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Differentiation of Line, Surface, and Volume Integrals

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DIFFERENTIATION OF LINE, SURFACE,
AND VOLUME INTEGRALS

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DIFFERENTIATION OF LINE, SURFACE,
AND VOLUME INTEGRALS

S.W. Maley

Abstract

It is occasionally necessary to differentiate a line or surface integral with respect to a parameter of the integrand or of the line or surface of integration. Differentiation, after evaluation of an integral, presents no difficulties but occasionally it is desirable in computation, or, more often, in theoretical derivation, to interchange the order of differentiation and integration. Procedures for such interchange are discussed herein.

INTRODUCTION

It is occasionally necessary to differentiate a line or surface integral with respect to a parameter of the integrand or of the line or surface of integration. Differentiation, after evaluation of an integral, presents no difficulties but occasionally it is desirable in computation, or, more often, in theoretical derivation, to interchange the order of differentiation and integration. Procedures for such interchange are discussed below.

Theory

The case of a simple integral in one dimensional space can be handled by the familiar Leibnitz rule, but a line or surface integral in 3-dimensional space requires an extension.

In this discussion it will be assumed that the parameter with respect to which differentiation is to be performed is time, t ; the results, however, are easily adapted to other choices. Consider the line integral

$$Q = \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell}$$

from point P_1 to point P_2 along the contour C . It is assumed that the integrand \bar{F} is a function of t , that contour C is in motion and that points P_1 and P_2 are in motion along the contour. The motion will be characterized by a finite velocity \bar{v} which is a function of position along contour C . (It is assumed that C may be changing shape and

position but that it remains continuous and the unit vector tangent to C remains a continuous function of distance along C .) \bar{v} can be resolved into components, \bar{v}_t tangential to, and \bar{v}_n normal to C ; thus $\bar{v} = \bar{v}_t + \bar{v}_n$. The tangential component, \bar{v}_t , at an interior point of C represents a stretching, or contraction, of the contour, with respect to an unmoving coordinate system; it therefore does not influence the value of the line integral. However, \bar{v}_t at the end points does influence the length of the contour and thus the line integral itself. It may be said that the motion of a contour, for determination of its influence on a line integral, may be characterized by \bar{v}_n at an interior point and \bar{v}_t at its end points.

The objective is an expression for $\frac{dQ}{dt}$, in which motion effects are expressed in terms of \bar{v}_n and \bar{v}_t , and which does not involve any coordinate system (i.e., it should be in vector notation).

Two of the terms in the expression for $\frac{dQ}{dt}$ involve the end points of C and $\frac{\partial \bar{F}}{\partial t}$; they are similar to their counterparts in the Leibnitz rule and may be written

$$[\bar{F}(P_2) \cdot \bar{v}_{t2} - \bar{F}(P_1) \cdot \bar{v}_{t1}] + \int_C^{P_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell}$$

where \bar{v}_{t1} and \bar{v}_{t2} are \bar{v}_t at points P_1 and P_2 respectively.

Another contribution to $\frac{dQ}{dt}$ involves the rate of change of \bar{F} due to the motion of C . The rate of change of \bar{F} , at a point on C , is the directional derivative of \bar{F} in the direction of \bar{v}_n multiplied by the magnitude of \bar{v}_n . In vector notation this is given by the scalar operator $(\bar{v}_n \cdot \nabla)$ operating on \bar{F} (∇ is the familiar del, or nabra, operator).

The final contribution to $\frac{dQ}{dt}$ involves the influence of motion upon $d\bar{\ell}$. To express this in terms of \bar{v}_n first express $d\bar{\ell}$ as $d\bar{\ell} = \bar{a}_t d\ell$ where \bar{a}_t is the unit vector tangent to C , in the direction of integration, and $d\ell$ is the scalar differential of length. Next note that \bar{a}_t can be expressed as $\bar{a}_t = \frac{\partial \bar{R}}{\partial \ell}$ where \bar{R} is the position vector of a point on C (from an arbitrary origin); and ℓ is length along C . Finally, since \bar{v}_n can be expressed as $\bar{v}_n = \frac{\partial \bar{R}}{\partial t}$, the rate of change of $d\bar{\ell}$ with t , due to \bar{v}_n , is

$$\frac{\partial}{\partial t} \frac{\partial \bar{R}}{\partial \ell} d\ell = \frac{\partial}{\partial \ell} \frac{\partial \bar{R}}{\partial t} d\ell = \frac{\partial \bar{v}_n}{\partial \ell} d\ell.$$

