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SCIENTIFIC REPORT NO. 82  
EXPLICIT APPROXIMATIONS  
FOR THE STATIC CAPACITANCE  
OF A MICROSTRIP PATCH  
OF ARBITRARY SHAPE

by

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## ABSTRACT

Closed-form expressions for the static capacitance of a microstrip patch of arbitrary shape are derived for both large and small patch size relative to the substrate thickness. For these formulas, only the capacitance of the isolated patch in free space, or the area, perimeter, and Neumann self-inductance of the perimeter are required. Comparison with existing data obtained by rather extensive numerical computation shows good agreement in the appropriate ranges of validity. The results of this report have direct application to microstrip lumped circuit elements and to the evaluation of the quasistatic portion of the input impedance of a microstrip patch antenna.

## I. HISTORICAL PERSPECTIVE

It is common practice by the authors of introductory texts on electromagnetic fields to provide a heuristic justification for the classical formula

$$C = \frac{\epsilon S}{d} \quad (1.1)$$

for the electrostatic capacitance (in MKS units) of two identical plates of area  $S$ , separated by a distance  $d$  and embedded in a homogeneous region of permittivity  $\epsilon$  (Fig. 1). It is usually stated that "fringing fields" are neglected, and that the fields in the region between the plates together with the charge density on the plates are uniform. Somewhat more advanced textbooks may give a treatment of the fringing fields for the edge of a semi-infinite, two dimensional parallel-plate system as shown in Fig. 2, using conformal mapping. A connection with the fringing fields of the capacitor in Fig. 1 will probably be mentioned, though no details will be given. That the excess charge (the difference between the actual charge on the plates and the uniformly distributed charge in the approximation of eqn. (1.1) is infinite in Fig. 2 may not be mentioned at all. Eqn. (1.1), furthermore, is presented as an approximation for sufficiently small  $d$ . The case when  $d$  is large is seldom mentioned.

In fact, of all finite-size capacitors only rectangular and circular plate capacitors have received much attention in the literature, although the history of the circular plate capacitor is indeed a venerable one. Clausius [1852] was apparently the first to tackle the problem of improving the estimate (1.1) when  $d$  is small, working on the case when the plates are circular discs of radius  $a$ . He realized that uniform charge density on the plates could hold only in a zero-order approximation, and attempted to find a correction term which would describe the nonuniformity of charge near the

edges of the plates. This he did, but unfortunately the correction term he found possessed a nonintegrable singularity at the edge so that the total charge on a plate would be infinite. Seeking to rectify this, he made an empirical modification to his expression to achieve what he knew must be the proper, integrable (inverse square root) singularity at the edge. This involved an arbitrary constant  $\alpha$  which Clausius determined by requiring that his modified charge distribution still produce the proper voltage at the plate edge. When rearranged to eliminate all his infinite series, Clausius' equation for  $\alpha$  becomes

$$\text{Li}_2(-\alpha^4) + \frac{7}{6} \pi^2 + 4 \ln \alpha \ln\left(\frac{1+\alpha^4}{\alpha^2}\right) + 2\pi \tan^{-1} \left[ \frac{2^{3/2} \alpha + 3\alpha^2 + 2^{3/2} \alpha^3 - \alpha^4}{1 + 2^{1/2} \alpha - 3\alpha^2 + 2^{1/2} \alpha^3} \right] = 0 \quad (1.2)$$

where  $\text{Li}_2$  represents the dilogarithm function [Lewin, 1981]. Clausius' solution,  $\alpha \approx 0.247$  seems accurate when checked by modern computing methods and led him to the following approximate formula for the capacitance:

$$C \approx \epsilon a \left[ \pi \frac{a}{d} + \ln \left( \frac{8a}{\alpha^2 d} \right) - 2 \right] \quad (1.3)$$

Although the form of the correction term,  $\ln(a/d) + \text{constant}$ , in the square brackets was correct, the numerical value of the constant was not quite correct, as would be shown later by Kirchhoff [1877] in his great memoir. A more readily accessible account of Kirchhoff's method can be found in Kottler [1927]. Kirchhoff's approach is, in a somewhat less systematic form, what would today be called the method of matched asymptotic expansions. Approximate solutions for the potential are developed for three separate regions: inside the parallel plates but away from the edge (interior), outside the plates but away from the edge (exterior), and the immediate vicinity of the edge itself (edge). The solutions in

the first two regions are elementary, that in the edge region can be obtained by conformal mapping as mentioned previously. These three solutions are then matched to each other in their mutual ranges of validity. There results the expression for the capacitance,

$$C \cong \epsilon a \left[ \pi \frac{a}{d} + \ln \left( \frac{16\pi a}{d} \right) - 1 \right] \quad (1.4)$$

which was proved many years later by Hutson [1963] to be asymptotically correct to the order of terms retained.

In his later book on electricity, Clausius [1893] notes Kirchhoff's solution, and the fact that (1.4) would follow from (1.3) if we use  $\alpha = (2\pi\epsilon)^{-\frac{1}{2}} \cong 0.242$  instead of the solution of (1.2). Clausius remarks that his value of  $\alpha = 0.247$  was obtained by approximating the value of a series development, and indeed, a number of authors since have held or implied that the only difference between Clausius' and Kirchhoff's results is due to an inadequate numerical calculation by Clausius [Magini, 1906; von Guttenberg, 1953]. In point of fact, however, Clausius' numerical procedure was sound, and it was his analytical technique which did not suffice to reproduce the proper constant term in (1.3).

The analytical task of determining this constant, as well as higher terms in the asymptotic expansion whose leading terms are given by (1.4), has proved challenging. Ignatowsky [1932] used an infinite series involving Legendre polynomials which, like so many other methods, becomes inefficient for small plate separations. His result was

$$C \cong \epsilon a \left[ \pi \frac{a}{d} + \ln \left( \frac{8a}{d} \right) - \frac{1}{2} \right]$$

and works out to be much farther from the correct Kirchhoff expression than was that of Clausius, even though it can be shown to be a lower bound [Pólya and Szegő, 1951] by means of variational techniques which approximate the charge density on the plates by constants. Noble [1958], somewhat

in the same spirit as Clausius, modified the trial function for the charge density to reflect the proper singularity at the edges. He thus obtained an improved lower bound, though no better determination of the proper constant in (1.4). An upper bound obtained by Fikhmanás and Fridberg [1974] does nothing to determine this constant either. Cooke [1958] has also used a technique related to that of matched asymptotic expansions, citing as inspiration a paper by Maxwell [1866] on a somewhat different problem. Cooke obtains the expression

$$C \cong \frac{\epsilon d}{4\pi} (\tau_2 - \tau_1)^2 \quad (1.6)$$

where  $\tau_1$  and  $\tau_2$  are the two positive real roots of the transcendental equation

$$\tau - \ln \tau = 2\pi a/d + 1 \quad (1.7)$$

When  $a/d \gg 1$ , eqns. (1.6) and (1.7) simplify to

$$C \cong \epsilon a \left[ \frac{\pi a}{d} + \ln \left( \frac{2\pi a}{d} \right) + 1 \right] \quad (1.8)$$

The constant term in (1.8) is still not in agreement with that of Kirchhoff, and is not even quite as close as (1.3) of Clausius. Ironically, however, if no approximation to the roots of (1.7) is made, eqn. (1.6) yields values much closer to numerically "exact" values (see below) than does the asymptotically correct Kirchhoff result (1.4) when  $d/a$  has only moderately small values.

Another thread which can be traced through the attacks on this problem has been that of an integral equation formulation. The variety of integral equations to which the circular parallel-plate problem reduces is truly astounding. Serini [1920] was apparently the first to derive such an integral equation, although his later attempt to obtain an approximation for  $C$  when  $a/d \gg 1$  was not asymptotically correct [Serini, 1922].

Nicholson [1924] considered various equivalent representations for the problem, though unfortunately his work contains invalidating errors, as point out by Love [1949]. This latter paper presents an alternative integral equation for the problem which has been re-derived by others [Cooke, 1956; Kloit-Dashinskii and Minkov, 1959]. Nomura [1941] has produced some numerical results based on one of these representations, which were subsequently improved upon by Cooke [1958]. Hutson [1963] used the Love-Cooke integral equation to prove the asymptotic nature of (1.4).

Serini's approach also leads to a dual integral equation for this problem [Noble, 1958]. Sneddon [1966] provides a summary of much of this work up to 1965, though a number of misprints and inaccurate statements are made in his treatment [Wintle and Kurylowicz, 1985]. The latter authors also present further improvements of the Nomura-Cooke numerical results.

All these integral equations relate in one way or another to the unknown charge density distribution on one of the discs. Leppington and Levine [1970] devised a new integral equation in which the unknown function is the radial derivative of the potential in the plane of one of the discs and beyond its edge. This formulation allowed them to obtain not only the Kirchhoff expansion(1.4), but also the next higher-order term in the asymptotic expansion of  $C$ . Fikhmanas and Fridberg later used this type of integral equation as the basis of a variational approach to the problem.

The more systematic and rigorous modern apparatus of matched asymptotic expansions allowed Shaw [1970] to obtain still further terms in the expansion of  $C$ . A comment by Wigglesworth [1972] considerably simplified one of the terms, and a later paper by Chew and Kong [192] corrected a term of order  $d/a$  by properly evaluating an integral. The result is:



$$C \cong \epsilon a \left\{ \frac{\pi a}{d} + \ln\left(\frac{16\pi a}{d}\right) - 1 + \frac{d}{4\pi a} \left[ \ln^2\left(\frac{16\pi a}{d}\right) - 2 \right] \right\} \quad (1.9)$$

More recently, Soibel'man [1981, 1982, 1984] has used the same method to evaluate the asymptotic capacitance of a parallel-plate capacitor whose plates have arbitrary shape. Although the details of his method are most readily available in Soibel'man [1984], there are unfortunately, many misprints in this paper and the correct formulas are only available in a report difficult to obtain in the West [Soibel'man, 1982; also Soibel'man, 1985, personal communication]. When his formulas are corrected, the result for a pair of identical simply-connected plates is, in the notation of this report,

$$C \cong \frac{\epsilon S}{d} + \frac{\epsilon P}{2\pi} \left[ B + 1 + \ln\left(\frac{2\pi}{d}\right) \right] + \frac{\epsilon d}{4\pi} \ln^2\left(\frac{2P}{d}\right) \quad (1.10)$$

where  $S$  is the area of one of the plates,  $P$  its perimeter,  $d$  the separation between the plates, and  $B$  is a constant given by a rather elaborate expression involving multiple integrals of a projected area function on one of the plates.\* It can be noted by comparison with (1.9) that the error in (1.10) is of order  $d \ln d$ .

This systematic use of matched asymptotic expansions is very important from the standpoint of mathematical rigor, since it permits a proof of the asymptotic nature of (1.4), (1.9) and (1.10). A look at any of these papers, however, shows how cumbersome the method is beyond a relatively low order of approximation. In Soibel'man's work, moreover, the mathematical

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\*The expression (22) given for  $B$  in Soibel'man [1984] is incorrect. It can, however been shown that

$$B = \frac{J_p}{2} + \ln\left(\frac{4P}{\pi}\right)$$

where  $J_p$  is defined by eqn. (B.8) in Appendix B.

notation used is not likely to be understandable to many engineers and physicists who wish to follow the derivation. This is compounded by the lack of physical insight into his expansions and constructions--matched asymptotic expansions can generally be motivated on a fairly intuitive basis. Nevertheless, Soibel'man's result suggests that a similar formula might be obtained for the capacitance of an arbitrary microstrip patch, and was the inspiration behind the present work.

Indeed, the recent renewed interest in this class of problems is primarily due to the importance of microstrip disk capacitors and patch antennas. Smaller size disks are used as lumped elements in the tuning of microstrip circuits, while the static capacitance of a larger patch antenna forms an important contribution to the impedance it presents to a feed line. Again, the most thoroughly studied example has been that of the circular disk, because of its rotational symmetry.

Minkov [1960] was apparently the first to formulate this problem, and obtained a set of dual integral equations for a pair of circular disks loaded with a dielectric slab which generalized those of Serini. Later formulations followed [Borkar and Yang, 1975; Coen and Gladwell, 1977; Chew and Kong, 1980; Braveras et al., 1981; Leong et al., 1981a,b; Hongo and Takahashi, 1982], whose solutions were carried out either by a Galerkin-like method or by Noble's variational method.

The method of matched asymptotic expansions was made possible for the air-filled capacitor because of the availability of a solution (via conformal mapping) of the two-dimensional potential problem near the edge. This same problem for the case when a dielectric layer is present has been solved by Lebedev [1958] and by Chew and Kong [1981, 1982] using

the Wiener-Hopf technique. The latter two papers have thus been able to extend the work of Leppington and Levine, Shaw, and Wigglesworth to the case of a circular microstrip disc.

Other than the above-mentioned work of Soibel'man, little analytical work has been done on plate shapes other than circular. The edge correction for the case of rectangular plates is treated in an empirical fashion by Scott and Curtis [1939], Kortel [1955], Wolff [1973], and Wolff and Knoppik [1974]. In these works, plausible edge effects based upon the edge effects of two-dimensional strip capacitors are put forward. A further empirical correction for the effects of the corners has been given without derivation by Bedair [1985]. None of these formulas is rigorously established to be asymptotically valid.

