Implicit hybrid simulation of magnetic reconnection and the ion-temperature-gradient-driven instability

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Implicit hybrid simulation of magnetic reconnection and the ion-temperature-gradient-driven instability

by

Jianhua Cheng

B.A., University of Science and Technology of China, 2005

A thesis submitted to the
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has been approved for the Department of Physics

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Date ______________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
Cheng, Jianhua (Ph.D., Physics)

Implicit hybrid simulation of magnetic reconnection and the ion-temperature-gradient-driven instability

Thesis directed by Prof. Scott E. Parker

This thesis presents a second-order accurate, semi-implicit $\delta f$ hybrid simulation with Lorentz force ions and fluid electrons. This model includes full ion kinetic effects and is suitable for studying magnetohydrodynamics (MHD) scale physics. We report the first particle-in-cell simulation of nonlinear ion Landau damping of ion acoustic waves, and the results agree with theory. The numerical damping associated with the implicit time advance is also analyzed. We have investigated the full evolution of resistive tearing mode. The linear growth rate is in reasonable agreement with resistive MHD theory. The nonlinear growth and saturation stage has been observed and compared quantitatively with the resistive MHD theory. The simulation shows that current sheets of large aspect ratio tend to develop multiple islands and eventually coalesce to a single elongated island. During this process, significant ion heating inside the island was observed. The simulation shows that over 50% of the dissipated magnetic energy is converted into the kinetic energy of ions for a current sheet with sufficiently large aspect ratios, which is comparable with previous experimental measurements. We also compared the simulation to the extended-MHD NIMROD code for the ion-temperature-gradient-driven instability. The hybrid kinetic and fluid calculations agree well near the marginal stability point, but disagree as $k_\perp \rho_i$ or $\rho_i/L_T$ increases where the kinetic effects become important. Good agreement between the models for the shape of the unstable global eigenfunction is reported. The results help quantify how far fluid calculations can be extended accurately into the kinetic regime.
Dedication

To my parents.
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Chapter 1

Introduction

Magnetic fields are observed at all scales in the universe, from the magnetic dipole field at subatomic scales to solar corona and the interstellar medium. It is, therefore, important to understand how magnetic fields affect the characteristics of plasmas. Magnetic reconnection, a process involving topological rearrangement of magnetic field lines, has been proposed as a key mechanism for understanding the interaction between magnetic fields and plasmas. Magnetic reconnection has been observed in various space and laboratory plasmas such as solar flares, Earth’s magnetosphere, and sawtooth crash in tokamaks (see reviews in [1, 2, 3]). Take solar flares as an example, through X-ray pictures, we can visualize the topology change of magnetic field-line configurations [4]. As shown in transition region and coronal explorer (TRACE) satellite data (Figure 1.1), the topologies of X-ray images are observed to change within a time scale of minutes or hours in the solar atmosphere where the magnetic diffusion time scale is about $10^6$ years. These observations indicate the existence of an efficient channel from which the abundant magnetic field energy can be converted to kinetic energy of particles. In the 1950s, Sweet and Parker suggested that magnetic reconnection can provide such a mechanism for energy conversion [5, 6]. Since then, the theory of magnetic reconnection has attracted extensive research efforts. In this chapter, I first present a short review on the magnetic reconnection theory is given. And then I discuss the organization of this dissertation.
1.1 A short history of the magnetic reconnection theory

We start from the ideal magnetohydrodynamic (MHD) descriptions of the plasma, where the bulk plasma flow is determined by the $E \times B$ drift

$$u = \frac{E \times B}{B^2}. \quad (1.1)$$

This is equivalent to Ohm’s law

$$E + u \times B = 0. \quad (1.2)$$

Taking the curl of Ohm’s law together with Faraday’s law

$$\frac{\partial B}{\partial t} = -\nabla \times E, \quad (1.3)$$

we have

$$\frac{\partial B}{\partial t} - \nabla \times (u \times B) = 0. \quad (1.4)$$
Now we consider the time behavior of the magnetic flux, $\Phi$, through a closed curve $C$ around an open surface $S$ as depicted in Figure 1.2. The magnetic flux is defined as

$$\Phi = \int_S B \cdot dA. \quad (1.5)$$

The time variation of the magnetic flux $\Phi$ consists of two parts: the time variation of the magnetic field $B$ and the change of the curve $C$ due to plasma motion $u$. The total time derivative of the magnetic flux is

$$\frac{d\Phi}{dt} = \int_S \frac{\partial B}{\partial t} \cdot dA - \int_C u \times B \cdot dl. \quad (1.6)$$

Using Stokes’ theorem and Eq. (1.4), Eq. (1.6) becomes

$$\frac{d\Phi}{dt} = \int_S \frac{\partial B}{\partial t} \cdot dA - \int_S \nabla \times (u \times B) \cdot dA = 0. \quad (1.7)$$

This means that the magnetic flux through a closed fluid element remains constant. Therefore, the magnetic field lines move with the fluid and the magnetic fields are said to be “frozen-in” to the plasma. A direct consequence of the “frozen-in” condition is that two crossed or sheared flux tubes can not pass through each other, which means no topological change of the field lines is
allowed. Therefore, the change of the topology of the magnetic field lines associated with magnetic reconnection breaks the “frozen-in” condition.

Generally, there are three types of reconnection which are determined by the mechanisms leading to the breakdown of the “frozen-in” condition (Figure 1.3). The first one is the resistive reconnection. In a collisional plasma, the presence of electron-ion collisions will dissipate the current and convert magnetic energy into heat through Ohmic heating. This effect is represented by the resistivity \( \eta = m_e \nu_{ei} / n e^2 \), where \( \nu_{ei} \) stands for the electron-ion collision frequency. The second one is inertial reconnection, which is associated with the finite mass of electrons. The third one also occurs in collisionless plasma but is caused by the anomalous resistivity associated with the turbulence and particle-wave interactions. Magnetic reconnection is an important process that breaks the magnetic topology and releases the magnetic energy into the particle kinetic energy of the plasma. It is important in various plasma phenomena such as solar flares, coronal heating, and magnetic sub-storms in the earth magnetosphere [1, 2, 3]. Two key physics issues in magnetic reconnection are (1) mechanisms causing magnetic reconnection occur spontaneously and (2) the way that the magnetic energy is converted to particle kinetic energy of the plasma. The first issue, the so-called spontaneous magnetic reconnection, is widely believed to be initiated by some instabilities, the most relevant one being the tearing mode instability [7]. The second issue, often referred as the energy conversion mechanism, however, remains unclear because of the limited accessibility of reconnection events in both naturally occurring and laboratory plasmas. The details of this process are fundamental to plasma physics since reconnection is a virtually unavoidable process in all magnetized plasmas. Since local reconnection changes global magnetic field topology, overall plasma equilibrium and confinement properties can be profoundly altered.
In this thesis, we focus on the resistive reconnection. In the presence of a resistivity $\eta$, Ohm’s law becomes

$$E + u \times B = \eta j.$$  \hfill (1.8)

Taking the curl and using Faraday’s law yields

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \frac{\eta}{\mu_0} \nabla^2 B.$$  \hfill (1.9)

Suppose we have a one dimensional current sheet separating two parts of sheared magnetic fields as depicted in Figure 1.4. Near the center of the sheet, $B \to 0$, Eq. (1.9) is reduced to

$$\frac{\partial B}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 B.$$  \hfill (1.10)

This is simply the diffusion equation of the magnetic field, which is characterized by the resistive time scale $\tau_\eta = (\frac{\eta}{\mu_0 \delta^2})^{-1}$. This dissipation annihilates the magnetic fields in the center of the current and reconnects the oppositely directed field lines, leading to an $x$-line configuration. The newly reconnected field lines are bent and the magnetic tension forces the plasma away from the $x$-line. As the density in the center lowers, more plasma from the upstream flows in and the process continues. This process presents a simple picture of magnetic reconnection. In Figure 1.4, the diffusion region (dashed box) is $2L$ long and $2\delta$ wide, where $L \gg \delta$. For simplicity, we assume that the system is symmetric. The top and bottom represents the inflow region and the left and right represents the outflow region where the plasma is pushed away from the $x$-point. 

Figure 1.3: Schematic review of magnetic reconnection.
Figure 1.4: A schematic plot of the Sweet-Parker reconnection model [8]. The dashed box stands for the diffusion region where field lines reconnect.

Following the Sweet and Parker’s analysis [5, 6], for a steady state, the electric field is uniform and points out of the plane. In the inflow region,

$$E \sim u_{in} B_0,$$  \hspace{1cm} (1.11)

where $B_0$ is the asymptotic magnetic field strength far from the diffusion region. Inside the diffusion region, the magnetic field vanishes, and Ohm’s law becomes

$$E \sim \eta j.$$  \hspace{1cm} (1.12)

If the current thickness is taken as $2\delta$, Ampere’s law gives

$$j \sim \frac{B_0}{\mu_0 \delta}.$$  \hspace{1cm} (1.13)
Combining the above three equations, we have the approximate current thickness

$$\delta \sim \frac{\eta}{\mu_0 u_{in}}.$$  \hspace{1cm} (1.14)

Assuming the plasma is incompressible, the conservation of mass gives

$$u_{in} L = u_{out} \delta.$$  \hspace{1cm} (1.15)

The electromagnetic energy inflow rate per unit area is given by the Poynting flux

$$|S| = |E \times H| = \frac{EB_0}{\mu_0} = \frac{u_{in} B_0^2}{\mu_0}.$$  \hspace{1cm} (1.16)

Because the outflow plasma takes the kinetic energy away from the diffusion region ($u_{out} \gg u_{in}$), the change rate of energy per unit area in the incident flow is

$$\frac{1}{2} nm u_{in} (u_{out}^2 - u_{in}^2) \sim \frac{1}{2} nm u_{in} u_{out}^2.$$  \hspace{1cm} (1.17)

To enforce energy conservation, equating Eq. (1.16) and (1.17), we have

$$u_{out}^2 \sim 2 V_A^2,$$  \hspace{1cm} (1.18)

where $V_A = B_0^2/\mu_0 nm$ is the asymptotic Alfvén speed. The inflow speed $u_{in}$ follows Eq. (1.15)

$$u_{in} \sim \frac{\delta}{L} V_A.$$  \hspace{1cm} (1.19)

Combining with Eq. (1.14), we have

$$u_{in} \sim S^{-1/2} V_A,$$  \hspace{1cm} (1.20)

where $S = \sqrt{\frac{\mu_0 L V_A}{\eta}}$ is the Lundquist number. Unfortunately, for solar flares, the Lundquist number can be very large ($> 10^{12}$), resulting in the reconnection rate ($u_{in}/u_{out}$) to be very slow compared to observations. For $S = 10^{12}$, Sweet-Parker model predicts a reconnection time of about 3 months, which is far longer than the observed 10s of minutes from satellite data. This huge discrepancy has prompted scientists to explore new mechanisms to explain fast reconnection. For example, Petschek proposed a new scheme where most of the plasma does not need to flow through the diffusion region in order to be accelerated [9]. Instead, it can be accelerated through the standing shocks that are
connected to the diffusion region. These shocks can squeeze the dissipation region into very small scale, allowing $\delta/L \sim 1$. Therefore, much faster reconnection rates can be achieved. However, the Petschek model has been challenged by theory [10] and numerical simulations in the presence of uniform resistivity [11, 12]. Recent progress suggests that Hall processes and fluctuation-induced anomalous-resistive processes may provide answers to the physics of fast reconnection [13, 14]. More detailed discussion can be found in the review article of M. Yamada et al. [3].

1.2 Organization of the thesis

In the study of magnetic reconnection, because of the fast development of supercomputers, computer simulation has assumed a dominant role in addition to the laboratory experiments and space observation from satellites. In this thesis, we present a hybrid simulation of magnetic reconnection. We focus on two fundamental questions in magnetic reconnection:

(1) Why does magnetic reconnection occur spontaneously?

(2) How is the magnetic energy converted to thermal energy during the process of magnetic reconnection?

In Chapter 2, we present a “current closure” hybrid model with Lorentz force ions and fluid electrons along with the model equations and the numerical schemes employed. For benchmarks, we have studied the shear and compressional Alfvén waves, whistler waves and ion acoustic waves including ion Landau damping. The simulations are in good agreement with theory. In particular, we have performed simulations of the nonlinear Landau damping of an ion acoustic wave, which to our knowledge has not been reported in the literature. The numerical damping associated with the implicit time advance is discussed in Section 2.4.3 concerning whistler waves. The simulation agrees well with the derived numerical dispersion relation.

In Chapter 3, we present the full evolution of the resistive tearing mode, which can trigger spontaneous magnetic reconnection [7]. In the presence of resistivity, the current sheet will break into a series of current filaments due to the instability of resistive tearing mode. As a result,
magnetic islands start to build up and magnetic reconnection proceeds. Our simulation shows reasonable agreement with the resistive MHD theory [15, 16] on the linear growth rate and eigenmode structure of tearing mode. Important nonlinear tearing mode dynamics such as the Rutherford stage and saturation are also captured. The Rutherford growth rate and saturation island width are consistent with previous MHD studies [17, 18, 19, 20, 21, 22].

In Chapter 4, we show island coalescence in a current sheet with large aspect ratios. In this process, we have observed ion heating using various ion diagnostics. It is also shown that the amount of energy converted to ion kinetic energy increases as the aspect ratio becomes large, which means the energy conversion is more efficient in a thinner current sheet. Our simulation shows over 50% dissipated magnetic energy goes to the kinetic energy of ions for a sufficiently large aspect ratio current sheet, which is consistent with the experimental measurements. The ion pressure shows significant anisotropy around the $x$-points, which may be attributed to the elongation of the islands leading to coalescence.

In Chapter 5, we present the kinetic theory of the slab ion-temperature-gradient-driven instability (ITG) in its linear stage with the “local approximation” which holds when the characteristic length scale associated with the temperature gradient is small compared to the length scale of the system. For the electrostatic case, the local kinetic dispersion relation and the corresponding form in the fluid limit are derived. The Hybrid simulation agrees well with the kinetic dispersion relation. We have also compared the simulation to the extended MHD calculation and the NIMROD simulation [23]. The hybrid kinetic and fluid calculations agree well near the marginal stability point, but diverge as $k_L \rho_i$ or $\rho_i/L_T i$ increases where the kinetic effects becomes important. Good qualitative agreement between the models for the shape of the unstable global eigenfunction is reported. The results quantify how far fluid calculations can be extended accurately into the kinetic regime for the case of linear ITG. For the electromagnetic case, the local dispersion relation is obtained and some preliminary global simulation results are presented. We have observed clear linear growth and good eigenfunction structure for the perturbed density. Further comparison with the derived dispersion relation is yet to be carried out.
Possible implementation schemes in NIMROD (NonIdeal MHD with Rotation - Open Discussion) is discussed in Chapter 6, and conclusions are drawn in Chapter 7.
Chapter 2

Simulation model and algorithm

2.1 Computer simulation of plasmas

A plasma is a gas in which a significant fraction of the atoms are ionized, so that the electrons and ions are separately free. When these ions and electrons interact with electromagnetic fields, plasmas exhibit rich physics which has drawn intensive research interests. Traditional approaches to studying plasma physics include experiments (or observations) and theoretical analysis based on fundamental physical laws. Over the past five decades, due to the enormous gains in computing power, computer simulation has become more and more important in plasma research. Naively, using computer simulations, one can integrate the dynamics for every single particle moving in the electromagnetic fields governed by Maxwell’s equations. Unfortunately, dealing with the huge number of particles is simply beyond the capability of today’s computers. To make practical simulations of the system, various simplified models have been suggested [24]. Based on hierarchies of physics content, most simulations fall into categories of kinetic, hybrid and fluid models.

Let us start with the usual basis for analytic treatments of a collisionless plasma (collision can be included by introducing appropriate collision operators), the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0,$$

(2.1)

where $f(x, v, t)$ is the particle distribution function, and the electric and magnetic fields $\mathbf{E}, \mathbf{B}$ are determined by Maxwell’s equations:

$$\nabla \cdot (\epsilon \mathbf{E}) = \sigma,$$

(2.2)
\[ \nabla \times \mathbf{B} = \mu j + \mu \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{2.3} \]
\[ \nabla \cdot \mathbf{B} = 0, \tag{2.4} \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.5} \]

where \( \sigma \) and \( j \) are the charge density and current density. \( \varepsilon \) and \( \mu \) represent permittivity and permeability, respectively. The Vlasov equation represents a hyperbolic partial differential equation in a six-dimensional phase space plus time. To fully resolve the phase space requires a huge amount of computer memory. Thus directly solving the Vlasov equation is often restricted to two or three phase space dimensions. A more practical approach is using macroparticles to represent the particle distribution function \( f \), which is equivalent to solving the Vlasov equation by the characteristics method. To avoid the computation cost of calculating Coulomb interactions pairwise, a spatial grid on which particles’ charges and current densities are deposited through an interpolation scheme is introduced. The field equations are solved on this grid, and forces acting on the particles are obtained by interpolating the fields back to the particles’ positions. This is the particle-in-cell (PIC) method. Generally, PIC models perform well when the fields vary slowly compared to grid sizes. Detailed reviews of the PIC simulation can be found in [25, 26].

In kinetic simulations, both ions and electrons are treated as particles whose motions in the self-consistent electric and magnetic fields are strictly followed. The fields are determined by Maxwell’s equations. Such a model covers the full range of collisionless plasma physics and represents the most accurate description of real plasmas of all simulation models. However, due to the fine length (the Debye length \( \lambda_D \)) and time scales (the gyro-frequency \( \Omega_{i,e} \)) involved, the kinetic model is often restricted by the simulation domain and ion to electron mass ratio. In some special circumstances, when the plasma is strongly magnetized, the Vlasov equation can be reduced to the gyrokinetic equation which has only five phase space dimensions [27]. The corresponding gyrokinetic particle simulations were invented by Lee [28, 29]. However, the gyrokinetic equations rely on the validity of the gyrokinetic ordering, and their application is often limited when studying the tokamak edge and magnetic reconnection. In the fluid approach, the equations are obtained by
taking moments of the kinetic equation. The particle density, flow velocity, pressure, stress tensor, and heat flux are defined as

\[ n = \int f d^3v, \]  
\[ n u = \int v f d^3v, \]  
\[ P = (\Gamma - 1) \frac{1}{2} m \int (v - u)(v - u) f d^3v, \]  
\[ \Pi = m \int v v f d^3v, \]  
\[ q = \frac{1}{2} m \int (v - u)^2(v - u) f d^3v, \]

where \( \Gamma \) is the ratio of specific heats. With these quantities, we can write the corresponding continuity equation, momentum equation, and energy equation as

\[ \frac{\partial n}{\partial t} + \nabla \cdot (n u) = 0, \]  
\[ n m \frac{d u}{d t} = n q (E + u \times B) - \nabla \cdot \Pi, \]  
\[ \frac{n}{\Gamma - 1} \frac{\partial T}{\partial t} + \frac{n}{\Gamma - 1} u \cdot \nabla T + n T \nabla \cdot u + \Pi : \nabla u + \nabla \cdot q = 0. \]

The closure of this set of equations requires higher moments. This has been discussed in [30]. The fluid approach greatly reduces the complexities in the kinetic model, and it can be used to explore long time plasma phenomena for larger systems. However, the fluid equations eliminate the velocity space information and important kinetic effects may be missing.

For modeling low frequency phenomena, one plausible approach is to include kinetic ions within an MHD model [31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]. In one kinetic MHD hybrid approach [31, 32], a multi-fluid model is closed with a pressure tensor calculated from the particle distribution function, while the ion density and flow velocity are calculated through the continuity equation and the ion momentum equation as in Eq. (2.11) and (2.12). In this scheme, ions are usually divided into two parts: a core bulk background component and a hot component. The hot component has a relatively smaller density compared to the bulk component, but has a much higher temperature leading to a pressure comparable to the bulk pressure. Therefore, it is
natural to treat the bulk plasma as fluid and the hot ions as particles. The kinetic effects from hot ions are captured by including the pressure calculated from the particle distribution function in the momentum equation. Another approach, the so-called “current closure”, uses Lorentz force ions and fluid electrons. The ion density and ion current density are calculated directly from the ion distribution function governed by the Vlasov equation [33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]. In typical hybrid models, the electric and magnetic fields are calculated from the generalized Ohm’s law, Faraday’s law, and Ampere’s law in which the displacement current is often neglected [46]. The generalized Ohm’s law comes from the fluid electron momentum equation

$$\frac{\partial n_e m_e u_e}{\partial t} = -e n_e (E + u_e \times B) - \nabla \cdot P_e + \eta j,$$

(2.14)

where $u_e$ is the electron flow velocity, and $\eta$ is the resistivity. The electron pressure tensor $P_e$ is usually closed by an equation of state. In these hybrid models, plasma is often assumed to be quasi-neutral and Poisson’s equation is replaced by the generalized Ohm’s law. The kinetic treatments of ions follow the position $x_i$ and velocity $v_i$ of a single ion based on the following equations of motion

$$\frac{dx_i}{dt} = v_i,$$

(2.15)

$$\frac{dv_i}{dt} = \frac{q_i}{m_i} (E + v_i \times B) - \frac{q_i}{m_i} \eta j.$$

(2.16)

The last term on the right-hand-side (rhs) of Eq. (2.16) balances the corresponding term in the electron momentum equation Eq. (2.14). Such hybrid codes can adequately reproduce ion physics over thousands of $\Omega_i^{-1}$ and is enough to recover relatively low-frequency phenomena. For numerical implementations of hybrid models, some good reviews can be found in [47, 48, 49].

In this chapter, I present a “current closure” hybrid model with Lorentz force ions and fluid electrons similar to [37]. When quasi-neutrality is assumed, the electron density and flow needed in the generalized Ohm’s law can be calculated by the ion density and ion flow obtained directly from the particles. Our model includes the electron inertia effect. The particle ions are treated with the $\delta f$ method [29, 50], which enables us to obtain clean linear results of various instabilities. Based
on previous efforts [51, 52], a second-order accurate semi-implicit algorithm has been developed to eliminate the constraint on the timestep imposed by the fast compressional wave. The code has been developed by coding within the three-dimensional (3D) general geometry toroidal gyrokinetic code GEM [53, 54]. Since we use real space to avoid Fourier convolutions due to the inhomogeneous equilibrium fields, the field solver involves large matrix inversion. Assuming a grid of \( n_1 \times n_2 \times n_3 \), for one-dimensional (1D) inhomogeneous equilibrium in the \( x \) direction, the matrix size is \( 3n_1 \times 3n_1 \). To accommodate the inhomogeneities of the equilibrium in the other two directions, the matrix size will be \( 27n_1n_2n_3 \times 27n_1n_2n_3 \) which may be extremely large. For demonstration of the code, all simulations presented in this thesis assume a slab geometry with a 1D inhomogeneous equilibrium.

