The Dynamics of a Three-Dimensional Heton

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The Dynamics of a Three-dimensional Heton

by

Adhithiya Sivakumar

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The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
Hetons are defined, in two-layer quasigeostrophy, as tilted counter-rotating baroclinic vortex pairs with each vortex present in a different layer. The study of hetons is motivated by their usage within the context of two-layer quasigeostrophic theory to model the transport of heat in a number of geophysical flows including, perhaps most famously, advection in the open ocean. A number of variations and generalizations of the heton concept exist in literature. Here, following the work of V.M. Gryanik, we investigate the three-dimensional point vortex heton. We start with the derivation of a non-canonical Hamiltonian system of $2n$ ODEs corresponding to point vortex solutions of the Quasigeostrophic Potential Vorticity Equation in an unbounded three-dimensional domain, where $n$ is the number of point vortices. We then show that three-dimensional hetons arise naturally as solutions of this system when $n = 2$. The dynamics of a single three-dimensional heton in a comoving frame are then discussed. Fixed points and bifurcations in the Lagrangian trajectories are then catalogued using various analytical and numerical techniques, and finally, the volume trapped by a single three-dimensional heton is calculated numerically for various values of the parameter $Z$ corresponding to the vertical distance between the counter-rotating vortices that compose the heton.
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Chapter 1

Introduction

The present work concerns itself with the development and explication of a theory pertaining to the dynamics of a single three-dimensional heton in an unbounded domain. These three-dimensional hetons arise naturally as singular solutions to the quasigeostrophic potential vorticity equation for a continuously stratified fluid, and have seen considerable development (in both the continuously stratified and two-layer cases) in the works of V.M. Gryanič, Xavier Carton, William Young, and others. They may be thought of as idealizations of coherent vortex structures that are known to exist and participate in various geophysical processes involving the transport of heat. Indeed, the literature on hetons (and geophysical point-vortices in general) is abundant with applications of these models to turbulent convection and other such phenomena \[15\] \[9\] \[13\]. The fact that hetons transport heat is connected to the existence of a trapping region around them. We are hence interested in understanding the structure of fixed points and invariant manifolds that contribute to this region with a view to calculating its volume. This report is therefore organized as follows. A review of the literature relating to hetons is presented in the following chapter in three parts – the first two parts reviewing the theory of point-vortices and quasigeostrophy respectively. The third part then summarizes relevant work on quasigeostrophic point vortex theory. The third chapter presents a point-vortex formulation of the quasigeostrophic potential vorticity equation in an unbounded domain. N-vortex ODEs are briefly discussed as solutions to the aforementioned equations before solving the ODEs for the case \( N = 2 \) analytically. This yields two solutions, one of them being a three-dimensional heton. A detailed presentation of the dynamics of a passive particle
influenced by the heton follows, and the chapter ends with the description of a fairly rudimentary numerical algorithm used to compute the volume of the trapping region. The final chapter details several numerical calculations of the trapping volume, and relates them to the dipole length of the heton (the separation between its constituent vortices). Tests on the accuracy and convergence of the algorithms used are also presented. This is followed by a section in which the model of heton derived in this work is compared to the two-layer model in [22] and the continuously stratified model in [9]. The report then ends with proposals and discussions relating to possible future work on this topic.
Being that the objective of the present work is to study a specific configuration (namely, the three-dimensional heton) of three-dimensional point-vortex solutions to the quasi-geostrophic potential vorticity equation for a continuously stratified fluid, we present in this chapter a summary of previous work that motivated and influenced it. We do this in three sections. The first section is a brief summary of classical point vortex theory as it relates to the study of two point vortices. The second section serves as a short introduction to quasi-geostrophic theory following primarily the treatment given by Vallis in [21]. Here we present the quasi-geostrophic potential vorticity equation in two-layer and continuously stratified flows and talk about various related concepts. The final section is a history of point-vortex models of quasi-geostrophy. We run through the early work of Gryanik and the some relevant numerical experiments conducted by Charney. We also review the literature on hetons, starting with Hogg and Stommel, all the way to present-day applications of heton theory.

2.1 Classical Point-vortex Theory

The study of vortex motion began with Helmholtz in 1858. He states, in a paper he published that year, his now famous theorems of vorticity [11] for incompressible inviscid flow. He also constructs in Section V of the same paper perhaps the first point-vortex model of a fluid phenomenon. Here, the point-vortex is envisioned as the point of intersection between an infinitely long “vortex filament” and a plane perpendicular to it [3].
Relevant to the present work is the conclusion that two point-vortices in an unbounded domain will cause each other to move in a direction perpendicular to the line connecting them, and therefore rotate about their common center of gravity [11]. If these vortices have circulation of the same sign, the center of gravity lies between them. If they have circulation of opposite signs and unequal magnitudes, the center of gravity lies on the extension of the line connecting them, and if they have equal and opposite circulations, the center of gravity is infinitely far away, i.e., the point-vortices do not rotate but translate [11]. It must be noted that the term ‘center of gravity’ comes from an analogy Helmholtz drew in his paper between point vortices and point masses. The term has seen use in a lot of the literature since then, and is therefore used in this study too. Properly defined, the center of gravity is an average of vortex position co-ordinates weighted by vortex strength – a center of vorticity [22]. That point-vortices are weak or generalized solutions to the Euler Equations in two dimensions has been known for a long time. One can find references dating as far back as Lamb (1932) and Sommerfeld (1964) [19]. Note, however, that is the the velocity field in three dimensions is incompressible, then point-vortices can not exist, in the traditional sense, in 3D [2]. Following Aref [1], we present here a simple 2D point vortex model.

We start with the vorticity equation for two-dimensional incompressible, inviscid flow, for which the streamfunction $\Psi$ can be shown to exist:

$$\frac{\partial \zeta}{\partial t} - \frac{\partial \Psi}{\partial x} \frac{\partial \zeta}{\partial y} + \frac{\partial \Psi}{\partial y} \frac{\partial \zeta}{\partial x} = 0,$$

(2.1)

where $\zeta = -\Delta \Psi$ is the vorticity.

We postulate singular sources of vorticity $\zeta = \sum_{k=1}^{N} \Gamma_{k} \delta(x - x_{k}(t)) \delta(y - y_{k}(t))$, with each vorticity source (or point-vortex) at $(x_{k}, y_{k})$. Moving to the complex plane, i.e, defining $w := x + iy$, and inverting the equations (2.1), (2.2), gives us the trajectories of these point vortices in the complex plane:

$$\dot{w}^{*} = \frac{1}{2\pi i} \sum_{l=1, l\neq k}^{N} \Gamma_{l}(w_{k} - w_{l})^{-1},$$

where the asterisk denotes a complex conjugate.
Without moving to the complex plane, i.e. in $\mathbb{R}^2$ and without boundary conditions we may invert the Poisson Equation (2.2) and get:

$$
\Psi(x, y) = \frac{1}{2\pi} \sum_{k=1}^{N} \Gamma_k \log \sqrt{(x - x_k)^2 + (y - y_k)^2}.
$$

(2.3)

The point vortex trajectories are then given by the relations:

$$
\dot{x}_k(t) = -\frac{\partial \Psi}{\partial y} \bigg|_{x=x_k,y=y_k},
$$

(2.4)

$$
\dot{y}_k(t) = \frac{\partial \Psi}{\partial x} \bigg|_{x=x_k,y=y_k},
$$

(2.5)

with singular terms removed. Kirchhoff in 1876 proved that point-vortices in an unbounded domain form a Hamiltonian system. It has since been shown that the equations for any number of point-vortices in an arbitrary domain are Hamiltonian [1]. In an unbounded domain, the maximum number of vortices for which the equations (2.4), (2.5) are integrable is 3. The primary feature in the present work, systems of two vortices, are therefore always integrable in an unbounded domain.

Aref presents corresponding results for more general domains in [1]. The study of point-vortices has led to interesting developments not only in its field of origin, i.e. fluid mechanics, but also in more general areas of physics and applied mathematics. Of particular interest in such studies are stationary and steadily translating configurations of point-vortices. Such configurations can be derived from solutions of certain ordinary differential equations, and examples of such configurations are presented in [5]. Newton’s 2014 paper [17] presents a somewhat comprehensive account of other such work on point-vortex theory, mainly by H. Aref and his contemporaries. Of note is the idea that the distribution of point vortices along a straight line and will be in equilibrium if they are placed at the zeroes of certain special polynomials. Aref himself was a prolific reviewer of studies in vortex theory, and talks about this in detail in [4] and [3]. Even though closer approximations to real life vortex phenomena exist, viz. vortex patches and vortex sheets [1] [7], the study of point-vortices continues to be relevant because they provide extremely simplified descriptions of physical phenomena which are nevertheless useful in that their amenability to analysis using the machinery of Hamiltonian dynamics and other aspects of ODE theory enables the facile extraction of a large
quantity of useful information regarding the underlying dynamic processes from these models. Classical fluid dynamics aside, point-vortex models have found wide applicability in the study of superfluids [5], and perhaps more relevant to the present work, in geophysical fluid dynamics. Point-vortex models in geophysical fluid dynamics are the subject of the final section of this chapter.