The result can be expressed as follows:

If

$$Q = \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell}.$$

along contour C , then

$$\begin{aligned} \frac{dQ}{dt} &= \bar{F}(P_2) \cdot \bar{v}_{t2} - \bar{F}(P_1) \cdot \bar{v}_{t1} \\ &+ \int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} + \int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \cdot d\bar{\ell} + \int_{P_1}^{P_2} \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial \ell} d\ell \end{aligned} \quad (1)$$

It is noted that, if contour C is closed, the first two terms, involving the end points, cancel.

The case of a surface integral follows similar reasoning but the details are significantly more involved. The motion of the surface is characterized by velocity, \bar{v} , which is a function of position on

surface S . It is resolved into components, \bar{v}_t tangential to, and \bar{v}_n normal to S . Thus $\bar{v} = \bar{v}_t + \bar{v}_n$. As in the case of the line integral, \bar{v}_t does not influence the surface integral at interior points of the surface. Therefore, the motion of a surface, for determination of its influence on a surface integral, may be characterized by \bar{v}_n at an interior point and \bar{v}_t on the boundary contour, C , of the surface. In this discussion, it is assumed the surface S is bounded by a single contour, C ; but the results are easily extended to surfaces bounded by multiple contours.

Consider the surface integral

$$Q = \int_S \bar{F} \cdot d\bar{S}.$$

The first term in the expression for $\frac{dQ}{dt}$ involves the tangential component, \bar{v}_t , of velocity of the contour C which bounds surface S ; it corresponds to the expression $\bar{F}(P_2) \cdot \bar{v}_{t2} - \bar{F}(P_1) \cdot \bar{v}_{t1}$ in the case of the line integral. To find a suitable expression for this term, assume temporarily that $\bar{v}_n = 0$ and $\frac{\partial \bar{F}}{\partial t} = 0$. Next introduce orthogonal coordinates u_1, u_2, u_3 and corresponding unit vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3$ such that \bar{a}_1 and \bar{a}_2 are tangent to surface S , such that contour C is defined by $u_1 = \text{constant}$, and \bar{a}_3 is normal to S . Then Q may be written

$$Q = \int_S F_3 du_2 du_1$$

where F_3 is the component of \bar{F} in the direction of \bar{a}_3 . dQ resulting from \bar{v}_t during time interval dt may be written

$$dQ = \oint_C (v_t \cos \theta dt) F_3 du_2$$

where v_t is the magnitude of \bar{v}_t , θ is the angle between \bar{v}_t and \bar{a}_1 , and the integration symbol implies a closed contour, C . The direction of integration along C is related to the positive direction on S by the right hand rule. \bar{v}_t can be expressed as $\bar{v}_t = v_t \bar{a}_t$ where \bar{a}_t is a unit vector on C , tangent to S but not necessarily tangent to C . It is easily shown that $\bar{F} \times \bar{a}_t \cdot \bar{a}_2 = F_3 \cos \theta$. Then, since the differential of length along C can be written $d\bar{\ell} = \bar{a}_2 du_2$, $\frac{dQ}{dt}$ becomes

$$\frac{dQ}{dt} = \oint_C \bar{F} \times \bar{v}_t \cdot d\bar{\ell}$$

Although a coordinate system was introduced in the discussion the result is independent of coordinates. Now the assumptions $\bar{v}_n = 0$ and $\frac{\partial \bar{F}}{\partial t} = 0$ will be relaxed leading to other terms of $\frac{dQ}{dt}$. The terms involving $\frac{\partial \bar{F}}{\partial t}$ and the rate of change of \bar{F} due to \bar{v}_n are of the same form as in the case of the line integral.