### 2.2 Model equations

In this section, we present the governing equations, including the ion and electron models and the field equations. The ions follow the equations of motion

\[
\frac{dx_i}{dt} = v_i, \quad (2.17)
\]

\[
\frac{dv_i}{dt} = \frac{q_i}{m_i} (E + v_i \times B) - \frac{q_i}{m_i} \eta j, \quad (2.18)
\]

where \( \eta \) is the resistivity and \( j \) is the current density. The ion charge and mass are noted as \( q_i \) and \( m_i \). Traditionally, the third term on the rhs of Eq. (2.18) is used to provide the momentum balance. The field equations are Ampere’s law and Faraday’s law

\[
\nabla \times B = \mu_0 j = \mu_0 (q_i n_i u_i - e n_e u_e), \quad (2.19)
\]

\[
\nabla \times E = -\frac{\partial B}{\partial t}, \quad (2.20)
\]

Here, \( u_i \) and \( u_e \) stand for the ion and electron flow velocity, respectively. Computationally, it is convenient to solve for the electric field from the generalized Ohm’s law that arises from the electron momentum equation

\[
en_e (E + u_e \times B) = en_e \eta j - \nabla \cdot \Pi_e - m_e \frac{\partial (n_e u_e)}{\partial t}, \quad (2.21)
\]
where electron stress tensor is $\mathbf{\Pi}_e = \int f_e m_e \mathbf{v} \mathbf{v} d\mathbf{v}$, and it is determined by the selection of electron models. Upon substituting in Ampere’s equation $\mathbf{j} = \mathbf{j}_i - \mathbf{j}_e = \nabla \times \mathbf{B}/\mu_0$, Eq. (2.21) can be rewritten as

$$en_e \mathbf{E} = -\mathbf{j}_i \times \mathbf{B} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{en_e}{\mu_0} \eta (\nabla \times \mathbf{B}) - \nabla \cdot \mathbf{\Pi}_e - m_e \frac{\partial (n_e \mathbf{u}_e)}{\partial t}. \quad (2.22)$$

The last term on the rhs represents the electron inertial effect, which is calculated from Ampere’s law in combination of the ion momentum equation [55]. First, we take the time derivative of Ampere’s law, yielding

$$\mu_0 (q_i \frac{\partial n_i \mathbf{u}_i}{\partial t} - e \frac{\partial n_e \mathbf{u}_e}{\partial t}) = \nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \nabla \times \mathbf{E}. \quad (2.23)$$

The first term on the left hand side (lhs) is obtained from the ion momentum equation

$$m_i \frac{\partial n_i \mathbf{u}_i}{\partial t} = q_i n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla \cdot \mathbf{\Pi}_i - q_i n_i \frac{\eta}{\mu_0} \nabla \times \mathbf{B}, \quad (2.24)$$

where $\mathbf{\Pi}_i = \int f_i m_i \mathbf{v} \mathbf{v} d\mathbf{v}$ is the ion stress tensor. With Eqs. (2.23) and (2.24), the electron inertia term takes the form

$$m_e \frac{\partial (n_e \mathbf{u}_e)}{\partial t} = \frac{m_e}{m_i} \frac{q_i}{e} \left( q_i n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla \cdot \mathbf{\Pi}_i - q_i n_i \frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right)$$

$$+ \frac{m_e}{\mu_0 e} \nabla \times (\nabla \times \mathbf{E}). \quad (2.25)$$

Using Eq. (2.25) and quasi-neutrality $n_i = n_e$, the generalized Ohm’s law Eq. (2.22) becomes

$$en_i (1 + \frac{m_e q_i^2}{m_i e^2}) \mathbf{E} + \frac{m_e}{\mu_0 e} \nabla \times (\nabla \times \mathbf{E})$$

$$= - (1 + \frac{m_e q_i}{m_i e}) \mathbf{j}_i \times \mathbf{B} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$+ \eta \frac{en_i}{\mu_0} (1 + \frac{m_e q_i^2}{m_i e^2}) \nabla \times \mathbf{B} - \nabla \cdot \mathbf{\Pi}_e + \frac{m_e q_i}{m_i e} \nabla \cdot \mathbf{\Pi}_i, \quad (2.26)$$

In general, one needs a kinetic electron model to obtain the electron stress tensor $\mathbf{\Pi}_e$. In this paper, however, the electrons are assumed isothermal and the electron pressure tensor reduces to

$$\mathbf{\Pi}_e = n_e T_e = n_i T_e, \quad (2.27)$$
where $T_e$ is the constant electron temperature and no bulk electron flow is present. Therefore, the above equations are closed with ion density and current density obtained directly from the particle ions.

To include more electron physics, more sophisticated models such as the drift kinetic electron model can be included. For a possible implementation of the drift kinetic electrons [52], the electrons are described by the drift kinetic equation for electron distribution function $f_e(x, \varepsilon, \mu)$

$$\frac{\partial f_e}{\partial t} + \mathbf{V}_G \cdot \nabla f_e + \frac{d\varepsilon}{dt} \frac{\partial f_e}{\partial \varepsilon} = 0,$$

where the guiding center velocity is

$$\mathbf{V}_G = v_{\parallel} \hat{b} + \mathbf{V}_D + \mathbf{V}_E$$

with $\hat{b} = \mathbf{b} + \mathbf{B}_{\perp}/B_0$ and $\mathbf{b}$ represents the unit vector along the guide field $\mathbf{B}_0$. $\mathbf{V}_D$ and $\mathbf{V}_E$ stand for the grad-$\mathbf{B}$ and $\mathbf{E} \times \mathbf{B}$ drift. The kinetic energy is $\varepsilon = \frac{1}{2} m_e v^2$ and the magnetic moment is $\mu$.

The equations of motion for electrons are

$$\frac{d\mathbf{x}}{dt} = \mathbf{V}_G,$$

$$\frac{d\varepsilon}{dt} = -e \mathbf{V}_G \cdot \mathbf{E} + \mu \frac{\partial B}{\partial t},$$

$$\frac{d\mu}{dt} = 0.$$  \hspace{1cm} (2.30)

The Ampere’s equation becomes

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J}_i - e n_e (\mathbf{U}_{e\perp} + \mathbf{u}_{\parallel} \mathbf{b})).$$

where

$$\mathbf{U}_{e\perp} = \frac{1}{B} \mathbf{E} \times \mathbf{b} - \frac{1}{enB} \mathbf{b} \times \nabla P_{\perp e}.$$  \hspace{1cm} (2.31)

To close the system, the corresponding electron parallel flow and pressure are provided as

$$n_e \mathbf{u}_{\parallel e} = \int f_e v_{\parallel} dv_{\parallel} d\varepsilon,$$

and

$$P_{\perp e} = \int f_e \mu B dv_{\parallel} d\varepsilon.$$  \hspace{1cm} (2.33)
2.3 Numerical method

To study complex kinetic plasma behavior, PIC simulation is a very important tool. These PIC codes follow trajectories of the macroparticles in electromagnetic fields calculated self-consistently from Maxwell’s equations. The time integration schemes of these codes usually fall into two categories: the explicit method and implicit method. The explicit method is easy to implement because it does not need to predict future time level quantities to advance the equations. However, the explicit method needs to resolve very fast time scales associated with the high-frequency waves in the system in order to avoid numerical instabilities [26]. This poses a serious limitation on the size of timestep, which becomes a troublesome issue especially when the phenomena of interest occurred on much longer scales. To relax these timestep restrictions, implicit methods are introduced [56, 57, 58, 59, 60]. In general, the implicit methods exhibit the charge and current densities on the electric fields and linearize them to solve for field quantities. In this section, we present the general implicit method and the detailed numerical scheme of updating the particle velocity and position, the \( \delta f \) method, and the field solver in this work.

2.3.1 The implicit PIC method

As discussed in Section 2.1, the PIC method is one of the most used numerical method for solving the Vlasov-Maxwell system because of its capability to deal with six-dimensional phase space and its simple implementation. In the PIC method, the distribution function \( f(x,v,t) \) is described as a collection of \( N \) macroparticles. Without loss of generality, here we consider an electrostatic case in a simple one-dimension configuration [61]. Thus the solution to the Vlasov equation

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{q}{m} E \frac{\partial f}{\partial v} = 0
\]  

(2.34)

can be expressed as

\[
f(x,v,t) = \sum_{j=1}^{N} S(x - x_j) \delta(v - v_j),
\]

(2.35)
where $\delta$ is Dirac’s delta function. The shape function $S(x - x_j)$ is usually defined as

$$
S(x - x_j) = \begin{cases} 
\frac{1}{\Delta x} \left( 1 - \frac{|x - x_j|}{\Delta x} \right), & |x - x_j| \leq \Delta x \\
0, & |x - x_j| \geq \Delta x 
\end{cases}
$$

where $\Delta x$ is the grid size. This is essentially a linear interpolation method. Therefore, for a particle at position $x_j < x < x_{j+1}$, the density deposited at the neighboring $j$th and $j+1$th grids are (Figure 2.1)

$$
n(x_j) = \frac{x_{j+1} - x}{\Delta x},
$$

$$
n(x_{j+1}) = \frac{x - x_j}{\Delta x}.
$$

![Figure 2.1: Schematic plot showing the deposition of particle density $n(x)$ in a grid as described by the shape function $S(x - x_j)$.]

And the evolution of the macroparticles are described by

$$
\frac{dx_j}{dt} = v_j,
$$

$$
\frac{dv_j}{dt} = \frac{q_j}{m_j} E_j,
$$

where $E_j$ is calculated from

$$
E_j = \int E(x) S(x - x_j) dx.
$$
Explicitly, based on the standard “leap frog” method [26], the particle difference equations are

\[ v^{n+1/2} = v^{n-1/2} + \frac{q\Delta t}{m} E^n, \]  
\[ x^{n+1} = x^n + \Delta t v^{n+1/2}, \]

(2.41)
(2.42)

where \( \Delta t \) is the timestep. Note that to obtain \( x^{n+1} \), we only need to use field values at time level \( n \) and earlier. This method is attractive because it is second-order accurate and easy to implement as long as the acceleration depends on the position. The numerical stability of the PIC method has been studied extensively using the numerical dispersion relations [26]. Due to the time and spatial discretization of the Vlasov equation, various numerical modes can result in numerical instability if not controlled. Research has shown that the application of explicit PIC methods is restricted by severe numerical stability constraints. To illustrate the constraints on the timesteps, we can test the explicit scheme using a simple harmonic motion

\[ \frac{d^2x}{dt^2} = -\omega_0^2 x. \]

(2.43)

Substitute the following equations

\[ x^n = x_0 \exp (i n \omega \Delta t), \]  
\[ v^n = v_0 \exp (i n \omega \Delta t), \]

(2.44)
(2.45)

into Eq. (2.41) and (2.42), we have the following numerical dispersion relation

\[ (\omega_0 \Delta t)^2 = 4 \sin^2 \left( \frac{\omega \Delta t}{2} \right). \]

(2.46)

For \( \omega_0 \Delta t \ll 1 \), Eq. (2.46) can be reduced to

\[ \omega \approx \omega_0 \left( 1 + \frac{1}{24} (\omega_0 \Delta t)^2 \right), \]

(2.47)

which is second-order accurate in time. According to Eq. (2.46), the stability condition for the “leapfrog” scheme is

\[ (\omega_0 \Delta t)^2 \leq 4, \]

(2.48)
which gives $\omega_0 \Delta t \leq 2$. Therefore, the corresponding constraint on timestep is set by the highest frequency in the system $\omega_h \Delta t \leq 2$. Usually, the highest frequency in plasmas is the electron plasma frequency, which makes this timestep constraint a substantial obstacle for studying longer time scale phenomena.

When studying kinetic plasma at the relatively long MHD time scale, the implicit PIC method becomes a more appropriate choice [57, 58, 59, 60]. In the direct implicit method, an implicit solution of the field equations is achieved by relating linear increments to the charge and current density directly to the change in the particle motion induced by the fields or their increments at the future time level [58]. In this scheme, the particles are advanced as

$$v^{n+1/2} = v^{n-1/2} + \frac{q\Delta t}{2m} [E^{n-1} + E^{n+1}(x^{n+1})],$$

$$x^{n+1} = x^n + \Delta t v^{n+1/2}. \quad (2.49)$$

Analyzing the scheme using the simple harmonic motion, we obtain the following numerical dispersion relation

$$(\omega_0 \Delta t)^2 + 2(e^{i\omega \Delta t} - 1)^2 = 0, \quad (2.51)$$

where the following relation is used

$$\frac{q}{m} E^n = -\omega_0^2 x^n.$$

For $\omega_0 \Delta t \ll 1$, the numerical dispersion relation reduces to

$$\omega = \omega_0 [1 + O(\omega_0 \Delta t)^2] = iO(\omega_0 \Delta t)^3]. \quad (2.52)$$

Again, the implicit scheme is second order accurate. The imaginary part, which is negative and higher order, introduces a small amount of damping at low frequencies. For large $\omega_0 \Delta t$, the oscillations are subject to strong numerical damping. This damping removes high frequency modes that is not required to be accurately resolved in the simulation. Therefore, there is no unstable range for $\omega_0$.

Although unconditionally stable, it is difficult to solve the implicit scheme because the new electric field in Eq. (2.49) depends on the new particle position $x^{n+1}$. Also, to obtain the new
particle position, the information of the new electric field is required. The most general idea is to iterates on all these equations together. However, the high computational cost makes it no better than the simple explicit PIC methods. Therefore, the problem reduces to making good approximations of the quantities at a future time level. Notice that without knowing the new electric field at \( n + 1 \), we can partially advance the particles to an intermediate position \( \hat{x} \), that is the full advance without the contribution from \( E^{n+1} \):

\[
\hat{x} = x^n + \Delta t v^{n+1/2} + \frac{q \Delta t^2}{2m} E^{n-1}.
\] (2.53)

And the particle position \( x^{n+1} \) at time level \( n + 1 \) is

\[
x^{n+1} = \hat{x} + \frac{q \Delta t^2}{2m} E^{n+1}(x^{n+1}).
\] (2.54)

Now we proceed to determine \( E^{n+1}(x) \). Charge is deposited on the grid from particle position \( x_j^{n+1} \), which provides the charge density \( \rho^{n+1} \):

\[
\rho^{n+1}(x) = q \sum_j S(x - x_j^{n+1}),
\] (2.55)

where \( j \) is the particle index, and \( S \) is the shape function. Expand the shape function \( S \) gives

\[
S(x - x_j^{n+1} \approx S(x - \hat{x}_j) + (x_j^{n+1} - \hat{x}_j) \frac{\partial S(x - \hat{x}_j)}{\partial \hat{x}_j},
\] (2.56)

and from Eq. (2.61),

\[
x_j^{n+1} - \hat{x}_j = \frac{q \Delta t^2}{2m} E^{n+1}(x_j^{n+1}).
\] (2.57)

The error is approximated as

\[
\frac{q \Delta t^2}{2m} |\nabla E| \ll 1.
\] (2.58)

A correction to the charge density at the future time step \( n + 1 \) is given by

\[
\rho^{n+1}(x) = \hat{\rho}(x) + \delta\rho(x).
\] (2.59)

The change in charge density caused by a small displacement is

\[
\delta\rho(x) \approx -\frac{\partial}{\partial x}(\hat{\rho}(x) \delta x(x)),
\] (2.60)
which is consistent with charge continuity equation. Note these calculations can be easily generalized to higher dimensions by replacing \( \frac{\partial}{\partial x} \) with the operator \( \nabla \). The small displacement \( \delta x \) is simply the difference between \( x^{n+1} \) and \( \hat{x} \),

\[
\delta x = \frac{q \Delta t^2}{2m} E^{n+1}(x).
\]  

(2.61)

Define the implicit susceptibility \( \chi(x) \) as

\[
\chi(x) = \frac{q \Delta t^2}{2m} \hat{\rho}(x),
\]  

(2.62)

which provides a measure of how “implicit” the spatial region in the method is. Thus the Poisson’s equation at the future time level is

\[
\frac{\partial}{\partial x} E^{n+1} = \hat{\rho}(x) + \delta \rho(x),
\]

\[
= \hat{\rho}(x) - \frac{\partial}{\partial x} (\chi E^{n+1}),
\]  

(2.63)

where we have used Eq. (2.60), (2.61), and (2.62). Further simplification yields

\[
\frac{\partial}{\partial x} (1 + \chi) E^{n+1} = \hat{\rho}(x)
\]  

(2.64)

Using the more conventional electric potential \( E = -\frac{\partial}{\partial x} \phi \), the implicit Poisson’s equation Eq. (2.64) can be written as

\[
\frac{\partial}{\partial x} (1 + \chi) \frac{\partial}{\partial x} \phi = -\frac{1}{\epsilon_0} \hat{\rho}.
\]

(2.65)

Eq. (2.65) may be expressed in finite difference as

\[
\frac{1}{\Delta x^2} [(1 + \chi_{i+1/2}) \phi_{i+1}^{n+1} - (2 + \chi_{i-1/2} + \chi_{i+1/2}) \phi_i^{n+1} + (1 + \chi_{i-1/2}) \phi_{i-1}^{n+1}] = -\frac{1}{\epsilon_0} \hat{\rho}_i.
\]

(2.66)

After using Eq. (2.66) to calculate \( \phi^{n+1} \), we can obtain the electric field by

\[
E_i^{n+1} = -\frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x}.
\]

(2.67)

With the field at the future time level known, we can advance the particles from \( \hat{x} \) to \( x^{n+1} \).

Another popular approach is the implicit moment method developed by Brackbill et al. [57], in which the implicitness is derived from the fluid-moment equations. This improves the hybrid
models in that all species are now kinetic and implicitly coupled to the fields. The implicit moment method has good stability properties. In this scheme, the fluid equations in difference from are

\[ n_s^{n+1} = n_s^n - \frac{\Delta t}{q_s} \frac{\partial}{\partial x} J_s^{n+1/2}, \]  
\[ J_s^{n+1/2} = J_s^{n-1/2} + \frac{q_s \Delta t}{m_s} \left( -\frac{\partial}{\partial x} P_s^n + q_s n_s^n E^* \right), \]  

where \( P_s \) is the stress tensor, \( n_s \) and \( J_s \) are implicit predictions of the fluid number and current densities. And the predicted \( E^* \) is

\[ E^* = \theta E^{n+1} + \frac{1 - \theta}{4} (E^{n+1} - 2E^n + E^{n-1}), \]

where \( \theta \) (0 < \( \theta \) < 1) is the centering parameter. The electric field is determined by Poisson’s equation

\[ \frac{\partial}{\partial x} E^{n+1} = \frac{1}{\epsilon_0} \sum_s q_s n_s^{n+1}. \]

### 2.3.2 \( \delta f \) Method

In PIC simulations, there are two ways to treat the particles: the “full \( f \)” method and \( \delta f \) method. In the “full \( f \)” method, the distribution function is described by macroparticles so that it requires a huge amount of particles to suppress the statistical noises. In the \( \delta f \) method, however, the distribution function is divided by an equilibrium distribution and a small perturbation, where the equilibrium distribution is usually taken to be Maxwellian. By evolving only the perturbed part of the distribution function, one can remove the noise associated with the equilibrium distribution due to the use of finite number of particles. Thus it is well suited to study various plasma instabilities. Here we utilize the usual nonlinear \( \delta f \) method [29, 50], which is effective in reducing particle noise to obtain clean linear results with relatively a small number of particles. The \( \delta f \) method originates from the Klimontovich representation of the particle distribution function

\[ f(x, v, t) = \sum_j \delta(x - x_j(t))\delta(v - v_j(t)), \]

which is a natural solution to the Vlasov equation

\[ \frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q}{m} (E + v \times B) \cdot \frac{\partial f}{\partial v} = 0. \]
If the distribution function $f$ can be divided into an equilibrium part $f_0$ and a perturbed part $\delta f$ as $f = f_0 + \delta f$, the above equation becomes
\[
\frac{d\delta f}{dt} = -\frac{df_0}{dt}.
\] (2.74)

If we assign a particle weight to each particle $w_j \approx \delta f/f$, $\delta f$ can be then represented as
\[
\delta f = \sum_j w_j \delta(x - x_j(t))\delta(v - v_j(t)).
\] (2.75)

The evolution of the particle weight is described by the so-called weight equation
\[
\frac{dw}{dt} = -\left(\frac{f}{g} - w\right)\frac{d\ln f_0}{dt},
\] (2.76)

where $g$ is the marker distribution loaded in the simulation. Usually, if the particle weight $w \ll 1$, the above equation can be simplified as
\[
\frac{dw}{dt} = -\frac{d\ln f_0}{dt}.
\] (2.77)

Once the particle weights are known, the perturbed particle density and current can be calculated as follows
\[
\delta n = \sum_j w_j S(x - x_j),
\] (2.78)
\[
\delta j = \sum_j w_j v_j S(x - x_j),
\] (2.79)

where $S(x - x_j)$ is the shape function.

### 2.3.3 The particle advancing

For $\rho_i$ scale instabilities in tokamaks, $k\rho_i \sim 1$ and $\beta \sim 0.01$, thus the compressional wave frequency is very high ($\omega/\Omega_{ci} \geq 10$), which places a severe constrain on the time step $\Omega_{ci} \Delta t \leq 0.01$. To eliminate the strict timestep constraint, we have developed a second-order accurate semi-implicit method with an adjustable centering parameter.