2.2 A Brief Review of Quasigeostrophic Theory

We concern ourselves in this section with a continuously stratified fluid (in the Boussinesq sense) on the f-plane, i.e. we assume meridional variations of the Coriolis parameter are small and can be neglected, so that $\vec{f} = f_0 \hat{k}$. In this context, the quasigeostrophic equations can be thought of as describing flows that are almost in geostrophic balance – the idealized state where Coriolis forces are perfectly in balance with forces due to the pressure gradient. These flows are characterized by low Rossby number ($Ro = \frac{U}{f_0L} << 1$), where $U$ and $L$ are typical velocity and length scales. Being that the Rossby number parametrizes advective qualities of a motion relative to the effects of planetary rotation, a low Rossby number is characterized by the domination of rotation over advection, of Coriolis forces over inertial forces. Note that for a fully geostrophic flow, inertial forces are absent, and the Rossby number is therefore zero.

Under the assumptions of low Rossby number, advective time scaling, and small variations in stratification, the quasigeostrophic equations may be derived from the Boussinesq equations by non-dimensionalizing/scaling them and expanding the dependent variables in an asymptotic series in the Rossby number. A complete derivation can be found in Vallis, Ch. 5 [21], or Pedlosky, Ch. 6. Here, we merely state that in the first order, the momentum equations are in full geostrophic balance and the horizontal velocity is divergence-free. They can therefore be recast using a streamfunction $\Psi_0$. These equations are diagnostic in the sense that they do not contain any information about the evolution of low-order fields. The momentum equation in the next order ($O(Ro)$) yields a prognostic equation for the first order velocity, which depends upon a higher-order ageostrophic velocity. Elimination of the ageostrophic velocity from the $O(Ro)$ mass and momentum equations
leads to the quasigeostrophic potential vorticity equation for a continuously stratified fluid:

\[
\frac{D_0 Q}{D_0 t} = 0, \quad Q = \zeta_0 + \frac{\partial}{\partial z} \left( f_0^2 \frac{\partial \Psi}{N^2 \partial z} \right)
\] (2.6)

where \( Q \) is the quasigeostrophic potential vorticity, \( N \) is the Brunt-Väisälä frequency, \( \zeta_0 = \Delta H \Psi_0 \) is the first-order relative vorticity, the remaining term corresponds to vorticity stretching, and

\[
\frac{D_0}{D_0 t} = \frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla, \text{ a material derivative in the } O(1) \text{ geostrophic velocity.}
\]

Here \( \Delta_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) and \( N^2 = -\frac{g}{\rho_0} \frac{\partial \rho_0(z)}{\partial z} \) is proportional to the vertical derivative of the mean density, and therefore a measure of stratification. Closure of (2.6) requires vertical boundary conditions corresponding to advection of buoyancy.

Also of interest are the quasigeostrophic potential vorticity equations corresponding to a two-layer shallow water system. The two-layer system has seen considerable development over the years due to its being a reasonably good approximation of density stratification as it occurs in the ocean – two layers of more or less constant, but distinct, densities \[21\]. The quasigeostrophic potential vorticity (\( Q^m_S \), where \( m = 1, 2 \) indexes the layers) equation for two shallow layers on the f-plane is presented here in the form that it appears in \[22\]:

\[
\frac{\partial Q^m_S}{\partial t} + \frac{\partial \Psi^m_S}{\partial x} \frac{\partial Q^m_S}{\partial y} - \frac{\partial \Psi^m_S}{\partial y} \frac{\partial Q^m_S}{\partial x} = 0,
\] (2.7)

\[
Q^m_S = \Delta_H \Psi^m_S + \frac{(-1)^n}{2\lambda^2} (\Psi^1_S - \Psi^2_S).
\] (2.8)

Here \( \Psi^m_S \) is the streamfunction for each shallow layer and \( \lambda = \frac{1}{f_0} \sqrt{\frac{g' H_1 H_2}{H_1 + H_2}} \) is the internal deformation radius (\( g' = g \frac{\rho_2 - \rho_1}{\rho_1} \) is the reduced gravity) \[12\]. The second term in (2.8) represents vorticity due to interfacial stretching.

### 2.3 Point-vortex Models of Quasigeostrophy

One of the earliest, if not the earliest, generalizations of two-dimensional point vortex theory (as described in the first section of this chapter) to geophysical phenomena dominated by gravitational and Coriolis forces comes from Morikawa \[16\]. In the 1960 paper titled Geostrophic
**Vortex Motion**, Morikawa derives point-vortex solutions to the quasigeostrophic potential vorticity equation for a single shallow layer. The streamfunction in this case is shown to be a modified zeroth order Bessel function of the second kind, and the trajectories of these solutions are shown to be Hamiltonian in the sense of Kirchhoff. Also discussed are possible applications to the prediction of fully developed hurricane tracks and the simulation of upper-air cyclogenesis, albeit in restricted physical conditions. In 1963, J.G. Charney extended these results to continuously stratified flows [14], remarking that the conservation of potential vorticity (see eqn. (2.6)) enables the approximation of the three-dimensional continuum mechanics of the atmosphere and oceans by point vortices, which are ordinarily restricted to two-dimensional flows.

In this paper, titled **Numerical Experiments in Atmospheric Hydrodynamics**, Charney presents a derivation of a three-dimensional point-vortex model corresponding to the continuously stratified case with a single rigid boundary, followed by numerical experiments investigating the stability and dynamics of various configurations of point-vortices in a barotropic fluid. The corresponding extension of Morikawa’s work to two-layer quasigeostrophic systems was done by Gryanik in 1983 [12]. In another paper published the same year [8], Gryanik investigated the dynamics of a point-vortex model closely resembling Charney’s and states analytical solutions to the case of two point-vortices. Self similar solutions indicating vortex collapse are also presented. The theory of two-layer and continuously stratified quasi-geostrophic point-vortex models has seen considerable development since. Of importance to us is the introduction of the heton by Hogg and Stommel [12] in 1984. Hetons, traditionally defined, are baroclinic pairs of oppositely signed vortices present in different layers. When these two vortices are close enough, they transport heat in the direction of movement, and are therefore named accordingly. The heat flux is shown to reduce as the quantity $\alpha/\lambda$, the ratio of pair separation to interfacial deformation radius, becomes larger, and the interactions between two hetons are discussed. Studies on the trapping of passive particles by translating baroclinic pairs were presented by W.R. Young in the same year [22]. This paper also contains a discussion of various interaction between pairs of hetons, particularly focusing slip-through collisions – those in which initial heton structures are preserved. Lim and Majda in
2000 demonstrated the existence of long-lived heton clusters in the two-layer model \[15\], and in the same paper, derived the heton equations for a coupled surface/interior quasigeostrophic system.

A number of generalizations of the heton concept exist – a comprehensive review of these is provided by Gryanik et al in \[10\]. The three-dimensional heton is an extension of the heton concept to the continuously stratified fluid, and was studied by Gryanik in \[9\]. The paper is a staggeringly detailed account of the generation, propagation, and interaction between populations of three-dimensional hetons (a term that cannot be used unless the constituent vortices of the heton have a horizontal separation smaller than the barotropic Rossby radius) with reference to lateral heat/buoyancy transport in rotation-dominated localized turbulent convection in continuously stratified fluids of finite depth. Some results pertaining to the trapping region of a single three-dimensional heton and how these regions are related to buoyancy transport are also presented.