By contrast, much work has gone into purely numerical solutions of such problems. Reitan [1959] used a moment method to treat the case of a rectangular air-filled parallel-plate capacitor, and a host of researchers have tackled the rectangular microstrip capacitor in this way [Farrar and Adams, 1971; 1972; Itoh, 1973; Alexopoulos et al., 1980] or using finite elements [Benedek and Silvester, 1972]. A spectral domain moment method has been used to find the capacitance of triangular [Sharma, 1979] and hexagonal [Sharma and Hofer, 1982] microstrip discs.

Most of the foregoing work has been intended for the case when the separation of the discs is small compared with their diameter. There has also, however, been a small amount of work aimed at the case when the plate separation is large. Boulgakov [1904], as a limiting case of a different problem, found that the capacitance of a circular plate capacitor when  $d \gg a$  is

$$C \cong \frac{2\pi\epsilon a}{\tan^{-1}(d/a)} \quad (1.11)$$

This result has also been obtained by later authors [Ollendorff, 1932, p. 282; Waldmann, 1937]. Taking the limit of (1.11) when  $d \gg a$ , and noting that  $C_\infty = 8\epsilon a$  is the capacitance of an isolated circular disc (cf. Appendix A), this becomes

$$C \cong \frac{C_\infty/2}{1 - C_\infty/4\pi\epsilon d} \quad (1.12)$$

Iossel' et al., [1969, p. 117] give this same expression for the case of rectangular plates, if  $C_\infty$  is now understood to be the capacitance of an isolated rectangular plate. Pólya and Szegő [1951, p. 86], in fact, give a slightly approximated version of (1.12) as applicable to any shape of plate. A method proposed in Bozdarenko et al. [1967] and in Kolybel'nikova and Simonenko [1971] makes possible the extension of this type of result to microstrip geometries, though unfortunately the formulas for capacitance of a circular disc presented by Bozdarenko et al. are in error.

Finally, we should mention the ingenious work of Wheeler [1982], who not only devised a more accurate version of (1.11) for the circular plate capacitor, but also deduced an empirical formula valid for any value of  $d/a$ , correctly reproducing the limiting behaviors of (1.4) and (1.12). He further adapts his formula to give an approximate capacitance for the circular microstrip disc as well. In this connection, however, we should mention that von Guttenberg [1953] has also presented empirical formulas for the air-filled capacitor, and that Cooke's formula (1.6)/(1.7) tracks reasonably well with exact values of  $C$  even down to  $a/d = 0$ , although it does have the drawback of involving solution of the implicit equation (1.7).

In this report, we will obtain analytical approximations for the static capacities of a microstrip patch of arbitrary shape. One approximation will be valid for relatively small patches, and is a generalization of the Boulgakov/Ollendorff/Waldmann result (1.11). The other approximation generalizes the Kirchhoff formula (1.4) for relatively large patches. Comparison with numerical results will show that quite accurate results are obtained using only these simple formulas, and also points to some possible inaccuracy in previously reported numerical data for this class of problems.

## II. CAPACITANCE OF A SMALL MICROSTIP PATCH

The method of this section is due to Bozdarenko et al. [1967] and Kolybel'nikova and Simonenko [1971], although the former paper on the circular patch contains some errors, while the latter paper on the arbitrarily shaped patch did not address the microstrip patch problem specifically.

The geometry of the problem is illustrated in Fig. 3. A perfectly conducting, infinitely thin patch occupying the area  $S$  lies on top of a grounded dielectric substrate of thickness  $d$  and permittivity  $\epsilon_r$  as shown. In this section, we begin by formulating an integral equation for the electrostatic charge distribution  $\sigma_m(\bar{\rho})$  on the patch. This takes the form:

$$\frac{1}{\epsilon_0} \int_S G(\bar{\rho}; \bar{\rho}') \sigma_m(\bar{\rho}') dS' = 1 \quad (\bar{\rho} \in S) \quad (2.1)$$

where  $\bar{\rho} = x\bar{a}_x + y\bar{a}_y$ ,  $\bar{\rho}' = x'\bar{a}_x + y'\bar{a}_y$ ,  $\bar{a}_x$ ,  $\bar{a}_y$  and  $\bar{a}_z$  are the unit vectors in Cartesian coordinates, and (see, e.g., Smythe [1968]; or Benedek and Silvester [1972]):

$$\begin{aligned} G &= \frac{1}{2\pi(\epsilon_r+1)} \int_0^\infty J_0(\lambda R) \frac{1 - e^{-2\lambda d}}{1 + \delta_\epsilon e^{-2\lambda d}} d\lambda \\ &= \frac{1}{2\pi(\epsilon_r+1)} \left\{ \frac{1}{R} + \frac{2\epsilon_r}{\epsilon_r-1} \sum_{m=1}^\infty \frac{(-\delta_\epsilon)^m}{\sqrt{R^2 + (2md)^2}} \right\} \end{aligned} \quad (2.2)$$

in which  $J_0$  is the Bessel function, and

$$R = |\bar{\rho} - \bar{\rho}'| = \sqrt{(x-x')^2 + (y-y')^2} \quad ; \quad \delta_\epsilon = \frac{\epsilon_r - 1}{\epsilon_r + 1} \quad (2.3)$$

The second form of  $G$  in (2.2) follows from the first by expanding the denominator of the integrand in powers of  $\delta_\epsilon$ , integrating term-by-term and re-arranging slightly, having used the identity

$$\frac{1}{\sqrt{R^2+z^2}} = \int_0^\infty J_0(\lambda R) e^{-\lambda|z|} d\lambda \quad (2.4)$$

Once  $\sigma_m(\bar{\rho})$  is solved for, the capacitance  $C$  between the patch and ground plane is obtained from

$$C = \int_S \sigma_m(\bar{\rho}) dS \quad (2.5)$$

Let the diameter  $D$  of  $S$  be defined by

$$D = \max_{\bar{\rho}, \bar{\rho}' \in S} \{R\} \quad (2.6)$$

and assume that  $D \ll d$ . Then we can approximate  $G$  by

$$\begin{aligned} G &\cong \frac{1}{2\pi(\epsilon_r+1)} \left\{ \frac{1}{R} + \frac{2\epsilon_r}{\epsilon_r-1} \sum_{m=1}^{\infty} \frac{(-\delta_\epsilon)^m}{2md} \right\} \\ &= \frac{1}{2\pi(\epsilon_r+1)} \left\{ \frac{1}{R} - \frac{\epsilon_r}{d(\epsilon_r-1)} \ln \left( \frac{2\epsilon_r}{\epsilon_r+1} \right) \right\} \end{aligned} \quad (2.7)$$

Note that if  $d \rightarrow \infty$  and  $\epsilon_r = 1$ , then  $G$  reduces to  $1/4\pi R$ , the static Green's function of free space. In this limit, eqn. (2.1) reduces to

$$\frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma_\infty(\bar{\rho}')}{R} dS' = 1 \quad (\bar{\rho} \in S) \quad (2.8)$$

where  $\sigma_\infty$  is the static charge distribution on the same arbitrary patch  $S$ , now isolated in free space. The capacitance of this isolated patch with respect to an infinitely distant point is then

$$C_{\infty} = \int_S \sigma_{\infty}(\bar{\rho}) dS \quad (2.9)$$

But the kernel (2.7) is simply a multiple of free-space kernel plus a constant. When (2.7) is used in (2.1), it can be rearranged into the form

$$\begin{aligned} \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma_m(\bar{\rho}')}{R} dS' &= \frac{\epsilon_r + 1}{2} + \frac{\int_S \sigma_m(\bar{\rho}') dS'}{4\pi\epsilon_0 d} \left[ \frac{\epsilon_r}{\epsilon_r - 1} \ln \left( \frac{2\epsilon_r}{\epsilon_r + 1} \right) \right] \\ &= \frac{\epsilon_r + 1}{2} + \frac{C}{4\pi\epsilon_0 d} \left[ \frac{\epsilon_r}{\epsilon_r - 1} \ln \left( \frac{2\epsilon_r}{\epsilon_r - 1} \right) \right] \quad (\bar{\rho} \in S) \end{aligned} \quad (2.10)$$

Except for a different constant on the right side, eqn. (2.10) is identical to (2.8). We conclude that

$$\sigma_m(\bar{\rho}) = \sigma_{\infty}(\bar{\rho}) \left\{ \frac{\epsilon_r + 1}{2} + \frac{C}{4\pi\epsilon_0 d} \left[ \frac{\epsilon_r}{\epsilon_r - 1} \ln \left( \frac{2\epsilon_r}{\epsilon_r + 1} \right) \right] \right\} \quad (2.11)$$

and thus, by (2.9) and (2.5) that

$$C = \frac{\frac{\epsilon_r + 1}{2} C_{\infty}}{1 - \frac{C_{\infty}}{4\pi\epsilon_0 d f(\epsilon_r)}} \quad (2.12)$$

where

$$f(\epsilon_r) = \frac{\epsilon_r - 1}{\epsilon_r \ln \left( \frac{2\epsilon_r}{\epsilon_r + 1} \right)} \quad (2.13)$$

The function  $f(\epsilon_r)$  is a well-behaved quantity, varying from 2 at  $\epsilon_r = 1$  to  $1/\ln 2 \cong 1.4427$  as  $\epsilon_r \rightarrow \infty$ . We thus have an explicit approximation to  $C$  which requires only knowledge of the value of  $C_{\infty}$ , the capacitance of the patch isolated in free space. This capacitance  $C_{\infty}$  is discussed in Appendix A. It must in general be determined numerically, but values



of it have been calculated for a variety of shapes, some of which are presented in Table A-1.

Following eqn. (A.1) of Appendix A, we define the equivalent radius of the patch to be  $a_e = C_\infty / 8\epsilon_0$ , whence (2.12) becomes

$$C = \frac{4\epsilon_0(\epsilon_r+1)a_e}{1 - \frac{2a_e}{\pi df(\epsilon_r)}} \quad (2.14)$$

Equations (2.12) and (2.14) can be derived for the case of a circular patch from the results of Borkar and Yang [1975], but appear to be new for the general case. We should point out that the approximation in (2.7) has a relative error of  $O(D^2/d^2)$ , not just  $O(D/d)$ , so that we can expect a fair degree of accuracy without requiring  $D/d$  to be inordinately small.

We observe the similarity of (2.12) to the approximation (1.12) of formula (1.11) for the circular disk capacitor. In fact, both formulas (1.11) and (1.12) have the same order of accuracy as does (2.12) based on the error in (2.7). However, eqn. (1.11) is to be preferred in practice because while (1.12) becomes infinite when  $d = C_\infty / 4\pi\epsilon$ , and negative for smaller values of  $d$ , (1.1) remains finite and positive for all  $d > 0$ . Furthermore, it is observed to track the correct numerical value of  $C$  better for smaller  $d$  than does (1.12). Based on these observations, we suggest the following empirical modification of (2.14), which we have found to be in closer agreement with numerically "exact" data:

$$C = \frac{2\pi\epsilon_0(\epsilon_r+1)a_e}{\tan^{-1}[df(\epsilon_r)/a_e]} \quad \begin{array}{l} \text{Small} \\ \text{Patch} \\ \text{Formula} \end{array} \quad (2.15)$$

Numerical examples based on (2.15) will be given in section IV.

### III. CAPACITANCE OF A LARGE MICROSTRIP PATCH

In this section we will treat the case of a patch whose characteristic dimensions are large compared to the substrate thickness. Our method is somewhat similar to that used by Leppington and Levine [1970] in their analysis of the case of an air-filled, circular plate capacitor. When the patch has arbitrary shape, we need to be careful in defining precisely the parameters of the patch which must be large compared to  $d$ . We can do this following the lead of Aleksandrov [1969] in his analysis of an elastostatic contact problem.

We will first of all require that the radius of curvature at each point of the perimeter  $P$  of the patch must be large compared to  $d$  (though it is probably possible to relax this condition a bit to allow for a small number of corners). Second, we require that the minimum normal diameter of the patch be large compared to  $d$ . This quantity is defined by reference to Fig. 4. Choose any point  $M$  on the perimeter of the patch, and draw a line normal to  $P$  and  $M$ . This line will intersect the perimeter at one or more points denoted  $M_1, M_2$ , etc., as shown. Let  $D_N(M)$  be the smallest of the distances  $MM_1, MM_2, \dots$ . The minimum normal diameter of  $S$  is the minimum value of  $D_N(M)$  for all  $M$  on  $P$ . Roughly speaking the condition that the minimum normal diameter be large compared with  $d$  means that:

- (a) narrow waists are not allowed;
- (b) the patch may not bend around in a U-shape and nearly touch its ends together.

The formulation of the large patch problem, by contrast with that of the small patch problem, is best done as an integral equation for the tangential electric field on the portion  $S_A$  of the top of the substrate

$z = d$  which is not occupied by the patch (Fig. 3). We will refer to  $S_A$  as the aperture, and indeed for a large patch (in the sense of the preceding paragraph), it is expected that the aperture field will be similar to that near a straight edge at each point of  $P$ . This field is fairly well concentrated within a few  $d$  of the edge, and forms the basis of our approximate technique in this section.

