symbols of message $W_i$ are transmitted in the nullspace of $\mathcal{N}(H_{ij})$. Thus, we have that

$$m_i - m_{i,j} - m_{i,k} \geq -(m - 2n).$$

Similar to the analysis of case 3, we have that

$$2m_1 + 2m_2 + 2m_3 + 3m_{123} \leq 3n + (3n - m) + 3(m - 2n)$$

$$= 2m$$

Consequently, we obtain the outer bound on the linear DoF region as

$$2d_1 + 2d_2 + 2d_3 + 3d_{123} \leq 2m.$$ 

Case 5: $m > 3n$.

In this case, the transmitter has more antennas than the receivers together have. The DoF region of the system is given by the cutset bound $d_i + d_{123} \leq n$ $(i = 1, 2, 3)$. Thus

$$2d_1 + 2d_2 + 2d_3 + 3d_{123} \leq 6n < 2m$$

is also valid.

In sum, we have that an outer bound on the linear DoF region of the three-user broadcast channel with message sets $(W_1, W_2, W_3, W_0)$ is given by constraint (B.9).
B.2 Generic transformation

**Lemma 18.** Consider two matrices $A$ and $B$. If $A \in \mathbb{C}^{n \times m}$ is a generic matrix and $B \in \mathbb{C}^{m \times p}$ has full column rank, and $p \leq n \leq m$, then $AB$ is a full column rank matrix almost surely.

**Proof.** We prove this Lemma by contradiction.

Suppose $AB$ is not full column rank, then there exist a vector $\beta \in \mathbb{C}^{p \times 1}$ and $\beta \neq 0$, such that $AB\beta = 0$. Since $B$ has full column rank, we have that $B\beta$ is a $m \times 1$ non-zero vector, and we denote it as $\alpha = B\beta$, and $\alpha$ belongs to the space spanned by the column vectors of matrix $B$, i.e., $\alpha \in \text{Span}(B)$. Moreover, since $A\alpha = 0$, we have that $\alpha$ belongs to the null space of transformation $A$, i.e., $\alpha \in \mathcal{N}(A)$. Consequently, we have that $\text{Span}(B)$ and $\mathcal{N}(A)$ intersect with each other at a subspace with at least rank 1.

Since $A$ is a generic matrix, it will be full rank almost surely, and the rank of its nullspace is equal to $m - n$. Again, since $A$ is generic, the column vectors of $\mathcal{N}(A)$ will be linearly dependent of the columns of $B$ only if they have to be. Since the sum of the rank of $\mathcal{N}(A)$ and the rank of $\text{Span}(B)$ is no greater than the total dimension of the available subspace, i.e., $(m - n) + p \leq m$, it is almost surely that $\mathcal{N}(A)$ and $\text{Span}(B)$ will be linearly independent with each other. Thus, this contradicts with previous argument that $\text{Span}(B)$ and $\mathcal{N}(A)$ intersect with each other at a subspace with at least rank 1. Consequently, such a vector $\beta$ does not exist almost surely. In other words, $AB$ is a full column rank matrix almost surely. \qed
Appendix c

MIMO 2-user BC-CM

C.1 Proof of Lemma 17

Here we restate Lemma 17 before proving it.

**Lemma 17.** For the 2-user MIMO BC-PM with hybrid CSIT of type ‘PN’, if \( N_2 \leq \min(M, N_1) \), considering any linear coding scheme as described as in Section 4.2, if \( V_1^{(T)} \) is decodable (i.e., the symbols of message \( W_1 \) are all decodable) at receiver 1, we have that

\[
\frac{\text{rank}(H_1^{(T)} V_1^{(T)})}{\text{rank}(H_2^{(T)} V_1^{(T)})} \leq \frac{\min(M, N_1)}{N_2} \tag{C.1}
\]

for arbitrary \( T \) and \( V_1^{(T)} \).

**Proof.** It is worth noting that we only consider the precoding matrix \( V_1^{(T)} \) for message \( W_1 \) and its transformations at both receivers in the statement of this lemma, so that the matrices \( V_0^{(T)} \) and \( V_2^{(T)} \) are non-existent here in the analysis without any impact on its validity. In other words, no symbols of message \( W_2 \) and \( W_0 \) are transmitted in the channel in the following analysis. This setting is crucial in this proof, as we will show later.