The term that involves the rate of change of $d\bar{S}$ can be found by a procedure similar to that used for $d\bar{\ell}$ in the line integral. Using the coordinates introduced above, $d\bar{S} = \bar{a}_3 du_1 du_2$; but $\bar{a}_3 = \bar{a}_1 \times \bar{a}_2$; $\bar{a}_1 = \frac{\partial \bar{R}}{\partial u_1}$, and $\bar{a}_2 = \frac{\partial \bar{R}}{\partial u_2}$ in terms of a position vector \bar{R} to a point on S . Differentiation with respect to t now gives

$$\frac{d(d\bar{S})}{dt} = \left[\frac{\partial(\frac{\partial \bar{R}}{\partial t})}{\partial u_1} \times \frac{\partial \bar{R}}{\partial u_2} + \frac{\partial \bar{R}}{\partial u_1} \times \frac{\partial(\frac{\partial \bar{R}}{\partial t})}{\partial u_2} \right] du_1 du_2 .$$

and since $\frac{\partial \bar{R}}{\partial t} = \bar{v}_n$

$$\frac{d(d\bar{S})}{dt} = \left[\frac{\partial \bar{v}_n}{\partial u_1} \times \frac{\partial \bar{R}}{\partial u_2} + \frac{\partial \bar{R}}{\partial u_1} \times \frac{\partial \bar{v}_n}{\partial u_2} \right] du_1 du_2$$

$$= \left[\frac{\partial \bar{v}_n}{\partial u_1} \times \bar{a}_2 + \bar{a}_1 \times \frac{\partial \bar{v}_n}{\partial u_2} \right] du_1 du_2 .$$

This can be expressed in a form not involving a coordinate system;

$$\frac{\partial}{\partial t} \overline{dS} = - (\overline{a}_n \times \nabla) \times \overline{v}_t dS ,$$

where $dS = du_1 du_2$ is the scalar differential of area and \overline{a}_n is the unit vector normal to S and in the direction of \overline{dS} . Since $\overline{a}_n dS = \overline{dS}$, an alternative expression for the right hand side is $-(\overline{dS} \times \nabla) \times \overline{v}_n$.

The results can be expressed as follows:

If

$$Q = \int_S \overline{F} \cdot \overline{dS}$$

then

$$\begin{aligned} \frac{dQ}{dt} = & \oint_C \overline{F} \times \overline{v}_t \cdot d\overline{\ell} + \int_S \frac{\partial \overline{F}}{\partial t} \cdot \overline{dS} \\ & + \int_S [(\overline{v}_n \cdot \nabla) \overline{F}] \cdot \overline{dS} + \int_S \overline{F} \cdot [(-\overline{dS} \times \nabla) \times \overline{v}_n] \end{aligned} \quad (2)$$

This is the desired result. It is noted that, if surface S is closed, the term involving the line integral, along contour C , vanishes.

Line Integrals

The above results for differentiation of line and surface integrals are useful for a variety of applications; however, they can be expressed in simpler forms which are preferable for most uses. Such forms will now be found.

Consider the integral

$$Q = \oint_C \bar{F} \cdot d\bar{\ell}$$

around the closed contour C . Using Stokes Theorem this may be written in terms of a surface integral

$$\oint_C \bar{F} \cdot d\bar{\ell} = \int_S \nabla \times \bar{F} \cdot d\bar{S}$$

where surface S is bounded by contour C but is otherwise arbitrary. Differentiation with respect to t gives

$$\frac{d}{dt} \oint_C \bar{F} \cdot d\bar{\ell} = \frac{d}{dt} \int_S \nabla \times \bar{F} \cdot d\bar{S} .$$

Equation (2) may now be applied to the right hand side. The arbitrariness of surface S permits its choice such that $\bar{v}_n = 0$. The velocity \bar{v} of contour C is then the tangential velocity \bar{v}_t in eq. (2). Thus

$$\frac{d}{dt} \oint_C \bar{F} \cdot d\bar{\ell} = \oint_C (\nabla \times \bar{F}) \times \bar{v} \cdot d\bar{\ell} + \int_S \frac{\partial}{\partial t} (\nabla \times \bar{F}) \cdot d\bar{S} .$$

Consideration is restricted to functions \bar{F} such that $\frac{\partial}{\partial t} (\nabla \times \bar{F}) = \nabla \times \frac{\partial \bar{F}}{\partial t}$; then using Stokes Theorem the result may be written

$$\frac{d}{dt} \oint_C \bar{F} \cdot d\bar{\ell} = \oint_C (\nabla \times \bar{F}) \times \bar{v} \cdot d\bar{\ell} + \oint_C \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} .$$