We begin by constructing two Green's functions for the regions  $V_1(z > d)$  and  $V_2(0 < z < d)$ , shown in Fig. 5. For  $z > d$ , we look for  $G_1$  satisfying

$$\begin{aligned} \nabla^2 G_1 &= -\delta(\bar{r}-\bar{r}') = -\delta(x-x')\delta(y-y')\delta(z-z') \quad (\text{in } V_1) \\ \frac{\partial G_1}{\partial z'} \Big|_{z'=d} &= 0 \end{aligned} \quad (3.1)$$

By standard methods we find this to be

$$G_1 = \frac{1}{4\pi} \left[ \frac{1}{\sqrt{|\bar{\rho}-\bar{\rho}'|^2 + (z-z')^2}} + \frac{1}{\sqrt{|\bar{\rho}-\bar{\rho}'|^2 + (z+z'-2d)^2}} \right] \quad (3.2)$$

where again  $\bar{\rho} = x\bar{a}_x + y\bar{a}_y$ ,  $\bar{\rho}' = x'\bar{a}_x + y'\bar{a}_y$ , while  $\bar{r} = \bar{\rho} + z\bar{a}_z$  and  $\bar{r}' = \bar{\rho}' + z'\bar{a}_z$ . Likewise, the function  $G_2$  satisfying

$$\begin{aligned} \nabla^2 G_2 &= -\delta(\bar{r}-\bar{r}') \quad (\text{in } V_2) \\ \frac{\partial G_2}{\partial z'} \Big|_{z'=0,d} &= 0 \end{aligned} \quad (3.3)$$

is found to be

$$G_2 = -\frac{1}{2\pi d} \ln \frac{|\bar{\rho}-\bar{\rho}'|}{d} + G_2^c \quad (3.4)$$

where

$$G_2^c = -\frac{1}{\pi d} \sum_{m=1}^{\infty} \cos \frac{m\pi z}{d} \cos \frac{m\pi z'}{d} K_0 \left( \frac{m\pi |\bar{\rho} - \bar{\rho}'|}{d} \right) \quad (3.5)$$

and  $K_0$  is the modified Bessel function. We have the estimates

$$\left. \begin{aligned} G_1 &= O\left(\frac{1}{r}\right) \quad \text{as } r = |\bar{r}| \rightarrow \infty \quad \text{in } V_1 \\ G_2 &= -\frac{1}{2\pi d} \ln \frac{\rho}{d} + O\left(\frac{1}{\rho}\right) \quad \text{as } \rho = |\bar{\rho}| \rightarrow \infty \quad \text{in } V_2 \end{aligned} \right\} \quad (3.6)$$

We now invoke a variant of a vector Green's theorem discussed by Harrington [1961], eqn. (3-50). In a homogeneous, source-free region  $V$ , let  $\bar{E}$  be a static electric field with  $\nabla \times \bar{E} = 0$  and  $\nabla \cdot \bar{E} = 0$ . If  $G$  is a Green's function satisfying

$$\nabla^2 G = -\delta(\bar{r} - \bar{r}')$$

in  $V$ , and  $\bar{c}$  is a constant vector, then

$$\bar{c} \cdot \bar{E}(\bar{r}) = \oint_S [(\bar{E}(\bar{r}') \cdot \nabla' G) \bar{c} - (\bar{E}(\bar{r}') \cdot \bar{c}) \nabla' G - (\bar{c} \cdot \nabla' G) \bar{E}(\bar{r}')] \cdot \bar{a}'_n dS' \quad (3.7)$$

where  $S$  is the surface bounding  $V$ , and  $\bar{a}'_n$  is the unit outward normal vector to  $S$  at the integration point  $\bar{r}'$ .

Let  $V$  be  $V_1$  and  $S$  be  $S_1$  as in Fig. 5. We choose  $\bar{c} = \bar{a}_z$  and  $G$  to be  $G_1$ , and apply (3.7). From the boundary conditions on  $\bar{E}$  and  $G_1$  as well as the conditions as  $|\bar{r}| \rightarrow \infty$ , most of the terms in the surface integral vanish, and we are left with

$$E_z(\bar{r}_1) = - \int_{S_A} \bar{E}(\bar{r}') \cdot \nabla' G_1(\bar{r}_1, \bar{r}') dS' \Big|_{z'=d} \quad (\bar{r}_1 \in V_1) \quad (3.8)$$

where, again,  $S_A$  is the "aperture" portion of the plane  $x = d$ , not occupied by the patch. Similarly, application of (3.7) to  $\bar{E}$ ,  $G_2$ ,  $S_2$  and  $V_2$  gives

$$E_z(\bar{r}_2) = \int_{S_A} \bar{E}(\bar{r}') \cdot \nabla' G_2(\bar{r}_2, \bar{r}') dS' \Big|_{z'=d} \quad (\bar{r}_2 \in V_2) \quad (3.9)$$

Now, if the field  $\bar{E}$  corresponds to a voltage  $V$  between patch and ground plane,

$$\int_{\substack{\text{patch} \\ \text{edge}}} \bar{E}(\bar{r}) \cdot d\bar{\ell} \Big|_{z=d} = V$$

then the capacitance  $C$  of the patch is given by

$$C = \frac{\epsilon_0}{V} \int_S [E_z(x, y, d+) - \epsilon_r E_z(x, y, d-)] dS \quad (3.10)$$

By taking  $z_1 \rightarrow d+$  in (3.8) and  $z_2 \rightarrow d-$  in (3.9) and plugging the result into (3.10) we have

$$C = - \lim_{\substack{z_1 \rightarrow d+ \\ z_2 \rightarrow d-}} \frac{\epsilon_0}{V} \int_{S_A} dS' [\bar{E}(\bar{r}') \cdot \nabla' (G_1 + \epsilon_r G_2)] \Big|_{z'=d} \quad (3.11)$$

Since  $\nabla' G_{1,2} = -\nabla G_{1,2}$ , then interchanging the order of the integrations and using the two-dimensional divergence theorem, we have

$$\begin{aligned} C &= \lim_{\substack{z_1 \rightarrow d+ \\ z_1 \rightarrow d-}} \frac{\epsilon_0}{V} \int_{S_A} dS' \bar{E}(\bar{r}') \cdot \int_A dS \nabla_T (G_1 + \epsilon_r G_2) \Big|_{z'=d} \\ &= \frac{\epsilon_0}{V} \int_{S_A} dS' \bar{E}_S(\bar{\rho}') \cdot \oint_P \bar{a}_n \left[ -\frac{\epsilon_r}{2\pi d} \ln \frac{|\bar{\rho} - \bar{\rho}'|}{d} + K_1(\bar{\rho}, \bar{\rho}') \right. \\ &\quad \left. + \epsilon_r K_2(\bar{\rho}, \bar{\rho}') \right] d\ell \end{aligned} \quad (3.12)$$

where  $\nabla_T = \bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \frac{\partial}{\partial y}$ ,  $\bar{a}_n$  is the outward unit normal to  $P$  in the  $xy$ -plane,

$$E_s(\bar{\rho}) = \bar{E}_{\tan}(\bar{r}) \Big|_{z=d} \quad (3.13)$$

is the tangential electric field in the aperture, and

$$K_1(\bar{\rho}, \bar{\rho}') = G_1(\bar{r}, \bar{r}') \Big|_{z=z'=d} = \frac{1}{2\pi|\bar{\rho}-\bar{\rho}'|} \quad (3.14)$$

$$\begin{aligned} K_2(\bar{\rho}, \bar{\rho}') &= G_2^C(\bar{r}, \bar{r}') \Big|_{z=z'=d} \\ &= \frac{1}{\pi d} \sum_{m=1}^{\infty} K_0\left(\frac{m\pi}{d}|\bar{\rho}-\bar{\rho}'|\right) \end{aligned} \quad (3.15)$$

The evaluation of the capacitance now consists of evaluating the double integral in (3.12). The term involving the logarithmic kernel in (3.12) can be treated exactly. We have

$$\begin{aligned} C_1 &= -\frac{\epsilon_0 \epsilon_r}{2\pi d V} \int_{S_A} dS' \bar{E}_s(\bar{\rho}') \cdot \oint_P \bar{a}_n \ln \frac{|\bar{\rho}-\bar{\rho}'|}{d} d\ell \\ &= -\frac{\epsilon_0 \epsilon_r}{2\pi d V} \int_{S_A} dS' \bar{E}_s(\bar{\rho}') \cdot \int_S \nabla_T \ln \frac{|\bar{\rho}-\bar{\rho}'|}{d} dS \\ &= \frac{\epsilon_0 \epsilon_r}{2\pi d V} \int_S dS \int_{S_A} \bar{E}_s(\bar{\rho}') \cdot \nabla_T' \ln \frac{|\bar{\rho}-\bar{\rho}'|}{d} dS' \end{aligned} \quad (3.16)$$

But  $\bar{E}_s = -\nabla_T \phi_s$ , where  $\phi_s \Big|_S = V$ , and also

$$\nabla_T^2 \ln \frac{|\bar{\rho}-\bar{\rho}'|}{d} = 2\pi\delta(\bar{\rho}-\bar{\rho}') \quad (3.17)$$

so, since  $\bar{E}_s \equiv 0$  on  $S$ ,

$$\begin{aligned} &\int_{S_A} \bar{E}_s(\bar{\rho}') \cdot \nabla_T' \ln \frac{|\bar{\rho}-\bar{\rho}'|}{d} dS' \\ &= - \int_{S+S_A} \nabla_T' \phi_s \cdot \nabla_T' \ln \frac{|\bar{\rho}-\bar{\rho}'|}{d} dS' \end{aligned}$$

$$\begin{aligned}
&= \int_{S+S_A} \phi_S \nabla_{\perp}^2 \ln \frac{|\bar{\rho}-\bar{\rho}'|}{d} dS' \\
&= 2\pi \phi_S(\bar{\rho})
\end{aligned} \tag{3.18}$$

and thus

$$C_1 = \frac{\epsilon_0 \epsilon_r}{2\pi dV} 2\pi \int_S \phi_S(\bar{\rho}) dS = \frac{\epsilon_0 \epsilon_r S}{d} \tag{3.19}$$

which is simply the classical formula (1.1) found by neglecting the fringing fields.

The other two contributions to  $C$  in (3.12) will be evaluated approximately. Consider first

$$C_2 \equiv \frac{\epsilon_0 \epsilon_r}{V} \int_{S_A} dS' \bar{E}_S(\bar{\rho}') \cdot \oint_P \bar{a}_n K_2(\bar{\rho}, \bar{\rho}') d\ell \tag{3.20}$$

We introduce an  $(\ell, n)$  coordinate system for points near  $P$  as shown in Fig. 6. By  $\ell$ , we denote the arc length measured along  $P$  from some arbitrarily chosen origin, and by  $n$  we mean the normal distance of a point  $\bar{\rho}(\ell, n)$  from the point  $\bar{\rho}_0(\ell, 0)$  on  $P$ . We take  $n > 0$  for points in  $S_A$ , and  $n < 0$  for points in  $S$ .

Under the assumptions at the beginning of this section on the patch  $S$ , if  $n = 0(d)$ , the curve  $P$  can be considered to be approximately a straight line in the neighborhood of  $\ell = \ell'$ . This idea has been used by other authors in a variety of problems [Panych, 1950, 1953; Ivanov, 1966, Mitzner, 1967]. For eqn. (3.20), this is permissible because for  $|\ell - \ell'| = 0(d)$ ,  $d \rightarrow 0$ , we have (since  $n = 0$ )

$$\left. \begin{aligned}
|\bar{\rho}-\bar{\rho}'| + \sqrt{(x-x')^2 + (y-y')^2} &\cong \sqrt{(\ell-\ell')^2 + (n')^2} ; \\
\bar{a}_n &\cong \bar{a}'_n
\end{aligned} \right\} \tag{3.21}$$

while for  $|\ell - \ell'| \gg d$  when this is no longer true, the kernel  $K_2$  in (3.20) is exponentially small and negligible. The assumption that the minimum normal diameter is large compared to  $d$  assures that  $K_2$  is negligible for  $|\ell - \ell'| \gg d$  for either the exact value of  $|\bar{\rho} - \bar{\rho}'|$  or its approximation (3.21). By (3.21), then, we have

$$\oint_P \bar{a}_n K_2(\bar{\rho}, \bar{\rho}') d\ell \cong \frac{\bar{a}'_n}{\pi d} \sum_{m=1}^{\infty} \oint_P K_0\left(\frac{m\pi}{d} \sqrt{(\ell - \ell')^2 + (n')^2}\right) d\ell \quad (3.22)$$

But again due to the exponential decay of the integrand, we can replace the integral over  $P$  with an integral from  $\ell = -\infty$  to  $+\infty$  with the same overall accuracy. Then by [Gradshteyn and Ryzhik, 1965, no. 6.596.3], we have

$$\begin{aligned} \oint_P \bar{a}_n K_2(\bar{\rho}, \bar{\rho}') d\ell &\cong \frac{\bar{a}'_n}{\pi d} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} K_0\left(\frac{m\pi}{d} \sqrt{(\ell - \ell')^2 + (n')^2}\right) d\ell \\ &= \frac{\bar{a}'_n}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} e^{-m\pi n'/d} \\ &= -\frac{\bar{a}'_n}{\pi} \ln(1 - e^{-\pi n'/d}) \end{aligned} \quad (3.23)$$

Hence we have

$$C_2 \cong -\frac{\epsilon_0 \epsilon_r}{\pi V} \int_{S_A} \bar{E}_S(\rho') \cdot \bar{a}'_n \ln(1 - e^{-\pi n'/d}) dS' \quad (3.24)$$

Now  $\bar{E}_S \cdot \bar{a}'_n$  is concentrated primarily in a narrow region along the edge  $P$  of the patch, and decays reasonably quickly to zero beyond this strip. We therefore approximate the field near the edge of the patch by that of the two-dimensional edge geometry shown in Fig. 7. This problem has been solved using modified Wiener-Hopf techniques, and its solution is available in closed form [Lebedev, 1958; Chew and Kong, 1981, 1982]. For a voltage difference of  $V$  between upper conductor and ground plane, we denote its aperture field solution by  $\bar{E}_T(x, d) \equiv \bar{a}'_x E_0(x)$ . Then we put



$$\bar{E}_s(\bar{\rho}') \cong \bar{a}'_n E_0(n')/[1 + n'\kappa(\ell')] \quad (3.25)$$

where  $\kappa(\ell')$  is the curvature of P at the point  $(\ell', 0)$ . The denominator of (3.25) is included to account for the spreading of the field lines as we move away from the curved edge of the patch. On the other hand,

$$\int_{S_A} (\cdot) dS' \rightarrow \int_0^\infty dn' \oint_P (\cdot) [1 + n'\kappa(\ell')] d\ell'$$

so that (3.24) becomes

$$C_2 \cong - \frac{\epsilon_0 \epsilon_r P}{\pi V} \int_0^\infty E_0(n') \ln(1 - e^{-\pi n'/d}) dn' \quad (3.26)$$

Finally, consider

$$C_3 = \frac{\epsilon_0}{V} \int_{S_A} dS' \bar{E}_s(\rho') \cdot \oint_P \bar{a}_n K_1(\bar{\rho}, \bar{\rho}') d\ell \quad (3.27)$$

Unlike the integrand in the left side of (3.22), the integrand of the contour integral in (3.27) does not decay rapidly enough as  $|\ell - \ell'|$  gets large to permit quite such casual approximations; more careful treatment is needed.