The difficulty of the proof is that the channel matrices \( H_1^{(T)} \) and \( H_2^{(T)} \) are not generic matrices. They are block-diagonal matrices. Many nice properties of generic matrices cannot be directly used here. To deal with the block-diagonal channels, we first show that there exists a block-diagonal
matrix \( \hat{V}_1^{(T)} \), which has the same size as \( V_1^{(T)} \) and can be written as

\[
\hat{V}_1^{(T)} = \begin{bmatrix}
\hat{V}_1(1) & 0 & \cdots & 0 \\
0 & \hat{V}_2(2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{V}_2(T)
\end{bmatrix}
\] (C.2)

where \( \hat{V}_1(i) \), \( i = 1, \ldots, T \) are \( M \times m_1(i) \) matrices with full column rank and \( \sum_{i=1}^{T} m_1(i) = m_1^{(T)} \).

Furthermore, the beamformers chosen from \( \text{Span}(\hat{V}_1^{(T)}) \) are all decodable at Receiver 1, and the following two constraints are satisfied

\[
\text{rank}(H_1^{(T)}V_1^{(T)}) = \text{rank}(H_1^{(T)}\hat{V}_1^{(T)}) = m_1^{(T)} \quad \text{(C.3)}
\]

\[
\text{rank}(H_2^{(T)}V_1^{(T)}) \overset{a.s.}{\geq} \text{rank}(H_2^{(T)}\hat{V}_1^{(T)}). \quad \text{(C.4)}
\]

A detailed approach to constructing such a \( \hat{V}_1^{(T)} \) is given in Appendix C.2. From constraints (C.3) and (C.4) we have that

\[
\frac{\text{rank}(H_1^{(T)}V_1^{(T)})}{\text{rank}(H_2^{(T)}V_1^{(T)})} \overset{a.s.}{\leq} \frac{\text{rank}(H_1^{(T)}\hat{V}_1^{(T)})}{\text{rank}(H_2^{(T)}\hat{V}_1^{(T)})}. \quad \text{(C.5)}
\]

In order to prove (C.1), it therefore suffices to prove that

\[
\frac{\text{rank}(H_1^{(T)}\hat{V}_1^{(T)})}{\text{rank}(H_2^{(T)}\hat{V}_1^{(T)})} \leq \frac{\min(M, N_1)}{N_2}. \quad \text{(C.6)}
\]

Since \( H_1^{(T)} \), \( H_2^{(T)} \) and \( \hat{V}_1^{(T)} \) are all block diagonal, the image subspaces at each receiver corresponding to different time slots are orthogonal to each other. Thus, we have that \( \text{Span}(H_r^t\hat{V}_1^{(T)}) \) \( (t = 1, \ldots, T) \) are linearly independent with each other for \( r = 1, 2 \), where \( H_r^t \) is the \( t \)-th block-row of matrix \( H_r^{(T)} \). Since only the values of \( \hat{V}_1^{(T)} \) during the \( t \)-th time slot contribute to \( \text{Span}(H_r^t\hat{V}_1^{(T)}) \), we have that \( \text{rank}(H_r^t\hat{V}_1^{(T)}) = \text{rank}(H_r(t) \cdot \hat{V}_1(t)) \).

Because the transmitter has no CSI from Receiver 2 and the channel matrix \( H_2(t) \) is generic, the least amount of alignment will occur at Receiver 2. If \( m_1(t) < N_2 \), i.e., the number of \( W_1 \) symbols transmitted at time slot \( t \) is fewer than the total available dimension at Receiver 2, we have that \( \text{rank}(H_1(t)\hat{V}_1(t)) \overset{a.s.}{\leq} \text{rank}(H_2(t)\hat{V}_1(t)) \). In general we have that \( \text{rank}(H_1(t)\hat{V}_1(t)) \leq \text{rank}(H_2(t)\hat{V}_1(t)) \).
rank(\(\hat{V}_1(t)\)) \leq m_1(t), \text{ for all } t. \text{ However, since message } W_1(t) \text{ needs to be decodable at Receiver 1, we cannot have strict inequality for any } t, \text{ for if we did, summing over all } t \text{ we would have}
\sum_{t=1}^{T} \text{rank}(H_1(t)\hat{V}_1(t)) = \text{rank}(H_1^{(T)}\hat{V}_1^{(T)}) < \sum_{t=1}^{T} m_1(t) = m_1^{(T)}, \text{ contradicting (C.3). Hence, we have that rank}(H_1(t)\hat{V}_1(t)) = \text{rank}(\hat{V}_1(t)) = m_1(t), \forall t. \text{ Also, we have that rank}(H_2(t)\hat{V}_1(t)) \leq \text{rank}(\hat{V}_1(t)) = m_1(t). \text{ Consequently, we have that rank}(H_2(t)\hat{V}_1(t)) \overset{a.s.}{=} \text{rank}(H_1(t)\hat{V}_1(t)) = m_1(t).