This result suggests the possibility of a similar simplified version of eq. (1) for the more general case of a non-closed contour C . This,

in fact, is possible but requires a more intricate argument. Let Q be given by the line integral

$$Q = \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell} ,$$

along contour C . Using eq. (1), the derivative $\frac{dQ}{dt}$ can be expressed as

$$\frac{dQ}{dt} = \bar{F}(P_2) \cdot \bar{v}_{t2} - \bar{F}(P_1) \cdot \bar{v}_{t1} + \int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} + \left(\frac{dQ}{dt}\right)_{\bar{v}_n}$$

where $\left(\frac{dQ}{dt}\right)_{\bar{v}_n}$ is the contribution to $\frac{dQ}{dt}$ due to the normal component,

\bar{v}_n , of velocity \bar{v} on contour C . $\left(\frac{dQ}{dt}\right)_{\bar{v}_n}$ is the sum of the last two

terms in eq. (1).

Let contour C' be joined to contour C to make a closed contour denoted $C+C'$. Define C' so no discontinuity occurs in the unit tangent vector \bar{a}_ℓ at points P_1 and P_2 . C' is chosen to be motionless except in vanishingly short regions adjacent to P_1 and P_2 where its normal velocity \bar{v}_n must make smooth transitions from zero to the values assumed by \bar{v}_n on contour C at end points P_1 and P_2 .

Consider the integral

$$\oint_{C+C'} \bar{F} \cdot d\bar{\ell} = \int_C^{P_2} \bar{F} \cdot d\bar{\ell} + \int_{P_2}^{P_1} \bar{F} \cdot d\bar{\ell}$$

Using Stokes Theorem this can be written

$$\int_{C'}^{P_2} \bar{F} \cdot d\bar{\ell} + \int_C^{P_1} \bar{F} \cdot d\bar{\ell} = \int_S \nabla \times \bar{F} \cdot d\bar{S} .$$

Let C have only normal motion \bar{v}_n ; let C' have motion as described above and let $\frac{\partial \bar{F}}{\partial t} = 0$. Then differentiation gives

$$\left(\frac{dQ}{dt}\right)_{\bar{v}_n} + \frac{d}{dt} \int_{C'}^{P_1} \bar{F} \cdot d\bar{\ell} = \frac{d}{dt} \int_S \nabla \times \bar{F} \cdot d\bar{S} .$$

Equation (1) may be applied to the term involving contour C' . The velocity specified for C' causes all but the last term on the right hand side of eq. (1) to vanish. That term is

$$\int_{C'}^{P_1} \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial \ell} d\ell .$$

The integrand vanishes except in the transition zone near P_2 where \bar{v}_n makes a smooth transition from \bar{v}_{n2} to zero and near P_1 where \bar{v}_n makes a transition from zero to \bar{v}_{n1} . As the length of these transition zones approaches zero the contribution to the integral near P_2 is $\bar{F}(P_2) \cdot (-\bar{v}_{n2})$ and that near P_1 is $\bar{F}(P_1) \cdot \bar{v}_{n1}$. Thus

$$\left(\frac{dQ}{dt}\right)_{\bar{v}_n} - \bar{F}(P_2) \cdot \bar{v}_{n2} + \bar{F}(P_1) \cdot \bar{v}_{n1} = \frac{d}{dt} \int_S \nabla \times \bar{F} \cdot d\bar{S} .$$

Next, eq. (2) is applied to the term involving the surface integral. The surface S is arbitrary except that it is bounded by $C+C'$. Let it

be chosen so that $\bar{v}_n = 0$ (It may be said the motion of contour $C+C'$ is generating surface S). The tangential component of velocity for surface S , on its contour C , is thus the normal component \bar{v}_n of velocity for contour C . For this choice, only the first term on the right hand side of eq. (2) is non-zero so

$$\left(\frac{dQ}{dt}\right)_{\bar{v}_n} - \bar{F}(P_2) \cdot \bar{v}_{n2} + \bar{F}(P_1) \cdot \bar{v}_{n1} = \oint_{C+C'} (\nabla \times \bar{F}) \times \bar{v}_n \cdot d\bar{\ell}.$$