The velocity is normalized to the ion sound speed $c_s^2 = T_e/m_i$, length to the ion sound gyroradius $\rho_s = m_i c_s / eB_0$ and time to $\Omega_{ci}^{-1} = m_i/eB_0$. The charge and mass are normalized to that of
a proton. $\beta_e = \mu_0 n_0 T_e / B_0^2$ is defined upon the uniform background plasma. We now write down the equations in numerical form. The equations of motion are

\begin{align}
\frac{x_{n+1} - x^n}{\Delta t} &= (1 - \theta) v^n + \theta v^{n+1}, \\
\frac{v_{n+1} - v^n}{\Delta t} &= (1 - \theta) a^n + \theta a^{n+1}, \\
\frac{w_{n+1} - w^n}{\Delta t} &= -(1 - \theta) (v^n \cdot \nabla + a^n \cdot \frac{\partial}{\partial v}) \ln f_0(x^n, v^n) \\
&\quad - \theta (v^{n+1} \cdot \nabla + a^{n+1} \cdot \frac{\partial}{\partial v}) \ln f_0(x^{n+1}, v^{n+1}),
\end{align}

where $a = \frac{q_i}{m_i}[(E + v \times B) - \eta \nabla \times B / \beta_e]$. If $f_0$ is Maxwellian and no resistivity is present, the weight equation Eq. (2.82) reduces to

\begin{align}
\frac{v_{n+1} - v^n}{\Delta t} &= \frac{q_i}{T_{i0}} ((1 - \theta) (E^n \cdot v^n) + \theta (E^{n+1} \cdot v^{n+1}))
\end{align}

which is used in Section 4 for linear comparisons.

To integrate the particle orbits in the magnetic field, extra care should be taken since it involves future field information. We carried out the particle push as follows. In the predictor push, the particles are integrated based on the known field information.

\begin{align}
\frac{x^* - x^n}{\Delta t} &= (1 - \theta) v^n \\
\frac{v^* - v^n}{\Delta t} &= \frac{q}{m} (1 - \theta) E^n \\
\frac{v^* - v'}{\Delta t} &= \frac{q}{m} (1 - \theta) v' \times B^n
\end{align}

After the fields $E, B$ are updated, the corrector push follows

\begin{align}
\frac{v^* - v^*}{\Delta t} &= \frac{q}{m} \theta v^* \times B^{n+1}(x^*) \\
\frac{v^{n+1} - v^*}{\Delta t} &= \frac{q}{m} \theta E^{n+1}(x^*) \\
\frac{x^{n+1} - x^*}{\Delta t} &= \theta v^{n+1}
\end{align}

For the Lorentz force orbit integration, we introduce two rotations to preserve the gyro-radius. If the centering parameter $\theta = 1/2$, this method works similar as the Boris scheme [62] to the first order of $\Delta t$. 
2.3.4 Field solver

Faraday’s law is solved using

\[
\frac{\delta B^{n+1} - \delta B^n}{\Delta t} = -[(1 - \theta) \nabla \times E^n + \theta \nabla \times E^{n+1}],
\]

and the generalized Ohm’s law is

\[
(n_i^{0} + \delta n_i^{n+1})(1 + \frac{m_e}{m_i} q_i^2)E^{n+1} + \frac{m_e}{m_i} \beta_e \nabla \times (\nabla \times E^{n+1}) = -\frac{1}{\beta_e} (\nabla \times \delta B^{n+1}) \times B_0 \\
+ \frac{1}{\beta_e} (\nabla \times (B_0 + \delta B^{n+1})) \times \delta B^{n+1} + \frac{\eta}{\beta_e}(1 + \frac{m_e}{m_i} q_i^2)(n_i^{0} + \delta n_i^{n+1}) \nabla \times \delta B^{n+1} \\
- \nabla \delta n_i^{n+1} + \frac{m_e}{m_i} q_i \cdot P_i^{n+1},
\]

where \(n_i^{0}, B_0\) stands for the equilibrium ion density and magnetic field respectively. The bulk current distribution is assumed to be in diffusive equilibrium \(E_0 = \eta j_0\). The off-diagonal terms of ion stress tensor are neglected here, hence the ion stress tensor \(\Pi_i\) becomes the isotropic ion pressure \(P_i\). If \(\theta = 0\), the method is a first-order accurate explicit scheme (forward Euler), while if \(\theta = 1\), it is first-order accurate implicit (backward Euler), and when \(\theta = 0.5\), it becomes the Crank-Nicolson method [63].

The previous first-order scheme by Chen and Parker [52] solves the fields in Fourier space and hence can not handle inhomogeneous equilibrium fields because of Fourier convolutions. This difficulty is overcome in the present second-order scheme by solving the fields in real space along the inhomogeneous direction for each Fourier mode in the uniform direction. The generalized Ohm’s law is separated into an equilibrium part and a perturbed part. This adds additional complexity into the coding, but it simplifies the implicit field solver. The equilibrium part can be written as a large linear equation and solved by direct matrix inversion. Note that the matrix is static for each Fourier mode, therefore the matrix only needs to be LU-decomposed once at the preconditioning stage. This significantly reduces the computation time. The perturbed part is solved iteratively.
Figure 2.2: The iteration times needed for field solver to converge to an accuracy order of 
$\Delta E_x/E_x = 10^{-9}$ (triangles) and $10^{-7}$ (diamonds).

First we advance the particle position and velocity using the known field quantities $E^n$ and $B^n$,

$$x^* = x^n + \Delta t (1 - \theta) v^n,$$
$$v^* = v^n + \Delta t (1 - \theta) a^n. \quad (2.92)$$

And the particle weight is

$$w^* = w^n - \Delta t (1 - \theta) (v^n \cdot \nabla + a^n \cdot \frac{\partial}{\partial v}) \ln f_0(x^n, v^n). \quad (2.93)$$

Usually the marker particle distribution is loaded as Maxwellian, therefore we can approximate the
ion current as
\[
\delta j_{i}^{n+1} = q_i \sum_j w_{j}^{n+1} v_{j}^{n+1}
\]
\[
= \delta j_{i}^{*} + q_i \theta \Delta t \sum_j \frac{q_i}{T_i} E_{n}^{n+1} (x_{j}^{n+1}) \cdot v_{j}^{n+1} v_{j}^{n+1}
\]
\[
\simeq \delta j_{i}^{*} + \theta \Delta t \frac{q_i^2}{m_i} E_{n}^{n+1} \equiv J'_i. \tag{2.94}
\]

For accuracy, we iterate on the differences between $\delta j_{i}^{n+1}$ and $J'_i$. Typically, in linear simulations, 5 iterations are enough. Figure 2.2 shows the needed iteration times to make the field solver converge to the desired order of accuracy for various values of $\delta n/n_0$.

Once the $E_{n}^{n+1}$ is calculated from the generalized Ohm’s law, one can go back to Eqs. (2.80), (2.81), (2.82) and (2.90) to advance the particle information and update the magnetic fields.

## 2.4 Benchmarks

This section presents simulations of the shear and compressional Alfvén waves, the whistler waves, and the ion acoustic wave with Landau damping. Unless explicitly stated otherwise, all simulations assume an ion-to-electron mass ratio of $m_i/m_e = 1837$. Boundary conditions are periodic in all three dimensions. Unless explicitly mentioned otherwise, the grids is $2 \times 32 \times 32$ and the simulation domain is $l_x \times l_y \times l_z = 628\rho_i \times 628\rho_i \times 1000\rho_i$. The total number of particles used is 131072. The timestep is set as $\Omega_i \Delta t = 0.05$.

### 2.4.1 Shear and compressional Alfvén waves

In this section, we discuss two important cold-plasma waves: shear and compressional Alfvén waves. We start with the ideal MHD fluid equations
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{2.95}
\]
\[
\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = \mathbf{j} \times \mathbf{B} - \nu_{th}^2 \nabla \rho, \tag{2.96}
\]
and the field equations
\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \tag{2.97}
\]
\[ \nabla \times E = -\frac{\partial B}{\partial t}, \quad (2.98) \]
\[ E + u \times B = 0, \quad (2.99) \]

where \( \rho, u, \) and \( j \) are plasma density, flow velocity, and current density, respectively. We now make a perturbation expansion about the equilibrium values

\[ \rho = \rho_0 + \rho_1 e^{ik \cdot x - i\omega t}, \]
\[ B = B_0 + B_1 e^{ik \cdot x - i\omega t}, \]
\[ u = u_1 e^{ik \cdot x - i\omega t}, \]

where the background flow \( u_0 = 0 \). With these assumptions, the linearized MHD equations read

\[ \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot u_1 = 0, \quad (2.100) \]
\[ \rho_0 \frac{\partial u_1}{\partial t} = \frac{1}{\mu_0} (\nabla \times B_1) \times B_0 - v_{th}^2 \nabla \rho_1, \quad (2.101) \]
\[ \frac{\partial B_1}{\partial t} - \nabla \times (u_1 \times B_0) = 0. \quad (2.102) \]

Differentiate Eq. (2.101) with respect to time, we have

\[ \rho_0 \frac{\partial^2 u_1}{\partial t^2} = \frac{1}{\mu_0} (\nabla \times \frac{\partial B_1}{\partial t}) \times B_0 - v_{th}^2 \nabla \frac{\partial \rho_1}{\partial t}. \quad (2.103) \]

After substituting Eq. (2.100) and (2.102), Eq. (2.103) becomes

\[ \rho_0 \frac{\partial^2 u_1}{\partial t^2} = \frac{1}{\mu_0} [\nabla \times (\nabla \times (u_1 \times B_0))] \times B_0 + \rho_0 \nabla (\nabla \cdot u_1). \quad (2.104) \]

Replace the derivatives

\[ \nabla \rightarrow ik \quad \frac{\partial}{\partial t} \rightarrow -i\omega, \]

and neglect the pressure term (cold-plasma), Eq. (2.104) reduces to

\[ \omega^2 u_1 = V_A^2[(k \cdot u_1 - k_{\parallel} u_{1\parallel})k + k_{\parallel}^2 u_1 - k_{\parallel}^2(k \cdot u_1)b], \quad (2.105) \]

where the Alfvén speed is \( V_A^2 = B_0^2/\mu_0 \rho_0 \) and \( b \) is the unit vector along \( B_0 \).
Figure 2.3: Comparison of simulated shear Alfvén wave frequency $\omega$ vs $\beta_e$ (diamonds) and the dispersion relation (solid line).

For shear Alfvén waves, $\mathbf{k} = k_\parallel \mathbf{b}$, the dispersion relation yields

$$\omega^2 = k_\parallel^2 V_A^2,$$  \hspace{1cm} (2.106)

which, reduces to $\omega^2 = k_\parallel^2 / \beta_e$ in our scaled quantities. For results presented here, there is a uniform background magnetic field $\mathbf{B}_0$ along $z$ direction. Figure 2.3 shows the simulation results for a mode with $k_\perp = 0$, $k_\parallel \rho_i = 0.00628$, which agrees with the dispersion relation very well.

For compressional Alfvén waves, $\mathbf{k} \perp \mathbf{b}$, the dispersion relation is $\omega = k_\perp V_A$ or $\omega = k_\perp / \sqrt{\beta_e}$ in dimensionless quantities. If $\Omega_i \Delta t$ is small enough, this fast wave can be accurately captured. As shown in Figure 2.4, the simulation for $k_\parallel = 0$, $k_\perp \rho_i = 0.01$ agrees with the theory very well.
Figure 2.4: Comparison of simulated compressional Alfvén wave frequency $\omega$ vs $\beta_e$ (diamonds) and the dispersion relation (solid line).

2.4.2 Ion acoustic waves and Landau damping

The simulation of ion acoustic wave is compared with the kinetic dispersion relation. Following the discussion in [16], for linearized Vlasov equation

$$\frac{\partial f_{1s}}{\partial t} + v \cdot \nabla f_{1s} + \frac{q_s}{m_s} E \frac{\partial f_{0s}}{\partial v} = 0,$$

(2.107)

where $f_s = f_{0s} + f_{1s}$, $s$ stands for species, and we assume plane-wave solutions

$$E = e_z E(t)e^{ikz},$$

$$f_{1s}(z,v,t) = f_{1s}(v,t)e^{ikz}.$$
After taking the Fourier transform of the linearized Vlasov equation in space, and using the Laplace transform in time, we obtain

\[(s + ikv)\mathcal{F}_1(s, v, t) + \frac{q_s}{m_s} \mathcal{E}(s) \frac{\partial f_{0s}}{\partial v} = f_{1s}(v, 0),\] (2.108)

where the Laplace transform is defined as

\[\mathcal{F}_1(s, v) = \int_0^\infty f_{1s}(v, t)e^{-st}dt,\]
\[\mathcal{E}(s) = \int_0^\infty E(t)e^{-st}dt,\]

Substitute the solution of Eq. (2.108) into the Laplace transform of the Poisson equation

\[ik\mathcal{E}(s) = \sum_q \int_0^\infty \mathcal{F}_1(s, v)dv,\] (2.109)

we obtain the following dispersion relation for the ion acoustic wave

\[D(k, \omega) \equiv 1 + \sum_s \frac{q_s^2}{\epsilon_0 km_s} \left( Pr \int_{-\infty}^{+\infty} \frac{\partial f_{0s}}{\partial v} dv \frac{\omega - kv}{\omega^2} - \frac{\pi i}{k} \frac{\partial f_{0s}}{\partial v} \right) = 0,\] (2.110)

where the summation is over species and \(Pr\) stands for the Cauchy principal value. In the regime of

\[kv_{thi} \ll \omega \ll kv_{the},\]

proper approximations for the two principle value integrals can be made. For ions, expand the factor in the integrand in Eq. (2.110)

\[\frac{1}{\omega - kv} \sim \frac{1}{\omega} + \frac{kv}{\omega^2},\] (2.111)

we obtain

\[Pr \int_{-\infty}^{+\infty} \frac{\partial f_{0}/\partial v}{\omega - kv} dv \sim -\frac{nk}{\omega^2}.\] (2.112)

For electrons, we have

\[\frac{1}{\omega - kv} \sim -\frac{1}{kv},\] (2.113)

and

\[Pr \int_{-\infty}^{+\infty} \frac{\partial f_{0}/\partial v}{\omega - kv} dv \sim \frac{n}{kv_{the}^2}.\] (2.114)
With Eq. (2.112) and (2.114), keeping the imaginary terms from both ions and electrons, Eq. (2.110) becomes

\[
D(k, \omega) \equiv 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{k^2 v_{the}^2} + i \sqrt{\frac{\pi}{2}} \left[ \frac{\omega_{pi}^2}{k^3 v_{thi}^3} \exp \left( -\frac{\omega^2}{2k^2 v_{thi}^2} \right) + \frac{\omega_{pe}^2}{k^3 v_{the}^3} \right] = 0, \tag{2.115}
\]

where \(\omega_{pe}\) and \(\omega_{pi}\) are plasma frequency for electrons and ions, respectively. For wavelengths much longer than the Debye length, \(k \lambda_{De} = k v_{the}/\omega_{pe} \ll 1\), the dispersion relation \(D(k, \omega) = 0\) gives \(\omega \approx k C_s - i \gamma\) where \(C_s = (T_e/m_i)^{1/2}\) is the sound speed. And the corresponding damping rate \(\gamma\) is given by

\[
\gamma \approx \frac{1}{2} \sqrt{\frac{\pi}{2}} k C_s \left[ \sqrt{\frac{m_e}{m_i}} + \left( \frac{T_e}{T_i} \right)^{3/2} \exp \left( -\frac{T_e}{2T_i} \right) \right]. \tag{2.116}
\]

According to the dispersion relation, ion acoustic wave is damped due to the resonance of particles with speeds close to the phase speed \(\omega/k\). This kinetic effect is known as Landau damping. From Eq. (2.116), the Landau damping from electrons is relatively small compared to that from ions by a factor of \(\sqrt{m_e/m_i}\), therefore only the ion Landau damping is considered here. And the Landau damping from ions is small only if \(T_e \gg T_i\), otherwise the undamped ion acoustic waves do not exist due to the strong ion Landau damping. The dispersion relation is solved numerically for the case of \(T_i/T_e = 0.2\) and the result is shown as the solid line in Figure 2.5. The measured real frequency and damping rate for ion acoustic wave in the simulation are also shown in Figure 2.5 (squares and pluses). The simulation is in good agreement with the kinetic theory.
Figure 2.5: Simulated ion acoustic wave frequency $\omega$ (diamonds) and ion Landau damping rate $\gamma$ (pluses) vs $k v_{ti}$ compared with the dispersion relation (solid lines). The ion temperature is $T_i/T_e = 0.2$.

Linearly, the initially excited waves will damp exponentially. Nonlinearly, the nonlinear energy interchange between waves and the trapped particles eventually stops this damping process [64]. As the phase-mixing of resonant particles becomes more complete, the less energy interchange occurs. The wave amplitude eventually saturates at a nonzero value which is directly related to the initial perturbation. Figure 2.6 compares the linear and nonlinear runs with the same parameters. In the nonlinear regime, the wave amplitude reaches a constant residual value higher than the recurrence value in the linear run [65, 66]. Figure 2.7 shows the saturated electric field amplitude as a function of the initial perturbation amplitude. The dash line corresponds to the analytical solution of Ref. [67]. Although Lancellotti et al.’s analysis [67] is based on the nonlinear Landau
damping of the electron plasma wave, the governing equations are the same as in the present case. Therefore, the results apply in the present case without much modification. Consider the one-dimensional nonlinear Vlasov-Poisson (VP) system,

\[ \frac{\partial f_s}{\partial t} + v \frac{\partial f_s}{\partial x} + \frac{q_s}{m_s} E \frac{\partial f_s}{\partial v} = 0, \]  
\[ \frac{\partial E}{\partial x} = \frac{1}{\epsilon_0} \sum_s \int dv f_s, \]

where \( f_s(x,v,t) \) is the distribution function for particle species \( s \) and \( E \) is the longitudinal electric field. The electric field can be divided into a transient part and an asymptotic part

\[ E(x,t) = T(x,t) + A(x,t), \]

where \( T(x,t) \) and \( A(x,t) \) are spatially periodic. Asymptotically, \( T(x,t) \) approaches zero and \( A(x,t) \) is an almost periodic function of \( t \). Substitute the above equation into Eq. (2.118) and separate the asymptotic and transient parts of the resulting equation provides

\[ \frac{\partial A}{\partial t} = \frac{1}{\epsilon_0} P_s \sum_s \int dv f_s (A + T), \]
\[ \frac{\partial T}{\partial t} = \frac{1}{\epsilon_0} (I - P_s) \sum_s \int dv f_s (A + T), \]

where \( P_s \) is an operator that only keeps the asymptotic part of an asymptotically periodic function. Given the initial condition of

\[ f_s(x,v,0) = F_s(v) + h_s(x,v), \]

where \( h_s(x,v) \) stands for the initial perturbation to \( F_s(v) \) (usually a Maxwellian). After taking the Fourier-Bohr transformation

\[ g_{k,\omega} = \frac{1}{\pi} \lim_{\sigma \to \infty} \frac{1}{\sigma} \int_0^\sigma dt \int_{-\pi}^\pi dx e^{-ikx-i\omega t} G(x,t) \]

to Eq. (2.120), and integrate by parts in \( t \) (the particle trajectories are taken as straight lines \( x = vt \)), we have

\[ a_{k,\omega} = \frac{1}{2\pi \epsilon_0 k} \sum_s \frac{q_s^2}{m_s} \lim_{\sigma \to \infty} \frac{1}{\sigma} \int_0^\sigma dt \int_{-\pi}^\pi dx e^{-ikx-i\omega t} A(x,t) Pr \int dv \frac{f_s'(x,v,t)}{\omega + kv} \]
and

\[ D(k, \omega) = 1 - \frac{1}{\epsilon_0 k} \sum_s \frac{q_s^2}{m_s} Pr \int dv \frac{F_s^{T_0}(v)}{\omega + kv}, \quad (2.125) \]

where \( F_s^{T_0}(v) \) is the asymptotic Vlasov equilibrium determined by the corresponding transient field \( T_0 \). In the small amplitude limit, the asymptotic field is

\[ A(x, t) = \sum_{j=1}^M a_{k,\omega_j} e^{ikx+i\omega_j(k)t} + o(a_{k,\omega_j}). \quad (2.126) \]

Now we approach to solve \( f_s \) by taking an approximated form of Eq. (2.117)

\[ \frac{\partial f_s}{\partial t} + v \frac{\partial f_s}{\partial x} + q_s m_s A \frac{\partial f_s}{\partial v} = -q_s m_s T \frac{\partial F_s}{\partial v}, \quad (2.127) \]

where \( F = f_s(x, v, 0) \). Take the characteristic time scale for the decay of \( T \) as \( \tau_T \) and \( \tau_b \) as the typical trapping time scale, the above equation holds for \( \tau_T \ll \tau_b \). Physically, if \( A = 0 \), the system yields a linearized Landau damping solution. If \( T = 0 \), the system is described by an undamped strongly trapping wave as discussed by O’Neil [64]. Using the sequence of canonical transformations introduced in [68], the equation can be solved as

\[ f_s(x, v, t) = \mathcal{F}_s[x_0^A(x, v, t), v_0^A(x, v, t)] - \frac{q_s}{m_s} \int_0^t \int \left( T \frac{\partial F_s}{\partial v} \right)[x_\tau^A(x, v, t), v_\tau^A(x, v, t)] \]

where \([x_\tau^A(x, v, t), v_\tau^A(x, v, t)]\) is the trajectory of the particle that arrives at \((x, v)\) at time \(t\). Substitute this solution to Eq. (2.120) and (2.121), we can solve \( T \) and \( A \). Take a simple sinusoidal perturbation to a Maxwellian distribution

\[ \mathcal{F}_s(x, v) = (1 + \epsilon \cos x) F_s(v), \quad (2.129) \]

and the asymptotic field can be described as a pair of traveling wave

\[ A(x, t) = a \sin(x - v_{ph}t) + a \sin(x + v_{ph}t), \quad (2.130) \]

where \( v_{ph} \) is the phase velocity of the wave. As \( t \to \infty \), if the wave amplitude \( a \neq 0 \), spatially periodic plateaus develop on the distribution functions at \( v = \pm v_{ph} \). Substitute the solution of Eq. (2.128), \( f_s \), into Eq. (2.120) and apply the Fourier-Bohr transform at \((k, \omega) = (1, v_{ph})\), we obtain a nonlinear equation for \( a \). An asymptotic expansion in \( a \) of the equation gives

\[ [\beta \epsilon - \Gamma(\epsilon)]a^{1/2} + \sigma(\epsilon)a^{3/2} = o(a^2), \quad (2.131) \]
where $\beta, \Gamma(\epsilon)$ and $\sigma(\epsilon)$ are determined by the initial distribution function and $\Gamma(0) = 0$. Expand the equation in $\epsilon - \epsilon^0$ yields

$$a = -\frac{\beta - \Gamma'(\epsilon^0)}{\sigma(\epsilon^0)}(\epsilon - \epsilon^0) + o[(\epsilon - \epsilon^0)^{3/2}],$$