The ratio \(\frac{\text{rank}(H_1(t)\hat{V}_1(t))}{\text{rank}(H_2(t)\hat{V}_1(t))} \overset{a.s.}{=} 1 \leq \frac{\min(M,N_1)}{N_2}.

Next, consider the case that \(m_1(t) \geq N_2\). Since the number of \(W_1\) symbols is greater than the total available dimension at Receiver 2 at time slot \(t\), \(H_2(t)\hat{V}_1(t)\) will almost surely span the entire receiver subspace, i.e., \(\text{rank}(H_2(t)\hat{V}_1(t)) \overset{a.s.}{=} N_2\). Meanwhile, the decodability of message \(W_1\) requires that \(m_1(t) \leq \min(M,N_1)\). Hence, we have

\[\frac{\text{rank}(H_1(t)\hat{V}_1(t))}{\text{rank}(H_2(t)\hat{V}_1(t))} \overset{a.s.}{=} \frac{m_1(t)}{N_2} \leq \frac{\min(M,N_1)}{N_2}. \tag{C.7}\]

Thus, for both cases, we have that inequality (C.7) is always true.

From (C.7), we obtain that
\[\text{rank}(H_2(t)\hat{V}_1(t)) \geq \frac{N_2}{\min(M,N_1)} \cdot \text{rank}(H_1(t)\hat{V}_1(t)).\]

Consequently, we have that
\[\sum_{t=1}^{T} \text{rank}(H_2(t)\hat{V}_1(t)) \geq \frac{N_2}{\min(M,N_1)} \cdot \sum_{t=1}^{T} \text{rank}(H_1(t)\hat{V}_1(t)),\]

which leads to
\[\sum_{t=1}^{T} \text{rank}(H_2^{(T)}\hat{V}_1^{(T)}) \geq \frac{N_2}{\min(M,N_1)} \cdot \sum_{t=1}^{T} \text{rank}(H_1^{(T)}\hat{V}_1^{(T)})\]

and
\[\text{rank}(H_2^{(T)}\hat{V}_1^{(T)}) \geq \frac{N_2}{\min(M,N_1)} \cdot \text{rank}(H_1^{(T)}\hat{V}_1^{(T)}). \tag{C.8}\]

which is the same as (C.6). Hence the proof is complete.

Remark 14. In Lemma 17, \(\hat{V}_1^{(T)}\) is constructed only to assist the proof of inequality (C.1). It does not mean that by directly replacing \(V_1^{(T)}\) in the original system with \(\hat{V}_1^{(T)}\), the original system
can still work. This is because $\hat{V}_{1}^{(T)}$ may conflict with $V_{2}^{(T)}$ and $V_{0}^{(T)}$, and make some messages undecodable. However, in the analysis of Lemma 17, this does not matter, because the other messages are non-existent.

Remark 15. In Lemma 17, if $N_2 > \min(M, N_1)$, the LHS of (C.1) will be almost surely equal to 1. This follows directly from the fact that $m_1(t) \leq \min(M, N_1)$ and is always less than $N_2$, such that $\text{rank}(H_2(t)\hat{V}_1(t)) \overset{a.s.}{=} \text{rank}(H_1(t)\hat{V}_1(t)) = m_1(t)$ is always true.

C.2 The way to construct $\hat{V}_{1}^{(T)}$

In this appendix, we systematically construct the $\hat{V}_{1}^{(T)}$ of equation (C.2) step-by-step from $V_{1}^{(T)}$, and then show that its aforementioned properties in constraints (C.3) and (C.4) hold.

Step 1: consider the first block-row in $V_{1}^{(T)}$, i.e., $V_{1}(1)$. Let $a = \text{rank}(V_{1}(1))$. Then, we can express the beamformer subspace $\text{Span}(V_{1}^{(T)})$ using another set of basis vectors, such that they are column vectors of the following block triangular matrix

\[
V_{1,\text{step1}}^{(T)} = \begin{bmatrix}
V_{1,a}(1) & 0 \\
V_{1,a}(2) & V_{1,b}(2) \\
\vdots & \vdots \\
V_{1,a}(T) & V_{1,b}(T)
\end{bmatrix}.
\]  

(C.9)

where the size of sub-matrix $V_{1,a}(i)$ is $M \times a$, and the size of sub-matrix $V_{1,b}(i)$ is $M \times (m_{1}^{(T)} - a)$.