More recent work has focused on the usage of finite core vortices and other less ideal approximations to simulate real-world geophysical phenomena. However, point-vortex methods continue to be developed and discussed. Xavier Carton, in his 2001 survey of vortex methods used to model the dynamics of mesoscale coherent vortices in the ocean, presents discussions of point-vortex approximations to these phenomena. A number of recent works have also focused on the stability of interacting quasigeostrophic hetons (whether they be singular or finite core), \[18\], for instance. The present work seeks to explicate the dynamics of a three-dimensional heton in an unbounded domain, and therefore continues in the tradition of the early works by Gryanik and his contemporaries.
Chapter 3

A Point Vortex Formulation of Quasigeostrophic Dynamics

A formulation of quasigeostrophic dynamics using three-dimensional point vortices is presented in this chapter. The derivation of the aforementioned point vortex model is done in the context of a continuously stratified fluid in an infinite domain with zero background flow and a constant Coriolis frequency. The resulting system of ODEs is shown to be Hamiltonian, albeit non-canonically so, and a few conserved quantities are introduced and briefly discussed. Analytical solutions are presented for the very simple case of two point vortices. A detailed discussion of the dynamics of a single counter-rotating vortex pair (a three-dimensional heton, so to speak) follows. The chapter ends with a discussion of the numerical methods used to calculate the volume trapped by a single 3D heton in a co-moving frame.

3.1 Derivation of the Point Vortex ODEs

Following [9], we start with the Quasigeostrophic Potential Vorticity Equation for a continuously stratified fluid.

\[ \frac{\partial Q}{\partial t} + \frac{\partial \Psi}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial \Psi}{\partial y} \frac{\partial Q}{\partial x} = 0, \]

\[ Q = \Delta_H \Psi + \frac{\partial}{\partial z} \left( \frac{f^2}{N^2} \frac{\partial \Psi}{\partial z} \right), \]

where \( Q \) is the quasigeostrophic potential vorticity, \( \Psi \) is the streamfunction, \( N \) is the Brunt-Väisälä frequency defined in 2.2, \( f \) is the Coriolis frequency, and \( \Delta_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \).
Assuming \( f \) and \( N \) are constant (i.e. that mean buoyancy, and therefore the mean density, varies linearly in the vertical), we scale \( z \) by \( \frac{f}{N} \), thus eliminating the term \( \frac{f^2}{N^2} \) from (2), leaving behind a familiar Poisson-type equation for \( \Psi \).

\[
Q = \Delta \Psi. \tag{3.3}
\]

We now seek point vortex solutions of the form

\[
Q = \sum_{i=1}^{N} \Gamma_i(t) \delta(x - x_i(t)) \delta(y - y_i(t)) \delta(z - z_i(t)), \tag{3.4}
\]

where \( N \) is the number of point vortices, \( \Gamma_i \) is the strength of each point vortex, and \( x_i, y_i, z_i \) are their coordinates. Substituting (3.4) into (3.1), and equating the terms with delta functions and those without, we obtain the following relations \([8][9]\).

\[
\dot{x}_i(t) = -\frac{\partial \Psi}{\partial y} \bigg|_{x=x_i, y=y_i, z=z_i}, \tag{3.5}
\]

\[
\dot{y}_i(t) = \frac{\partial \Psi}{\partial x} \bigg|_{x=x_i, y=y_i, z=z_i}, \tag{3.6}
\]

\[
\dot{z}_i(t) = 0,
\]

\[
\dot{\Gamma}_i(t) = 0.
\]

With \( Q \) defined as above, the fundamental solution to the Poisson Equation (3.3) for point sources at \( x_j, y_j, z_j \) is given by

\[
\Psi(x, y, z) = \frac{1}{4\pi} \sum_{i=1}^{N} \frac{\Gamma_i}{\left((x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2\right)^{\frac{3}{2}}}. \tag{3.7}
\]

Then, using the ODEs (3.5) and (3.6), we derive equations of motion for the point vortices:

\[
\dot{x}_i(t) = -\sum_{j=1, j \neq i}^{N} \frac{\Gamma_j}{4\pi r_{ij}^3} (y_i - y_j), \tag{3.8}
\]

\[
\dot{y}_i(t) = \sum_{j=1, j \neq i}^{N} \frac{\Gamma_j}{4\pi r_{ij}^3} (x_i - x_j), \tag{3.9}
\]

where \( r_{ij} = \left((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right)^{\frac{1}{2}} \).
The equations above can be interpreted as saying that each point vortex undergoes motion induced by every other point vortex apart from itself.

### 3.2 A Survey of Conserved Quantities

The system of ODEs above (3.5), (3.6) is Hamiltonian for each point vortex. The conservation of the streamfunction can be seen explicitly through the following calculation of the total derivative of the streamfunction:

\[
\frac{d\Psi}{dt} \bigg|_{x=x_i, y=y_i, z=z_i} = \frac{\partial \Psi}{\partial x} \frac{dx}{dt} \bigg|_{x=x_i, y=y_i, z=z_i} + \frac{\partial \Psi}{\partial y} \frac{dy}{dt} \bigg|_{x=x_i, y=y_i, z=z_i} = -\frac{\partial \Psi}{\partial x} \frac{\partial \Psi}{\partial y} \bigg|_{x=x_i, y=y_i, z=z_i} + \frac{\partial \Psi}{\partial y} \frac{\partial \Psi}{\partial x} \bigg|_{x=x_i, y=y_i, z=z_i}
\]

\[= 0.\]

Following V.M. Gryanik in [8], we present the following list of conserved quantities - in order, the linear momentum in the \(x\)-direction, the linear momentum in the \(y\)-direction, the angular momentum, and the energy of interaction between point vortices or the Hamiltonian.

\[
p_x = \sum_{i=1}^{N} \Gamma_i x_i
\]

\[
p_y = \sum_{i=1}^{N} \Gamma_i y_i
\]

\[
I = \sum_{i=1}^{N} \Gamma_i (x_i^2 + y_i^2)
\]

\[
H = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{\Gamma_i \Gamma_j}{4\pi r_{ij}}
\]

We prove here that these quantities are indeed conserved by computing their derivatives with respect to time. Therefore, consider the time derivative of the linear momentum of in \(x\).

\[
\frac{dp_x}{dt} = \sum_{i=1}^{N} \Gamma_i \dot{x}_i
\]

\[= \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} D_{ij}
\]

where \[D_{ij}^{px} = -\frac{\Gamma_i \Gamma_j (y_i - y_j)}{4\pi r_{ij}^3}.\]
Three points are to be noted here. First, that since the ODEs (3.5) and (3.6) have a singularity when \( x_i, y_i, z_i = x_j, y_j, z_j \), the expression \( D_{ij}^{p_x} \) is only valid when \( i \neq j \). Second, the sum above has \( N(N-1) \) terms, a number that is always even. Third, that \( D_{ji}^{p_x} = -D_{ij}^{p_x} \). Because the total number of terms is even and there are no terms \( D_{ij}^{p_x} \), there has to exist for every \( D_{ij}^{p_x} \) a corresponding \( D_{ji}^{p_x} \). And since for every \( i, j \) where it is defined, \( D_{ij}^{p_x} + D_{ji}^{p_y} = 0 \), the sum above is zero, which proves the conservation of \( p_x \). The conservation of \( p_y \) and \( I \) can be proved in a similar manner, with \( D_{ij}^{p_y} \) and \( D_{ij}^I \) being defined as \( \frac{\Gamma_i \Gamma_j (x_i - x_j)}{4\pi r_{ij}} \) and \( \frac{\Gamma_i \Gamma_j (y_i x_j - x_i y_j)}{2\pi r_{ij}} \) respectively. That \( H \) is conserved follows from conservation of the streamfunction \( \Psi \), and can be seen from the following calculation.

\[
\frac{dH}{dt} = -\frac{1}{2} \frac{d}{dt} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \frac{\Gamma_i \Gamma_j}{4\pi r_{ij}}
\]

\[
= -\frac{1}{2} \sum_{i=1}^{N} \frac{d\Psi_i}{dt}
\]

\[
= 0
\]

where \( \Psi_i = \Psi(x_i, y_i, z_i) \)

These quantities are useful primarily because - first, they are indicative of translational and rotational symmetry of the governing equations in the \( xy \)-plane, a fact that is likely to help in simplifying future analysis. Second, they serve as an anchor in numerical simulations of the point vortices where analytical solutions are not always available for accuracy and precision tests.

### 3.3 Analytical Solutions to the Two Vortex Problem

The case of two point vortices is of particular import because it happens to be the simplest configuration of \( N \) point vortices for which the dynamics – encompassing revolution of vortices around a common center as well as translating pairs of vortices (AKA three-dimensional hetons) – are not entirely trivial. Furthermore, analytical solutions to the ODEs governing this particular configuration can be found without too much difficulty, making it an ideal vehicle to explore and codify ideas which may be useful in discussions of more complex configurations to which analytical solutions are not as easily found. This section contains a derivation of solutions to problems with
two co-orbiting vortices with a stationary center of gravity \((\Gamma_1 + \Gamma_2 \neq 0)\) \[8\], and two counter-rotating vortices \((\Gamma_1 = -\Gamma_2)\).