(2.132)

where

$$\beta\epsilon^0 = \Gamma(\epsilon^0).$$

Through some length calculations, the amplitude $a$ can be determined by

$$a = \frac{4\pi^2 \sum_s q_s \sqrt{|q_s|^2/m_s}^{1/2} h_s(v_{ph})}{\sum_s q_s^2 |q_s|^{1/2} 2dF_s(v_{ph})/d\epsilon^2} (\epsilon - \epsilon^0).$$

(2.133)

In our units, the saturation amplitude of the wave is determined by

$$E_{fi} = \frac{4\pi^2 v_{th}^2}{(v_{ph}^2/v_{th}^2 - 1)}(E_{in} - E_{in}^*) \approx 0.31(E_{in} - 0.51)$$

(2.134)

given $v_{th}^2 = 0.2, v_{ph} = 1.35, k = 0.1$ in the simulation. Here $E_{in}^*$ is the critical value of the initial perturbation amplitude which distinguishes the linear and nonlinear regimes. The simulation agrees well with the theory. Note that in order to reduce the wave amplitude at which the recurrence phenomenon occurs, many more particles (2097152) are used in this simulation.
Figure 2.6: Comparison of the linear and nonlinear Landau damping in a semilog plot of the time history of the wave amplitude. $k_z \rho_i = 0.1, T_i/T_e = 0.2$. 
2.4.3 Numerical damping associated with whistler waves

By dropping ion dynamics, the model is simplified and it becomes possible to investigate the numerical damping associated with the implicit time advance. To work out a numerical dispersion relation, we drop the ion current and the electron inertia in the generalized Ohm’s law. The reduced generalized Ohm’s law yields

\[ E = \frac{1}{\beta_e} (\nabla \times \delta B) \times B_0. \]  

(2.135)

Numerically, the above equation reads,

\[ E^{n+1} = \frac{1}{\beta_e} (\nabla \times \delta B^{n+1}) \times B_0. \]  

(2.136)
And the numerical Faraday’s law is
\[
\frac{\delta B^{n+1} - \delta B^n}{\Delta t} = -[\theta \nabla \times E^{n+1} + (1 - \theta) \nabla \times E^n].
\] (2.137)

Assume the uniform background \( B_0 \) is along \( z \) direction, we consider the perturbed fields \( E^n, \delta B^n \) of the form
\[
E^n = E_c e^{i(kz-n\omega \Delta t)}
\]
\[
\delta B^n = B_c e^{i(kz-n\omega \Delta t)},
\]
where \( E_c \) and \( B_c \) are uniform. A Von Neumann stability analysis provides the corresponding numerical dispersion relation for whistler wave
\[
\begin{aligned}
\tan(\omega_r \Delta t) &= \frac{k^2 \Delta t}{\beta_e} \\
\gamma \Delta t &= -\frac{1}{2} \ln \left( \frac{1 - (\frac{k^2}{\beta_e} \Delta t)^2 \theta(1 - \theta)^2 + (\frac{k^2}{\beta_e} \Delta t)^2}{1 + (\frac{k^2}{\beta_e} \Delta t)^2 (1 - \theta)^2} \right).
\end{aligned}
\] (2.138)

By neglecting terms with an order of \((\Delta t)^2\) or higher \((\Delta t \ll 1)\), the above equations can be simplified as
\[
\begin{aligned}
\omega_r &= \frac{k^2}{\beta_e}, \\
\gamma &= (\frac{1}{2} - \theta)(\frac{k^2}{\beta_e})^2 \Delta t.
\end{aligned}
\]

We can see that the centering parameter \( \theta \) does not affect the real frequency (up to \((\Delta t)^2\)). But the imaginary part of the frequency \( \gamma \) depends on \( \theta \) directly. The implicit time advance damps the whistler wave when \( \theta > 0.5 \) and drives it unstable when \( \theta < 0.5 \). When \( \theta = 0.5 \), the time-centered scheme introduces negligible numerical damping. Figure 2.8 shows the numerical damping under different centering parameters for \( k_\perp = 0, k_\parallel \rho_i = 0.0628, \beta_e = 0.004 \). The results agree with the numerical dispersion relation Eq. (2.138) very well. And as expected, different centering parameter produces the same real frequency, which is not shown here. Also omitted is the unstable regime of \( \theta < 0.5 \). The grids used here is \( 16 \times 16 \times 32 \) and the total number of particles is 131072. The simulation domain is \( l_x \times l_y \times l_z = 628 \rho_i \times 628 \rho_i \times 100 \rho_i \).
Figure 2.8: Comparison of simulated numerical damping rates of whistler wave vs. timesteps (stars for $\theta = 1$, diamonds for $\theta = 0.75$ and triangles for $\theta = 0.5$) and the numerical dispersion relation (solid lines).
Chapter 3

Resistive tearing mode instability

The tearing mode is an important instability that occurs in plasmas with a sheared magnetic field, where small perturbations of the field lines would lead the plasma to a lower magnetic energy state via field bending and form magnetic islands. There are various mechanisms to break the local magnetic flux conservation resulting in a magnetic topology change, which makes tearing mode a natural cause of magnetic reconnection [7]. Analysis of the tearing mode is extensive including resistive MHD, two-fluid and kinetic theories [15, 69, 70, 71, 72]. The simplest analysis of the tearing mode is done in the presence of finite resistivity, where the instability develops on a resistive time scale and is the so called resistive tearing mode.

3.1 Resistive MHD theory

The linear theory of resistive tearing mode instability is well established by Furth, Killeen and Rosenbluth, the FKR theory [15]. Here we follow the simple analysis discussed in [69]. Consider the reduced MHD model, ignoring the variation in z-direction the field in the xy-plane can be described by the flux function $\psi$

$$B_\perp = e_z \times \nabla \psi,$$

(3.1)

and the flow in xy-plane is described by the stream function $\phi$

$$v_\perp = e_z \times \nabla \phi.$$  

(3.2)
The reduced momentum equation reads
\[ \rho \left( \frac{\partial w}{\partial t} + v \cdot \nabla w \right) = B \cdot \nabla j, \] (3.3)
where \( w = e_z \cdot \nabla \times v \) represents the vorticity of the flow in \( z \)-direction and \( \rho \) is the plasma density.

For small perturbations of the equilibrium, the magnetic term breaks into two parts
\[ \delta(B \cdot \nabla j) = B_0 \cdot \nabla \delta j + \delta B \cdot \nabla j_0. \] (3.4)

The first term on the rhs is associated with the Alfvén wave. The second term stands for the free energy associated with the current density \( j_0 \), which drives the magnetic instabilities including the tearing mode. In general, resistive instabilities are relatively slow compared to the Alfvén time, hence the associated motion is almost incompressible in the perpendicular plane. Equations for the flux function \( \psi \) and vorticity \( w \) can then be written as
\[ \frac{\partial \psi}{\partial t} - B \cdot \nabla \phi = \frac{\eta}{\mu_0} \nabla^2 \psi, \] (3.5)
\[ \rho \left( \frac{\partial w}{\partial t} + v \cdot \nabla w \right) - B \cdot \nabla j = 0. \] (3.6)

Assume the flux function has the form of
\[ \psi(x, y) = \psi_0(x) + \psi_1(x, y) \] (3.7)
and in the vicinity of the neutral line (the resistive layer)
\[
\psi_1(x, y) = \psi_1(x)e^{iky+\gamma t}, \\
\phi_1(x, y) = \phi_1(x)e^{iky+\gamma t},
\]
we can linearize Eq. (3.5) and (3.6) as
\[ \gamma \psi_1 = B_0 \cdot \nabla \phi_1 + \frac{\eta}{\mu_0} \nabla^2 \psi_1, \] (3.8)
and
\[ \gamma \rho \nabla^2 \phi_1 = B_0 \cdot \nabla j_1 + B_1 \cdot \nabla j_0. \] (3.9)
The resistivity is only important in the resistive layer \(-\delta < x < \delta\) around the neutral line defined by \(k \cdot B_0 = 0\) \((x = 0)\). Outside this layer, the system can be well approximated by the ideal MHD equations. To find out the growth rate \(\gamma\), one needs to match the solutions inside and outside the resistive layer asymptotically. The magnetic perturbation \(\psi_1\) can diffuse across the resistive layer and holds a finite value at \(x = 0\). The derivative of \(\psi_1\) is very small, but it exhibits a jump characterized by the tearing mode parameter

\[
\Delta' = \frac{1}{\psi_1(0)} \left( \frac{d\psi_1}{dx} \bigg|_{0+} - \frac{d\psi_1}{dx} \bigg|_{0-} \right).
\]  

(3.10)

Because \(\psi_1' \ll \psi_1/\delta\), one can adopt the constant-\(\psi\) approximation by assuming \(\psi_1\) is constant in the resistive layer.

Inside the resistive layer, to the first order, the magnetic field is approximated as \(B_{y0}(x) \sim B'_{y0}x\) and \(\nabla^2 \sim \partial^2/\partial x^2\). Eq. (3.8) and (3.9) are simplified as

\[
\gamma \psi_1 = i\delta k B'_{y0} \phi_1 + \frac{\eta}{\mu_0} \psi_1'',
\]  

(3.11)

and

\[
\gamma \rho \phi_1'' = \frac{i\delta k B'_{y0}}{\mu_0} \psi_1'' - i k j'_{0} \psi_1.
\]  

(3.12)

For tearing mode, \(\phi_1\) has odd parity and \(\psi_1\) has even parity. Using further approximations

\[
\phi_1'' \sim -\phi_1/\delta^2,
\]

\[
\psi_1'' \sim \Delta' \psi_1/\delta,
\]

one equates the lhs to the second term on the rhs of Eq. (3.11) to obtain the resistive layer width

\[
\delta \sim \frac{\eta}{\mu_0} \frac{\Delta'}{\gamma}.\]

(3.13)

At the boundary of the resistive layer \(x = \delta\), Eq. (3.12) reads

\[
-\gamma \rho \phi_1'' \sim i\delta k B'_{y0} \Delta' \frac{\Delta'}{\mu_0 \delta^2}
\]  

(3.14)

by neglecting the second term on the rhs. And Eq. (3.11) becomes

\[
\gamma \psi_1 = i\delta k B'_{y0} \phi_1.
\]  

(3.15)
Eq. (3.14) and (3.15) together yield a semi-quantitative solution of \( \gamma \)

\[
\gamma \sim \frac{\Delta^{4/5} \eta^{3/5} (kB_y'0)^{2/5}}{\varrho^{1/5} \mu_0^{4/5}},
\tag{3.16}
\]

and the resistive layer width \( \delta \)

\[
\delta \sim \frac{\varrho^{1/5} \Delta^{1/5} \eta^{2/5}}{\mu_0^{1/5} (kB_y'0)^{2/5}}.
\tag{3.17}
\]

A more rigorous treatment is given below [16]. Keeping in mind that the \( x \) derivatives dominate over the \( y \) derivatives inside the resistive layer, neglecting the second term on the rhs of Eq. (3.12) yields

\[
\gamma \varrho \phi_1'' = \frac{ixkB_y'0}{\mu_0} \psi_1''.
\tag{3.18}
\]

Substituting for \( \psi_1'' \) from Eq. (3.11), Eq. (3.18) becomes

\[
\gamma \eta \varrho \phi_1'' = xkB_y'0 (i\gamma \psi_1 + xkB_y'0 \phi_1).
\tag{3.19}
\]

Invoking the constant-\( \psi \) approximation, Eq. (3.19) can be solved to find an explicit solution for \( \phi_1(x) \). However, the solution cannot be given in terms of analytic functions but must be evaluated partially numerically. It is apparent from Eq. (3.19) that \( \phi_1(x) \) has odd parity. Let’s first determine a more accurate characteristic width of the resistive layer. By balancing the term on the lhs of Eq. (3.19) against the second term on the rhs, we have the characteristic width

\[
\delta_w = \frac{(\gamma \eta \varrho)^{1/4}}{(kB_y'0)^{1/2}}.
\tag{3.20}
\]

The resistive layer becomes thinner as resistivity \( \eta \) decreases.

To simplify Eq. (3.19), it is convenient to introduce two scaled variables \( X \) and \( \Phi \), which are defined as

\[
X = \frac{x}{\delta_w},
\tag{3.21}
\]

\[
\Phi = \frac{(\gamma \eta \varrho)^{1/4} (kB_y'0)^{1/2} \phi_1}{i\gamma \psi_1}.
\tag{3.22}
\]

In terms of these variables, Eq. (3.19) becomes

\[
\frac{\partial^2 \Phi}{\partial X^2} = X(1 + X \Phi).
\tag{3.23}
\]
The solution $\Phi(X)$ will be an odd function of $X$ and, as long as $\frac{\partial^2 \Phi}{\partial X^2}$ is well-behaved asymptotically as $X \to \infty$, $\Phi \to -X^{-1}$ as $X \to \infty$. This implicitly assumes that the solution of the inhomogeneous Eq. (3.23) is unique, i.e. that the homogeneous equation obtained by omitting the term $X$ on the rhs has no permitted solutions. This can be easily justified by multiplying the homogeneous equation by $\Phi^*$ and integrating from $-\infty$ to $\infty$ which results in the following expression

$$- \int |\frac{\partial \Phi}{\partial X}|^2 dX = \int X^2 |\Phi|^2 dX,$$

(3.24)

where integration by parts is used and $\Phi \to 0$ as $X \to 0$ is assumed. It is obvious that the condition for Eq. (3.24) to hold is $\Phi(X) = 0$.

An explicit solution for Eq. (3.23) can be written in an integral form

$$\Phi(X) = -\frac{X}{2} \int_0^{\pi/2} \exp \left( -\frac{X^2}{2} \cos \theta \right) \sin^{1/2} \theta d\theta.$$

(3.25)

By substituting Eq. (3.25) into Eq. (3.23), one can easily verify that this is indeed the exact solution. With this explicit solution, we can now proceed to find the exact expression for tearing mode growth rate $\gamma$. By integrating Eq. (3.12) across the resistive layer, we have

$$\left[ \frac{\partial \psi_1}{\partial x} \right]_{x=0} = \frac{\mu_0}{\eta} \int (i\gamma \psi_1 + x k B_0') \phi_1 dx.$$  

(3.26)

Divide by $\phi_1$ and convert to the scaled variables $X$ and $\Phi$, Eq. (3.26) becomes

$$\Delta' = \frac{\mu_0}{\eta} \frac{5/4}{(k B_0')^{1/4}} \int_{-\infty}^{\infty} (1 + X \Phi) dX,$$

(3.27)

where the definition of $\Delta'$ as in Eq. (3.40) is used and that the integration limits in Eq. (3.26) are taken as ±$\infty$ on the scale of resistive layer width. In terms of the exact solution of $\Phi(X)$ as in Eq. (3.25), the integral on the rhs of Eq. (3.27) can be carried out numerically as follows:

$$\int_{-\infty}^{\infty} (1 + X \Phi) dX = \frac{1}{2} \int_{-\infty}^{\infty} dX \int_0^{\pi/2} \exp \left( -\frac{X^2}{2} \cos \theta \right) \sin^{1/2} \theta (3 \cos \theta - X^2 \cos^2 \theta) d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin^{1/2} \theta d\theta \int_{-\infty}^{\infty} dX \int_0^{\pi/2} \exp \left( -\frac{X^2}{2} \cos \theta \right) (3 \cos \theta - X^2 \cos^2 \theta)$$

$$= \sqrt{2\pi} \int_0^{\pi/2} \sin^{1/2} \theta \cos^{1/2} \theta d\theta$$

$$\approx 2.12,$$

(3.28)
which then gives a factor of 0.55 in the expression for $\gamma$

$$\gamma = 0.55 \frac{\Delta'^{4/5} \eta^{3/5} (kB'_y \eta)^{2/5}}{\rho^{1/5} \mu_0^{4/5}}. \quad (3.29)$$

The condition for the resistive tearing mode to be unstable is $\Delta' > 0$. For a wave-like (in y-direction) magnetic perturbation, the annihilation of magnetic fields around $x = 0$ lowers the magnetic energy and is thus energetically favored. However, this process also produces the corresponding $B_x$ component which bends the magnetic field lines. It in turn increases the magnetic energy, especially as the wavelength decreases. Thus the resistive tearing mode is unstable only for sufficiently long wavelengths for which the energy released by field annihilation exceeds that needed for field bending.

The quantity $\Delta'$ is obtained from the solution in the external region. The tearing mode growth is slow compared to the Alfvén time but faster than resistive diffusion time

$$\gamma \sim \tau^{-3/5} \tau_A^{-2/5} \ll \tau_A^{-1},$$

where $\tau = \frac{\mu_0 \rho}{\eta}$ is the resistive diffusion time scale and $\rho$ is the characteristic length for the current layer. Therefore, inertia effect can be neglected in Eq. (3.9), which yields

$$B_0 \cdot \nabla j_1 + B_1 \cdot \nabla j_0 = 0. \quad (3.30)$$

In the resistive layer, Eq. (3.30) reads

$$\psi''_1 - \frac{\kappa}{x} \psi_1 = 0, \quad (3.31)$$

where $\kappa = \mu_0 j'_0 / B'_y$. The solution takes the form

$$\psi_1(x) = [f(x) + Cg(x)]\psi_1(0), \quad (3.32)$$

where

$$f(x) = 1 + \kappa x \ln |x| + o(x^2),$$

$$g(x) = x + o(x^2).$$

Due to the singularity of $\psi'_1$ across the resistive layer,

$$\Delta' = C_{0+} - C_{0-}, \quad (3.33)$$
Where \( C_{0+} \) and \( C_{0-} \) are determined by the outer boundary conditions. Outside the resistive layer, Eq. (3.30) becomes

\[
\psi_1'' - \left( k^2 + \frac{\mu_0 j_0'}{B_{y0}(x)} \right) \psi_1 = 0.
\] (3.34)

As an example, for an equilibrium field (Figure 3.1)

\[
B_{y0}(x) = B_{y0} \tanh(x/a),
\] (3.35)

and the equilibrium current density

\[
j_0(x) = \left( \frac{B_{y0}}{\theta_0 a} \right) \text{sech}^2\left( \frac{x}{a} \right),
\] (3.36)

the general solution of Eq. (3.34) is

\[
\psi_1 = c_1 e^{-kx} \left( 1 + \frac{1}{ka} \tanh(x/a) \right) + c_2 e^{kx} \left( 1 - \frac{1}{ka} \tanh(x/a) \right),
\] (3.37)

where \( c_1, c_2 \) are constants yet to be determined by the boundary conditions. For the boundary condition \( \psi_1 = 0 \) at \( x = \pm l_x/2 \), the solution is

\[
\psi_1 = \begin{cases} 
  e^{-kx} \left( 1 + \frac{1}{ka} \tanh(x/a) \right) - A e^{kx} \left( 1 - \frac{1}{ka} \tanh(x/a) \right), & x \geq 0 \\
  e^{kx} \left( 1 - \frac{1}{ka} \tanh(x/a) \right) - B e^{-kx} \left( 1 + \frac{1}{ka} \tanh(x/a) \right), & x < 0
\end{cases}
\]

where

\[
A = e^{-kl_x} \frac{1 + \frac{1}{ka} \tanh(l_x/2a)}{1 - \frac{1}{ka} \tanh(l_x/2a)},
\]

\[
B = e^{-kl_x} \frac{1 - \frac{1}{ka} \tanh(l_x/2a)}{1 + \frac{1}{ka} \tanh(l_x/2a)}.
\]

The corresponding \( \Delta' \) is

\[
\Delta' = \frac{2 + A + B}{a} \left( \frac{1}{ka} - ka \right).
\] (3.38)

If \( l_x \to \infty, A = B = 0 \), and the solution is reduced to the asymptotic form (Figure 3.2)

\[
\psi_1 = e^{-k|x|} \left( 1 + \frac{1}{ka} \tanh(|x|/a) \right).
\] (3.39)

Therefore the asymptotic tearing mode parameter \( \Delta' \) is

\[
\Delta' = \frac{2}{a} \left( \frac{1}{ka} - ka \right).
\] (3.40)
Figure 3.1: The schematic plot of the Harris equilibrium magnetic field and current density.
In the nonlinear stage, Rutherford’s analysis exhibits an algebraic growth rate for tearing mode [17]. And the saturation of tearing mode has been discussed extensively [18, 19, 20, 21]. Recently, Escande et al. [20] and Militello et al. [21] reported a simple rigorous calculation. Following Escande et al.’s discussion [20], one expands the flux function to a higher-order of the small parameter $\delta_n \gg \delta$

$\psi(x, y, t) = \psi_0(x) + \delta_n^2 \cos(ky) + \delta_n^\alpha \tilde{\psi}(\xi, ky, \delta_n, t), \tag{3.41}$

where $\alpha > 2, \xi = x/\delta_n$, and $\tilde{\psi}$ is $2\pi$-periodic in $ky$. And the corresponding current density is

$j(x, y, t) = j_0(x) - \delta_n^2 \cos(ky) + \delta_n^{\alpha-2} \frac{\partial^2 \tilde{\psi}}{\partial^2 \xi}. \tag{3.42}$

In the resistive layer,

$\psi_0(x) = 1 - \frac{1}{2} sx^2 + \frac{1}{24} rx^4 + o(x^6), \tag{3.43}$
where \( s, r \) are constants. Substitute the expansion of \( \psi \) and \( j \) into Eq. (3.3) (without the inertia effect), one obtains (to the leading order of \( \delta_n \))

\[
\xi \partial^2_{\xi} \partial_{\chi} \bar{\psi} - \frac{1}{s} \sin(ky) \partial^3_{\xi} \bar{\psi} \sim \delta_n^{\alpha-\alpha} \lambda^2 \xi \sin(ky),
\]

(3.44)

where \( \chi = ky \) and \( \lambda^2 = r/s - k^2 \). Let

\[
\bar{j} = \partial^2_{\xi} \bar{\psi},
\]