We will split up the integrand into the sum of two terms, one of which can be treated exactly, and in the other of which we can make approximations (3.21).

We write

$$\begin{aligned} \oint_P \bar{a}_n K_1(\bar{\rho}_0, \bar{\rho}') d\ell &= \frac{1}{2\pi} \oint_P \frac{\bar{a}_n d\ell}{\sqrt{|\bar{\rho}_0 - \bar{\rho}'|^2 + d^2}} \\ &+ \frac{1}{2\pi} \oint_P \left[ \frac{1}{|\bar{\rho}_0 - \bar{\rho}'|} - \frac{1}{\sqrt{|\bar{\rho}_0 - \bar{\rho}'|^2 + d^2}} \right] \bar{a}_n d\ell \end{aligned} \quad (3.28)$$

where  $\bar{\rho}'_0$  is the point  $(\ell', 0)$  on P corresponding to the point  $\bar{\rho}'$  whose coordinates are  $(\ell', n')$  in  $S_A$ . We have emphasized that  $\bar{\rho}$  is on P by relabeling it as  $\bar{\rho}_0$ .

If  $n' = 0(d)$ , then the integrand of the second term in (3.28) is  $O(d^2/|\rho_0 - \rho'|^3)$  when  $|\bar{\rho}_0 - \bar{\rho}'| \gg d$ . In this term, therefore, the approximations (3.21) can be made, and we follow the same sequence of arguments as for  $C_2$ :

$$\begin{aligned}
& \frac{1}{2\pi} \oint_p \left[ \frac{1}{|\bar{\rho}_0 - \bar{\rho}'|} - \frac{1}{\sqrt{|\bar{\rho}_0 - \bar{\rho}'|^2 + d^2}} \right] \bar{a}_n d\ell \\
& \approx \frac{\bar{a}'_n}{2\pi} \oint_p \left[ \frac{1}{\sqrt{(\ell - \ell')^2 + (n')^2}} - \frac{1}{\sqrt{(\ell - \ell')^2 + d^2}} \right] d\ell \\
& \approx \frac{\bar{a}'_n}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{(\ell - \ell')^2 + (n')^2}} - \frac{1}{\sqrt{(\ell - \ell')^2 + d^2}} \right] d\ell \\
& = -\frac{\bar{a}'_n}{\pi} \ell n \frac{n'}{d}
\end{aligned} \tag{3.29}$$

The relative error of (3.29) is not exponentially small in  $1/d$  as was true of (3.22) and (3.23), but is still acceptably small of order  $d^2 \ell n d$ . Hence,

$$\oint_p \bar{a}_n K_1(\bar{\rho}_0, \bar{\rho}') d\ell \approx \frac{1}{2\pi} \oint_p \frac{\bar{a}_n d\ell}{\sqrt{|\bar{\rho} - \bar{\rho}'|^2 + d^2}} - \frac{\bar{a}'_n}{\pi} \ell n \frac{n'}{d} \tag{3.30}$$

Once again, we will make the approximation (3.25) for  $\bar{E}_s(\bar{\rho}')$ , so that from (3.27) and (3.30) we have

$$C_3 \approx -\frac{\epsilon_0 P}{\pi V} \int_0^\infty E_0(n') \ell n \frac{n'}{d} dn' + \frac{\epsilon_0 P J}{2\pi} \tag{3.31}$$

where  $J$  is the normalized Neumann mutual inductance integral defined in (B.2). By (B.7), for small  $d$ ,

$$J \approx 2 \ell n \frac{4P}{\pi d} + J_p \tag{3.32}$$

where  $J_p$  is a shape constant given by (B.8). Specific values of  $J_p$  for various shapes P are given in Table B-1.

Collecting together (3.26) and (3.31) we obtain

$$C_2 + C_3 \cong \frac{\epsilon_0^P}{\pi} \left\{ \ln \frac{4P}{\pi d} + \frac{J_p}{2} - \frac{1}{V} \int_0^\infty E_0(n') \left[ \ln \frac{n'}{d} + \epsilon_r \ln(1 - e^{-\pi n'/d}) \right] dn' \right\} \quad (3.33)$$

But in Kuester et al. [1985] it is proved that:

$$\begin{aligned} & - \frac{1}{V} \int_0^\infty E_0(n') \left[ \ln \frac{n'}{d} + \epsilon_r \ln(1 - e^{-\pi n'/d}) \right] dn' \\ & = \epsilon_r \left[ \ln 2\pi - 2Q_0(-\delta_\epsilon) \right] + 1 - \ln 2 \end{aligned} \quad (3.34)$$

where

$$Q_0(-\delta_\epsilon) = \sum_{m=1}^{\infty} (-\delta_\epsilon)^m \ln m$$

and  $\delta_\epsilon = (\epsilon_r - 1)/(\epsilon_r + 1)$ . An accurate closed-form expression for  $Q_0$  has been given by Kuester [1984]. Therefore, from (3.19), (3.33) and (3.34), we finally obtain:

$$C \cong \frac{\epsilon_0 \epsilon_r S}{d} + \frac{\epsilon_0^P}{\pi} \left\{ \ln \frac{2P}{\pi d} + 1 + \frac{J_p}{2} + \epsilon_r \left[ \ln 2\pi - 2Q_0(-\delta_2) \right] \right\} \quad (3.35)$$

Large Patch Formula

This formula is the generalization to the case of a microstrip patch of arbitrary shape loaded with a substrate different from air, of the first two terms of Soibel'man's formula (1.10). In fact, for  $\epsilon_r = 1$ , eqn. (3.35) reduces to:\*

\*The correct form (3.36) of (1.10) for an air-filled microstrip patch is obtained by letting  $d \rightarrow 2d$  and multiplying the resulting formula by 2.

$$C \Big|_{\epsilon_r=1} \cong \frac{\epsilon_0 S}{d} + \frac{\epsilon_0 P}{\pi} \left( \ln \frac{4P}{d} + 1 + \frac{J_p}{2} \right) \quad (3.36)$$

For the case of a circular patch, (3.35) also correctly reproduces the results of Chew and Kong [1981, 1982] for  $\epsilon_r \neq 1$ .

A word is in order here about the fringing correction to the capacitance  $C_2 + C_3$ . The term  $C_2$ , which came from the parallel-plate region's cutoff modes through the term  $K_2$ , was proportional to the perimeter  $P$ . It could thus be accounted for by adjoining a small strip region of width  $\delta n$  around the perimeter; thus making a new patch of area  $S + P\delta n$ , and merely computing  $C_1 + C_2$  as  $\epsilon_0 \epsilon_r (S + P\delta n)/d$  for a suitable  $\delta n$ . If the patch were made of triplate or symmetric stripline rather than microstrip (Fig. 8), then  $C_3$  would have the same form as  $C_2$ , and the entire capacitance  $C$  could be described in terms of an equivalent edge extension as for  $C_1 + C_2$  above. This was known over a hundred years ago to Maxwell [1866; 1954, vol. 1, pp. 299-301], and has been used in similar problems since then [Aleksandrov, 1964, 1969]. Most recently it has become common to use an equivalent edge extension in symmetric stripline problems [Okoshi, 1985].

While the aforementioned usage is appropriate, we can see from the form  $C_3$  for the microstrip patch, (3.31)-(3.32), that a simple edge extension of the patch cannot properly account for the fringing capacitance. This is because of the term  $P \ln(P/d)$  as well as the shape-dependent factor  $J_p$  which enter into it. It is thus incorrect to use, for example, an "edge extension" inferred from the formula for a circular patch to predict the capacitance of a rectangular patch. The same thing is true for the use of the edge extension for dynamic problems involving resonant modes of the patch. This latter point will be treated in more detail in a future report.

The appearance of the Neumann inductance term  $J$  might seem surprising in a static capacitor problem. However, its appearance is easily explained. The aperture field  $\bar{E}_s(\bar{\rho})$  is very nearly equivalent to a magnetic line source located at the perimeter  $P$  of the patch. The interaction of this magnetic current with its own fields through the upper half-space  $V$  is mathematically virtually identical with that of an electric current loop and its own magnetic field, and so produces the term involving  $J$  in (3.31), as well as a further term resulting from the fact that  $\bar{E}_s(\bar{\rho})$  is not completely concentrated at the edge.

As a final comment: in the next section we will apply this theory to the case of several polygonal patches. At the vertices, of course, our assumption of large radius of curvature is violated as is the assumption of a large minimum normal diameter. We conjecture that if there are not "too many" vertices along the perimeter of the patch, our results in this section remain valid, because most of the fringing capacitance is still produced by the smooth portions of the edge. A rigorous proof that this conjecture is even asymptotically correct would undoubtedly be formidable. However, the agreement of our formulas with independently obtained numerical results does suggest that our formulas are valid.

#### IV. NUMERICAL RESULTS AND DISCUSSION

The most extensively studied patch structure from a numerical standpoint has been the circular patch of radius  $a$ . Leong et al. [1981b] and Hongo and Takahashi [1982] have presented what are probably the most accurate numerical values obtained to date for the circular patch. These are compared with results from our approximate formulas (2.15) and (3.35) in Figs. 9-11. In these figures and those which follow, we plot the normalized capacitance

$$C_n \equiv \frac{Cd}{\epsilon_0 \epsilon_r S} \quad (4.1)$$

Also, we have made use here of the values of  $a_e$  from Table A-1 and the values of  $J_p$  from Table B-1 for use in (2.15) and (3.35) respectively. It is observed that below  $d/a = 0.5$  or so, formula (3.35) is most accurate, while for larger  $d/a$ , equation (2.15) is to be preferred. The two approximate curves (which are upper and lower bounds in this case) are closest to each other roughly where this changeover point occurs. The maximum error is about 15% when  $\epsilon_r = 1$ , but shrinks to only about 6% when  $\epsilon_r = 9.6$ .

Rectangular patches have been treated both by the moment method and by finite-element methods [Farrar and Adams, 1971, 1972; Benedek and Silvester, 1972; Alexopoulos et al. 1980].\* A comparison of numerical data from different sources often shows substantial differences (e.g., Fig. 12), and it is difficult to know which to use as a reference. It is probably the case that inadequate handling of the charge singularity at the edges leads to these discrepancies (see Appendix A), and we must await publication of data whose error can be bounded to resolve this point. In their respective ranges of validity, (2.15) and (3.35) are at least as accurate as the uncertainty in the numerical results for the case of a square patch (Figs. 12-13).

\* Dr. E. Arvas [personal communication, 1986] has kindly supplied the author with numerical data for the case shown in Fig. 12. These data turn out to be indistinguishable from those of Benedek and Silvester [1972].

Also shown in Fig. 12 is a computation from an empirical formula due to Wolff [1973]; see also Wolff and Knoppik [1974]:

$$C_n \cong 1 + \left[ \frac{dC_\ell(\epsilon_r, d, a)}{\epsilon_0 \epsilon_r a} - 1 \right] + \left[ \frac{dC_\ell(\epsilon_r, d, b)}{\epsilon_0 \epsilon_r b} - 1 \right] \quad (4.2)$$

where  $a$  and  $b$  are the lengths of the sides of the rectangular patch, and  $C_\ell(\epsilon_r, d, w)$  is the capacitance per unit length of a microstrip line of width  $w$ , substrate thickness  $d$ , and substrate permittivity  $\epsilon_r$ . The idea is that the fringe-field correction due to each parallel set of edges can be accounted for by multiplying the correction of the infinitely long microstrip by the length of the edges. This concept was used by Scott and Curtis [1939] (although, as Wintle and Kurylowicz [1985] point out, Scott and Curtis use an incorrect approximation for  $C_\ell$  which seriously degrades the accuracy of their formula) and by Iossel' et al. [1969, eqn. (4-25)], and is expected to be most accurate when  $a$  and  $b$  are large compared to  $d$ . As can be seen from Fig. 12, the accuracy of (4.2) is indeed acceptable for the square patch for  $d/a \leq 1$ ,\*, but is inadequate for larger  $d$ . The empirical correction due to Bedair [1985] which accounts for corner effects is also shown in Fig. 12, and can be seen to track with the numerical solutions much better for larger  $d/a$ . For such values of  $d/a$ , (2.15) can also be used.