To begin, the ODEs governing the two vortex problem are written down in full, and some notation is introduced. Let the instantaneous locations of the vortices of strengths \(\Gamma_1\) and \(\Gamma_2\) be \((x_1(t), y_1(t), z_1)\) and \((x_2(t), y_2(t), z_2)\). Then, using (3.8), (3.9), the motion of two point vortices is formulated as the following initial value problem:

\[
\begin{align*}
\dot{x}_1(t) &= -\Gamma_2 \frac{y_1 - y_2}{4\pi r_{12}^3} \\
\dot{y}_1(t) &= \Gamma_2 \frac{x_1 - x_2}{4\pi r_{12}^3} \\
\dot{x}_2(t) &= -\Gamma_1 \frac{y_2 - y_1}{4\pi r_{12}^3} \\
\dot{y}_2(t) &= \Gamma_1 \frac{x_2 - x_1}{4\pi r_{12}^3}
\end{align*}
\]

\(x_1(0) = x_{10}, \quad y_1(0) = y_{10}, \quad x_2(0) = x_{20}, \quad y_2(0) = y_{20}.\) \(3.10\) \(3.11\) \(3.12\) \(3.13\)

### 3.3.1 Co-orbiting Vortices

Let \(X(t)\) and \(Y(t)\) be the coordinates of the center of gravity of the system \[8\]. They are defined thus:

\[
\begin{align*}
X(t) &= \frac{\Gamma_1 x_1 + \Gamma_2 x_2}{\Gamma_1 + \Gamma_2} \\
Y(t) &= \frac{\Gamma_1 y_1 + \Gamma_2 y_2}{\Gamma_1 + \Gamma_2}
\end{align*}
\]

\(X(0) = X_0 = \frac{\Gamma_1 x_{10} + \Gamma_2 x_{20}}{\Gamma_1 + \Gamma_2} \quad Y(0) = Y_0 = \frac{\Gamma_1 y_{10} + \Gamma_2 y_{20}}{\Gamma_1 + \Gamma_2}.
\]

Using (3.10) – (3.13), we see that

\[
\dot{X} = 0, \quad \dot{Y} = 0,
\]

i.e. the center of gravity of the system remains stationary, meaning that \(X(t) = X_0\) and \(Y(t) = Y_0\).

Now, we define more variables \(\tilde{x} = x_1 - x_2, \tilde{y} = y_1 - y_2,\) and \(\tilde{z} = z_1 - z_2\) \[8\]. Recasting the IVP
\[
\hat{x} = -\frac{K\hat{y}}{(\hat{x}^2 + \hat{y}^2 + k_z)^{3/2}}, \quad \hat{x}(0) = \hat{x}_0, \quad (3.14)
\]
\[
\hat{y} = \frac{K\hat{x}}{(\hat{x}^2 + \hat{y}^2 + k_z)^{3/2}}, \quad \hat{y}(0) = \hat{y}_0, \quad (3.15)
\]

where \( k_z = \hat{z}^2 \) and \( K = \Gamma_1 + \Gamma_2 \).

Let \( \hat{x} = R \cos(\theta) \) and \( \hat{y} = R \sin(\theta) \). Expressed another way, \( R = \sqrt{\hat{x}^2 + \hat{y}^2} \) and \( \theta = \arctan \frac{\hat{y}}{\hat{x}} \).

Under this change of variables, the equations (3.14), (3.15) become:

\[
\hat{R} = 0 \quad R(0) = R_0, \quad (3.16)
\]
\[
\hat{\theta} = \frac{K}{(R^2 + k_z)^{3/2}} \quad \theta(0) = \theta_0. \quad (3.17)
\]

Solutions to (3.16), (3.17) are found by direct integration.

\[
R(t) = R_0, \quad \theta(t) = \theta_0 + \frac{Kt}{(R_0^2 + k_z)^{3/2}}.
\]

Physically, this indicates rotation at a constant speed around a common axis. Defining the common angular velocity of the vortices \( \omega = \frac{K}{(R_0^2 + k_z)^{3/2}} \), and expressing the solutions found above in cartesian co-ordinates, we get

\[
\hat{x}(t) = \sqrt{\hat{x}_0^2 + \hat{y}_0^2} \cos (\theta_0 + \omega t),
\]
\[
\hat{y}(t) = \sqrt{\hat{x}_0^2 + \hat{y}_0^2} \sin (\theta_0 + \omega t).
\]

Now,

\[
\cos (\theta_0) = \cos (\arctan (\frac{\hat{y}_0}{\hat{x}_0})) = \frac{\hat{x}_0}{\sqrt{\hat{x}_0^2 + \hat{y}_0^2}},
\]
\[
\sin (\theta_0) = \sin (\arctan (\frac{\hat{y}_0}{\hat{x}_0})) = \frac{\hat{y}_0}{\sqrt{\hat{x}_0^2 + \hat{y}_0^2}},
\]

and therefore:

\[
\hat{x}(t) = \hat{x}_0 \cos (\omega t) - \hat{y}_0 \sin (\omega t),
\]
\[
\hat{y}(t) = \hat{x}_0 \sin (\omega t) + \hat{y}_0 \cos (\omega t).
\]
Letting $C = \cos(\omega t)$ and $S = \sin(\omega t)$, and writing the solutions found for $(X,Y,\tilde{x},\tilde{y})$ in terms of the original variables $(x_1, x_2, y_1, y_2)$, we get the following system of linear equations:

\begin{align*}
\Gamma_1 x_1 + \Gamma_2 x_2 &= \Gamma_1 x_{10} + \Gamma_2 x_{20}, \\
\Gamma_1 y_1 + \Gamma_2 y_2 &= \Gamma_1 y_{10} + \Gamma_2 y_{20}, \\
(x_1 - x_2) &= (x_{10} - x_{20})C - (y_{10} - y_{20})S, \\
y_1 - y_2 &= (x_{10} - x_{20})S + (y_{10} - y_{20})C.
\end{align*}

This is a system of four equations in four unknowns, the determinant of which is given by $-(\Gamma_1 - \Gamma_2)^2$ -- a quantity which, by assumption, is nonzero. Solving the above system of equations yields explicit trajectories for the two vortices.

\begin{align*}
x_1(t) &= \frac{1}{\Gamma_1 + \Gamma_2} \left[ \Gamma_1 x_{10} + \Gamma_2 x_{20} + (x_{10} - x_{20})\Gamma_2 C - (y_{10} - y_{20})\Gamma_2 S \right], \\
y_1(t) &= \frac{1}{\Gamma_1 + \Gamma_2} \left[ \Gamma_1 y_{10} + \Gamma_2 y_{20} + (x_{10} - x_{20})\Gamma_2 S + (y_{10} - y_{20})\Gamma_2 C \right], \\
x_2(t) &= \frac{1}{\Gamma_1 + \Gamma_2} \left[ \Gamma_1 x_{10} + \Gamma_2 x_{20} - (x_{10} - x_{20})\Gamma_1 C + (y_{10} + y_{20})\Gamma_1 S \right], \\
y_2(t) &= \frac{1}{\Gamma_1 + \Gamma_2} \left[ \Gamma_1 y_{10} + \Gamma_2 y_{20} - (x_{10} - x_{20})\Gamma_1 S - (y_{10} - y_{20})\Gamma_1 C \right].
\end{align*}

It can be verified by direct substitution that this solution does in fact conserve linear momentum, angular momentum, and the Hamiltonian. To illustrate typical features of this solution, we consider the trajectories (see Fig. 3.1) of two point vortices of strengths $\Gamma_1 = -1$ and $\Gamma_2 = 2$ placed initially at $x_1 = 1, y_1 = 1, z_1 = 5$ and $x_2 = 0.5, y_2 = 0.5, z_2 = 6$. Notice that the two vortices revolve around an axis passing through the common, and as expected, stationary, center of gravity. As the ODEs dictate, the vortices ‘see’ each other, but do not ‘see’ themselves. And therefore, the radius of the circular orbit described by each vortex is proportional to the strength of the other vortex.

In the present scenario, this is shown by the aforementioned radius being larger for the first (and weaker) vortex. Finally, note that the common direction of revolution appears to be influenced by the circulation of the stronger vortex. Positive circulations indicate counterclockwise revolution, and vice versa.
Note that this solution is only valid when $\Gamma_1 \neq -\Gamma_2$, which motivates the second case – counter-rotating vortices.