(3.45)

and the corresponding solution is

\[
\bar{j} \sim -\lambda^2 \cos(\chi) + \Phi(\nu),
\]

(3.46)

where \( \nu = s\xi^2/2 - \cos \chi \) and \( \Phi(\nu) \) is an arbitrary function yet to be determined. Substituting the above equation into Eq. (3.42) yields

\[
j(x, y, t) \sim j_0(x) - (k^2 \delta_n^2 + \delta_n^{\alpha-2} \lambda^2) \cos ky + \delta_n^{\alpha-2} \Phi(\nu).
\]

(3.47)

Introducing a new function \( \bar{\phi}(\rho, \chi, \delta_n, t) \) such that

\[
\phi(x, y, t) = \eta \bar{\phi}(\rho(\xi, ky), \chi, \delta_n, t),
\]

Eq. (3.5) becomes

\[
\partial_{\chi} \bar{\phi} \sim -\Lambda \frac{\delta_n^{\alpha-3}}{k} \left[ \left( \frac{r}{s} + \frac{2\delta_n^{\alpha-3}}{\eta} \partial_t \delta_n \right) \cos \chi + \Phi(\nu) \right] (\nu + \cos \chi)^{-1/2},
\]

(3.48)

where \( \Lambda = (2s)^{-1/2} \) for \( \xi > 0 \) and \( \Lambda = -(2s)^{-1/2} \) for \( \xi < 0 \). Note \( \bar{\phi}(\rho, \chi, \delta_n, t) \) is periodic in \( \chi \), when integrating over \([-\pi, \pi]\), the lhs of the above equation is zero. To obtain the saturated tearing mode, we set \( \partial_t \delta_n = 0 \), which gives

\[
\Phi(\nu) = -\frac{r}{s} \frac{I(\nu)}{D(\nu)}
\]

\[
I(\nu) = \int_{-\pi}^{\pi} \frac{\cos \chi d\chi}{(\nu + \cos \chi)^{1/2}}
\]

\[
D(\nu) = \int_{-\pi}^{\pi} \frac{d\chi}{(\nu + \cos \chi)^{1/2}}
\]
Let $\psi_{11}$ be the cos $\chi$ component of the Fourier expansion of $\psi - \psi_0$ in $\chi$. For $\xi \gg 1$, the asymptotic expansion in $\xi$ gives

$$\psi_{11} \sim \delta_n^2 + \delta_n^4 \left[ -\frac{1}{2} \lambda^2 \xi^2 - \frac{|\xi|}{2\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} \Phi(s\xi^2/2 - \cos \chi) \cos \chi d\chi d\xi \right]. \quad (3.49)$$

By matching with the outer region solution of $\psi_{11}^{\text{out}}$

$$\psi_{11} \propto 1 + \Delta' |\xi|, \quad (3.50)$$

we have

$$\Delta' = -\frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} \Phi(s\xi^2/2 - \cos \chi) \cos \chi d\chi d\xi. \quad (3.51)$$

Taking out the integration, we have the time dependent solution

$$\Delta' = \left( \frac{2}{s} \right)^{1/2} \frac{K_0}{\pi} \left[ r \delta_n + \frac{2}{\eta} \partial_t \delta_n \right], \quad (3.52)$$

where

$$K_0 = \int_{-1}^{\infty} \frac{I^2(\nu)}{D(\nu)} d\nu \approx 3.654.$$ 

By introducing the island width $w = 4s^{-1/2}\delta_n$, one obtains the familiar expression

$$\frac{dw}{dt} \approx 1.22\eta (\Delta' - \alpha' w), \quad (3.53)$$

where $\alpha' = 0.41r/s$. For the equilibrium $B_{y0}(x) = B_{y0} \tanh(x/a)$, $\alpha' = 0.82$. If $w \ll 1$, Eq. (3.53) becomes the well known Rutherford equation [17]

$$\frac{dw}{dt} \approx 1.22\eta \Delta'. \quad (3.54)$$

For the saturation of tearing mode, the island growth rate becomes zero as the island width approaches the saturation island width $w_s$

$$\frac{dw_s}{dt} \approx 1.22\eta (\Delta' - \alpha' w_s) = 0, \quad (3.55)$$

which yields

$$w_s = \frac{1}{\alpha'} \Delta'. \quad (3.56)$$
Again, for the standard tanh equilibrium, the corresponding saturation island width is

\[ w_s = 1.22\Delta'. \] (3.57)

Previous resistive MHD simulations have shown good agreement with the resistive MHD theory [11, 22, 73]. In the following sections, our hybrid simulation results are presented and compared to the resistive MHD theory in the small \( \Delta' \) regime.

### 3.2 Simulation setup

The simulation starts with an initial configuration of Harris sheet equilibrium [74], the equilibrium magnetic field is

\[ B_0 = e_z B_z + e_y B_{y0} \tanh(x/a), \] (3.58)

where \( B_z \) represents the uniform guide field, \( B_{y0} \) is the asymptotic Harris magnetic field, and \( a \) stands for the width of the current sheet. The coordinates are chosen as \( x \) in the direction of the gradients, \( y \) along the sheared Harris magnetic field, and \( z \) along the equilibrium current. The equilibrium distribution function is

\[
\begin{align*}
  f_{0s} &= n_{h0} \text{sech}^2 \left( 2\pi T_s \left( \frac{v_x^2}{m_i} + \frac{v_y^2 + (v_z - v_{ds})^2}{2T_s} \right) \right)^{-\frac{1}{2}} \\
  &\quad \times \exp \left[ -m(v_x^2 + v_y^2 + (v_z - v_{ds})^2) \right] \\
  &\quad + n_b \left( \frac{2\pi T_s}{m_i} \right)^{-\frac{3}{2}} \exp \left[ -\frac{mv^2}{2T_s} \right], \quad (3.59)
\end{align*}
\]

where \( v_{ds} = 2T_s/q_s B_{y0} a \) is the drift velocity, and \( n_h = n_{h0} \text{sech}^2(x/a) \) and \( n_b \) stands for the Harris particle density and uniform background density respectively. The temperature of the Harris and background particles are chosen to be the same. Since the equilibrium current density is carried by the ion and electron drift velocity, the Ampere’s law \( \nabla \times B_y = en_i (v_{di} - v_{de}) \) leads straightforwardly to the pressure balance \( B_{y0}^2/2\mu_0 = n_{h0} (T_i + T_e) \). The marker ion distribution is loaded as \( g_0 = (2\pi T_{i0}/m_i)^{-3/2} \exp(-mv^2/2T_{i0}) \), and according to Eq. (2.77), the weight equation
reads
\[
\frac{dw_i}{dt} = \frac{q_i}{T_i} \left( (E - \eta \beta_e \nabla \times B) \cdot v_i \left( \frac{f_h}{g_0} + n_b \right) \right.
\]
\[
- v_d \cdot (E + v_i \times \delta B - \eta \beta_e \nabla \times B) \left( \frac{f_h}{g_0} \right).
\] (3.60)

The boundary conditions are periodic in \( y \) and \( z \) directions and a perfect conducting impenetrable wall in \( x \). Particles are reflected when they hit the wall. \( E_\parallel = B_\perp = 0 \) at the wall. The results presented here use a mass ratio of \( m_i/m_e = 400 \). The equations are solved using finite difference in \( x \)-direction and Fourier series in \( y \) and \( z \)-directions (only \( k_z = 0 \) is kept to enforce the 2D simulation). In the following simulations, \( l_x \) is fixed and \( l_y \) is varied to obtain different \( \Delta' \). The most unstable mode occurs at the first harmonic \( k = 2\pi/l_y \).

### 3.3 Simulation results

We now investigate the dependence of the linear growth rate on the wave number \( k \) in \( y \) direction. The tearing mode develops a discontinuity in the derivative of magnetic flux in the vicinity of the neutral layer, which is usually characterized by the tearing instability parameter

\[
\Delta' = \frac{1}{\psi_1(0)} \left( \frac{d\psi_1}{dx}|_{x^+} - \frac{d\psi_1}{dx}|_{x^-} \right),
\]

where \( \psi_1 \) is the perturbed flux eigenfunction outside the inner region. In most cases, \( \Delta' \) can only be calculated numerically. Fortunately, for Harris sheet equilibrium, there is an asymptotic analytic value of

\[
\Delta' = \frac{2}{a} \left( \frac{1}{ka} - ka \right).
\]

Moreover, if the finite \( l_x \) (length between the walls) is considered, the expression for \( \Delta' \) becomes Eq. (3.38) (Figure 3.3)

\[
\Delta' = \frac{2}{a} \left( \frac{1}{ka} - ka \right) \frac{ka - \tanh(l_x/2a) \tanh(kl_x/2)}{ka \tanh(kl_x/2) - \tanh(l_x/2a)}.
\] (3.61)

In dimensionless quantities, the resistive-MHD linear growth rate can be written as [15, 16]

\[
\gamma = 0.55 \left( \frac{1}{\beta_e} \right)^{1/5} \Delta'^{4/5} \eta^{3/5} (k B'_y)^{2/5}.
\] (3.62)
Therefore, the Harris sheet is unstable with respect to tearing when $\Delta' > 0$, while it remains stable when $\Delta' < 0$.

Figure 3.3: $\Delta'$ as a function of $ka$ with $l_x/a = 12.8$ as in Eq. (3.61).

Figure 3.4 shows the full evolution of island width with various values of $\Delta'$. Note the instability threshold is $\Delta' > 0$, which is confirmed by the simulation. For $\Delta' = 0$ (blue line in Figure 3.4), the system shows no observable growth rate. For $\Delta' > 0$ (black, red, and green lines in Figure 3.4), we can clearly identify the linear growing, nonlinear growing and saturation stages. The island shrinking after saturation indicates that the free energy stored in the initial configuration has been exhausted. Figure 3.5 shows snapshots of the mode structure and the corresponding magnetic island at two different times. The magnetic island width remains at a relatively small value indicating the simulation is in the resistive MHD regime (Figure 3.5 (a) (c)). The current density (color) peaks at the X-point in the linear stage (Figure 3.5 (a)), which indicates that strong
magnetic reconnection occurs at that point. However, in the saturation stage, the current density peaks at the \( O \)-point (Figure 3.5 (c)), which suggests a slow magnetic reconnection rate and the system approaches a steady state. In the linear stage, the discontinuity of the derivative of \( \psi_1 \) at the middle of the current sheet manifests itself as the “dent” in the middle of the \( \psi_1 \) profile (Figure 3.5 (b)). As the tearing mode saturates, the discontinuity gradually disappears as the instability develops (shown in Figure 3.5 (d)), which stands as a signature of the tearing mode saturation.

In Figure 3.6, we show the linear growth rates as a function of \( ka \). The simulation (diamond) is in good agreement with theory (solid line). When resistivity gets smaller than 0.001, the electron inertia effects becomes important and the results deviate from the simple resistive MHD theory.

The spatial convergence properties of the eigenfunction (perturbed \( A_z \), the flux function) is shown in Figure 3.7. The two grids produce similar results, which suggests that the mesh size of \( 128 \times 16 \times 32 \) is good enough for the simulations presented here.
Figure 3.4: The evolution of island width with various $\Delta'$. Other parameters are $\frac{a}{\rho_i} = 1.0, \beta_c = \frac{\mu_0\eta a T_a}{B_0^2} = 0.5, \frac{B_G}{B_0} = 0, \frac{B_i}{T_i} = 1, \frac{l_c}{\rho_i} = 12.8, \eta \frac{\rho_{na}}{B_0} = 0.0015.$
Figure 3.5: Two snapshots of the island structure (a,c), line represents magnetic field lines and color represents the current density. Two snapshots of the corresponding eigenmode structure (perturbed $A_z$, the flux function) (b,d). (a)(b) $t = 233\Omega_i^{-1}$, (c)(d) $t = 495\Omega_i^{-1}$. The grid is $128 \times 32 \times 64$, a total of 8388608 particles are used. Other parameters are $\frac{a}{\rho_i} = 1.0$, $\beta_e = \frac{\mu_0 n_e T_e}{B_0^2} = 0.5$, $\eta^{ema}_{B_0} = 0.0015$, $\frac{B_0}{B_0} = 0$, $\frac{T_i}{T_e} = 1$, $\frac{I_x}{\rho_i} = 12.8$, $\frac{I_y}{\rho_i} = 25.12$. 
Figure 3.6: The tearing mode linear growth rates vs. $k$ (diamonds). Solid line is the resistive MHD theory as in Eq. (3.62). Other parameters are $\frac{a}{\rho_i} = 1.0$, $\beta_e = \frac{\mu_0 n_i T_e}{B_0^2} = 0.5$, $\frac{B_G}{B_0} = 0$, $\frac{T_i}{T_e} = 1$, $l_x = 12.8$, $\frac{\eta_{\text{rms}}}{B_0} = 0.0015$. 
Figure 3.7: Tearing mode eigenstructure for $A_z$ with different mesh sizes. (a) $128 \times 16 \times 32$ (b) $256 \times 16 \times 32$. Parameters for both cases are $\frac{\alpha}{\rho_i} = 1.0$, $\beta_e = \frac{\mu_0 n g T_e}{B_0} = 0.5$, $\eta_{\text{en}} = 0.0015$, $T_i/T_e = 1.0$, $\frac{L_x}{\rho_i} = 12.8$, $\frac{L_y}{\rho_i} = 25.12$. 
After a linear growth period, the tearing mode enters into the nonlinear regime. From Figure 3.4, we can identify the Rutherford regime (around $\Omega_{ci}t \sim 150 - 200$) followed by the saturation (after $t \sim 230$). The Rutherford regime is characterized by an algebraic growth rate of the island width [20]:

$$\frac{dw}{dt} = 1.22 \eta (\Delta' - \alpha'w),$$

(3.63)

which reduces to the conventional Rutherford equation when island width $w$ is small [17]. For Harris equilibrium, $\alpha' = 0.82$. As shown in Figure 3.8, the simulated island growth rate scales linearly with $\Delta'$. The simulation agrees well with the theory.

There are numerous theoretical studies of the nonlinear saturation of the tearing mode [11, 18, 19, 20, 21, 22, 73]. Among these results, one simple interpretation of saturation is $\frac{dw}{dt} = 0$. According to Eq. (3.63), the saturation island width is

$$w_s = 1.22\Delta'.$$

(3.64)

In Figure 3.9, the saturation island width is plotted for various values of $\Delta'$ calculated from Eq. (3.61). The saturation island width from the simulation (diamonds) is significantly smaller than the prediction (Eq. (3.64)) if we use the theoretical $\Delta'$ (Eq. (3.61)). However, if we use the measured $\Delta'$ from the eigenmode structure plots, the predicted saturation island width (stars) agrees much better with the simulation (Figure 3.10). The reason is because the theoretical $\Delta'$ expression comes from asymptotic approximation of small $\Delta'$. When we measured the $\Delta'$ from the eigenmode structure at the moment of saturation, it is more consistent with the simulation thus leads to better agreement.

For larger values of $\Delta'$ ($\geq 9$), secondary islands start to form and later coalesce with the main island [22]. This is discussed in details in the next chapter.
Figure 3.8: The growth rate of island width vs. resistivity for different $\Delta'$. The solid line stands for the theory. Other parameters are $\frac{a}{\rho_i} = 1.0, \beta_e = \frac{\mu_0 \eta_0 T_e}{B_0} = 0.5, \frac{B_{c2}}{B_0} = 0, T_i/T_e = 1, l_s/\rho_i = 12.8, \eta \frac{\mu_0}{\rho_i} = 0.0015$. 
Figure 3.9: The saturation island width vs $\Delta'$ (diamonds). The stars stand for measured $\Delta'$ from the eigen structure at the moment of saturation and the solid line is the fitted line of $1.22\Delta'_\text{measured}$.

$\frac{a}{\rho_i} = 1.0, \beta_e = \frac{i\omega_n T_e}{B_0} = 0.5, \frac{B_G}{B_0} = 0, \frac{T_i}{T_e} = 1, \frac{t_x}{\rho_i} = 12.8, \eta \frac{e\tau_0}{B_0} = 0.0015$
Figure 3.10: Measurement of $\Delta'_\text{measured}$ from the eigen structure obtained from the simulation. The theoretical $\Delta'$ are 0, 1.95, 2.97, 4.08 for (a), (b), (c) and (d) respectively. The tearing mode eigen mode develops in all cases except in (a) where there is no instability growth. The other parameters are $\frac{\alpha}{\rho_0} = 1.0$, $\beta_\text{e} = \frac{\nu_0 m n T_e}{B_0^2} = 0.5$, $\frac{B_G}{B_0} = 0$, $\frac{T_i}{T_e} = 1$, $l_\rho \alpha = 12.8$, $\eta = 0.0015$. 
4.1 Introduction

Magnetic reconnection is an important process that breaks the magnetic topology and releases the magnetic energy into the particle kinetic energy of the plasma. It is important in various plasma phenomena such as solar flares, coronal heating, and magnetic sub-storms in the earth magnetosphere [1, 2, 3]. The dynamic scales of magnetic reconnection vary over many orders of magnitudes. Various numerical methods, from MHD to fully kinetic simulations, have been used to study this phenomenon. While fully kinetic models are preferable, due to their computation cost, the fully kinetic simulations are often limited to relatively small domains and unrealistically large electron-to-ion mass ratios. The fluid approach, on the other hand, does not include important kinetic effects such as nonthermal particle acceleration. A promising way to get around these issues is a hybrid simulation which is suitable to study MHD scale physics while retaining full ion kinetic effects.

In numerical study of magnetic reconnection, most simulations start with an initial configuration of the Harris sheet which has been known to be unstable to tearing modes. By considering the growth rate of the tearing mode, for current sheets with small aspect ratios, the first harmonic is most unstable which indicates that only a single island is present. This phenomenon is well explained in resistive MHD theory and has been studied in Chapter 3. However, for current sheets with larger aspect ratios, the most unstable mode can occur at higher harmonics and multiple islands may form. The interaction between these islands may greatly influence the dynamic behav-
ior of magnetic reconnection. In this regime, particles ions may be subject to strong nonthermal acceleration. The simple resistive MHD model is inadequate to cover this physics. Hence, to study the kinetic ion effects which is crucial to understanding the heating mechanism, the hybrid model appears to be a good candidate.

In this chapter, we study the coalescence of magnetic island chains in a current sheet with large aspect ratios using a Lorentz force ion, fluid electron hybrid model. The model is described in Chapter 2. Our simulations exhibit the formation of magnetic island chains and their eventual coalescence to a single giant elongated island. During the coalescence, we have observed heating of the ions inside the island region. Detailed power spectra of ions are obtained. The dissipated magnetic energy per unit length in y direction increases as the aspect ratio increases. In addition, a larger fraction of dissipated magnetic energy is converted into the kinetic energy of ions as the aspect ratio increases. Asymptotically, the ratio reaches a little over 50% which is consistent with previous measurements in the Magnetic Reconnection Experiment (MRX) [75]. Possible ion acceleration mechanisms are discussed.

4.2 Equations and parameters

The simulation is based on our three-dimensional hybrid model which uses Lorentz force ions and fluid electrons [76]. To better understand the simulation results, we list the important equations used in the simulation. Full derivations are presented in Chapter 2. The ions follow the equations of motion

\[
\frac{dx_i}{dt} = v_i, \tag{4.1}
\]

\[
\frac{dv_i}{dt} = \frac{q_i}{m_i} (E + v_i \times B - \eta j), \tag{4.2}
\]

where the ion charge and mass are noted as \(q_i\) and \(m_i\). The \(\eta j\) term on the rhs of Eq. (4.2) represents the momentum balance. For particle ions, the usual \(\delta f\) method is used [29, 50]. Assume the ion distribution function can be divided by an equilibrium part and perturbed part as \(f = f_0 + \delta f\).
The distribution function satisfies the Vlasov equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q_i}{m_i} (E + v \times B) \cdot \frac{\partial f}{\partial v} = 0. \quad (4.3)$$

Thus the perturbed ion distribution function $\delta f$ evolves as

$$\frac{d\delta f}{dt} = -\frac{d f_0}{dt} = -\frac{q_i E}{m_i} \frac{\partial f_0}{\partial v}. \quad (4.4)$$

The latter equation holds when $f_0$ is Maxwellian. If we assign a particle weight to each particle $w_j \approx \delta f / f$, $\delta f$ can be then represented as

$$\delta f = \sum_j w_j \delta(x - x_j(t)) \delta(v - v_j(t)). \quad (4.5)$$

The evolution of the particle weight is described by the so-called weight equation

$$\frac{dw}{dt} = -\left( \frac{f}{g} - w \right) \frac{d \ln f_0}{dt}, \quad (4.6)$$

where $g$ is the marker distribution loaded in the simulation. Usually, if the particle weight $w \ll 1$, the above equation can be simplified as

$$\frac{dw}{dt} = -\frac{d \ln f_0}{dt}. \quad (4.7)$$

Once the particle weights are known, the perturbed particle density and current can be calculated as follows

$$\delta n = \sum_j w_j S(x - x_j), \quad (4.8)$$

$$\delta j = \sum_j w_j v_j S(x - x_j), \quad (4.9)$$

where $S(x - x_j)$ is the shape function.

The field equations are Ampere's law and Faraday's law:

$$\nabla \times B = \mu_0 j = \mu_0 (q_i n_i u_i - e n_e u_e), \quad (4.10)$$

$$\nabla \times E = -\frac{\partial B}{\partial t}. \quad (4.11)$$
Here, $\mathbf{u}_i$ and $\mathbf{u}_e$ stand for the ion and electron flow velocity, respectively. The generalized Ohm’s law is

\[
\begin{align*}
 en_i(1 + \frac{m_e q_i^2}{m_i e^2}) & E + \frac{m_e}{\mu_0 e} \nabla \times (\nabla \times E) \\
= & -(1 + \frac{m_e q_i}{m_i e}) \mathbf{j}_i \times \mathbf{B} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \\
+ & \eta \frac{en_i}{\mu_0}(1 + \frac{m_e q_i^2}{m_i e^2}) \nabla \times \mathbf{B} - \nabla \cdot \mathbf{\Pi}_e + \frac{m_e q_i}{m_i e} \nabla \cdot \mathbf{\Pi}_i,
\end{align*}
\]

The electrons are assumed to be isothermal and hence the electron pressure tensor $\mathbf{\Pi}_e$ reduces to

\[ P_e = n_e T_e = n_i T_e, \]

where $T_e$ is the constant electron temperature.