Since $\bar{v}_n = 0$ on C' (the vanishingly short transition zones near P_1 and P_2 make no contribution to this integral), this may be written

$$\left(\frac{dQ}{dt}\right)_{\bar{v}_n} - \bar{F}(P_2) \cdot \bar{v}_{n2} + \bar{F}(P_1) \cdot \bar{v}_{n1} = \int_{P_1}^{P_2} (\nabla \times \bar{F}) \times \bar{v}_n \cdot d\bar{\ell}.$$

Substitution of this into the above expression for $\frac{dQ}{dt}$ gives

$$\frac{dQ}{dt} = \bar{F}(P_2) \cdot \bar{v}_{t2} - \bar{F}(P_1) \cdot \bar{v}_{t1} + \bar{F}(P_2) \cdot \bar{v}_{n2} - \bar{F}(P_1) \cdot \bar{v}_{n1} + \int_{P_1}^{P_2} (\nabla \times \bar{F}) \times \bar{v}_n \cdot d\bar{\ell}.$$

The first four terms on the right can be combined into two using the relations $\bar{v}_2 = \bar{v}_{n2} + \bar{v}_{t2}$ and $\bar{v}_1 = \bar{v}_{n1} + \bar{v}_{t1}$. Since \bar{v}_t and $d\bar{\ell}$ are colinear, $(\nabla \times \bar{F}) \times \bar{v}_t \cdot d\bar{\ell} = 0$ so \bar{v}_n may be replaced by $\bar{v} = \bar{v}_n + \bar{v}_t$ in the line integral. The final result, for line integrals, may be expressed as a theorem.

Theorem 1

Let Q be the line integral

$$Q = \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell}$$

along contour, C , from point P_1 to point P_2 . Assume that the vector function, \bar{F} , is a function of time, t . Also assume the contour, C , is in motion with respect to the frame of reference with respect to which \bar{F} is defined. Further assume the end points P_1 and P_2 are in motion. Let the motion, with respect to the frame of reference, be defined by velocity, \bar{v} , which is a function of position along contour C . The derivative of Q with respect to t can be expressed as

$$\frac{dQ}{dt} = \bar{F}(P_2) \cdot \bar{v}_2 - \bar{F}(P_1) \cdot \bar{v}_1 + \int_{P_1}^{P_2} (\nabla \times \bar{F}) \times \bar{v} \cdot d\bar{l} + \int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{l}$$

where \bar{v}_2 is the velocity \bar{v} at end point 2 and \bar{v}_1 is the velocity at end point 1.

It is apparent that, if contour C is closed, the first two terms cancel and the result is expressed in terms of closed line integrals only.

This result is an improvement of eq. (1) in two important respects; first, it is simpler, and second, it is in terms of total velocity rather than tangential and normal components.

Surface and Volume Integrals

A similar simplification can be achieved in the case of surface integrals. Consider the integral

$$Q = \oint_S \bar{F} \cdot d\bar{S}$$

over the closed surface S . Using the divergence theorem this can be written in terms of a volume integral

$$\oint_S \bar{F} \cdot d\bar{S} = \int_V \nabla \cdot \bar{F} \, dv$$

where v is the volume bounded by closed surface S . Differentiation with respect to t gives

$$\frac{d}{dt} \oint_S \bar{F} \cdot d\bar{S} = \frac{d}{dt} \int_V \nabla \cdot \bar{F} dv .$$

Differentiation of a volume integral is discussed in App. 1; the above equation can be written

$$\frac{d}{dt} \oint_S \bar{F} \cdot d\bar{S} = \oint_S (\nabla \cdot \bar{F}) \bar{v} \cdot d\bar{S} + \int_V \frac{\partial}{\partial t} (\nabla \cdot \bar{F}) dv .$$

Restricting consideration to functions \bar{F} such that $\frac{\partial}{\partial t} (\nabla \cdot \bar{F}) = \nabla \cdot \frac{\partial \bar{F}}{\partial t}$ and using the divergence theorem the above result can be expressed as

$$\frac{d}{dt} \oint_S \bar{F} \cdot d\bar{S} = \oint_S (\nabla \cdot \bar{F}) \bar{v} \cdot d\bar{S} + \oint_S \frac{\partial \bar{F}}{\partial t} \cdot d\bar{S} .$$