### 3.3.2 Counter-rotating Vortices

Let $\Gamma_2 = \Gamma$ and $\Gamma_1 = -\Gamma$. The IVP (3.10) – (3.13) then becomes:

$$\begin{align*}
\dot{x}_1(t) &= -\Gamma \frac{\ddot{y}}{4\pi r_{12}^3} \quad x_1(0) = x_{10}, \\
\dot{y}_1(t) &= \Gamma \frac{\ddot{x}}{4\pi r_{12}^3} \quad y_1(0) = y_{10}, \\
\dot{x}_2(t) &= -\Gamma \frac{\ddot{y}}{4\pi r_{12}^3} \quad x_2(0) = x_{20}, \\
\dot{y}_2(t) &= \Gamma \frac{\ddot{x}}{4\pi r_{12}^3} \quad y_2(0) = y_{20},
\end{align*}$$

where $\ddot{x}$ and $\ddot{y}$ retain their previous definitions. Notice that since $\dot{x}_1 = \dot{x}_2$ and $\dot{y}_1 = \dot{y}_2$ in this case, $\dot{x} = \dot{y} = 0$, implying that $\ddot{x} = \ddot{x}_0$ and $\ddot{y} = \ddot{y}_0$. This means that we can define two new quantities $U$.
and \( V \) such that
\[
U = -\frac{\Gamma(y_{10} - y_{20})}{4\pi r_{12}^3}, \tag{3.22}
\]
\[
V = \frac{\Gamma(x_{10} - x_{20})}{4\pi r_{12}^3}, \tag{3.23}
\]
which makes it possible to rewrite the IVP above as:
\[
\dot{x}_1(t) = U \quad \quad x_1(0) = x_{10},
\]
\[
\dot{y}_1(t) = V \quad \quad y_1(0) = y_{10},
\]
\[
\dot{x}_2(t) = U \quad \quad x_2(0) = x_{20},
\]
\[
\dot{y}_2(t) = V \quad \quad y_2(0) = y_{20}.
\]
The solution to this system of differential equations, presented below, is fairly straightforward.
\[
x_1(t) = x_{10} + Ut; \quad \quad x_2(t) = x_{20} + Ut, \tag{3.24}
\]
\[
y_1(t) = y_{10} + Vt; \quad \quad y_2(t) = y_{20} + Vt. \tag{3.25}
\]
This indicates that both point vortices travel together in a straight line with velocity \([U, V]^T = U\hat{x} + V\hat{y}\) – a situation referred to by Grynik et al in [9] as a three-dimensional heton.

![Figure 3.2: Trajectories for counter-rotating vortices placed initially at \((0, 1, 0)\) and \((0, -1, 0)\).](image)
The typical trajectory described by this configuration of vortices is illustrated in Fig. 3.2 for point vortices of strengths $\Gamma_1 = -1$ and $\Gamma_2 = 1$, positioned initially at $x_1 = 0, y_1 = 1, z_1 = 0$ and $x_2 = 0, y_2 = -1, z_2 = 0$. $V = 0$ since $x_{10} = x_{20} = 0$ and therefore, as expected, the vortices move together in a straight line along the $x$-axis to the left (owing to the anticyclonic nature of the first vortex). Note that in this case, as in the previous one, point vortices that are positioned directly on top of each other remain stationary for all time.

### 3.4 The Dynamics associated with a Single Three-Dimensional Heton

In this section, we present a detailed discussion of the three-dimensional heton, focusing on the dynamical structure of the velocity field induced by it in a frame travelling with the heton. Fig. 4.7 is a schematic representation of a three-dimensional heton, presented here in order to establish the notation and conventions we will be using throughout the discussion.

![Figure 3.3: Schematic diagram of a three-dimensional heton with an anticyclonic point vortex (strength $-\Gamma$) at $(0, Y, Z)$ and a cyclonic point vortex (strength $\Gamma$) at $(0, -Y, -Z)$.

Figure 3.3: Schematic diagram of a three-dimensional heton with an anticyclonic point vortex (strength $-\Gamma$) at $(0, Y, Z)$ and a cyclonic point vortex (strength $\Gamma$) at $(0, -Y, -Z)$.](image)
We will consider exclusively hetons with constituent point vortices distributed symmetrically about the origin, on the \(yz\)-plane. We may restrict our discussions to this particular configuration without any loss of generality owing to the conservation of linear and angular momenta, which, according to Noether’s theorem \([9]\), corresponds to symmetries under translation in the \(xy\)-plane and rotation about the \(z\)-axis – meaning that any other configuration of the heton may be converted to the one in Fig. 3.3 by a suitable change of variables. The velocity of a heton in this configuration is given by:

\[
U = \frac{-\Gamma Y}{16\pi(Y^2 + Z^2)^{\frac{3}{2}}}, \quad V = 0.
\]

Consider the motion induced by this heton on a passive particle located at \((x(t), y(t), z)\). The particle’s trajectory is given by the following pair of ODEs:

\[
\dot{x} = \frac{\Gamma}{4\pi} \frac{y - Y}{[x^2 + (y - Y)^2 + (z - Z)^2]^\frac{3}{2}} - \frac{y + Y}{[x^2 + (y + Y)^2 + (z + Z)^2]^\frac{3}{2}}
\]

\[
\dot{y} = \frac{\Gamma}{4\pi} \frac{x}{[x^2 + (y + Y)^2 + (z + Z)^2]^\frac{3}{2}} - \frac{x}{[x^2 + (y - Y)^2 + (z - Z)^2]^\frac{3}{2}}.
\]

Under the change of variables \(x = x - Ut, \ y = y - Vt\), i.e., in a frame of reference moving with the heton, the ODEs above become:

\[
\dot{x} = \frac{\Gamma}{4\pi} \frac{y - Y}{[x^2 + (y - Y)^2 + (z - Z)^2]^\frac{3}{2}} - \frac{y + Y}{[x^2 + (y + Y)^2 + (z + Z)^2]^\frac{3}{2}} + \frac{\Gamma Y}{16\pi(Y^2 + Z^2)^{\frac{3}{2}}} \tag{3.26}
\]

\[
\dot{y} = \frac{\Gamma}{4\pi} \frac{x}{[x^2 + (y + Y)^2 + (z + Z)^2]^\frac{3}{2}} - \frac{x}{[x^2 + (y - Y)^2 + (z - Z)^2]^\frac{3}{2}}. \tag{3.27}
\]

These ODEs are also Hamiltonian, with the streamfunction \(\Psi(x, y, z)\) given by:

\[
\Psi = \frac{\Gamma}{4\pi} \frac{1}{[x^2 + (y - Y)^2 + (z - Z)^2]^\frac{3}{2}} - \frac{1}{[x^2 + (y + Y)^2 + (z + Z)^2]^\frac{3}{2}} - \frac{\Gamma y Y}{16\pi(Y^2 + Z^2)^{\frac{3}{2}}} \tag{3.28}
\]

We will proceed to discuss the velocity field \([\vec{x}, \vec{y}]^T\) in two subsections – in the first, we will consider the relatively simple case where \(Z = 0\), i.e., the heton’s constituent vortices are on the same horizontal plane. Next, we will discuss the tilted heton, where \(Z \neq 0\).
3.4.1 Case I, \( Z = 0 \)

We consider the case of a heton with its constituent vortices lying on the same horizontal plane separately because – firstly, the dynamics here are in a sense qualitatively different from the more general case of the tilted heton. For example, there are no homoclinic connections in the present case, whereas homoclinic connections between fixed points may be found in the second case. Furthermore, the dynamics are not only different, but simpler, which means the effects of various parameters \((\Gamma, Y, z)\) are easier to see. We exploit results from the study of this particular case in order to decide which parameters to eliminate during the study of the more complicated second case. Finally, the simplicity of this case also allows us to derive a rudimentary scaling law for the volume of the region trapped by the heton. We may use this scaling law to test that numerical results for the trapped volume are as we expect them to be. For this case the heton equations (3.26) – (3.28) take the form:

\[
\begin{align*}
\dot{x} &= \frac{\Gamma}{4\pi} \left[ \frac{y - Y}{[x^2 + (y - Y)^2 + z^2]^{3/2}} - \frac{y + Y}{[x^2 + (y + Y)^2 + z^2]^{3/2}} \right] + \frac{\Gamma}{16\pi Y^2} \quad (3.29) \\
\dot{y} &= \frac{\Gamma}{4\pi} \left[ \frac{x}{[x^2 + (y + Y)^2 + z^2]^{3/2}} - \frac{x}{[x^2 + (y - Y)^2 + z^2]^{3/2}} \right] \quad (3.30) \\
\Psi &= \frac{\Gamma}{4\pi} \left[ \frac{1}{[x^2 + (y - Y)^2 + z^2]^{3/2}} - \frac{1}{[x^2 + (y + Y)^2 + z^2]^{3/2}} \right] - \frac{\Gamma y}{16\pi Y^2} \quad (3.31)
\end{align*}
\]