The simulation starts with an initial configuration of Harris sheet equilibrium [74], in which the equilibrium magnetic field is

\[ \mathbf{B}_0 = e_z B_z + e_y B_{y0} \tanh(x/a), \]

where $B_z$ represents the uniform guide field, $B_{y0}$ is the asymptotic Harris magnetic field, and $a$ is the width of the current sheet. The coordinates are chosen as $x$ in the direction of the gradients, $y$ along the sheared Harris magnetic field, and $z$ along the equilibrium current. The equilibrium distribution function is

\[
\begin{align*}
 f_{0i} = n_{h0} \text{sech}^2 \left( \frac{x}{a} \right) \left( \frac{2\pi T_i}{m_s} \right)^{-\frac{3}{2}} \exp \left[ -\frac{m(v_x^2 + v_y^2 + (v_z - v_{ds})^2)}{2T_s} \right] + n_b \left( \frac{2\pi T_i}{m_s} \right)^{-\frac{3}{2}} \exp \left( -\frac{m v_{ds}^2}{2T_s} \right),
\end{align*}
\]

where $v_{ds} = 2T_s/q_s B_{y0} a$ is the drift velocity, and $n_h = n_{h0} \text{sech}^2(x/a)$ and $n_b$ are the Harris particle density and uniform background density, respectively. The temperature of the Harris and background particles are chosen to be the same. Since the equilibrium current density is carried by the ion and electron drift velocity, Ampere’s law $\nabla \times \mathbf{B}_y = en_h (v_{di} - v_{de})$ leads straightforwardly to the pressure balance $B_{y0}^2/2\mu_0 = n_{h0} (T_i + T_e)$. The marker ion distribution is loaded as $g_0 = \ldots$
\[(2\pi T_{i0}/m_i)^{-3/2} \exp(-mv^2/2T_{i0})\], thus the weight equation reads

\[
\frac{dw_i}{dt} = \frac{q_i}{T_i} \left( (E - \frac{\eta}{\beta_e} \nabla \times B) \cdot v_i \left( \frac{f_h}{g_0} + n_b \right) - v_d \cdot (E + v_i \times \delta B - \frac{\eta}{\beta_e} \nabla \times B) \frac{f_h}{g_0} \right). \tag{4.16}
\]

In the simulation, the velocity is normalized to the ion sound speed \(c_s^2 = T_e/m_i\), length to the ion sound gyro-radius \(\rho_s = m_ic_s/eB_0\) and time to \(\Omega_{ci}^{-1} = m_i/eB_0\). The charge and mass are normalized to that of a proton. Plasma beta \(\beta_e = \mu_0 n_0 T_e/B_0^2\) is defined upon the uniform background plasma. The periodic boundary conditions are enforced in \(y\) and \(z\) directions, and in \(x\) direction, a perfect conducting wall boundary condition is used. Particles are reflected when they hit the wall, and \(E_\parallel = B_\perp = 0\) at the wall. The mass ratio is \(m_i/m_e = 1836\). The resistivity is set as \(\eta n_0/B_0 = 0.0015\). To reach different aspect ratios, we keep \(l_x = 12.8\) fixed and change \(l_y\) from 12.8 to 125.6. The grid size is \(128 \times 64 \times 64\), and we have 16 particles per cell.

### 4.3 Islands chains forming and coalescing

For convenience reasons, we show only the middle half of the plane \((-3.2 < x < 3.2)\) where islands grow. To illustrate how reconnection proceeds for current sheets with various aspect ratios, we show the evolution of island width (Figure 4.1) and snapshots of the magnetic island structures at different times. As the aspect ratio increases, the corresponding tearing mode parameter \(\Delta'\) becomes larger, leading to larger magnetic islands. All these processes reach steady states where the magnetic islands cease to grow. Let us use the linear growth rate calculated from resistive MHD theory to estimate the most unstable mode. According to the discussion in Chapter 3, FKR theory finds that the linear growth rate for the tearing mode is

\[
\gamma = 0.55 \left( \frac{1}{\beta_e} \right)^{1/5} \Delta'^{4/5} \eta^{3/5} (k B'_y)^{2/5}, \tag{4.17}
\]

where in the presence of bounded walls at \(x = \pm l_x/2\)

\[
\Delta' = \frac{2}{a} \left( \frac{1}{ka} - ka \right) \frac{ka - \tanh(l_x/2a) \tanh(kl_x/2)}{ka \tanh(kl_x/2) - \tanh(l_x/2a)}.
\]
From Figure 4.2, we can see for $l_y/\rho_i = 25.12$, the most unstable mode occurs at the first harmonic. For $l_y/\rho_i = 100.48$, the most unstable mode occurs at the third harmonic. In Figure 4.3, only one island is observed for $l_y/\rho_i = 25.12$. In Figure 4.4, three islands can be identified in the linear stage though they coalesce into one island in the end. This is consistent with the estimation using Eq. (4.17).

Figure 4.1: Magnetic island width as a function of time for various aspect ratios. The parameters are $a/\rho_i = 1.0$, $\beta_e = \mu_{ena} T_e / B_0^2 = 0.5$, $T_e / T_i = 1$, $l_x / \rho_i = 12.8$, $\eta_{ena} / B_0 = 0.0015$. 
Figure 4.2: Linear growth rate of the tearing mode as a function of $ka$. The parameters are $\frac{a}{\rho_i} = 1.0, \beta_e = \frac{n_{ion} T_e}{B_0} = 0.5, \frac{T_i}{T_e} = 1, \frac{I_a}{\rho_i} = 12.8, \eta \frac{e_n}{B_0} = 0.0015$. 
In our simulations, for a Harris sheet with a small aspect ratio \( l_y = 25.12 \), the tearing mode instability is most unstable for the first harmonic and only a single island is observed (Figure 4.3). The magnetic island grows exponentially at first, and nonlinear effects slow down the growth and eventually lead the system to saturation. This is consistent with resistive MHD predictions.

For a Harris sheet with a large aspect ratio \( l_y = 100.48 \), the most unstable mode occurs at the third harmonic and multiple islands start to grow during the linear stage (Figure 4.4 (a)). As the islands grow larger, they tend to coalesce with each other (Figure 4.4 (a) (b)) and eventually merge into one large elongated island (Figure 4.4 (d)). Highly elongated magnetic islands have been observed in heliosheath based on the Voyager data [77]. Schoeffler et al. have also observed the existence of such elongated magnetic islands in their PIC simulation for a system of stacked current sheets which is analogous to the compressed heliospheric current sheet in the heliosheath [78]. In their paper, they conclude that the pressure anisotropy within the islands prevents full contraction of the islands, leading to elongated islands other than the more round islands commonly seen in the MHD simulations. In our simulation, we have evidence showing the strong pressure difference inside and outside the island. We will further investigate the pressure anisotropy along and perpendicular to the in-plane magnetic field and show how this anisotropy leads to the highly elongated island. This nonlinear development of the coalescence process, which has been investigated in resistive MHD theory and simulations [79, 80, 81, 82], is widely believed to be connected with various explosive magnetic events such as solar flares and magnetospheric sub-storms. In the next section, we show that as the aspect ratio increases, the total dissipated magnetic energy increases and a larger fraction of magnetic energy is converted into the kinetic energy of the ions.
Figure 4.3: Snapshots of the magnetic field lines (line) and out-of-plane current density (color) at four different times (a) $t = 182\Omega_i^{-1}$, (b) $t = 302\Omega_i^{-1}$, (c) $t = 412\Omega_i^{-1}$, (d) $t = 552\Omega_i^{-1}$. The parameters are $\frac{a}{\rho_i} = 1.0, \beta_e = \frac{\mu_0 n_0 T_e}{E_0^2} = 0.5, \frac{T_i}{T_e} = 1, \frac{L_i}{\rho_i} = 12.8, \frac{L_e}{\rho_e} = 25.12, \eta^{\text{seesaw}} = 0.0015$. 
Figure 4.4: Snapshots of the magnetic field lines (line) and out-of-plane current density (color) at four different times (a) \( t = 182\Omega_i^{-1} \), (b) \( t = 352\Omega_i^{-1} \), (c) \( t = 472\Omega_i^{-1} \), (d) \( t = 552\Omega_i^{-1} \). The parameters are \( \frac{\rho_i}{\rho_i} = 1.0, \beta_e = \frac{\mu_0 n_0 T_e}{T_i} = 0.5, \frac{T_i}{T_e} = 1, \frac{L_i}{\rho_i} = 12.8, \frac{T_i}{\rho_i} = 100.48, \eta^\text{em} \frac{T_i}{T_0} = 0.0015. \)
4.4 Ion heating and Energy conversion

In this section, we discuss the ion heating and energy conversion during magnetic reconnection. Long time observations of the solar corona using X-rays has indicated this is related to the source of fast solar winds [83]. X-ray bright spots usually endure 8 hours and consist of several small loops. The closed field regions are connected with the slow solar wind, whereas the fast solar wind comes from coronal holes, where the density and temperature reach $5 \times 10^{11} \text{ m}^{-3}$ and $1.6 \times 10^6 \text{ K}$ at a height of one solar radius. A particularly interesting kind of loops are the ones that join different active regions which are highly dynamic and with continual activity and a wide range of flows. In the active regions of solar corona, a combined radiative and conductive losses expressed by energy flow is around 5000 W/m$^2$. It is widely believed that magnetic energy is connected to this energy balance, especially in the active regions that are usually accompanied by strong magnetic fields. This is also where the exploration in magnetic reconnection began. In the active regions, for a plasma flow speed of 100 m/s and a field strength of 10 mT, the electromagnetic energy flow represented by Poynting flux is

$$|S| = |E \times H| \sim \frac{uB^2}{\mu} \sim 10^4 \text{W/m}^2.$$  

This energy flow is large enough to maintain the energy balance.

In our simulation, we have observed that the kinetic energy of ions increases in the island region and a larger fraction of magnetic energy is converted to the kinetic energy of ions. Figure 4.5 shows the initial and final magnetic energy (scaled by $l_y$) for current sheets with different aspect ratios. As the aspect ratio increases, the final magnetic energy (scaled by $l_y$) decreases, which means that a larger fraction of magnetic energy is released. This decrease in final magnetic energy is consistent with the saturated island widths (Figure 4.1). Since only one island exists in the saturation stage, the larger the island width, the more magnetic fields are annihilated and hence more magnetic energy is released. The ratio of the increase in the ion kinetic energy and the dissipated magnetic energy against the aspect ratio is shown in Figure 4.6. This ratio first grows roughly linearly with aspect ratios until reaching an asymptotic value which is a little over
50% for aspect ratios larger than 6. It is interesting to note that in the MRX experiment, Hsu et al. measured the local ion heating and their measured conversion rate at $48 \pm 21\%$, which is comparable with our results.

Figure 4.5: The initial and final magnetic energy $ME$ (scaled by $l_y$) for various aspect ratios ($l_y/l_x$).

The parameters are $\frac{a}{\rho_i} = 1.0$, $\beta_e = \frac{n_{ion}T_e}{B_0^2} = 0.5$, $\frac{T_i}{T_e} = 1$, $\frac{l_x}{\rho_i} = 12.8$, $\eta_{en}B_0 = 0.0015$. 
Figure 4.6: The ratio of the ion kinetic energy increase to the dissipated magnetic energy (scaled by $l_y$) for various aspect ratios ($l_y/l_x$). The parameters are $\frac{a}{\rho_i} = 1.0$, $\beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5$, $\frac{T_i}{T_e} = 1$, $\frac{l_x}{\rho_i} = 12.8$, $\eta_{\text{crit}} \frac{\rho_0}{B_0} = 0.0015$. 
Inside the island region \((-1.6 < x < 1.6)\), more evidence of ion kinetic energy increase is revealed by the perturbed distribution function. In Figure 4.7, the perturbed energy spectra inside the island region are shown for four different aspect ratios at four different times. As we can see, the perturbed ion distribution function decreases near zero and peaks at around \(2T_{i0}\) as time develops. This indicates the ions inside the island regions are heated to higher temperature than in the outside. And this ion heating becomes more significant as the aspect ratio increases.

For the large aspect ratio case \(l_y = 100.48\), the energy spectra (blue line) inside the island region \((-1.6 < x < 1.6)\) are shown in four different times in Figure 4.8 and compared to a Maxwellian energy spectra (green line). In Figure 4.8, the energy spectra displays a slightly larger tail in the relatively high energy part. This indicates the ions inside the island region is heated. The ion density, on the other hand, does not change much inside the island region (Figure 4.7). Therefore, this ion kinetic energy is unlikely the result of the plasma compressing and should be attributed to mechanisms leading to ion acceleration.
Figure 4.7: Snapshots of the perturbed ion distribution function inside the island region ($-1.6 < x < 1.6$) for different aspect ratios at 4 different times (a) $t = 0\Omega_i^{-1}$, (b) $t = 200\Omega_i^{-1}$, (c) $t = 400\Omega_i^{-1}$, (d) $t = 600\Omega_i^{-1}$. The energy unit is $T_i0$. The parameters are $\frac{a}{\rho_i} = 1.0, \beta_e = \frac{\mu_0\alpha_i T_e}{B_0^2} = 0.5, \frac{T_i}{T_e} = 1, \frac{\rho_e}{\rho_i} = 12.8, \eta \frac{\alpha_i}{B_0} = 0.0015$. 
Figure 4.8: Snapshots of the total distribution function of ions inside the island region ($-1.6 < x < 1.6$) for $l_y = 100.48$ at 4 different times (a) $t = 0\Omega_i^{-1}$, (b) $t = 200\Omega_i^{-1}$, (c) $t = 400\Omega_i^{-1}$, (d) $t = 600\Omega_i^{-1}$. The energy unit is $T_i0$. The blue line represents the energy spectra of the particles inside the island region and the red line shows the energy spectra of the Maxwellian distribution function.

The parameters are $\frac{a}{\rho_i} = 1.0, \beta_e = \frac{\nu_0 n_0 T_e}{B_0^2} = 0.5, \frac{T_i}{T_e} = 1, \frac{l_x}{\rho_i} = 12.8, \frac{l_y}{\rho_i} = 100.48, \eta e / B_0 = 0.0015$. 

4.5 Discussion

In the previous section, it was established that ion kinetic energy increases and the fraction of magnetic energy converted to ion kinetic energy approaches to an asymptotic value as the aspect ratio increases. In order to understand the energy conversion, let us start from the energy equation of the system. Taking the second moment of Eq. (4.3)

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q_i}{m_i} (E + v \times B) \cdot \frac{\partial f}{\partial v} = 0$$

and integrating over the phase space, we arrive at

$$\int v^2 \frac{df}{dt} d^3v d^3x + \int v^2 v \cdot \nabla f d^3v d^3x + \int v^2 \frac{q_i}{m_i} (E + v \times B) \cdot \frac{df}{dv} d^3v d^3x = 0. \quad (4.18)$$

The first term on the left hand side can be rewritten as

$$\frac{\partial}{\partial t} \left( \int v^2 d^3x \right) = \frac{\partial}{\partial t} \left( \frac{2}{m_i} KE \right), \quad (4.19)$$

which is just the time derivative of the kinetic energy $KE$. The second term vanishes and the third term reduces to

$$-\frac{2}{m_i} \int E \cdot j_i \, d^3x, \quad (4.20)$$

where $j_i$ is the ion current density. Now we can rewrite Eq. (4.18) as

$$\frac{\partial}{\partial t} (KE) = \int E \cdot j_i \, d^3x. \quad (4.21)$$

Next we take the cross product of Eq. (4.12) and $B$ to obtain $j_i$, then the rhs can be rewritten as

$$\int E \cdot j_i \, d^3x = \int \frac{1}{\mu_0} \nabla \times B \cdot E \, d^3x - \left\{ \int \frac{1}{B^2} \left[ \frac{1}{\mu_0} \nabla \times B - j_i \right] \cdot B \right\} B \cdot E \, d^3x + \int \frac{\eta n_0}{\mu_0} B^2 \nabla \times B \times B \cdot E \, d^3x - \int \frac{1}{B^2} \nabla \cdot \Pi_e \times B \cdot E \, d^3x - m_e \int \frac{1}{B^2} \frac{dn_e u_e}{dt} \times B \cdot E \, d^3x, \quad (4.22)$$

By integrating by parts, the first term on the rhs of Eq. (4.22) becomes

$$\int \frac{1}{\mu_0} \nabla \times B \cdot E \, d^3x = -\int_S E \times H \cdot dA - \int \frac{1}{\mu_0} \nabla \times E \cdot B \, d^3x, \quad (4.23)$$
where the first term on the rhs stands for the Poynting flux through boundary $S$, which vanishes as $E$ and $B$ approach zero at the boundary. Upon using Faraday’s law, the above equation becomes
\[
\int \frac{1}{\mu_0} \nabla \times B \cdot E \, d^3x = -\frac{\partial}{\partial t} \int \frac{B^2}{2\mu_0} \, d^3x.
\] (4.24)

Thus Eq. (4.22) can now be rewritten as the familiar form:
\[
\frac{\partial ME}{\partial t} = \int E \cdot j \, d^3x,
\] (4.25)

where
\[
ME = \int \frac{B^2}{2\mu_0} \, d^3x
\]
represents the magnetic energy density. Using the relation $j = j_i + j_e$, the physics meaning of Eq. (4.25) becomes more clear in the following form
\[
\frac{\partial ME}{\partial t} = \int E \cdot j_i \, d^3x + \int E \cdot j_e \, d^3x,
\] (4.26)
as the first term and second term on the rhs of Eq. (4.26) represent ion and electron energy variation, respectively.

To understand the energy conversion mechanism, several theoretical models have been proposed. Among these models, there are viscous heating, Hall effect, stochastic heating, and resonant wave-particle interactions [84, 85, 86]. Collisions are the chief mechanism in low-temperature or dense plasmas, although collisions amongst the same species of particles cannot increase the aggregate energy of the colliding particles, they can convert bulk kinetic energy to thermal energy through viscosity. The energy exchange with electrons is not tracked because electrons are assumed to be isothermal whose temperature is constant in the simulation (Eq. 4.26). In our simulation, the resistivity is kept rather low, so that the plasma is nearly collisionless. Therefore, the quadrupole fields associated with the Hall term is likely to play an role in the ion kinetic energy increase. In a simple picture of Hall-MHD, near the $X$ point, the electron density drops as the in-plane magnetic field decreases (electrons are strongly magnetized). However, the ions are not magnetized and their density remains almost the same. This inevitably creates a small in-plane electric field.
to pull the electrons along the field line inward to preserve charge neutrality. By Ampere’s law, the electric current associated with this electron flow generates a quadrupole out-of-plane magnetic field concentrated along the $X$ point. Further analysis shows that a bipolar in-plane electric field is generated due to this quadrupole out-of-plane magnetic field [86]. And this bipolar in-plane electric field can efficiently accelerate ions from in-flow region to out-flow region. Going back to our simulation, our natural next step is to figure out the pattern of this in-plane electric field and its associated quadrupole out-of-plane magnetic field near the $X$ point. And then take the contour plot of the ion kinetic energy changing rate $E \cdot j$, as expressed in Eq. (4.26), to determine how and where the ions are accelerated. From the in-plane flow pattern, we can choose appropriate tracer particles and in principle study the ion acceleration mechanism in Hall-MHD.

4.6 Summary

In this work, we have demonstrated the magnetic island chains formation and their eventual coalescence into a single elongated island in a current sheet with large aspect ratios. These highly elongated magnetic islands have been observed in heliosheath. It has been suggested that the pressure anisotropy prevents the full contraction of the islands [78]. We also observed ion kinetic energy increase inside the island region using various ion diagnostics including scans of ion distribution function. As the aspect ratio increases, more magnetic energy is dissipated and a larger fraction of magnetic energy is converted into ion kinetic energy. The ratio of the increase in ion kinetic energy and the dissipated magnetic energy grows linearly with aspect ratio until it reaches an asymptotic value of about 50%. This is comparable to the measurements from MRX [75]. We suspect this energy conversion and ion kinetic energy increase is connected to the bipolar electric field associated with the quadrupole out-of-plane magnetic field in Hall-MHD reconnection. Possible diagnostics of our simulation are proposed to investigate these phenomena.
In an inhomogeneous plasma, the system may deviate significantly from the thermodynamic equilibrium characterized by a homogeneous Maxwellian distribution. Such deviations can drive certain instabilities that generate turbulence leading to anomalous transports in tokamaks [87, 88], hence destroy the particle and energy confinement in the system. The most destructive instabilities fall into the category of relatively low frequency modes, the so-called microinstabilities, which may persist even when the large-scale MHD modes have been suppressed. One of the most studied microinstabilities is the ion-temperature-gradient-driven instability (ITG) [89, 90]. It was first suggested to be related to the anomalous transport discovered in the Alcator C Tokamak [91], which triggered extensive research interest in the ITG turbulence [92, 93]. Since the free energy source of the ITG is provided by the total ion pressure gradient, ITG can be driven unstable by both temperature gradient and density gradient. In a slab geometry, ITG appears to be a sound wave destabilized by the ion temperature gradient. Although a simple two fluid model can be used to describe the instability, in order to include essential kinetic effects such as the wave-particle interactions and finite Larmor Radius (FLR) effects, a kinetic treatment is required.

In this Chapter, I present the kinetic theory of the slab ITG in its linear stage with the “local approximation” which holds when the characteristic length scale associated with the temperature gradient is small compared to the length scale of the system. For the electrostatic case, the local kinetic dispersion relation and the corresponding form in the fluid limit are derived. Using the hybrid model described in Chapter 2, the slab ITG is investigated with important kinetic effects.
such as FLR effects and Landau damping. The hybrid simulation agrees well with the kinetic dispersion relation. We have further compared the simulation to the extended MHD NIMROD simulation [23]. The hybrid kinetic and fluid calculations agree well near the marginal stability point, but disagree as $k_{\perp} \rho_i$ or $\rho_i/L_{T_i}$ increase where the kinetic effects becomes important. Good qualitative agreement between the models for the shape of the unstable global eigenfunction is reported. These results serve as a good quantification of how far fluid calculations can be extended accurately into the kinetic regime for the case of linear ITG.