The way to obtain this new set of basis vectors is as follows. First, pick any $a$ column vectors of $V_{1}^{(T)}$ whose sub-vectors corresponding to the first time slots form a basis of $\text{Span}(V_{1}(1))$, and place them as the first $a$ columns of $V_{1,\text{step1}}^{(T)}$. This basis matrix corresponding to the first time slot is defined as $V_{1,a}(1)$ in (C.9), and we define the sub-matrix which contains the obtained first $a$ columns of $V_{1,\text{step1}}^{(T)}$ as $V_{1,a}^{1,\text{step1}}$. Next, for each of the rest $(m_{1}^{(T)} - a)$ column vectors in $V_{1}^{(T)}$, we denote generically (one by one) as $\begin{bmatrix}\vec{v}_1 \\
\vec{v}_R\end{bmatrix}$, where $\vec{v}_1$ contains the first $M$ rows (i.e., it corresponds to the first time slot), and $\vec{v}_R$ is the remaining part. If $\vec{v}_1 = 0$, we add this column vector directly as the next column vector of $V_{1,\text{step1}}^{(T)}$. If $\vec{v}_1 \neq 0$, then it can be rewritten as a linear combination of the
column vectors of $V_{1,a}(1)$, say $\vec{v}_1 = V_{1,a}(1)\vec{x}$, where $\vec{x}$ is the $a \times 1$ vector of coefficients. Then, we add $\begin{bmatrix} \vec{v}_1 \\ \vec{v}_R \end{bmatrix} - V_{1,\text{step}1}^{1:a} \vec{x}$ as the next column vector of $V_{1,\text{step}1}^{(T)}$, such that its top $M$ elements are zeros.

After processing all the rest of the $m_1^{(T)} - a$ unselected vectors in $V_{1}^{(T)}$ in this way, we finally obtain the $MT \times m_1^{(T)}$ dimensional new basis matrix $V_{1,\text{step}1}^{(T)}$. The column vectors of $V_{1,\text{step}1}^{(T)}$ are guaranteed to be mutually linearly independent since each of them contains a different independent basis vector from $V_{1}^{(T)}$. In linearly transforming $V_{1}^{(T)}$ to $V_{1,\text{step}1}^{(T)}$, it can be shown that the subspace spanned by the beamformers in $V_{1}^{(T)}$ remains unchanged, i.e., $\text{Span}(V_{1}^{(T)}) = \text{Span}(V_{1,\text{step}1}^{(T)})$. Consequenty, we have that $\text{rank}(H_{1}^{(T)}V_{1}^{(T)}) = \text{rank}(H_{1}^{(T)}V_{1,\text{step}1}^{(T)})$ and $\text{rank}(H_{2}^{(T)}V_{1}^{(T)}) = \text{rank}(H_{2}^{(T)}V_{1,\text{step}1}^{(T)})$.

Step 2: Let $n_1$ be the dimension of the intersection of the beamformer space spanned by the column vectors of $V_{1,a}(1)$ and the nullspace of channel $H_{1}(1)$, i.e., $n_1 = \text{rank}(\text{Span}(V_{1,a}(1)) \cap \text{null}(H_{1}(1)))$. Hence, if we only use the received signal at the first time slot, we have that only $a - n_1$ independent symbols of $W_1$ are decodable at receiver 1. Again, we perform a linear transformation of $V_{1,\text{step}1}^{(T)}$ to $V_{1,\text{step}2}^{(T)}$ given below

$$V_{1,\text{step}2}^{(T)} = \begin{bmatrix} V_{1,c}(1) & V_{1,d}(1) & 0 \\ V_{1,c}(2) & V_{1,d}(2) & V_{1,b}(2) \\ \vdots & \vdots & \vdots \\ V_{1,c}(T) & V_{1,d}(T) & V_{1,b}(T) \end{bmatrix},$$