For illustrative purposes we first fix the values of \( \Gamma \) and \( Y \) at 1, and consider the phase portraits of the system (3.29) – (3.30) for various values of \( z \), as in Fig. 3.4. It is clear that the integral curves of the vector field \([\vec{x}, \vec{y}]^T\) are given by contours of the streamfunction \( \Psi \), the streamlines. The phase portraits in Fig 3.4 are somewhat typical of conservative systems. We see the existence of four fixed points – two unstable fixed points (saddles) on the line \( y = 0 \) and two stable centers on the line \( x = 0 \) – for some values of the parameter \( z \). A separatrix cycle composed of heteroclinic connections between the saddle equilibria bounds the trapping region – the region that, in a stationary frame of reference, would be moving with the heton. These trapping regions have also been referred to in the literature as circulation cells \([9]\). We see that the trapping region shrinks as \( z \) grows in magnitude, eventually disappearing altogether – indicating a Hamiltonian saddle-center bifurcation.
Figure 3.4: Phase portrait of the heton ODEs when $Z = 0$, $Y = 1$, and $\Gamma = 1$ for various values of $z$ showing the presence and gradual disappearance of a trapping region bounded by the $\Psi = 0$ contour (coloured black in the figure).
Figure 3.5: Schematic diagram illustrating values of the streamfunction across the phase plane. The plot was generated for $z = 0$, $\Gamma = 1$ and $Y = 1$, but this pattern holds good for all values of $z,\Gamma$ and $Y$ where the trapping region exists.

We derive the fixed points of the system by finding the roots of $\dot{x}$ and $\dot{y}$. Equating (3.30) to zero gives us:

$$\frac{x}{[x^2 + (y + Y)^2 + z^2]^\frac{3}{2}} = \frac{x}{[x^2 + (y - Y)^2 + z^2]^\frac{3}{2}}.$$  

which, for nonzero $x$, yields $y = 0$. Plugging this in (3.29) and equating it to zero gives us, after a bit of algebra:

$$4Y^2 = x^2 + Y^2 + z^2 \implies x = \pm \sqrt{3Y^2 - z^2}.$$  

From Fig 3.4 we see that that the fixed points at $(\pm \sqrt{3Y^2 - z^2}, 0, z)$ correspond to saddles. Stable fixed points occur on the line $x = 0$. Everywhere on this line $\dot{y} = 0$, and since it is not obvious how to solve $\dot{x} = 0$ analytically, we make use of the Newton method in order to find zeroes numerically. The results of this computation for $Y = 1$ are shown in the plot to the left in Fig. 3.6. From the plot, it can be seen that the stable fixed points (as well as the unstable ones) move closer to each other as $z$ moves away from zero.
These fixed points (both stable and unstable) collide with each other when $z = \pm Y\sqrt{3}$ and vanish, meaning that beyond these values of $z$, a trapping region does not exist. We also note that since the trapping region is bounded, for each value of $z$ at which it exists, by heteroclinic connections between unstable fixed points, and since the stable fixed points are present inside the trapping region, the invariance of the unstable fixed points with $\Gamma$ implies that the positions of the stable fixed points and the size of the trapping region do not depend on $\Gamma$ either.

Figure 3.6: Bifurcation plot illustrating the generation/destruction of stable and unstable fixed points, respectively, on the $y$-axis (left) and on the $x$-axis (right) with the variation of $z$. Here, $Y = 0.5, 1, 2$ and the value of $\Gamma$ is immaterial. The bifurcation points are $z = \pm Y\sqrt{3}$.

Fig. 3.7 is a discrete representation of the three-dimensional volume trapped by the heton. This volume is diametrically widest at $z = 0$ and progressively shrinks as $z$ becomes larger in magnitude, finally disappearing when $|z| = Y\sqrt{3}$. We might justify a rough scaling law for the volume of the trapped region by constructing a cuboid that bounds this region. The length (measured on the $x$-axis) of this cuboid, defined by the maximum distance between saddles (which is $2Y\sqrt{3}$, occurring at $z = 0$) scales linearly with $Y$, and it can be shown that its breadth – which is defined
by the distance between the points on $z = 0$ at which the boundary of the trapping area intersects
the $y$-axis – approximately does so too. The height of this cuboid again scales linearly with $Y$.
Since this cube bounds the three-dimensional trapping region, we expect the volume of the trapping
region to scale approximately as $Y^3$.

Figure 3.7: Contour slices at $z = 0, 0.5, 1, 1.3, \sqrt{3}$, illustrating the presence of a three-dimensional
volume that is trapped by the heton and moves along with it. The trapping contours look exactly
the same for negative values of $z$ and hence are not plotted in this figure. The horizontal extent of
the trapped volume is determined by distance between the unstable fixed points at each value of
$z$, and its vertical extent is determined by the values of $z$ at which the fixed points vanish.
3.4.2 Case II, Z ≠ 0

The equations that completely describe a tilted heton are the ODEs (3.26), (3.27) and the streamfunction (3.28). As they are, these equations are difficult to analyse because they are non-linear and because they have three controlling parameters Γ, Y and Z. However, based on our analysis of the simpler case where Z = 0, we can eliminate some parameters in order to make the problem more tractable. In the previous subsection, we concluded that Γ has no effect on the fixed points or, consequently, on the region trapped by the heton. We also theorized that the volume of the trapping region scales roughly as Y^3, i.e., the effect of Y is to simply expand/contract the trapped volume. Since the effects of these two parameters appear to be fairly straightforward, we seek to eliminate them from the equations 3.26) – (3.28) using the following change of variables:

\[ x' = \frac{x}{Y}, \quad y' = \frac{y}{Y}, \quad z' = \frac{z}{Y}, \quad \Gamma' = \frac{\Gamma}{Y^3}, \quad Z' = \frac{Z}{Y}. \]

Under this change of variables, the equations become (after having dropped primes):

\[
\dot{x} = \frac{1}{4\pi} \left[ \frac{y - 1}{[x^2 + (y - 1)^2 + (z - Z)^2]^\frac{3}{2}} - \frac{y + 1}{[x^2 + (y + 1)^2 + (z + Z)^2]^\frac{3}{2}} \right] + \frac{1}{16\pi(1 + Z^2)^\frac{5}{2}}, \tag{3.32}
\]

\[
\dot{y} = \frac{1}{4\pi} \left[ \frac{x}{[x^2 + (y + 1)^2 + (z + Z)^2]^\frac{3}{2}} - \frac{x}{[x^2 + (y - 1)^2 + (z - Z)^2]^\frac{3}{2}} \right], \tag{3.33}
\]

\[
\Psi = \frac{1}{4\pi} \left[ \frac{1}{[x^2 + (y - 1)^2 + (z - Z)^2]^\frac{3}{2}} - \frac{1}{[x^2 + (y + 1)^2 + (z + Z)^2]^\frac{3}{2}} \right] - \frac{y}{16\pi(1 + Z^2)^\frac{3}{2}}. \tag{3.34}
\]

We proceed to discuss the dynamical structure of this system using both analytical and numerical techniques as in the previous case. To this end, we present two figures – Fig. 3.8 is a series of phase portraits of the system at Z = 1 and various values of z, and Fig. 3.9 is a representation of the same as contour slices in a box, with a view to demonstrating the extent of the physical, three-dimensional trapping region. Note that the trapping contours for this case are not always curves where Ψ = 0 – instead, they are curves where Ψ = Ψ_f, where Ψ_f is the value of the streamfunction at an unstable fixed point.
Figure 3.8: Phase portrait of the tilted heton ODEs when $Z = 1$ for various values of $z$ showing the presence and gradual disappearance of a trapping region bounded by the $\Psi = \Psi_f$ contour (coloured black in the figure), where $\Psi_f$ is the value of the streamfunction at an unstable fixed point. Vector field is normalized and scaled in order to emphasize direction over magnitude.
Figure 3.9: Contour slices at $z = 0, \pm 1, \pm \sqrt{3}, \pm 2.43, \pm 3.18$, illustrating the presence of a three-dimensional volume that is trapped by the heton and moves along with it. The trapping contours enclose the same area for corresponding positive and negative values of $z$, but are mirrored across $y = 0$ in orientation, as can be seen from separate the plots on the left ($z \geq 0$) and right ($z \leq 0$). The trapping contours (in black, as before), are curves where $\Psi = \Psi_f$. 
Inspection of Fig 3.8 gives us a rough idea of the dynamical structure of the equations under consideration. As we mentioned previously, the dynamics are qualitatively different from the previous case. Starting from the phase portrait at $z = 0$ (top left), we see, as before, the presence of two stable fixed points and two unstable ones, the heteroclinic connections between the latter bounding the trapping region. As we increase $z$, something interesting happens – the fixed points not only move closer, as in the previous case, but the unstable fixed points also move downwards on the $y$-axis (top right) – i.e, the fixed points do not all approach each other at the same rate like they did in the previous case; the unstable ones essentially move towards the bottom center and away from the top center as $z$ increases. Further increase in $z$ brings about a collision between the unstable fixed points and the bottom center, leading to the creation of another unstable fixed point, this time with a homoclinic orbit bounding the trapping region (center left). Increasing $z$ even further causes the newly created saddle and the surviving center to move towards each other – essentially shrinking the trapping region (center right), until these two fixed points collide (bottom left) and vanish (bottom right). Fig 3.9 shows these changes in the phase portrait mirrored when $z$ is decreased from zero.