This simple derivation can be extended to give a result applicable to integrals for which the surface of integration is not closed. Let Q be defined by the integral

$$Q = \int_S \bar{F} \cdot d\bar{S}$$

over surface S . Using eq. (2) the derivative $\frac{dQ}{dt}$ can be expressed as

$$\frac{dQ}{dt} = \oint_C \bar{F} \times \bar{v}_t \cdot d\bar{x} + \int_S \frac{\partial \bar{F}}{\partial t} \cdot d\bar{S} + \left(\frac{dQ}{dt} \right)_{\bar{v}_n}$$

where $\left(\frac{dQ}{dt} \right)_{\bar{v}_n}$ is the contribution to $\frac{dQ}{dt}$ due to the normal component,

\bar{v}_n , of velocity \bar{v} on surface S . $\left(\frac{dQ}{dt} \right)_{\bar{v}_n}$ is the sum of the last two terms in eq. (2).

Let surface S' be joined to surface S to make a closed contour denoted $S+S'$. Define S' so no discontinuity occurs in the unit normal vector \bar{a}_n at the boundary, C , between S and S' . S' is chosen to be motionless except in a vanishingly narrow strip along C where \bar{v}_n must make a smooth transition from zero to the value assumed by \bar{v}_n on the boundary C of surface S .

Consider the integral

$$\oint_{S+S'} \bar{F} \cdot d\bar{S} = \int_S \bar{F} \cdot d\bar{S} + \int_{S'} \bar{F} \cdot d\bar{S} .$$

Using the divergence theorem this can be written

$$\int_S \bar{F} \cdot d\bar{S} + \int_{S'} \bar{F} \cdot d\bar{S} = \int_V \nabla \cdot \bar{F} \, dv .$$

Let S have only normal motion \bar{v}_n ; let S' have motion as described above and let $\frac{\partial \bar{F}}{\partial t} = 0$. Then differentiation gives

$$\left(\frac{dQ}{dt}\right)_{\bar{v}_n} + \frac{d}{dt} \int_{S'} \bar{F} \cdot d\bar{S} = \frac{d}{dt} \int_V \nabla \cdot \bar{F} \, dv .$$

Equation (2) may be applied to the term involving surface S' . The velocity specified for S' causes all but the last term on the right hand side of eq. (2) to vanish. That term is

$$- \int_{S'} \bar{F} \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n] dS .$$

The integrand vanishes except in the transition region near C where \bar{v}_n makes a smooth transition from zero to the value assumed by the normal component of velocity on surface S at its boundary C .

As noted in the derivation of eq. (2), the above term may be written

$$+ \int_{S'} \bar{F} \cdot \left[\frac{\partial \bar{v}_n}{\partial u_1} \times \bar{a}_2 + \bar{a}_1 \times \frac{\partial \bar{v}_n}{\partial u_2} \right] du_1 du_2$$

where coordinate u_2 is distance along C and u_1 is distance normal to C . Let Δu_1 be the width of the transition region. As

$\Delta u_1 \rightarrow 0$ $\frac{\partial \bar{v}_n}{\partial u_1} \gg \frac{\partial \bar{v}_n}{\partial u_2}$ and the integral approaches

$$\oint_C \bar{F} \cdot \Delta \bar{v}_n \times \bar{a}_2 du_2$$

where $\Delta \bar{v}_n = \bar{v}_n$ is the normal component of velocity of S on its boundary C . Since $\bar{a}_2 du_2 = \bar{d}\ell$ and since $\bar{F} \cdot \bar{v}_n \times \bar{d}\ell = \bar{F} \times \bar{v}_n \cdot \bar{d}\ell$ the above expression may be written

$$\oint_C \bar{F} \times \bar{v}_n \cdot \bar{d}\ell .$$

The direction of integration along C is related to the outward directed normal to S' by the right hand rule which is opposite the direction of integration along C regarded as the boundary of surface S . Since the end result will be related to surface S , the sign will be changed. The result thus far is

$$\left(\frac{dQ}{dt} \right)_{\bar{v}_n} - \oint_C \bar{F} \times \bar{v}_n \cdot \bar{d}\ell = \frac{d}{dt} \int_V \nabla \cdot \bar{F} dv .$$