Sharma [1979] and Sharma and Hoefer [1982] have used a spectral-domain Galerkin method to analyze triangular and hexagonal patches. For small  $d/a$ , eqn. (3.35) again gives reasonable agreement with numerical results, but for large  $d/a$ , eqn. (2.15) begins to differ considerably. This seems to be due to the use of basis functions which are entirely inadequate to reproduce the charge singularity at the patch edge, which forms an important part of the total capacitance when  $d/a \geq 1$ .

\* Eqn.(4.2) is excellent in comparison with Benedek and Silvester [1972], while (3.35) tracks more closely with Farrar and Adams [1972 for  $d/a \leq 2$ ]. No further conclusion can be drawn about the accuracy of (4.2) at this point.

Using the aperture-field method, we have successfully generalized Soibel'man's formula (1.10) for the case of a microstrip patch. Our method is much simpler than Soibel'man's method of matched asymptotic expansions, although it is not obvious as to how higher-order terms in the approximation might be obtained in this way, nor how to rigorously establish its asymptotic nature. The success of the method leads us to believe that the same technique can be used to analyze the dynamic properties of an arbitrarily shaped patch. A first step in this direction has already been taken [Kuester et al. 1985], and further research is currently underway.



Fig. 1: Parallel-plate capacitor.

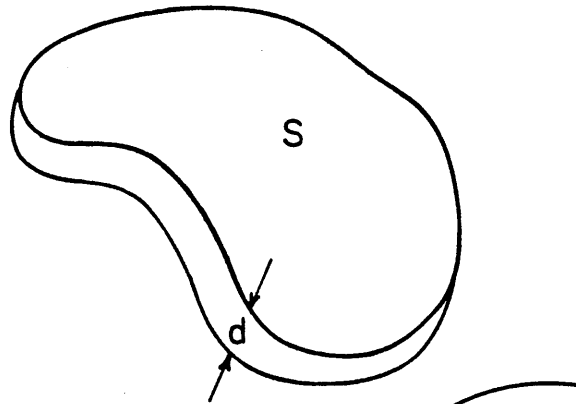


Fig. 2: Fringing fields at the edge of a semi-infinite parallel-plate structure.

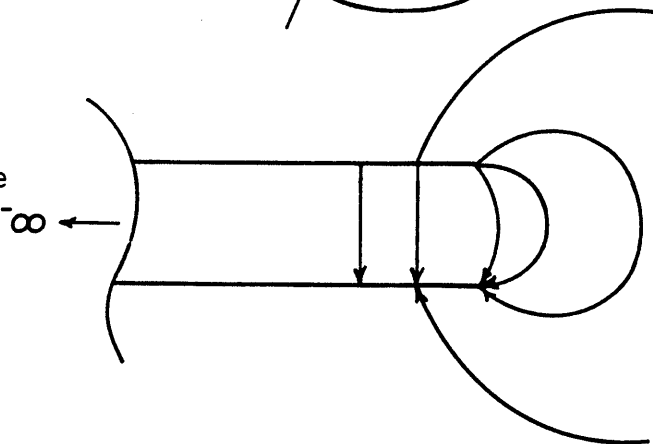
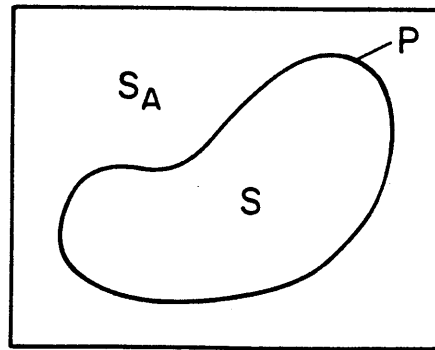
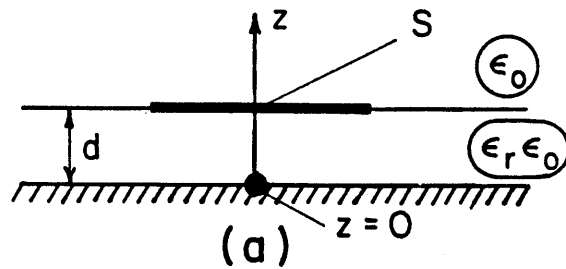


Fig. 3: Microstrip patch of arbitrary shape:  
(a) side view,  
(b) top view.



(b)

Fig. 4: Definition of the minimum normal diameter.

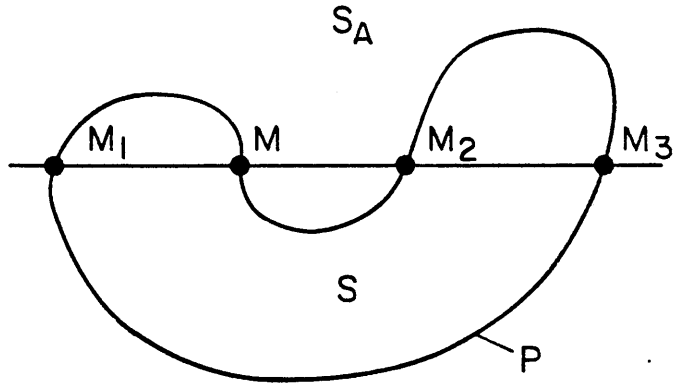


Fig. 5: Regions  $V_1$  and  $V_2$  for Green's theorem.

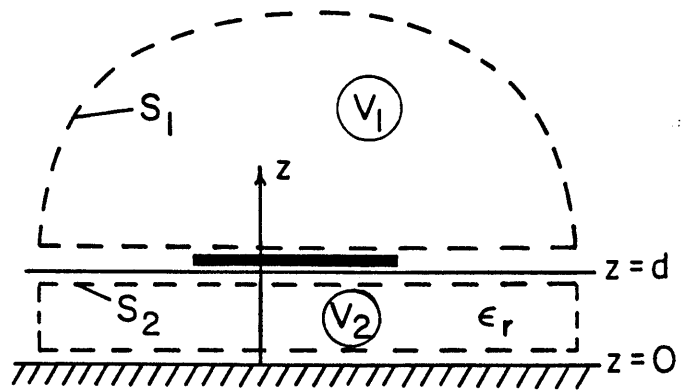


Fig. 6: The  $(\ell, n)$  coordinate system near  $P$ .

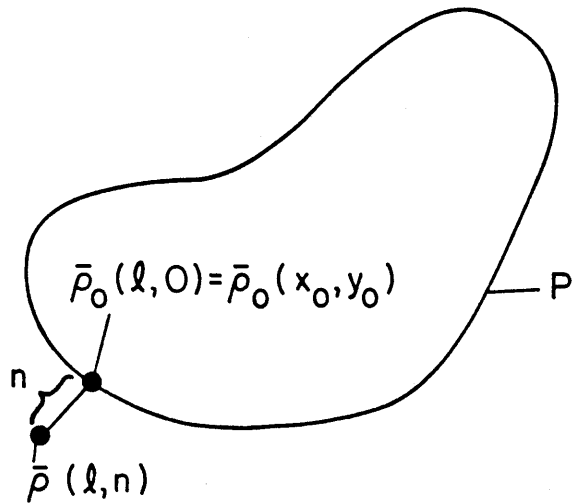


Fig. 7: Geometry for the semi-infinite two-dimensional patch.

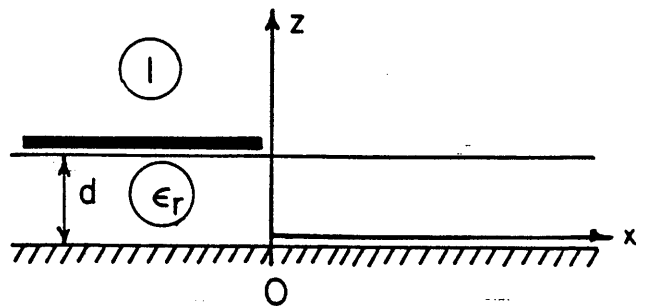
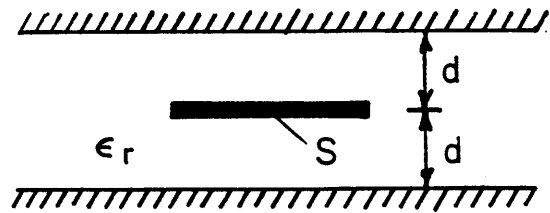


Fig. 8: A triplate patch.



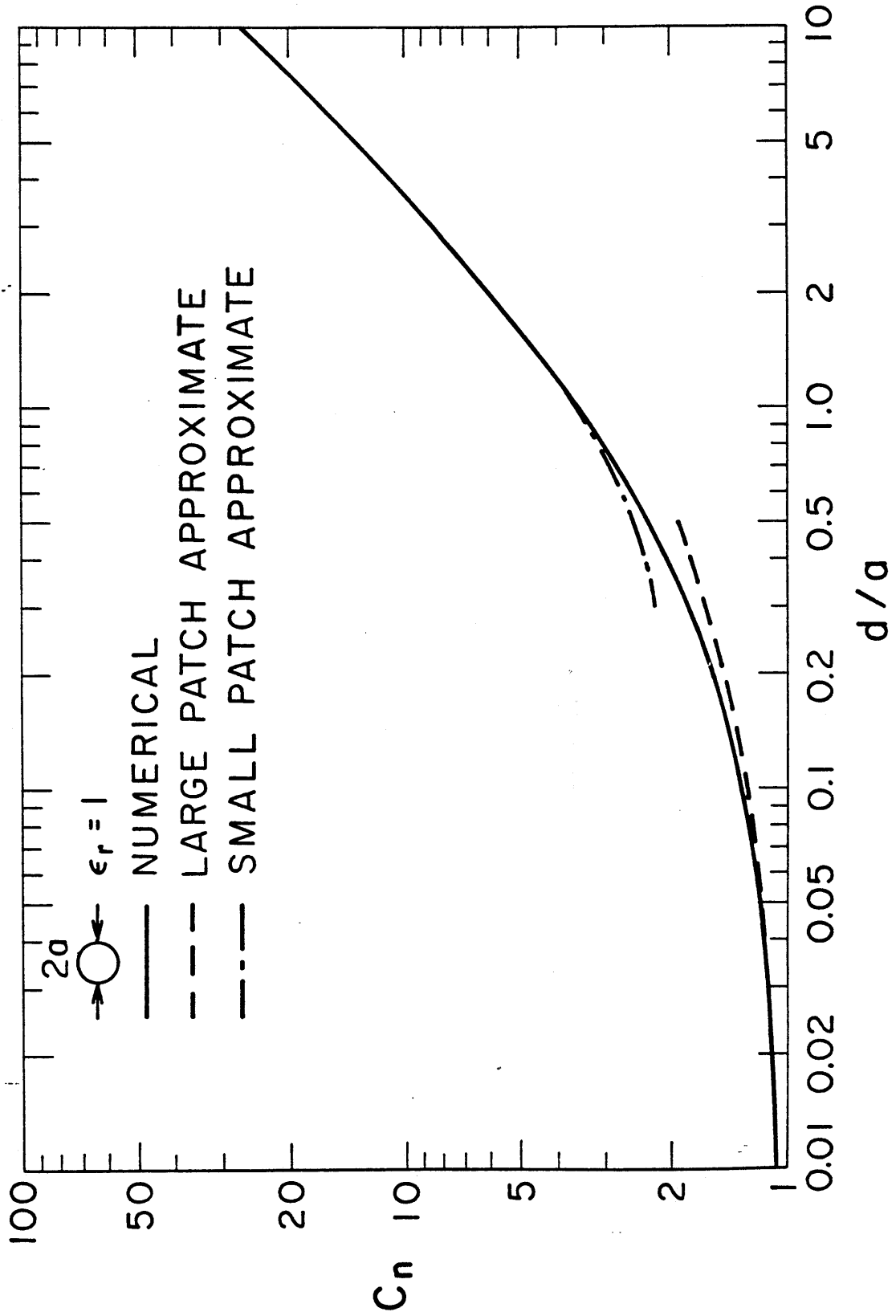


Fig. 9: Normalized capacitance of a circular patch ( $\epsilon_r = 1$ ): numerical values from Leong et al. [1981b] and Hongo and Takahashi [1982].