We may calculate the unstable fixed points analytically, as before, by finding the roots of $\dot{x}$ and $\dot{y}$ as defined in (3.32), (3.33) where $x \neq 0$. Repeating this process yields the roots $(\pm \sqrt{(3 - z^2)(1 + Z^2)}, -zZ)$ on the the $xy$-plane. Since the quantity inside the square root becomes negative for $|z| > \sqrt{3}$, we can conclude that the value $z = \pm \sqrt{3}$ corresponds to the subcritical pitchfork bifurcations involving the creation/destruction of a heteroclinic orbit. For expository purposes, we identify three 'regimes' in the dynamic structure of the tilted heton, viz. the heteroclinic regime; from $-\sqrt{3} \leq z \leq \sqrt{3}$, where the planar trapping region is bounded by heteroclinic orbits, and the first and second homoclinic regimes; where $z < -\sqrt{3}$ and $z > \sqrt{3}$, respectively, where the planar trapping region is bounded by homoclinic orbits. The heteroclinic regime has four fixed points, two saddles and two stable centers. Each homoclinic regime has two fixed points, a saddle and a center, their positions inverted between the first and second regimes. A collection of unstable fixed points for various values of $z$ in the heteroclinic regime is shown in Fig. 3.10.
Figure 3.10: TOP: A plot showing the location of unstable fixed points in the $xy$-plane for values of $z$ ranging from $-\sqrt{3}$ to $\sqrt{3}$. Unstable fixed points for the case where $Z = 0$ have also been included in order to emphasize the tilt away from the $xz$-plane of the collection of fixed points for the present case. BOTTOM: A projection of the previous plot onto the $yz$-plane with some new data added demonstrating increase in the tilt from the vertical and the spread of the fixed points with increase in $Z$. 
Fixed points on the \(y\)-axis, due to reasons mentioned in the previous subsection, are not amenable to analytical discovery, and so we try to find them using numerical methods. Since these fixed points occur purely on the \(yz\)-plane, they are presented in the 2D plots below for \(Z = 1, 3\):

![Figure 3.11: A collection of fixed points on the \(y\)-axis for \(Z = 1, 3\) in both the heteroclinic and homoclinic regimes. Note that in the heteroclinic regime, both such fixed points are stable (blue solid lines), while in the homoclinic regime, one point is unstable (red dotted line). The shift happens at \(z = \pm \sqrt{3}\).](image-url)

Figure 3.11: A collection of fixed points on the \(y\)-axis for \(Z = 1, 3\) in both the heteroclinic and homoclinic regimes. Note that in the heteroclinic regime, both such fixed points are stable (blue solid lines), while in the homoclinic regime, one point is unstable (red dotted line). The shift happens at \(z = \pm \sqrt{3}\).
As with the unstable fixed points in the heteroclinic regime, there are two fixed points on the $y$-axis at every value of $z$. In the heteroclinic regime both of them are centers, in the first homoclinic regime the one on the top is a saddle, and in the second homoclinic regime the one on the bottom is a saddle, as we can see from Figs. 3.9 and 3.11. While the vertical extent of the heteroclinic regime is independent of $Z$ – subcritical pitchfork bifurcations occur at $z = \pm \sqrt{3}$, whatever the value of $Z$ may be – the vertical extents of the homoclinic regions – determined by the value of $z$ at which the final saddle-center bifurcation occurs (say at $z = \pm z_b$) – appear to be growing with $Z$. $|z_b|$ is calculated numerically for various values of $Z$ (the procedure is described in the next section) and presented below in a logarithmic plot. It can be seen that initially $|z_b|$ appears to scale with $\sqrt{1 + Z^2}$, which corresponds to half the perpendicular distance between the counter-rotating vortices. For $Z >> 1$ though, the growth appears to be at least superlinear.

![Saddle-Node Bifurcation Points](image)

Figure 3.12: A logarithmic plot showing the variation in $z_b$, and hence the vertical extents of the homoclinic regimes for various values of $Z$. 
Having mapped out the dynamical structure of the three-dimensional heton, we proceed, in the next section, to describe an algorithm that can be used to calculate the trapped volume. We conclude this section with the following figure illustrating streamfunction values across the phase plane in various regimes.

Figure 3.13: Schematic diagram illustrating values of the streamfunction across the phase plane.

These relations are valid for all values of $z$ and $Z$ for which the trapping region exists.
3.5 An Algorithm to Calculate the Trapped Volume

That hetons are able to trap and carry with them a region of space as they move is what enables them to transport scalar quantities like heat/buoyancy or mass. The main challenge we face in calculating or estimating the size of this region analytically is the difficulty in defining, in explicit terms, the trapped region and its boundaries. Since, however, we know something about the values of the streamfunction inside and outside the trapped region – we may use this information in order to numerically delineate the trapped region for each value of \( z \) and then calculate the area of this region using some numerical integration scheme (here, we use the two-dimensional midpoint rule). We do this for successive values of \( z \) from 0 to a numerically calculated \( z_b \), i.e., until the trapped region vanishes, we then add the results according to some rule of quadrature (the rectangular rule in the present case). Due to conveniences granted to us by symmetry, we do not need to do the same for \( z < 0 \). In fact, for simpler cases like \( Z = 0 \), we may use symmetry to further simplify the problem. By way of conclusion, we now describe in detail the algorithm we use to calculate the trapped volume. In the following chapter, we present, among other things, some numerical experiments performed using this algorithm.

3.5.1 The Algorithm

(1) Pick values for the parameters \( Z \) and \( Y \). Notice that in our case, due to the scaling arguments made in 3.4.2, \( Y = 1 \) always when \( Z \neq 0 \).

(2) Pick grid separation distances \( \Delta x \), \( \Delta y \), and \( \Delta z \). Calculate numerically the value of \( z_b \). This is done in the present study using Newton’s method to find fixed points on the \( y \)-axis for successive values of \( z \) in the second homoclinic regime until the distance between the fixed points becomes small enough. Note that when \( Z = 0 \), there are no homoclinic regimes and \( z_b = \sqrt{3} \).

(3) Demarcate a two-dimensional search space in the \( xy \)-plane. This search space is determined by the bounds of the trapping region when \( z = 0 \), as this is the value of \( z \) at which the area
trapped by the heton tends to be the largest. When $Z = 0$, symmetry may be exploited to further reduce the search space. Generate a two-dimensional mesh of midpoints in $x$ and $y$.

(4) Loop through values of $z$ from 0 to $z_b$. For each value of $z$, calculate the value of the streamfunction at all points on the $xy$-mesh. Also calculate the value of $\Psi_f$, as defined in the previous section.

(5) Use the streamfunction relations in Fig. 3.13 to specify the trapped area. Count 'boxes' that satisfy these relations. Let this number be $C$.

(6) The trapped area is then given by $A_i = C \Delta x \Delta y$, where the $i$ is used to index values of $z$.

(7) Say there are $n$ values of $z$ between 0 and $z_b$; the trapped volume is then given, in general, by $V_T = 2 \Delta z \sum_{i=1}^{n} A_i$. 
Chapter 4
Results and Discussion

This chapter primarily concerns itself with numerical experiments conducted to estimate the volume trapped by a single three-dimensional heton. The convergence of the algorithm described in the previous chapter is verified, and the variation of trapping volume with vortex separation is shown and discussed. The second section of this chapter offers a comparison of the results obtained from the analysis of the three-dimensional heton in an unbounded domain to those obtained from the two-layer models as in [22] and continuously stratified models as in [9]. Further work is proposed and briefly discussed in the final section.