Utilizing the differentiation procedure given in App. 1 for volume integrals, this can be written

$$\left(\frac{dQ}{dt}\right)_{\bar{v}_n} - \oint_C \bar{F} \times \bar{v}_n \cdot d\bar{\ell} = \oint_{S+S'} (\nabla \cdot \bar{F}) \bar{v}_n \cdot d\bar{S} .$$

Since $\bar{v}_n = 0$ on S' (the vanishingly narrow transition region near C makes no contribution to the integral) and since $\bar{v}_n \cdot d\bar{S} = \bar{v} \cdot d\bar{S}$, this may be written

$$\left(\frac{dQ}{dt}\right)_{\bar{v}_n} - \oint_C \bar{F} \times \bar{v}_n \cdot d\bar{\ell} = \int_S (\nabla \cdot \bar{F}) \bar{v} \cdot d\bar{S} .$$

Substitution of this into the above expression for $\frac{dQ}{dt}$ gives

$$\frac{dQ}{dt} = \oint_C \bar{F} \times \bar{v}_t \cdot d\bar{\ell} + \oint_C \bar{F} \times \bar{v}_n \cdot d\bar{\ell} + \int_S \frac{\partial \bar{F}}{\partial t} \cdot d\bar{S} + \int_S (\nabla \cdot \bar{F}) \bar{v} \cdot d\bar{S} .$$

Since $\bar{v} = \bar{v}_t + \bar{v}_n$ the first two terms on the right hand side can be combined. The final result, for surface integrals, may be expressed as

Theorem 2

Let Q be the surface integral

$$Q = \int_S \bar{F} \cdot d\bar{S}$$

over the surface S which is bounded by the contour C . Assume that the vector function, \bar{F} , is a function of time, t . Also assume the surface S , is in motion with respect to the frame of reference with respect to which \bar{F} is defined. Further assume the contour, C , bounding surface S is in motion. Let the motion, with respect to the frame of reference, be defined by velocity, \bar{v} ,

which is a function of position on surface, S . The derivative of Q with respect to t can be expressed as

$$\frac{dQ}{dt} = \oint_C \bar{F} \times \bar{v} \cdot d\bar{\ell} + \int_S \frac{\partial \bar{F}}{\partial t} \cdot d\bar{S} + \int_S (\nabla \cdot \bar{F}) \bar{v} \cdot d\bar{S} .$$

It is apparent that, if surface S is closed, the first term on the right hand side, that is the line integral along closed contour C , vanishes. This result is an improvement of eq. (2) in two important respects; first, it is simpler and second, it is in terms of total velocity rather than tangential and normal components.

Example

As an example of the use of Theorem 2, consider the following expression:

$$\oint_C \bar{E} \cdot d\bar{\ell} = - \frac{d}{dt} \int_S \bar{B} \cdot d\bar{S} .$$

This is one of Maxwell's equations of the electromagnetic field, in integral form, in MKS units. The surface, S , is arbitrary except that C is its bounding contour. Suppose that \bar{B} is not an explicit function of time but that contour C is in motion with velocity \bar{v} which is a function of position along contour C . Theorem 2 is applicable to the right hand side. Since S is arbitrary it may be selected such that $\bar{v} = 0$ except on the boundary C then only the first term on the right hand side of Theorem 2 is non-zero. The result is

$$\oint_C \bar{E} \cdot d\bar{\ell} = - \oint_C \bar{B} \times \bar{v} \cdot d\bar{\ell} .$$

This is a well known result; it is usually derived from Maxwell's equation by a clever and intricate argument. The use of Theorem 2, however, reduces the argument to one simple step.

Extension to Other Types of Integrals

The analyses leading to Theorems 1 and 2 are adaptable to other similar types of integrals. In the case of line integrals the integral

$$\int_{P_1}^{P_2} \vec{F} \times d\vec{\ell}$$

can be handled by a sequence of steps similar to those used to obtain eq. (1) and Theorem 1. The only significant difference in procedure involves the use of a related identity rather than Stokes Theorem. The required identity is discussed in App. 2.

Similar results can also be obtained for the integral

$$\int_{P_1}^{P_2} F d\ell$$

The procedure involves use of another related identity rather than Stokes Theorem; this is also discussed in App. 2.