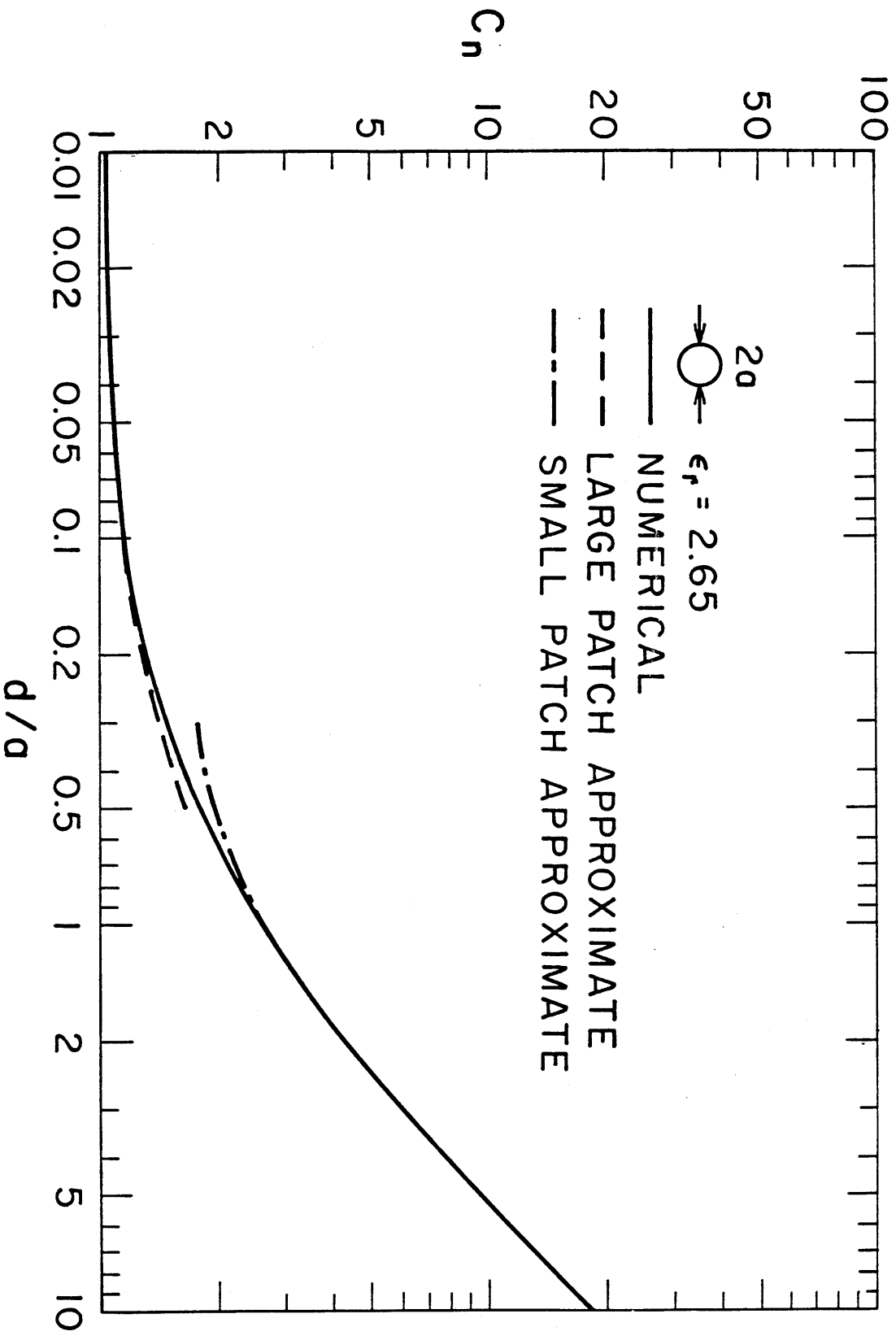


Fig. 10: Normalized capacitance of a circular patch ( $\epsilon_r = 2.65$ ): numerical values from Leong et al. [1981b] and Hongo and Takahashi [1982].

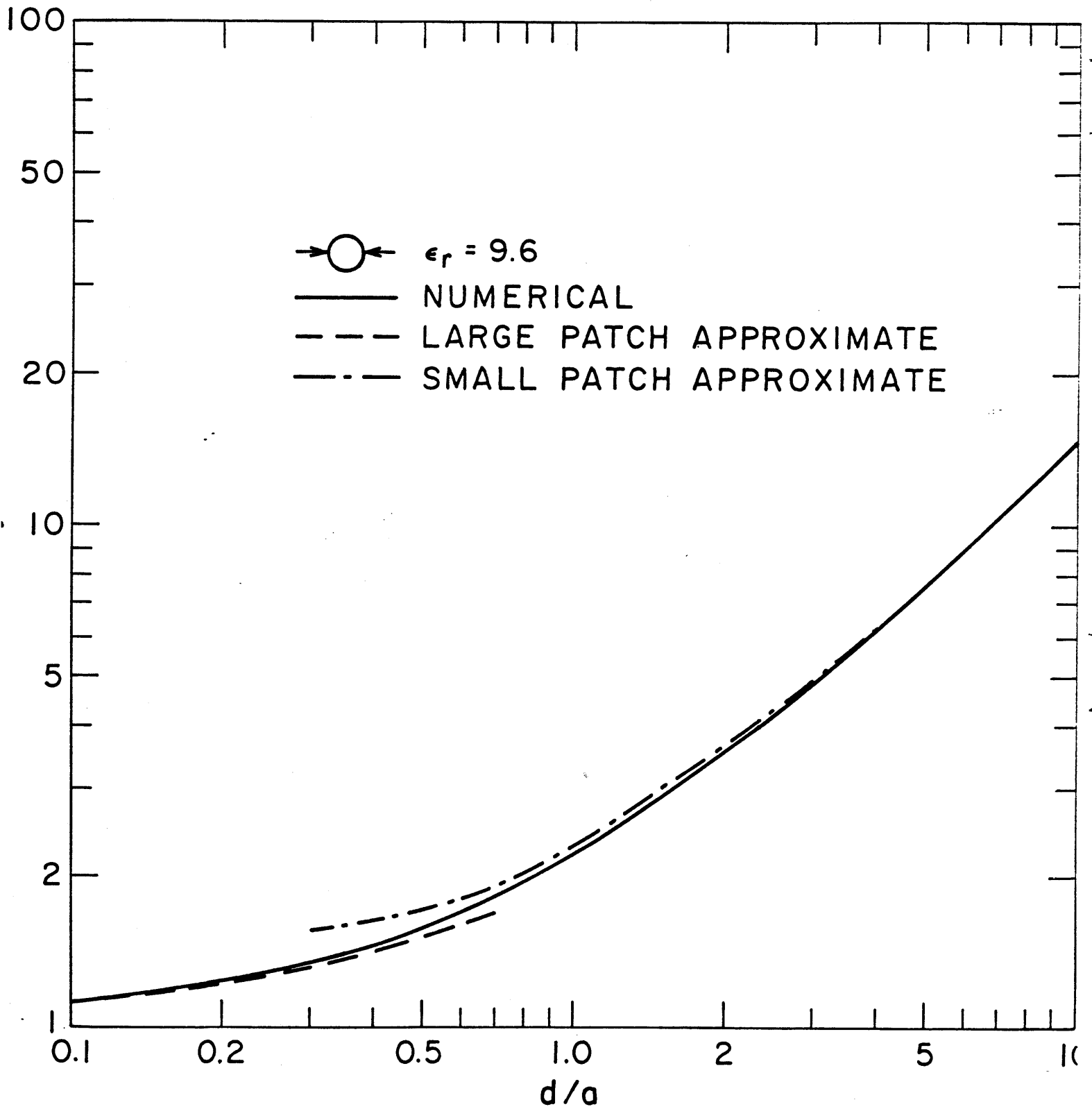


Fig. 11: Normalized capacitance of a circular patch ( $\epsilon_r = 9.6$ ): numerical values from Leong *et al.* [1981b] and Hongo and Takahashi [1982].

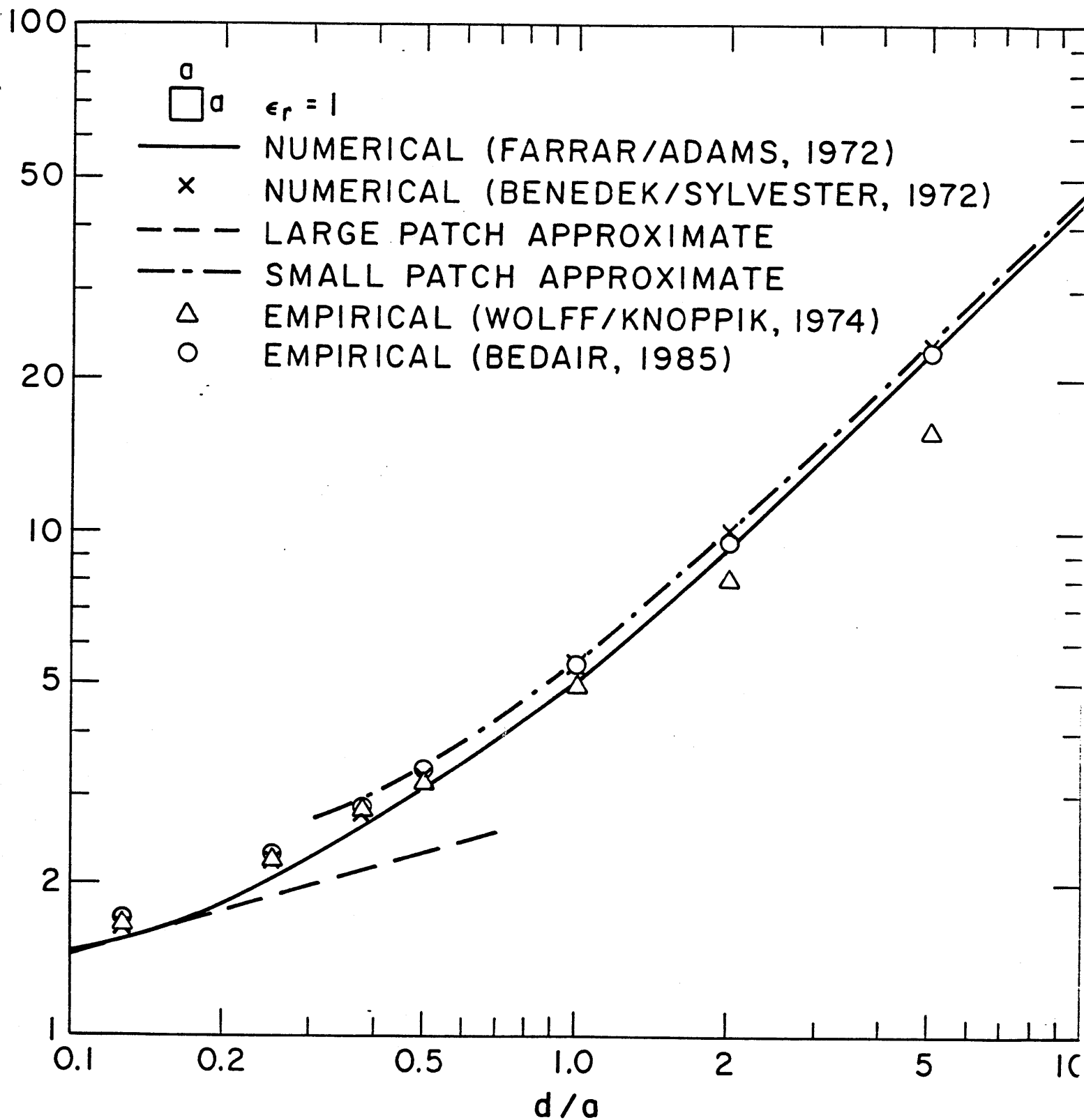


Fig. 12: Normalized capacitance of a square patch ( $\epsilon_r = 1$ ).

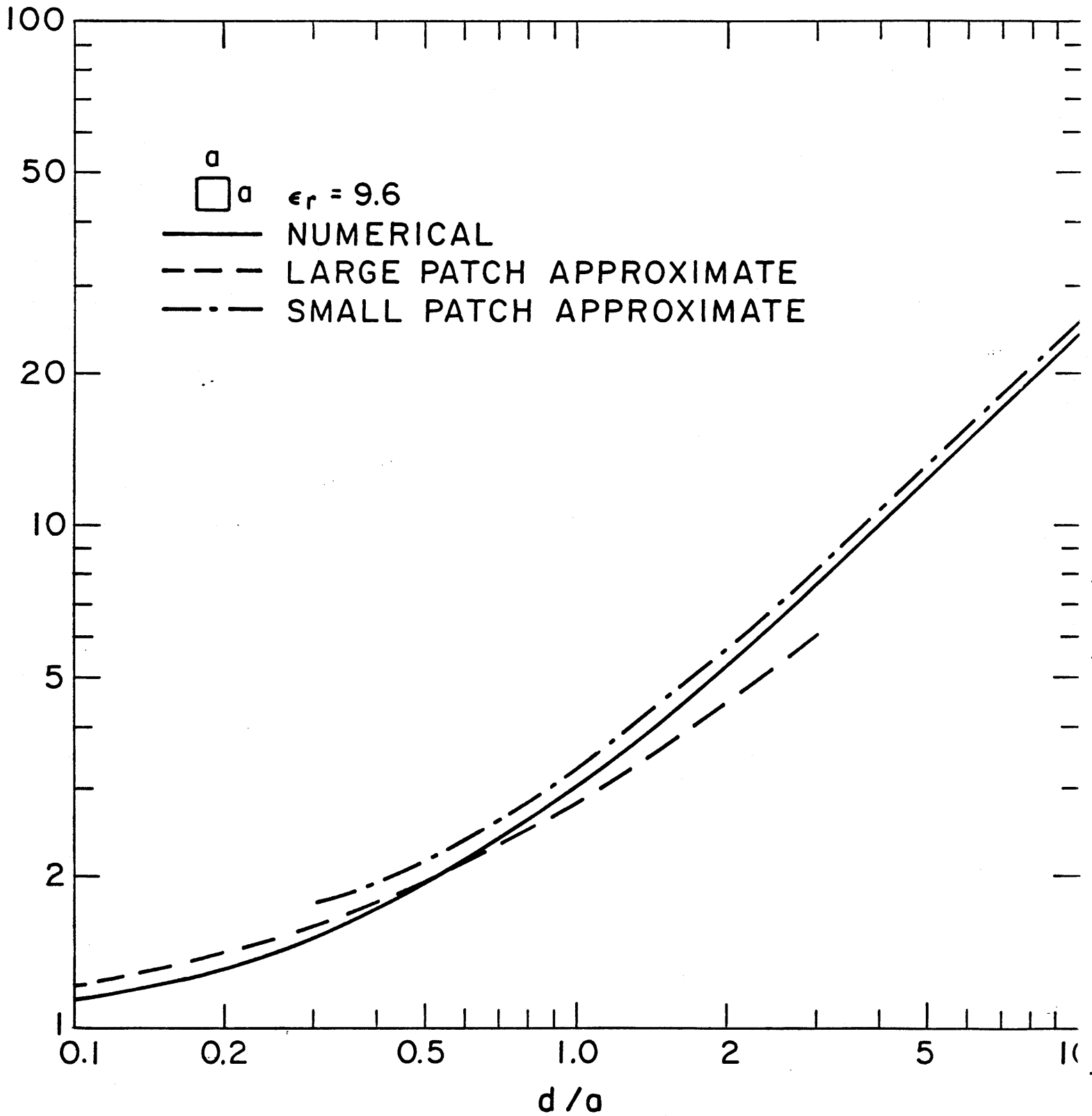


Fig. 13: Normalized capacitance of a square patch ( $\epsilon_r = 9.6$ ).