such that the size of $V_{1,c}(i)$ is $M \times (a - n_1)$ and the size of $V_{1,d}(i)$ is $M \times n_1$ and $\text{Span}(V_{1,d}(1)) \subset \text{null}(H_1)$. In other words, we linearly transform $V_{1,\text{step}1}^{(T)}$ such that the last $n_1$ column of its first block-column will be zero-forced at Receiver 1 in the first time slot. The transformation procedure is similar to that in Step 1, and so we omit the details for brevity. So far, the spanned beamformer subspace is still unchanged, i.e., $\text{Span}(V_{1}^{(T)}) = \text{Span}(V_{1,\text{step}2}^{(T)})$. As a result, we still have equalities that, $\text{rank}(H_{1}^{(T)}V_{1}^{(T)}) = \text{rank}(H_{1}^{(T)}V_{1,\text{step}2}^{(T)})$ and $\text{rank}(H_{2}^{(T)}V_{1}^{(T)}) = \text{rank}(H_{2}^{(T)}V_{1,\text{step}2}^{(T)})$. 
Step 3: We set $V_{1,d}(1)$ and $V_{1,c}(i), i = 2, ..., T$, in $V_{1,\text{step2}}^{(T)}$ to all-zero and obtain $V_{1,\text{step3}}^{(T)}$, i.e.,

$$V_{1,\text{step3}}^{(T)} = \begin{bmatrix} V_{1,c}(1) & 0 & 0 \\ 0 & V_{1,d}(2) & V_{1,b}(2) \\ \vdots & \vdots & \vdots \\ 0 & V_{1,d}(T) & V_{1,b}(T) \end{bmatrix}.$$  

The rationale is as follows: because the equivalent channel matrix $H_1^{(T)}$ is block-diagonal, the value of $V_{1,d}(1)$ in $V_{1,\text{step2}}^{(T)}$ can only affect Receiver 1 at the first time slot. Since $H_1(1)V_{1,d}(1) = 0$, the overall received signal at Receiver 1 is unchanged after we replace $V_{1,d}(1)$ with the all-zeros matrix (denoted simply as 0). Consequently, all the symbols of $W_1$ which were decodable continue to be decodable after this replacement. Recall that the messages $W_0$ and $W_2$ are empty, and only $W_1$ is transmitted over the channel. Since $V_{1,c}(1)$ is a full column rank matrix and has no intersection with the nullspace of $H_1(1)$, and $V_{1,c}(1)$ is decodable even if we just use the received signal from the first time slot\(^1\). Thus, no matter what value $H_1(i)V_{1,c}(i)$ $(i = 2, ..., T)$ may be, they can be eliminated after decoding $V_{1,c}(1)$. Consequently, we can, without loss of generality, set $V_{1,c}(i)$ $(i = 2, ..., T)$ to zero instead, and the resulting $V_{1,\text{step3}}^{(T)}$ is still decodable. As a result, we have that $\text{rank}(H_1^{(T)}V_{1,\text{step3}}^{(T)}) = \text{rank}(V_{1,\text{step3}}^{(T)}) = m_1^{(T)} = \text{rank}(H_1^{(T)}V_1^{(T)})$.

It can be shown from Lemma 19 and Remark 17 in the Appendix C.3 that $\text{rank}(H_2^{(T)}V_{1,\text{step2}}^{(T)}) \overset{a.s.}{\geq} \text{rank}(H_2^{(T)}V_{1,\text{step3}}^{(T)}).$

Step 4: we repeat Step 1-3 for the rest of the block rows (successively from the second block row to the last one), to finally obtain a block diagonal precoding matrix and name it $\hat{V}_1^{(T)}$.

From the construction of $\hat{V}_1^{(T)}$, we have that each of its column vectors are decodable at Receiver 1. Thus, we have that $\text{rank}(\hat{V}_1^{(T)}) = m_1^{(T)}$ and condition (C.3) is satisfied. We define the column rank of the $i$-th diagonal block in $\hat{V}_1^{(T)}$ as $m_1(i)$, and we have that $\sum_{i=1}^{T} m_1(i) = m_1^{(T)}$.

The condition (C.4) follows from the transitivity of the inequality relation, since with each transformation of the beamforming matrix $V_1^{(T)}$ in the sequence of transformations leading to $\hat{V}_1^{(T)},$

\(^1\) This may be not true if there are other messages in the system, since they will impact what Receiver 1 receives at each time slots. $V_{1,c}(1)$ may be aligned with other messages and thus it may be not decodable.
\text{rank}(H_2^{(T)}V_1^{(T)}) \text{ evolves in a monotonic non-increasing fashion to rank}(H_2^{(T)}\hat{V}_1^{(T)}).

\section{A useful lemma}

\textbf{Lemma 19.} Consider the matrix \( X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), where \( A, B, C \) and \( D \) are all sub-matrices whose sizes satisfied the concatenation requirement. If \( X \) has full column rank, then

\[
\text{rank} \left( \begin{bmatrix} H_1A & H_1B \\ H_2C & H_2D \end{bmatrix} \right) \overset{a.s.}{\geq} \text{rank} \left( \begin{bmatrix} H_1A & 0 \\ 0 & H_2D \end{bmatrix} \right),
\]

\[(C.10)\]

where \( H_1 \) and \( H_2 \) are two generic matrices independent with each other and also with \( X \).