5.1 The local kinetic dispersion relation of the ITG

In an inhomogeneous plasma, a temperature gradient $\frac{\partial T(x)}{\partial x}$ perpendicular to $B_0$ is a potential source of free energy. Since hot ions drift faster than the cold ones, this process produces the charge separation and the associated electric field. Through the $E \times B$ drift, hot ions are pushed to higher temperature $T(x)$ and cold ions toward lower temperature, driving the temperature gradient perturbation unstable. Following the discussion in [94], we now derive the local kinetic dispersion relation of ITG in a slab geometry. First we discuss the equilibrium configuration. Assuming the equilibrium magnetic field is $B = B_0 e_z$, the corresponding vector potential is thus $A = xB_0 e_y$ under Coulomb gauge. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}mv^2 + qv \cdot A.$$  \hspace{1cm} (5.1)

We then have two constants of motion: the conjugate momentum $p_y = mv_y + qxB_0$ and the Hamiltonian $H = \frac{1}{2}mv^2$. We can therefore write the equilibrium distribution function (with a uniform density) as

$$f_0(X,H) = \frac{n_0}{(2\pi T(X)/m)^{3/2}}e^{-H/T(X)},$$  \hspace{1cm} (5.2)

where $X = x + v_y/\Omega$ is the projection of guiding center along the $x$ direction and $\Omega = \frac{qB_0}{m}$ is the gyro-frequency. Assuming the inhomogeneity related to $T(x)$ is very small (the characteristic length $L$ is large compared to the gyro-radius $\rho$), the quasi-Maxwellian $f_0$ can be expanded to the
lowest order of the small parameter $\epsilon = \rho/L$ [95]

$$f_0(X, H) = f_0(x, H) + \frac{\partial f_0(x, H)}{\partial x} \frac{v_y}{\Omega},$$

$$f_0(x, H) + \frac{dT}{dx} \frac{\partial}{\partial T} f_0(x, H) \frac{v_y}{\Omega},$$

$$= \frac{n_0}{(2\pi T(x)/m)^{3/2}} e^{-mv^2/2T(x)} + \frac{d\ln T(x)}{dx} \left( \frac{mv^2}{2T(x)} - \frac{3}{2} \right) f_0 \frac{v_y}{\Omega}. \quad (5.3)$$

Take the first moment of this distribution, we obtain the plasma flow $U$

$$U = \frac{1}{n_0} \int d\mathbf{v} f_0(X, H)$$

$$= \frac{1}{n_0} \int d\mathbf{v} \left( f_0(x, H) + \frac{dT}{dx} \frac{\partial}{\partial T} f_0(x, H) \frac{v_y}{\Omega} \right)$$

$$= \frac{1}{qB_0} \frac{dT}{dx} \frac{e_y}{\Omega} \quad (5.4)$$

which is just the diamagnetic drift

$$V_d = -\frac{1}{qn_0B_0^2} \nabla p \times \mathbf{B}_0. \quad (5.5)$$

Consider a drift Maxwellian

$$f_d(x, H) = \frac{n_0}{(2\pi T(x)/m)^{3/2}} e^{-m(v-V_d)^2/2T(x)}, \quad (5.6)$$

if $V_d$ is small compared to the thermal velocity, we can expand $f_d$ as follows

$$f_d(x, H) = \frac{n_0}{(2\pi T(x)/m)^{3/2}} e^{-mv^2/2T(x)} \left( 1 + \frac{m}{T} \mathbf{v} \cdot \mathbf{V}_d \right)$$

$$= \frac{n_0}{(2\pi T(x)/m)^{3/2}} e^{-mv^2/2T(x)} \left( 1 + \frac{d\ln T(x)}{dx} \frac{v_y}{\Omega} \right)$$

$$= f_0 + \frac{d\ln T(x)}{dx} f_0 \frac{v_y}{\Omega} \quad (5.7)$$

This is similar to the expansion in Eq. (5.3).

Now we begin to solve the perturbed distribution function by integrating the linearized Vlasov equation along the unperturbed particle orbits. For simplicity, we only consider the electrostatic case. Assuming very weak inhomogeneity in $x$, only perturbations local to the surface $x = 0$ are
considered hereafter. The linearized perturbed electric potential $\phi$ and distribution function $\delta f$ can then be written as

$$
\phi = \tilde{\phi} e^{i(k_0 y + k_2 z - \omega t)},
$$

$$
\delta f = \tilde{\delta f} e^{i(k_0 y + k_2 z - \omega t)},
$$

(5.8)

where $f = f_0 + \delta f$. According to the linearized Vlasov equation

$$
\frac{D}{Dt} \delta f = \left[ \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + \frac{q}{m} (-\nabla \phi + v \times B) \cdot \frac{\partial}{\partial v} \right] \delta f = \frac{q}{m} \nabla \phi \cdot \frac{\partial f_0}{\partial v}.
$$

(5.9)

Integrating along the unperturbed particle orbits yields

$$
\delta f(r, v, t) = \frac{q}{m} \int_{-\infty}^{t} dt' \nabla \phi \cdot \frac{\partial f_0}{\partial v} |_{r', v', t'}.
$$

(5.10)

The unperturbed particle orbits are obtained by direct integrating the equations of motion

$$
\frac{d r'}{d t'} = v',
$$

$$
\frac{d v'}{d t'} = \frac{q}{m} (-\nabla \phi + v' \times B),
$$

(5.11)

with the initial conditions $r'(t' = t) = r, v'(t' = t) = v$. The particle motion is simply the gyro-motion, which can be expressed as

$$
r' = r + \frac{1}{\Omega} Q(t' - t) \cdot v,
$$

$$
v' = R(t' - t) \cdot v,
$$

(5.12)

where $Q$ and $R$ stands for

$$
Q(\tau) = \begin{pmatrix}
\sin(\Omega \tau) & 1 - \cos(\Omega \tau) & 0 \\
-1 + \cos(\Omega \tau) & \sin(\Omega \tau) & 0 \\
0 & 0 & \Omega \tau \\
\end{pmatrix},
$$

$$
R(\tau) = \begin{pmatrix}
\cos(\Omega \tau) & \sin(\Omega \tau) & 0 \\
-\sin(\Omega \tau) & \cos(\Omega \tau) & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
$$

Here $\tau = t' - t$. According to Eq. (5.3), to the lowest order of $\epsilon$,

$$
\frac{\partial f_0}{\partial v} \Rightarrow \left( \frac{e_y}{\Omega} \frac{dT}{dx} \frac{\partial}{\partial T} - \frac{v}{v_{th}^2} \right) f_0,
$$

(5.13)
where $v_{th} = \sqrt{T/m}$. The rhs of Eq. (5.9) can then be written as

$$\frac{q}{m} \nabla \phi \cdot \frac{\partial f_0}{\partial v} = \frac{q}{T} (i \omega_T' - i k \cdot v) f_0 \tilde{\phi} e^{i(k \cdot r - \omega t)},$$

(5.14)

where the drift frequency operator is

$$\omega_T' = \frac{T k y}{q B_0} \frac{dT}{dx} \frac{\partial}{\partial T}$$

and the wave vector $k = k_y e_y + k_z e_z$.

Notice that

$$\frac{df_0(r', v', t')}{dt'} = 0,$$

(5.15)

the integrand in Eq. (5.10) can be rewritten as

$$\frac{q}{m} \nabla \phi \cdot \left. \frac{\partial f_0}{\partial v} \right|_{r', v', t'} = \frac{q}{T} \left( i(\omega_T' - \omega) - \frac{d}{dt'} \right) f_0 \tilde{\phi} e^{i(k \cdot r' - \omega t')},$$

(5.16)

Integrating by parts, and using the definition

$$v_x = v_\perp \sin \theta, \quad v_y = v_\perp \cos \theta,$$

(5.17)

Eq. (5.10) becomes

$$\tilde{\delta f} = -\frac{q \tilde{\phi}}{T} \left[ 1 - i(\omega_T' - \omega) \int_{-\infty}^{t} dt' e^{i(k \cdot (r' - r) - \omega (t' - t))} \right] f_0$$

$$= -\frac{q \tilde{\phi}}{T} \left[ 1 - i(\omega_T' - \omega) \int_{-\infty}^{0} d\tau e^{i k_y v_\perp \Omega \sin(\Omega \tau + \theta) - \frac{k_y v_\perp}{n} \sin \theta + i k_z v_z - \omega \tau} \right] f_0.$$

(5.18)

Notice the generating function can be expanded as

$$e^{iz \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\theta}.$$

(5.19)

Carrying out the integration in Eq. (5.18), the amplitude of the perturbed distribution function is

$$\tilde{\delta f} = -\frac{q \tilde{\phi}}{T} \left[ 1 - (\omega_T' - \omega) \sum_{n, n'=-\infty}^{\infty} \frac{J_n \left( \frac{k_y v_\perp}{n} \right) J_{n'} \left( \frac{k_y v_\perp}{n'} \right) e^{i(n-n') \theta}}{k_z v_z + n \Omega - \omega} \right] f_0.$$

(5.20)

The next step is to calculate the perturbed density $\delta n$. By definition, the perturbed density $\delta n$ comes from the first moment of the Vlasov equation

$$\delta n = \int dv \delta f,$$

(5.21)
Using the following formula \[96, 97]\[
\int_0^\infty dx x e^{-\alpha x^2} J_n(\beta x) J_n(\gamma x) = \frac{1}{2\alpha} e^{-\frac{\beta^2 + \gamma^2}{4\alpha}} I_n(\frac{\beta \gamma}{2\alpha}),
\] (5.22)
and the recurrence relations of the Bessel functions
\[
\begin{align*}
\frac{2n}{x} J_n(x) &= J_{n-1}(x) + J_{n+1}(x), \\
\frac{2}{x} \frac{\partial J_n(x)}{\partial x} &= J_{n-1}(x) - J_{n+1}(x), \\
\frac{2}{x} \frac{\partial I_n(x)}{\partial x} &= I_{n-1}(x) + I_{n+1}(x),
\end{align*}
\] (5.23)
we can carry out the integration in Eq. (5.21) analytically. The perturbed density is therefore
\[
\delta n = -\frac{q n_0 \check{\phi}}{T} \left\{ 1 - \omega' T - \omega \sum_{n=-\infty}^{\infty} \frac{1}{\omega - n\Omega} \left[ W\left(\frac{\omega}{k_z v_{th}}\right) - 1\right] \Lambda_n(\xi) \right\},
\] (5.24)
where the plasma dispersion function is defined as
\[
W(z) = \frac{1}{\sqrt{2\pi}} \int_0^z dx \frac{x}{x-z} e^{-x^2/2},
\]
and the modified Bessel function is \(\Lambda_n(x) = e^{-x} I_n(x)\). The parameter \(\xi = (k y v_{th}/\Omega)^2\) represents the FLR effects.

5.2 The electrostatic dispersion relation

The electrostatic dispersion relation results in the Poisson equation
\[
k^2 \phi = \frac{1}{\epsilon_0} \sum_{\text{species}} q \delta n.
\] (5.25)
Substituting Eq. (5.24) into the above equation, we arrive at
\[
1 + \sum_{\text{species}} \frac{1}{(k\lambda_D)^2} \left\{ 1 + (\omega - \omega' \omega) \sum_{n=-\infty}^{\infty} \frac{1}{\omega - n\Omega} \left[ W\left(\frac{\omega}{k_z v_{th}}\right) - 1\right] \Lambda_n(\xi) \right\} = 0,
\] (5.26)
where the Debye length is defined by \(\lambda_D^2 = v_{th}^2/\omega_p^2 = \epsilon_0 T/n e^2\). For relatively low frequency phenomena, \(\omega \ll \Omega\), we can keep only one term \((n = 0)\) in Eq. (5.26). The local electrostatic kinetic dispersion relation is thus reduced to
\[
1 + \sum_{\text{species}} \frac{1}{(k\lambda_D)^2} \left\{ 1 + \frac{\omega - \omega' T}{\omega} \left[ W\left(\frac{\omega}{k_z v_{th}}\right) - 1\right] \Lambda_0(\xi) \right\} = 0,
\] (5.27)
5.2.1 Instability boundary

We work in the regime $|k_{\perp} v_{thi}| \ll 1$ so that electrons respond adiabatically, and $k\lambda_D \ll 1$ so that quasi-neutrality can be assumed. Therefore, the dispersion relation Eq. (5.27) becomes

$$\left(\frac{1}{(k\lambda_D) v_e^2} + \frac{1}{(k\lambda_D) v_i^2}\right)\left[1 + \left(1 - \frac{\omega_{T_i}}{\omega}\right)\left[W\left(\frac{\omega}{k_{\perp} v_{thi}}\right) - 1\right] \Lambda_0(\xi_i)\right] = 0. \quad (5.28)$$

The limits of instability are obtained by finding the conditions under which the solutions $\omega$ to Eq. (5.28) are real values. This sets the limit between damped and unstable modes.

Carrying out the derivative with respect to the ion temperature of the operator $\omega_{T_i}'$, Eq. (5.28) reads

$$\frac{T_i}{T_e} + \left\{1 + \left(1 + \frac{\omega_{T_i}}{2\omega}\right)[W - 1] \Lambda_0(\xi_i) - \frac{\omega_{T_i}}{\omega} \left[\frac{z_i^2}{2} W \Lambda_0(\xi_i) + (W - 1)\xi_i(\Lambda_1(\xi_i) - \Lambda_0(\xi_i))\right]\right\} = 0, \quad (5.29)$$

where $z_i = \frac{\omega}{k_{\perp} v_{thi}}, W = W(z_i)$. Note that $\omega_{T_i}'$ operates on both $W(z_i)$ through $v_{thi} = \sqrt{T_i/m_i}$ and $\Lambda_0(\xi_i)$ through $\xi_i = k_{\perp}^2 T_i/m_i \Omega_i^2$, the following relations are used

$$\omega_{T_i}' W(z_i) = -\frac{1}{2} \omega_{T_i}[(1 - z_i^2)W - 1],$$

$$\omega_{T_i}' \Lambda_0(\xi_i) = \omega_{T_i} \xi_i(\Lambda_1 - \Lambda_0),$$

where

$$\omega_{T_i} = \frac{T_i k_y d\ln T_i(x)}{eB \frac{dx}{dx}}. \quad (5.30)$$

To identify the instability boundary, one solves Eq. (5.28) under the condition $Im(\omega) = 0$. Using the following expression for $W(z)$ in terms of the complex error function:

$$W(z) = 1 - z e^{-z^2/2} \int_0^z dt e^{t^2/2} + i \sqrt{\frac{\pi}{2}} z e^{-z^2/2},$$

after some simple algebra, one obtains the following equations

$$\frac{T_i}{T_e} + 1 - \frac{\omega_{T_i} z_i^2}{\omega} \Lambda_0 = 0, \quad (5.31)$$

$$\left(1 + \frac{\omega_{T_i}}{2\omega}\right) \Lambda_0 - \frac{\omega_{T_i}}{\omega} \xi_i(\Lambda_1 - \Lambda_0) - \frac{\omega_{T_i} z_i^2}{\omega} \Lambda_0 = 0. \quad (5.32)$$
Further simplification yields

\[
\frac{z_i^2}{2} = \frac{1 + T_i/T_e}{\Lambda_0} \frac{\omega}{\omega_{T_i}} \geq 0. 
\]  
(5.33)

Therefore, the instability boundary is determined by

\[
\frac{\omega}{\omega_{T_i}} = \frac{1}{2} \frac{\Lambda_0 - 2\xi_i(\Lambda_1 - \Lambda_0)}{1 + T_i/T_e - \Lambda_0} \geq 0. 
\]  
(5.34)

Inserting \(z_i^2 = \omega^2/(k_z v_{thi})^2\) into Eq. (5.33) and (5.34) provides the limiting condition on \(k_z\) as

\[
k_z v_{thi} \leq \frac{\omega_{T_i}}{\sqrt{2}} \sqrt{\frac{\Lambda_0}{1 + T_i/T_e} \sqrt{\frac{\Lambda_0 - 2\xi_i(\Lambda_1 - \Lambda_0)}{1 + T_i/T_e - \Lambda_0}}}, 
\]  
(5.35)

which is plotted in Figure 5.1.

---

Figure 5.1: Upper limit on \(k_z L_T\) as a function of \(\xi_i\) where \(L_T\) represents the characteristic length scale for the temperature gradient. The blue line serves as the instability boundary for \(T_i/T_e = 1\).
5.2.2 The dispersion relation in the fluid limit

Starting from Eq. (5.28),

\[ \frac{1}{(k\lambda_D e)^2} + \frac{1}{(k\lambda_D i)^2} \left\{ 1 + \left( 1 - \frac{\omega'}{\omega} \right) \left[ W \left( \frac{\omega}{k_z v_{thi}} \right) - 1 \right] \Lambda_0(\xi) \right\} = 0. \] (5.36)

If the mode falls in the regime \( \left| \frac{\omega}{k_z v_{thi}} \right| \gg 1 \), we can use

\[ W(z) \approx -\frac{1}{z^2} + i \sqrt{\frac{\pi}{2}} z e^{-z^2/2}, \quad z \gg 1. \]

to expand the dispersion relation. Neglecting the FLR effects, \( \xi_i \ll 1 \), one obtains:

\[ \frac{1}{(k\lambda_D e)^2} + \frac{1}{(k\lambda_D i)^2} \left\{ 1 - \left( 1 - \frac{\omega'}{\omega} \right) \left[ 1 + \left( \frac{k_z v_{thi}^2}{\omega} \right)^2 \right] \right\} = 0. \] (5.37)

After carrying out the partial derivative with respect to \( \omega'_{Ti} \), the dispersion relation reduces to

\[ \frac{T_i}{T_e} - \left( \frac{k_z v_{thi}}{\omega} \right)^2 \left( 1 - \frac{\omega_{Ti}}{\omega} \right) = 0, \] (5.38)

where \( \omega_{Ti} = (T_i k_y / eB)(d \ln T_i / dx) \). If \( \omega \ll \omega_{Ti} \), the above equation becomes

\[ \frac{T_i}{T_e} + \frac{\omega_{Ti} (k_z v_{thi})^2}{\omega^3} = 0, \] (5.39)

with an unstable solution

\[ \omega = \left( \frac{1}{2} + \frac{i \sqrt{3}}{2} \right) \left( \omega_{Ti} (k_z v_{thi})^2 \frac{T_e}{T_i} \right)^{1/3}. \] (5.40)

This is a slightly more accurate expression than Eq. (1.44) in [94]. The initial assumptions \( v_{thi} \ll \omega / k_z \ll v_{the} \) yields

\[ \frac{1}{8} \sqrt{\frac{T_i}{T_e}} \left( \frac{m_e}{m_i} \right)^{3/2} \ll \frac{1}{\rho_i \sqrt{\ln T_i}} \frac{k_z}{k_y} \ll 8 \frac{T_e}{T_i}. \] (5.41)

The other assumption \( \omega \ll \omega_{Ti} \) imposes another constraint

\[ \frac{1}{\rho_i \sqrt{\ln T_i}} \frac{k_z}{k_y} \ll \sqrt{\frac{T_i}{T_e}}. \] (5.42)

In the two-fluid model [23], the slab ITG arises from the heat convection due to the ion temperature gradient. The ions are described by the continuity equation, the momentum equation
and the heat equation
\[
\begin{align*}
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i u_i) &= 0, \\
n_i m_i \left( \frac{\partial u_i}{\partial t} + u_i \cdot \nabla u_i \right) &= en_i (E + u_i \times B) - \nabla (n_i T_i), \\
\frac{\partial n_i T_i}{\partial t} + \nabla \cdot (n_i T_i V_E) &= 0,
\end{align*}
\]
(5.43)

where \( V_E = \frac{E \times B}{B^2} \). The electrons respond adiabatically,
\[ n_e = n_0 \exp \left( \frac{e \phi}{T_e} \right). \] (5.44)

And the quasi-neutrality \( n_i = n_e \) provides the closure. Consider a plasma with \( \nabla n_0 = 0 \) and \( \nabla T_i \neq 0 \), and we work in the relatively low frequency regime \( \omega / \Omega_i \ll 1 \). Assuming the density \( n_i = n_0 + \delta n \) and temperature \( T_i = T_{i0} + \delta T_i \), after linearizing the above equations, we have
\[
\begin{align*}
\frac{\partial \delta n}{\partial t} + n_0 \nabla \| u_i \| &= 0, \\
m_i \frac{\partial \| u_i \|}{\partial t} &= e E \| - \nabla \| \delta T_i, \\
\frac{\partial \delta T_i}{\partial t} + V_E \cdot \nabla T_{i0} &= 0, \\
\delta n &= n_0 \frac{e \phi}{T_e}. 
\end{align*}
\] (5.45)

If the perturbation has the form of \( E = -\nabla \phi \propto \exp(i k \cdot x - i \omega t) \), we then have
\[
\begin{align*}
-i \omega \delta n + n_0 i k_z u_{iz} &= 0, \\
-i \omega m_i u_{iz} &= -e i k_z \phi - i k_z \delta T_i, \\
-i \omega \delta T_i - \frac{ik_y \phi}{B} \frac{\partial T_{i0}}{\partial x} &= 0, \\
\delta n &= n_0 \frac{e \phi}{T_e}.
\end{align*}
\] (5.46)

By solving this set of reduced equations, we arrive at the same equation as Eq. (5.38).

5.3 Electrostatic simulation results

In this section, I first present the local simulation results and their comparison with the local dispersion relation (Eq. (5.28)). Then I proceed to show global simulations with a varying
temperature profile. These results have been used in the validation work with a two-fluid calculation and the NIMROD simulation [98, 23].