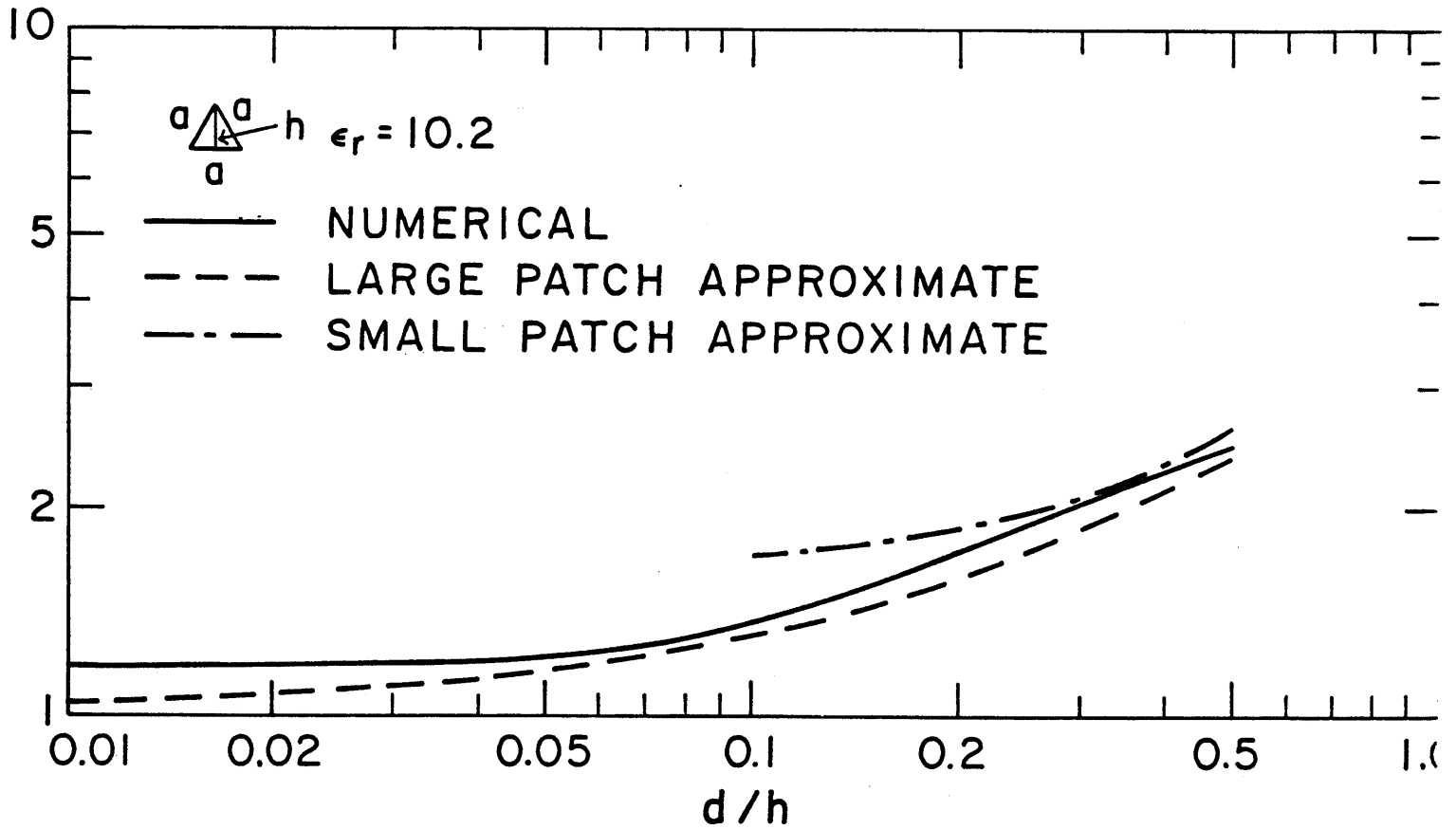


Fig. 14: Normalized capacitance of an equilateral triangular patch ( $\epsilon_r = 10.2$ ): numerical values from Sharma [1979].

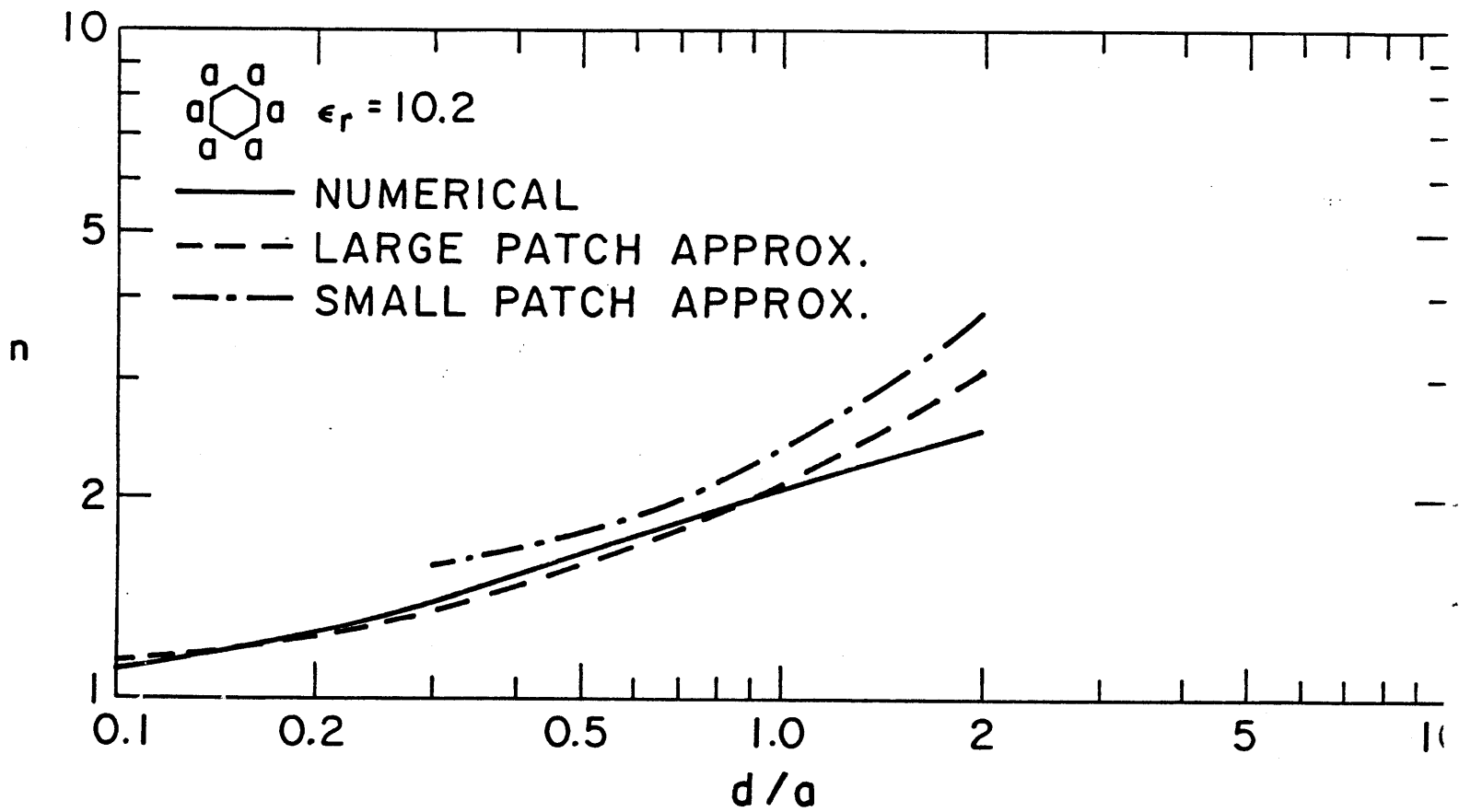


Fig. 15: Normalized capacitance of a regular hexagonal patch ( $\epsilon_r = 10.2$ ): numerical values from Sharma and Hofer [1982].

APPENDIX A

CAPACITANCES OF ISOLATED PLANAR  
DISCS OF VARIOUS SHAPES

The capacitance with respect to infinity,  $C_\infty$ , of planar discs of various shapes has been studied by numerous authors. In this Appendix, we present a summary of the best known results for certain of these shapes. For further results not included here, the reader should consult Pólya and Szegő [1951] or Iossel' et al. [1969].

Pólya and Szegő [1951, p. 3] give the classical result  $C_\infty = 8\epsilon_0 a$  for a circular disc of radius  $a$  in free space. In order not to encumber our formulas, we will give expressions for  $C_\infty$  for a general disc in terms of an equivalent radius  $a_e$  of the disc so that

$$C_\infty = 8\epsilon_0 a_e \tag{A.1}$$

Thus,  $a_e = a$  for a circular disc.

Pólya and Szegő [1951], p. 18] give the isoperimetric inequality

$$a_e \geq \sqrt{\frac{S}{\pi}} \tag{A.2}$$

where  $S$  is the area of the disc. An upper bound for  $a_e$  is also conjectured by them in the same place to be

$$a_e \leq \bar{r} \tag{A.3}$$

where  $\bar{r}$  is the "outer radius" of the disc, defined in terms of a certain conformal mapping, Eqn. (A.3), however, remains unproven.

When the edge of the disc is convex, Iossel' [1980] has proven

$$a_e < \frac{\pi P \gamma / 8}{\ln\left(\frac{1+\gamma}{1-\gamma}\right)}$$

where  $\gamma = \sqrt{1-32S/\pi P^2}$ , and  $P$  is the perimeter of the disc. Other, somewhat more complicated estimates are given by Kazantsev [1983] and Iossel' [1983], but these, like (A.4), are generally not as small as the unproven conjecture (A.3) of Pólya and Szegő, which we will assume here to be true for the shapes we consider.

Some numerical results are available for specific geometries using the method of moments. Reitan and Higgins [1956] compute results for the rectangle in this way, while Harrington [1968, pp. 27-28] and Ivliev and Iossel' [1983] give a result for the square of side  $a$ . De Meulenaere and Van Bladel [1977] provide results for a variety of shapes using a finite-difference method. It might be noted that such numerical results should be viewed with some caution because the charge density at the edge of a plate becomes infinite, and traditional moment methods using point matching do not reproduce this singularity well. As an example, Harrington's best numerical value for  $a_e$  (based on 100 sub-areas on the disc) is  $0.5577a$ , and is actually less than the simple lower bound given by (A.2). Methods have been suggested [Fuller and Chang, 1970]; Ruehli and Brennan, 1973; Balaban, 1973; Bogacheva et al., 1985] for example, to effectively deal with the edge singularity in numerical calculation. Alternatively, the point-matching solution can be markedly improved using a variational technique such as that of Noble [1960]. Okon and Harrington [1981], on the other hand, use a method similar to the Romberg algorithm to estimate the exact capacitance from computations using a finite number of basis functions.

In Table A-1, we present values of the lower bound  $a_\ell$  (A.2) and upper bound  $a_u$  (A.3) for discs of various shapes.\* There are two cases--the circle

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\* Since an exact expression for  $a_u$  in (A.3) does not seem to be available for an arbitrary rectangle, we have used  $a_\ell$  (A.4) for this purpose instead.

and the ellipse--for which exact expressions for  $a_e$  are available. It turns out for both of these that the exact value of  $a_e$  is precisely the arithmetic-geometric mean (AGM) of  $a_\ell$  and  $a_u$ :

$$\text{AGM}(a_\ell, a_u) = \frac{\pi a_u}{2K(\sqrt{1 - a_\ell^2/a_u^2})} \quad (\text{A.5})$$

where  $K$  is the complete elliptic integral of the first kind. It therefore seemed reasonable, when no appropriate numerical data was available, to use (A.5) as "best possible" estimate of  $a_e$ .