4.1 Numerical Experiments

In order to verify that the volume algorithm presented in 3.5.1 behaves as we expect it to, we compute the volume trapped by a three-dimensional heton with its constituent vortices lying on the same plane. This corresponds to $Z = 0$, the case discussed in 3.4.1. We compute the volume numerically for various values of the parameter $Y$ and test the results so obtained against the rough scaling law derived earlier for the trapping volume – that the volume scales roughly as $Y^3$. The logarithmic plot presented in Fig. 4.1 shows the expected behaviour, and therefore, we might conclude that the algorithm results in values that are reasonably accurate. Note that the application of the algorithm to this case is particularly simple due to the conveniences afforded by the symmetry of the three-dimensional trapping region about the origin (implying reductions in the search space) and the lack of a homoclinic regime.
Figure 4.1: A plot studying variations in the volume of three-dimensional trapping region with dipole length – the distance between the two counter-rotating vortices, defined as $2Y$ for this case. Calculations were carried out with $\Delta V_T = \Delta x \Delta y \Delta z = 0.001$. The plot shows reasonable agreement with the scaling law derived in 3.4.1.

Before presenting results on the trapping volume in the more complex tilted three-dimensional heton (i.e. $Z \neq 0$), we remark on the temporal costs and convergence properties of the algorithms used in this and the previous chapter, with a view to explaining computational decisions taken in this study. All fixed points in the homoclinic regimes are calculated using the standard Newton-Raphson method to a tolerance of $10^{-14}$. Computation of the trapping volume when $Z \neq 0$ requires the numerical calculation of a saddle-node bifurcation point in the second homoclinic regime (see Step 2 in 3.5.1). The tolerance, i.e. the distance between the saddle and the node at which the bifurcation point is declared ‘found’ is set to be $10^{-4}$. Fig 4.2 shows reasonable accuracy (at least up to one decimal point) for this value of the tolerance. Beyond this value, the temporal costs become prohibitively large.
Figure 4.2: TOP: A plot showing, in logarithmic scale, the time taken to execute the program to numerically calculate bifurcation points in the homoclinic regime for various tolerances. BOTTOM: Values of the bifurcation point for various values of the tolerance. Both plots were generated for $Z = 1$. 
Similar studies were performed on the code used to compute the trapping volume for \( Y = 1, \ Z = 1 \). The parameter under consideration is the box size \( \Delta V_T = \Delta x \Delta y \Delta z \). In all our computations, the grid separation is uniform in \( x, y, \) and \( z \), i.e. \( \Delta x = \Delta y = \Delta z \). Fig 4.3 and Fig 4.4 show the results of these studies.

Figure 4.3: TOP: A plot showing, in logarithmic scale, the time taken to execute the program to numerically calculate trapping volume for various grid separations. BOTTOM: Values of the trapping volume for various grid separations. Both plots were generated for \( Z = 1 \).
The plots above were generated by computing the trapping volume for grid separations ranging from \( \Delta x = 1 \) to \( \Delta x = 0.00390625 \), with each value of \( \Delta x \) being half of the previous value. The final value of the volume thus obtained is taken to be the ‘true’ volume \( (V^*_T) \), and is used for computations of absolute and relative error, seen in Fig 4.4, below.

![Convergence Study of Volume Code](image)

**Figure 4.4:** Plots of absolute error in volume, defined as \( |V_T - V^*_T| \), and relative error, defined as \( \frac{|V_T - V^*_T|}{V^*_T} \) with grid separation.

These plots indicate that the numerically computed values of volume converge, albeit somewhat slowly. Relatively low errors can be obtained with small grid separations. This would however, lead to very large temporal costs, as illustrated in Fig. 4.3 (top). For the purposes of this study, therefore, we choose a grid separation of 0.1, which leads to relative errors of a similar magnitude.
Using this value for grid separation, the trapping volume was computed for various values of the parameter $Z$ from $Z = 0$ to $Z = 10$. The results (depicted in Fig. 4.5) appear to be in accordance with those obtained for the the bifurcation point $z_b$ obtained in the previous chapter (see Fig. 3.12). Here, again, the quantity $\sqrt{1 + Z^2}$ represents the dipole length – the distance between the two vortices that constitute the heton. The plot shows that for small values of $Z$, the trapping volume appears to grow as the cube of the dipole length. At larger values, however, the volume grows faster than the cube of the dipole length.

![Variations in Trapping Volume](image)

**Figure 4.5:** A plot showing, in logarithmic scale, the trapping volume for various values of $Z$.

As $Z$ grows in magnitude, for a fixed $Y$, the three-dimensional heton tilts farther and farther away from the horizontal plane, becoming fully vertical as $Z$ tends to infinity. We remarked earlier that a vertically aligned heton does not move at all. Computation of the frame velocity (the heton velocity $U$ defined in 3.4.1) for the values of $Z$ previously considered shows that this is indeed the case (see Fig. 4.6). Because the trapping volume increases with $Z$, and the heton velocity decreases with $Z$, we are interested in the question of what happens to the volume flux $UV_T$ as $Z$ becomes
very large. A vertically aligned heton does not move, but induces a circulation that is symmetric about the axis of the heton. In this sense, it could be said, informally, that the volume trapped by the heton is infinite. The volume flux therefore, is undefined. The figure below show that the volume increases faster than the velocity decreases, corresponding to a net increase in the volume flux.

Figure 4.6: A plot showing, in logarithmic scale, the trapping volume, frame velocity, and volume flux for various values of $Z$.

4.2 Comparison with other models

A number of qualitative correspondences between the model of the heton presented in this study and those presented in previous studies – particularly in the two-layer model developed by W. Young in [22], and in the three-dimensional continuously stratified model presented by Gryanik et al in [9] – are briefly discussed. It is to be noted that these studies do not present calculations of the trapping volume, however they do present descriptions of the dynamical structure of a single heton, and that is where we draw our comparisons from. We begin with [9] – a study of three-
dimensional hetons derived from the quasigeostrophic potential vorticity equation in a domain with rigid vertical boundaries. The most significant difference between this model and ours is the nature of the streamfunction. The presence of boundaries at the surface and bottom (necessary by the context – turbulent oceanic convection) means that the Green’s function corresponding to the vorticity-streamfunction relation, and therefore the streamfunction, includes an infinite sum of image vortices. The heton speeds in our model (see Fig. 4.7 below), though much lower in magnitude than in their model, show comparable decay. For reference, see Figure 4 in [9].

![Heton Speed Study](image)

**Figure 4.7**: A plot showing the decay of heton speed for various values of $Y$ and $Z$.

Also similar are the trapping regions for various values of $z$ (Ref. Figure 3 in [9], compare with Fig. 3.8 of this report), although the paper does not provide a detailed description of the fixed points and invariant manifolds of the system. The independence of the trapping volume with respect to vortex strength is remarked upon, a property that is true for in case as well.
Figure 6 in [22], a development of heton theory in the context of the two-layer model, is a schematic diagram showing the formation of distinct homoclinic and heteroclinic regimes for a two-layer heton. The controlling parameter here, analogous to \( Z \) in our model, is \( \frac{d}{\lambda} \), described previously as the ratio between vertical pair separation and interfacial deformation radius. The shift between regimes occurs at \( \frac{d}{\lambda} = 0.86 \). Significantly, in the two-layer model, the regime shift is a function of vortex separation distance, whereas in our model, a regime shift occurs as we travel along the \( z \)-axis for all vortex separations \( Z \). For two counter-rotating vortices in the upper layer, which is comparable to the horizontally aligned heton (\( Z = 0 \)) presented in our study, the trapping region in the lower layer is shown to be very sensitive to the quantity \( \frac{d_2}{\lambda} \), disappearing when \( \frac{d_2}{\lambda} = 0.86 \). Our continuously stratified model does not show this effect.

### 4.3 Further Work

Owing to the wide applicability of point-vortex models in general and heton models in particular, the work presented in this report could be extended in any number of directions. A few of these are proposed here. Perturbation of the single heton system, either by introducing a second heton or by introducing some kind of background flow provides insight into the dynamics of multi-heton systems, and has been shown in previous studies to provide a reasonable idealization for real-world transport processes. Such work has been done previously by Gryanik [9] and Reinaud [18] for continuously stratified fluids and by Hogg and Stommel [13] [12] and Young [22] for a two-layer fluid. The development of faster and more efficient algorithms to calculate the trapping volume serves as another possible extension of the work presented here. Finally, one might consider the dynamics of other configurations of point-vortices that lead to steadily translating or well-defined states, a catalogue of these is available in [5].
Bibliography