\textit{Proof.} First, we prove that

\[
\text{rank} \left( \begin{bmatrix} H_1A & H_1B \\ H_2C & H_2D \end{bmatrix} \right) \overset{a.s.}{=} \text{rank} \left( \begin{bmatrix} H_1A \\ H_2C \end{bmatrix} \right) + \text{rank} \left( \begin{bmatrix} H_1B \\ H_2D \end{bmatrix} \right) \quad (C.11)
\]

Then, from the trivial facts that

\[
\text{rank} \left( \begin{bmatrix} H_1A \\ H_2C \end{bmatrix} \right) \geq \text{rank} (H_1A)
\]

\[
\text{rank} \left( \begin{bmatrix} H_1B \\ H_2D \end{bmatrix} \right) \geq \text{rank} (H_2D).
\]

and

\[
\text{rank} \left( \begin{bmatrix} H_1A & 0 \\ 0 & H_2D \end{bmatrix} \right) = \text{rank} (H_1A) + \text{rank} (H_2D),
\]

we have inequality \((C.10)\).

Consider \((C.11)\). It indicates that the Span \( \left( \begin{bmatrix} H_1A \\ H_2C \end{bmatrix} \right) \) and Span \( \left( \begin{bmatrix} H_1B \\ H_2D \end{bmatrix} \right) \) are linearly independent with each other almost surely. Suppose there exist a vector, \( \vec{v} \), which belongs to both Span \( \left( \begin{bmatrix} H_1A \\ H_2C \end{bmatrix} \right) \) and Span \( \left( \begin{bmatrix} H_1B \\ H_2D \end{bmatrix} \right) \). Then, there exist two non-trivial column vectors \( \vec{x} \)
and \( \vec{y} \), such that
\[
\vec{v} = \begin{bmatrix}
H_1 A \\
H_2 C
\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}
H_1 B \\
H_2 D
\end{bmatrix} \quad \vec{y}.
\]
Then, we have \( H_1 (A \vec{x} - B \vec{y}) = 0 \) and \( H_2 (C \vec{x} - D \vec{y}) = 0 \). Consequently, \( A \vec{x} - B \vec{y} = 0 \) or \( \in \text{null}(H_1) \), and \( C \vec{x} - D \vec{y} = 0 \) or \( \in \text{null}(H_2) \). Since \( X \) has full column rank, \( A \vec{x} - B \vec{y} = 0 \) and \( C \vec{x} - D \vec{y} = 0 \) cannot be true at the same time. If we select \( \vec{x} \) and \( \vec{y} \) such that \( A \vec{x} - B \vec{y} = 0 \), we need that \( C \vec{x} - D \vec{y} \) be zero-forced by \( H_2 \). However, since \( H_2 \) is a generic matrix independent of \( A, B, C \) and \( D \), the probability that \( C \vec{x} - D \vec{y} \) falls in the nullspace of \( H_2 \) is almost surely zero. Similarly, if we select \( \vec{x} \) and \( \vec{y} \) such that \( A \vec{x} - B \vec{y} \in \text{null}(H_1) \), it is almost sure that \( C \vec{x} - D \vec{y} \notin \text{null}(H_2) \). Consequently, such a vector \( \vec{v} \) does not exist almost surely. Thus, we have (C.11). □

Remark 16. The high block-dimension extension, i.e., \( X \) in the form of \( N \times N \) \((N > 2)\) sub-blocks, of Lemma 19 follows in the extra same way. We omit the detailed proof due to simplicity.

Remark 17. Also, Lemma 19 can be extended straightforwardly to the following case and higher block-dimension, under the same problem setting.

\[
\begin{align*}
\text{rank} & \begin{pmatrix}
H_1 A_1 & H_1 A_2 & H_1 A_3 \\
H_2 B_1 & H_2 B_2 & H_2 B_3 \\
H_3 C_1 & H_3 C_2 & H_3 C_3
\end{pmatrix} \geq \text{rank} \begin{pmatrix}
H_1 A_1 & 0 & 0 \\
0 & H_2 B_2 & H_2 B_3 \\
0 & H_3 C_2 & H_3 C_3
\end{pmatrix},
\end{align*}
\]

The detailed proof is left to the reader.