**Table A-1**  
**VALUES OF  $a_e$  FOR SOME PLANAR DISCS**

SHAPE	LOWER-BOUND <sup>2</sup> $a_l = \sqrt{S/\pi}$	UPPER-BOUND <sup>2</sup> $a_u = r$	BEST ESTIMATE of $a_e$
Circle, radius a	a	a	$a^1$
Ellipse, sem axes $a \geq b$ ; $e = \sqrt{1-b^2/a^2}$	$\sqrt{ab}$	$\frac{a+b}{2}$	$\frac{\pi a^1}{2K(e)}$
Semicircle, radius a	$a/\sqrt{2} \approx .707a$	.76980a	.7322a <sup>3</sup>
Rectangle, sides a and b	$\sqrt{\frac{ab}{\pi}}$	$\frac{\pi(a+b)\gamma}{4 \ln\left(\frac{1+\gamma}{1-\gamma}\right)}$ ; $\gamma = \sqrt{1 - \frac{8ab}{\pi(a+b)^2}}$ <sup>4</sup>	Fig.A-1 <sup>5</sup>
Square, side a	$a/\sqrt{\pi} = .5642a$	.59017a	.5765a <sup>6</sup>
Equilateral triangle, side a	.3713a	.4218a	.3961a <sup>7</sup>
45°-45°-90° triangle; short side a	.3989a	.47563a	.4364a <sup>7</sup>
30°-60°-90° triangle; long side a	.2625a	.3273a	.2940a <sup>7</sup>
Regular hexagon, side a	.9094a	.92039a	.9149a <sup>7</sup>

<sup>1</sup> Exact<sup>2</sup> Pólya and Szegő [1951] except as noted<sup>3</sup> Okon and Harrington [1981]<sup>4</sup> Eqn.(A.4) [Iossel', 1980].<sup>5</sup> De Meulenaere and Van Bladel [1977].<sup>6</sup> Noble [1960]<sup>7</sup> AGM, eqn. (A.5)

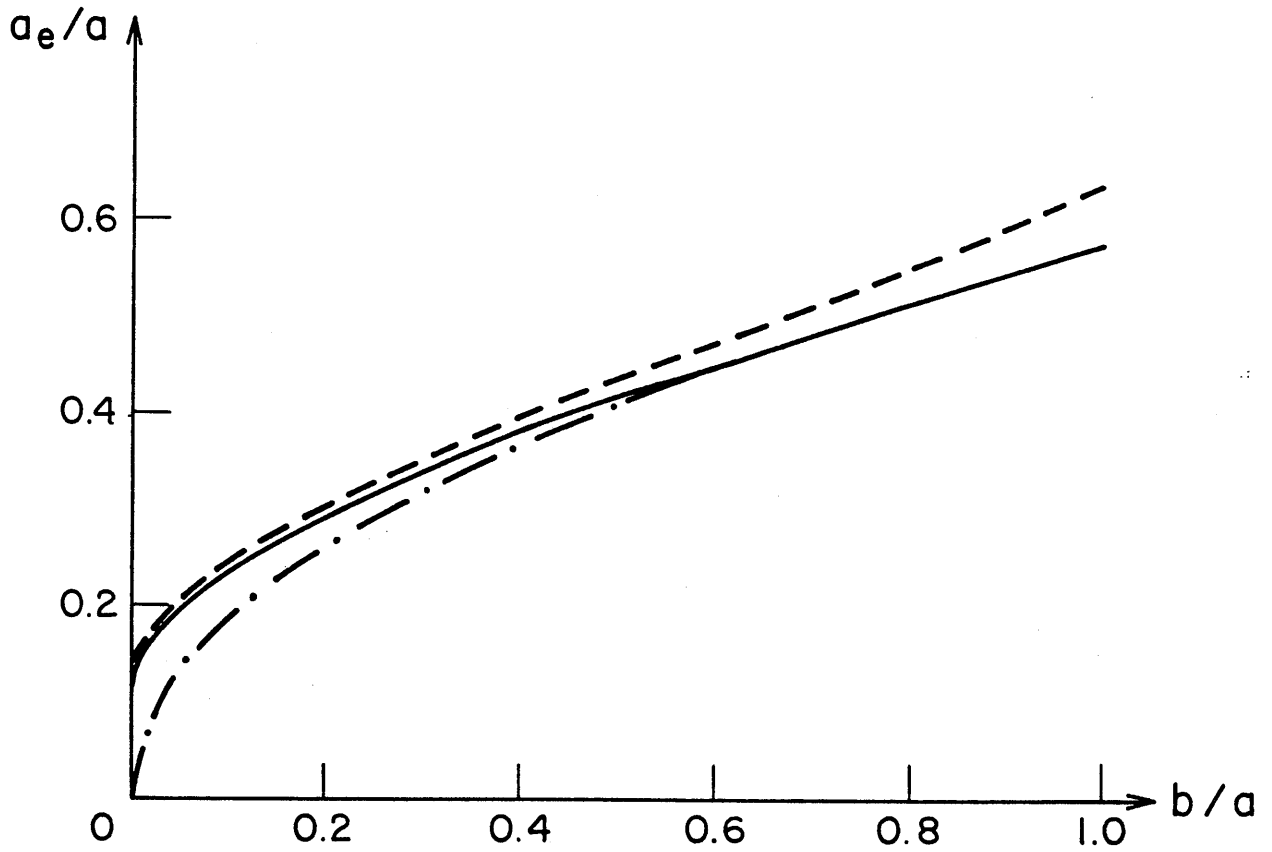


Fig. A-1: Equivalent radius of a rectangular disc of sides  $a$  and  $b$ ;  
 \_\_\_\_\_ Numerical [De Meulenaere and Van Bladel, 1977];  
 - - - -  $\sqrt{\pi/2 \ln(4a/b)}$  — small  $b/a$  approximation;  
 — . —  $0.5765 \sqrt{b/a}$  —  $b \approx a$  approximation.

## APPENDIX B

THE NEUMANN INDUCTANCE FORMULA  
FOR VARIOUS SHAPES

According to the classical result of Neumann (see, e.g., King [1963], pp. 418-421), the mutual inductance  $M$  between two filamentary loops of wire in the shape of a planar path  $P$ , parallel to each other and separated by a distance  $d$  (Fig. B-1) is:

$$M = \frac{\mu_0 P}{4\pi} J \quad (\text{B.1})$$

where the dimensionless quantity  $J$  is given by the double integral

$$J = \frac{1}{P} \oint_P \oint_P \frac{d\bar{\ell} \cdot d\bar{\ell}'}{\sqrt{|\bar{\rho}_0 - \bar{\rho}'|^2 + d^2}} \quad (\text{B.2})$$

Here,  $P$  is used to denote both the path itself as well as the perimeter of the path;  $|\bar{\rho}_0 - \bar{\rho}'|$  is the distance between two points  $\bar{\rho}_0(x_0, y_0)$  and  $\bar{\rho}'(x'_0, y'_0)$  on the path  $P$  (Fig. B-2):

$$|\bar{\rho}_0 - \bar{\rho}'| = \sqrt{(x_0 - x'_0)^2 + (y_0 - y'_0)^2} \quad (\text{B.3})$$

According to Maxwell [1954, vol 2, pp. 328-330] (see also Tseitlin [1946]), eqn. (B.1) also represents the self-inductance of a loop of wire of circular cross-section (of radius  $d$ ) whose axis follows the path  $P$ , if  $d$  is small compared to the characteristic dimensions of the loop (radius of curvature, minimum normal diameter-see section III). In this case, we can obtain an expression for  $J$  correct to leading order in  $d$  as follows. Let us write  $d\bar{\ell} \cdot d\bar{\ell}' = \bar{a}_n \cdot \bar{a}'_n d\ell d\ell'$ , where  $\bar{a}_n$  and  $\bar{a}'_n$  are outward unit normal vectors to  $P$  in the  $xy$ -plane as shown in Fig. B-2. Moreover, let



$$\frac{\bar{a}_n \cdot \bar{a}'_n}{\sqrt{|\bar{\rho}_o - \bar{\rho}'_o|^2 + d^2}} = \left\{ \frac{\bar{a}_n \cdot \bar{a}'_n}{\sqrt{|\bar{\rho}_o - \bar{\rho}'_o|^2 + d^2}} - \frac{\pi/P}{\sqrt{\sin^2\left[\frac{\pi(\ell - \ell')}{P}\right] + (\pi d/P)^2}} \right\} + \frac{\pi/P}{\sqrt{\sin^2\left[\frac{\pi(\ell - \ell')}{P}\right] + (\pi d/P)^2}} \quad (\text{B.4})$$

The term in curly brackets in (B.4) can be well approximated by setting  $d = 0$  (there is no singularity at  $\ell = \ell'$ ), while it proves possible to integrate the second term exactly:

$$\begin{aligned} & \int_P \frac{(\pi/P)d\ell'}{\sqrt{\sin^2\left[\frac{\pi(\ell - \ell')}{P}\right] + (\pi d/P)^2}} \\ &= \frac{\pi}{P} \int_{\ell - \frac{P}{2}}^{\ell + \frac{P}{2}} \frac{d\ell'}{\sqrt{\sin^2\left[\frac{\pi(\ell - \ell')}{P}\right] + (\pi d/P)^2}} \\ &= \int_{-\pi/2}^{\pi/2} \frac{dv}{\sqrt{\sin^2 v + (\pi d/P)^2}} \\ &= \frac{2}{\sqrt{1 + (\pi d/P)^2}} K\left(\frac{1}{1 + (\pi d/P)^2}\right) \end{aligned} \quad (\text{B.5})$$

by eqn. (282.00) of Byrd and Friedman [1971], where  $K$  is the complete elliptic integral of the first kind. For  $d/P \ll 1$ , the value of  $K$  is approximately [Byrd and Friedman, 1971, eqn. (112.01)]:

$$K\left(\frac{1}{1 + (\pi d/P)^2}\right) \cong \ell n \frac{4P}{\pi d} \quad (\text{B.6})$$

Therefore, approximating  $d/P$  by 0 in the curly brackets of (B.4), our expression for  $J$  in (B.2) becomes

$$J \cong 2 \ln \frac{4P}{\pi d} + J_p \quad (\text{B.7})$$

where

$$J_p = \frac{1}{P} \oint_P \oint_P \left[ \frac{\bar{a}_n \cdot \bar{a}_n'}{|\bar{\rho}_o - \bar{\rho}_o'|} - \frac{\pi/P}{|\sin \frac{\pi(\ell - \ell')}{P}|} \right] d\ell d\ell' \quad (\text{B.8})$$

is a parameter depending only on the shape of  $P$ , and not its length. From results given by Grover [1946], chapter 8, and King [1963], pp. 427-431, we compile the results shown in Table B-1 for the values of  $P$ ,  $J_p$  and the area  $S$  of various plane curves. Formulas for  $J$  for the case of an ellipse have been given by Cooke [1963].

**TABLE B-1**  
**VALUES OF P, S, AND  $J_p$**   
**FOR SOME PLANE CURVES**

CURVE	P	S	$J_p$
Circle of radius a	$2\pi a$	$\pi a^2$	-4
Equilateral triangle of side a	$3a$	$\frac{\sqrt{3}a^2}{4}$	-5.49127
Regular hexagon of a side a	$6a$	$\frac{3\sqrt{3}a^2}{2}$	-4.36969
Rectangle of sides a and b; the diagonal h = $\sqrt{a^2+b^2}$	$2(a+b)$	ab	$-4\left(1 - \frac{h}{a+b}\right)$ $+ \frac{2}{a+b} \left\{ a \ln \left[ \frac{\pi a(h-a)}{4b(a+b)} \right] + b \ln \left[ \frac{\pi b(h-b)}{4a(a+b)} \right] \right\}$

Fig. B-1: Two parallel planar loops of wire separated by a distance  $d$ .

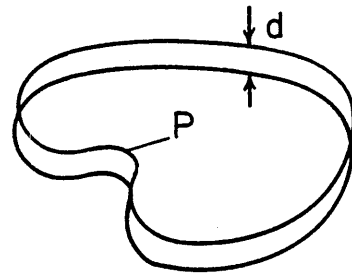
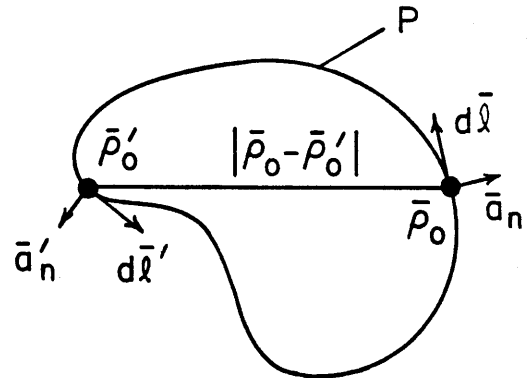


Fig. B-2: The points  $\bar{\rho}_o$  and  $\bar{\rho}'_o$  on the curve P, and the differential elements  $d\bar{\ell}$ ,  $d\bar{\ell}'$  and unit normals  $\bar{a}_n$ ,  $\bar{a}'_n$ .



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