5.3.1 The local simulation

In the hybrid simulation, the particles are loaded as $f_0(x, H)$ (Eq. (5.3)). The weight equation is therefore (to the lowest order in $\epsilon$)

$$
\frac{dw}{dt} = -\frac{q}{m} E \cdot \frac{\partial \ln f_0}{\partial v} = -\frac{q}{m} \left[ \frac{E_y}{\Omega} \frac{d\ln T(x)}{dx} \left( \frac{m v^2}{2 T(x)} - \frac{3}{2} \right) - \frac{m}{T(x)} E \cdot v \right] = -\frac{q}{m} \left[ \frac{E_y}{\Omega} \kappa T \left( \frac{m v^2}{2 T(x)} - \frac{3}{2} \right) - \frac{m}{T(x)} E \cdot v \right].
$$

(5.47)

Note that the first term on the rhs represents the drive for ITG instability. In the local simulations, we load the particles as Maxwellian with a constant given temperature $T$, and $\kappa T$ is a given parameter for the weight equation. By setting the constant temperature and a local temperature gradient represented by $\kappa T$, we essentially enforce the “local approximation” strictly. Excellent agreement between the local hybrid simulation and the local dispersion relation is observed. Figure 5.2 shows the real frequencies and growth rates scan over $\kappa T$ from the hybrid simulation and the dispersion relation. In this case, $k_y \rho_i = 0.3$, $k_z \rho_i = 0.003$, and $T_i/T_e = 1$. Figure 5.3 shows the real frequencies and growth rates scans over $k_z \rho_i$ with ion temperature kept at the same with electron temperature, and the temperature gradient fixed at $\kappa T = 0.1$. In Figure 5.3, the growth rate decreases for larger $k_z$ suggesting that $k_z$ must be below a critical value for the slab ITG to develop, which is consistent with Eq. (5.42).
Figure 5.2: Real frequencies and growth rates scan over $\kappa_T$ with $T_i/T_e = 1$, $k_z \rho_i = 0.003$, and $k_y \rho_i = 0.3$. The real frequency and growth rate are scaled to ion gyro-frequency. Red and blue squares and circles represent results from the local dispersion relation and local hybrid simulation respectively.
Figure 5.3: Real frequencies and growth rates scan over $k_z$ while $\kappa_T = 0.1$, $T_i/T_e = 1$, and $k_y \rho_i = 0.3$. The real frequency and growth rate are scaled to ion gyro-frequency. Red and blue squares and circles represent results from the local dispersion relation and local hybrid simulation respectively.
5.3.2 The global simulation

In the simplified slab geometry, the equilibrium quantities vary only in the $x$-direction, and the magnetic field is uniform in $z$-direction. The equilibrium fields are determined by force balance

$$E_0 + U_{0s} \times B_0 - \frac{1}{n_0 q_s} \nabla P_{0s} = 0,$$

(5.48)

where $q_s, U_{0s},$ and $P_{0s}$ are charge, flow, and pressure for particle species respectively. With the following temperature profile [23]

$$T(x) = T_{i0} \left(1 + 0.9 \tanh\left(\frac{x}{L_{Ti}}\right)\right),$$

(5.49)

there exists a diamagnetic drift for each species

$$V_{ds} = -\frac{\nabla P_{0s} \times B}{n_s q_s B^2} = \frac{1}{n_0 e B^2} \frac{dP_{0s}}{dx} e_y.$$

(5.50)

In Figure 5.4, the temperature profile, the ion drift velocity, and the instability drive $\kappa_T$ as a function of $x$ corresponding to Eq. (5.49) are plotted for three different $L_{Ti}$. For this particular temperature profile, the instability drive is not symmetric but rather peaks in the region $x < 0$. 
Figure 5.4: The temperature profile, diamagnetic drift, and the instability drive $\kappa_T$ as a function of $x$ as described in Eq. (5.49) for three different $L_{Ti}$.

By solving Eq. (5.28) numerically, the growth rate and real frequency can be obtained across the above temperature profile (see Figure 5.5). The location $x_{\text{max}}$ of the maximum growth rate and the corresponding real frequency is listed below
Table 5.1: The maximum growth rate and the corresponding real frequency obtained from the local dispersion solver

<table>
<thead>
<tr>
<th>$L_{Ti}/\rho_i$</th>
<th>10.</th>
<th>15.</th>
<th>20.</th>
<th>25.</th>
<th>30.</th>
<th>40.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{max}/\rho_i$</td>
<td>0.7</td>
<td>0.3</td>
<td>-0.7</td>
<td>-1.3</td>
<td>-2.5</td>
<td>-5.5</td>
</tr>
<tr>
<td>$\omega/\Omega_i(10^{-3})$</td>
<td>5.22</td>
<td>4.61</td>
<td>4.27</td>
<td>4.12</td>
<td>4.00</td>
<td>3.85</td>
</tr>
<tr>
<td>$\gamma/\Omega_i(10^{-3})$</td>
<td>3.71</td>
<td>2.82</td>
<td>2.22</td>
<td>1.79</td>
<td>1.45</td>
<td>0.95</td>
</tr>
</tbody>
</table>

The temperature and $\kappa_T = d\ln T(x)/dx$ at $x_{max}$ are shown in Table 5.2.

Table 5.2: The temperature and $\kappa_T = d\ln T(x)/dx$ at $x_{max}$ given by the local dispersion solver

<table>
<thead>
<tr>
<th>$L_{Ti}/\rho_i$</th>
<th>10.</th>
<th>15.</th>
<th>20.</th>
<th>25.</th>
<th>30.</th>
<th>40.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{max}/\rho_i$</td>
<td>0.7</td>
<td>0.3</td>
<td>-0.7</td>
<td>-1.3</td>
<td>-2.5</td>
<td>-5.5</td>
</tr>
<tr>
<td>$T$</td>
<td>1.063</td>
<td>1.018</td>
<td>0.9685</td>
<td>0.9530</td>
<td>0.9252</td>
<td>0.877</td>
</tr>
<tr>
<td>$\kappa_T \rho_i$</td>
<td>0.0843</td>
<td>0.0589</td>
<td>0.0464</td>
<td>0.0377</td>
<td>0.0322</td>
<td>0.0252</td>
</tr>
</tbody>
</table>

The real frequency and growth rate from the local and global simulations (in brackets) can be found in Table 5.3.

Table 5.3: The real frequency and growth rate from the local and global simulations.

<table>
<thead>
<tr>
<th>$L_{Ti}/\rho_i$</th>
<th>10.</th>
<th>20.</th>
<th>25.</th>
<th>30.</th>
<th>40.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega/\Omega_i(10^{-3})$</td>
<td>5.43(6.54)</td>
<td>4.25(4.34)</td>
<td>4.10(4.25)</td>
<td>4.04(4.12)</td>
<td>3.86(4.04)</td>
</tr>
<tr>
<td>$\gamma/\Omega_i(10^{-3})$</td>
<td>3.80(3.80)</td>
<td>2.24(2.18)</td>
<td>1.83(1.78)</td>
<td>1.47(1.47)</td>
<td>0.95(1.02)</td>
</tr>
</tbody>
</table>

The local growth rate and real frequency along the temperature is shown in Figure 5.5 for $L_{Ti}/\rho_i = 10, 15, 20, 25, 30, 40$. 
Figure 5.5: Local real frequency and growth rate on the profile. $L_{Ti}/\rho_i$. (a) $L_{Ti}/\rho_i = 10$. (b) $L_{Ti}/\rho_i = 15$. (c) $L_{Ti}/\rho_i = 20$. (d) $L_{Ti}/\rho_i = 25$. (e) $L_{Ti}/\rho_i = 30$. (f) $L_{Ti}/\rho_i = 40$. 
The growth rate and real frequency for the most unstable mode can be found in Figure 5.5 and is presented in the following figure. The local simulation agrees well with the dispersion relation for small $L_{Ti}/\rho_i$. Notice at $L_{Ti}/\rho_i = 10$, the real frequency diverges from the local dispersion relation. This is because the relatively large $\kappa_T$ breaks the assumption in the derivation of the local dispersion relation and there is no reason to expect the simulation to agree with the local dispersion relation at this parameter regime.

Figure 5.6: Real frequencies and growth rates from local dispersion solver, local and global simulations. Red and blue diamonds represent results from the dispersion relation and the local simulation at $x_{max}$ respectively, and green diamonds represent results from the global simulation.
The eigenmode structures for $L_{Ti}/\rho_i = 20$ and 30 are presented in Figure 5.7. From Figure 5.5, the ITG mode is mostly unstable at $x = -0.7$ and $x = -2.5$ for $L_{Ti}/\rho_i = 20$ and $L_{Ti}/\rho_i = 30$ respectively. The eigenmode plots show that the unstable modes are indeed peak at the predicted locations.
Figure 5.7: Eigenmode structure for $E_z$. (a) $L_{Ti}/\rho_i = 20$. (b) $L_{Ti}/\rho_i = 30$. 
5.3.3 Comparison with NIMROD

In Figure 5.8, we compare the local kinetic and fluid growth rates with the growth rate given by global NIMROD simulation. The growth rate is normalized to the ion gyro-frequency. The onset of instability occurs at $k_{\perp}\rho_i \sim 0.025$ ($k_y\rho_i$ in the simulation), indicating the necessity of FLR effects. All models agree well near the marginal point. The kinetic results are consistently lower than the fluid calculations which is justified in section 5.2.2. And the kinetic growth rate starts to decrease as $k_{\perp}\rho_i > 0.3$, and for even larger $k_{\perp}\rho_i$ the mode actually stabilizes. The fluid calculation does not show this stabilization effect as $k_{\perp}\rho_i$ increases due to the missing wave-particle interaction (e.g., Landau damping). The stabilization behavior observed in NIMROD results that is similar to the kinetic calculation is attributed to the global profile effect. The global kinetic simulation is compared with the NIMROD simulation in Figure 5.9. Local fluid results are evaluated using parameters at $x = 0$, and local kinetic values are taken at $x_{max}$ where the parameter $\kappa T$ reaches maximum. The local kinetic growth rate is systematically less than the local fluid growth rate, indicating the importance of wave-particle interaction effects. At and near the marginal stability point, the two calculations converge to reasonable agreement, but diverge as $k_{\perp}\rho_i$ or $\rho_i/L_{Ti}$ increases. This is not surprising because as $k_{\perp}\rho_i$ or $\rho_i/L_{Ti}$ increases, kinetic effects become more important and the fluid model is no longer validate. This study quantifies how far fluid calculations can be extended accurately into the kinetic regime for the case of linear ITG.
Figure 5.8: Growth rates from local calculation compared to the results from the local fluid calculation and the NIMROD simulation. The parameters are $T_e/T_i = 4$, $k_z \rho_i = 1.3 \times 10^{-4}$, $\rho_i/L_T i = 3.8 \times 10^{-3}$. 
Figure 5.9: Real frequencies and growth rates from local and global hybrid simulation compared with the two-fluid local calculation and the NIMROD simulation.
Chapter 6

Implementing the second-order accurate implicit hybrid scheme in an Extended MHD model

In this chapter, we discuss how one might implement the hybrid scheme discussed in Chapter 2 in the NIMROD code. The NIMROD resistive MHD code has the goal to obtain accurate and flexible modeling of nonlinear electromagnetic activity in computational domains that are realistic for a variety of laboratory plasmas. It is well suited for studying high temperature magnetized plasmas that are characterized by extremely anisotropic properties relative to the magnetic field direction [99, 100]. NIMROD solves 3D nonlinear non-ideal MHD equations using a semi-implicit scheme with 2D finite elements and finite Fourier series in the third direction. The self-adjoint operator along with variational approach to spatial discretization are synergistic provides good performance for simulation in extremely stiff conditions found in high temperature plasmas. The NIMROD field solver has been parallelized using distributed-memory of the SuperLU software library (http://acts.nersc.gov/superlu/).
Let us start with the extended MHD equations that NIMROD solves,

\[
\frac{\partial B}{\partial t} = -\nabla \times E + \kappa \text{div}_b \nabla \cdot B, \quad (6.1)
\]

\[
\nabla \times B = \mu_0 J, \quad (6.2)
\]

\[
E = -U \times B + \eta J + \frac{1}{en} J \times B + \frac{m_e}{ne^2} \left[ \frac{\partial J}{\partial t} + \nabla \cdot (JU + UJ) + \sum_{\alpha} \frac{q_{\alpha}}{m_{\alpha}} (\nabla p_{\alpha} + \nabla \cdot \Pi_{\alpha}) \right], \quad (6.3)
\]

\[
\frac{\partial n}{\partial t} + \nabla \cdot (nU) = \nabla \cdot D \nabla n, \quad (6.5)
\]

\[
nm \left( \frac{\partial U}{\partial t} + U \cdot \nabla U \right) = J \times B - \nabla p - \nabla \cdot \Pi, \quad (6.6)
\]

\[
\frac{n_{\alpha}}{\gamma - 1} \left( \frac{\partial T_{\alpha}}{\partial t} + U_{\alpha} \cdot \nabla T_{\alpha} \right) = -\nabla \cdot q_{\alpha} + Q_{\alpha} - p_{\alpha} \nabla \cdot U_{\alpha} - \Pi_{\alpha} : \nabla U_{\alpha}, \quad (6.7)
\]

where the heat flux is

\[
q = -n[\chi_\parallel \hat{b} \hat{b} + \chi_\perp (1 - \hat{b} \hat{b})] \cdot \nabla T
\]

and the source term from Ohmic and viscous heating is

\[
Q = \eta J^2 + \nu nm \nabla U^T : \nabla U.
\]

### 6.1 Including kinetic ion effects in NIMROD

For various MHD instabilities such as internal kink stabilization, sawtooth crash and toroidal Alfvén eigenmodes, ion kinetic effects are essential [101, 102, 103, 104, 105]. To include these key ion kinetic effects, we can implement particle ions and calculate the flow and pressure from ion distribution function as presented in this thesis. Notice that NIMROD solves a generalized Ohm’s law which is very similar to the one used in our hybrid model. If the kinetic particle pressure is comparable to the bulk plasma pressure, we can essentially evolve the ions as particles and calculate the density, flow, and pressure from the corresponding distribution functions directly. The moments calculated from particles can replace the continuity, momentum, and energy equation solved in NIMROD while retaining the NIMROD field solver. In this way, the equations we are solving become
\[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} + \kappa \text{div}_b \nabla \cdot \mathbf{B}, \quad (6.8) \]
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (6.9) \]
\[ \mathbf{E} = -\mathbf{U} \times \mathbf{B} + \eta \mathbf{J} + \frac{1}{en} \mathbf{J} \times \mathbf{B} \quad (6.10) \]
\[ + \frac{m_e}{n e^2} \left[ \frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot (\mathbf{JU} + \mathbf{UJ}) + \sum_{\alpha} \frac{q_{\alpha}}{m_{\alpha}} (\nabla p_{\alpha} + \nabla \cdot \Pi_{\alpha}) \right], \quad (6.11) \]
\[ n = \int f d^3v, \quad (6.12) \]
\[ nU = \int f v d^3v, \quad (6.13) \]
\[ \Pi_{\alpha} = \int f_{\alpha} \frac{1}{2} m_{\alpha} (\mathbf{v} - \mathbf{U})(\mathbf{v} - \mathbf{U}) d^3v, \quad (6.14) \]

where \( n, \mathbf{U}, \) and \( \Pi_{\alpha} \) are calculated directly from the particle distribution functions and can be inserted into the field solver without much modification. As we can see, this scheme incorporates key kinetic effects from the ion species in a relatively simple manner. This method appears to be a promising way to extend the NIMROD model to the kinetic regime.

### 6.2 PIC in finite element meshes

NIMROD uses a finite element mesh (FEM) in the poloidal plane and a finite Fourier series in the toroidal direction. The implementation of particles in higher-order quadrilateral finite elements has been reported in [106]. Follow the discussion in [106], consider the poloidal plane, we note the physical coordinates as \((R, Z)\), and the logical coordinates as \((p, q)\) which is the computational space of the finite element formalism. The mapping from the irregular Cartesian space to the uniform logical space is defined by
\[
R = \sum_{i=1}^{m^2} R_i N_i(p, q),
\]
\[
Z = \sum_{i=1}^{m^2} Z_i N_i(p, q), \quad (6.15)
\]
where \((R_i, Z_i)\) are the physical space coordinates of the nodes of the finite element, and \(N_i(p, q)\) represents the Lagrange polynomials which is usually quadratic degree in \((p, q)\) associated with
node $i$. A schematic plot of this mapping is shown in Figure 6.1.

![Figure 6.1: Schematic mapping from the irregular Cartesian space to the uniform logical space](image)

In the PIC method, only the physical coordinates of the particles ($\hat{R}, \hat{Z}$) are known. The logical coordinates of the particle $(p, q)$ must be determined by the mapping equations. Due to the nonlinear nature of Eq. (6.15), the Newton-Raphson method is used to solve for $(p, q)$ of particle at $(\hat{R}, \hat{Z})$. The Newton-Raphson method is expressed in the equation:

$$X_{k+1} = X_k - \frac{f(X_k)}{f'(X_k)},$$

(6.16)

where $X = (p, q)$ and $f$ is the mapping function (Eq. (6.15)). Substituting in these definitions, we obtain an iterative equation for $(p, q)$:

$$
\begin{bmatrix}
    p_{k+1} \\
    q_{k+1}
\end{bmatrix} =
\begin{bmatrix}
    p_k \\
    q_k
\end{bmatrix} +
\begin{bmatrix}
    \frac{\partial R}{\partial p} & \frac{\partial R}{\partial q} \\
    \frac{\partial Z}{\partial p} & \frac{\partial Z}{\partial q}
\end{bmatrix}
\begin{bmatrix}
    \hat{R} - \hat{R}_k \\
    \hat{Z} - \hat{Z}_k
\end{bmatrix},
$$

(6.17)

where $k$ is the iteration step, $(\hat{R}_k, \hat{Z}_k)$ are the particle coordinates obtained from $(p^k, q^k)$. We iterate Eq. (6.17) on the difference of $\hat{R} - \hat{R}_k$ and $\hat{Z} - \hat{Z}_k$. The Newton-Raphson method provides rather rapid converge. Once the logical coordinates of the particles are determined, particle searching and sorting can be done straightforwardly.
Now we discuss the deposition of particles onto the finite element grid. Using the perturbed particle pressure as an example:

$$\delta \Pi_\alpha = \int \delta f_\alpha \frac{1}{2} m_\alpha (v - U)(v - U) d^3v,$$

$$= \sum_{i=1}^{N} \frac{1}{2} m_\alpha (v_i - U)(v_i - U) w_i \delta (x - x_i),$$

where the $\delta f$ method is invoked and the particle weight $w_i$ is defined as

$$\delta f(x, v, t) = \sum_{i=1}^{N} w_i \delta (x - x_i) \delta (v - v_i).$$

First we perform a Fourier transform in the poloidal plane to Eq. (6.18),

$$\delta \Pi_{\alpha n}(R, Z) = \frac{1}{2\pi} \sum_{i=1}^{N} \frac{1}{2} m_\alpha (v_i - U)(v_i - U) w_i \delta (R - R_i) \delta (Z - Z_i) \exp \left(-in\phi_i\right) \frac{1}{R},$$

where $\exp \left(-in\phi_i\right)$ projects the particle’s $\phi_i$ to the Fourier mode $n$. Expand $\delta \Pi_{\alpha n}(R, Z)$ in the finite element basis $\sum_k \delta \Pi_{\alpha n}^k N^k(p, q)$ and project Eq. (6.20) onto the finite element space in the weak form:

$$\int N^j \delta \Pi_{\alpha n}^k N^k dRdZ = \int N^j \frac{1}{2\pi} \sum_{i=1}^{N} \frac{1}{2} m_\alpha (v_i - U)(v_i - U) w_i \delta (R - R_i) \delta (Z - Z_i) \exp \left(-in\phi_i\right) \frac{1}{R},$$

where $\delta \Pi_{\alpha n}^k$ are finite element pressure coefficients at node $k$, and $N^k(j)$ is the shape function associated with node $k(j)$. And further simplification yields

$$M \delta \Pi_{\alpha n}^k = \sum_{i=1}^{N} \frac{m_\alpha}{4\pi R_i} (v_i - U)(v_i - U) w_i \exp \left(-in\phi_i\right) N^k(p_i, q_i),$$

where $M$ is the corresponding finite element mass matrix. This formalism retains all conservation properties expected from PIC methods.
In this thesis, I have presented a hybrid model suitable for studying MHD scale plasmas using low order ion moments. This model has Lorentz force ions (i.e., fully kinetic ions) and fluid electrons. Using Lorentz ions is partly motivated by the invalidation of gyrokinetic turbulence simulations at both long and short wavelengths [53, 107]. Based on this model, a second-order accurate semi-implicit algorithm has been developed and implemented in a slab geometry. The model employs a semi-implicit scheme which eliminates the fast compressional waves and is numerically stable. The model advances the particle ions with fully electromagnetic equations, and fields are calculated through the generalized Ohm’s law and Faraday’s law. The electrons are assumed to be isothermal, and the quasi-neutrality provides the closure. To benchmark the code, we have simulated the linear shear and compressional Alfvén waves for a uniform plasma and the results agree with the theory. The ion acoustic wave and linear ion Landau damping are also correctly captured by the model. And we report the first PIC simulation of the nonlinear ion Landau damping in ion acoustic waves and the results are in good agreement with the theory. In order to compare the second-order and first-order schemes, a numerical dispersion relation for the whistler waves is derived and the simulation demonstrates that the time-centered second-order scheme introduces negligible numerical damping, which is in good agreement with the numerical analysis presented. Furthermore, the full evolution of the resistive tearing mode is investigated. The linear growth rate agrees with the resistive MHD prediction. Important nonlinear tearing dynamics including the Rutherford regime and the saturation are shown to be consistent with previous MHD studies.
We have also studied the coalescence of multiple islands in a Harris current sheet with large aspect ratios. We find that ion kinetic energy increases in the island region using particle ion diagnostics and a larger fraction of the dissipated magnetic energy is converted to ion kinetic energy in a current sheet with a larger aspect ratio. Asymptotically, the fraction of magnetic energy converted into ion kinetic energy reaches around 50%, which is comparable to the MRX measurements. For code verifications, we compare our hybrid model with the NIMROD two-fluid calculations on the slab ITG problem [23]. The results validate both models and quantify the appropriate parameter regimes for the two-fluid model on slab ITG. As a first step to extend NIMROD to kinetic regimes, we propose a simple scheme to incorporate particle ions while retaining the NIMROD field solver.

This work represents a first step in developing a simulation suitable for MHD scale physics with full kinetic ions. To address the kinetic electron physics in magnetic reconnection, more sophisticated electron models, e.g., drift-kinetic [52] and gyrokinetic electrons, should be incorporated. These are topics of ongoing and future work.
Bibliography