The line integrals

$$\int_{P_1}^{P_2} \vec{F} d\vec{\ell}$$

and

$$\int_{P_1}^{P_2} F d\ell$$

yield results similar to eq. (1) but it has not yet been possible to find results similar to Theorem 1. The results of these extensions to other types of integrals are given in App. 4.

The procedure leading to eq. (2) and to Theorem 2 on surface integrals can also be extended to other types of integrals. The integral

$$\int_S \bar{F} \times d\bar{S}$$

leads to a result similar to eq. (2) and, using an identity related to the divergence theorem, to a result similar to Theorem 2. The identity needed is discussed in App. 3.

The integral

$$\int_S F d\bar{S}$$

also leads to a result similar to eq. (2) and, using another identity related to the divergence Theorem, to a result similar to Theorem 2.

The surface integrals

$$\int_S \bar{F} dS$$

and

$$\int_S F dS$$

lead to results similar to eq. (2) but have not lead to results similar to Theorem 2. The results of these extensions to other types of surface integrals are given in App. 4.

One of the fundamental properties of the gradient operator dictates that $\nabla(\bar{F} \cdot \bar{v}) \cdot d\bar{\ell} = d(\bar{F} \cdot \bar{v})$ so

$$\int_C^{P_2} \nabla(\bar{F} \cdot \bar{v}) \cdot d\bar{\ell} = \bar{F}(P_2) \cdot \bar{v}_2 - \bar{F}(P_1) \cdot \bar{v}_1$$

so the terms, in Theorem 1, involving the end points, can be expressed in terms of an integral. Although the form explicitly involving the end points is preferable in most applications, that involving the integral is also listed among the results in App. 4.

A similar modification, in the case of surface integrals, involves a replacement of the closed line integral of Theorem 2 by a surface integral using Stokes Theorem. The form involving the line integral is preferable in most applications but both are listed among the results in App. 4.

After proof of Theorems 1 and 2 as discussed above, alternative proofs were found. These are similar to proofs frequently used for Stokes' theorem and for the divergence theorem. Since these may be more satisfactory for some persons, they are given in App. 5. Similar proofs can be given for several of the other relationships given in App. 4 for line and surface integrals.

APPENDIX I

DIFFERENTIATION OF VOLUME INTEGRALS

Let Q be a volume integral

$$Q = \int_V F \, dv$$

over a volume V bounded by the closed surface S . If S is in motion with velocity $\bar{v} = \bar{v}_t + \bar{v}_n$ where \bar{v}_t and \bar{v}_n are tangential and normal components respectively, only \bar{v}_n influences Q . dQ resulting from dt is seen to be

$$dQ = \oint_S F(v_n \, dt) \, dS + \int_V \left(\frac{\partial F}{\partial t} \, dt \right) \, dv .$$

Then since $v_n \, dS = \bar{v} \cdot \overline{dS}$

$$\frac{dQ}{dt} = \oint_S F(\bar{v} \cdot \overline{dS}) + \int_V \frac{\partial F}{\partial t} \, dv .$$

This result is also valid if F is replaced by \bar{F} ; thus

$$\frac{d\bar{Q}}{dt} = \oint_S \bar{F}(\bar{v} \cdot \overline{dS}) + \int_V \frac{\partial \bar{F}}{\partial t} \, dv .$$

APPENDIX 2

STOKES THEOREM AND RELATED IDENTITIES

Stokes Theorem is

$$\oint_C \vec{F} \cdot d\vec{\ell} = \int_S \nabla \times \vec{F} \cdot d\vec{S}$$

where S is any surface bounded by contour C .

Other relations of a similar type can be found; several of these are given below.

$$\oint_C \vec{F} \cdot d\vec{\ell} = \int_S \nabla \times \vec{F} \cdot d\vec{S}$$

$$= \int_S (d\vec{S} \times \nabla) \cdot \vec{F}$$

$$\oint_C \vec{F} \times d\vec{\ell} = - \int_S (d\vec{S} \times \nabla) \times \vec{F}$$

$$= \int_S (\nabla \cdot \vec{F}) d\vec{S} + \int_S (\nabla \times \vec{F}) \times d\vec{S} - \int_S (d\vec{S} \cdot \nabla) \vec{F}$$

$$\oint_C \vec{F} \cdot d\vec{\ell} = - \int_S \nabla \cdot \vec{F} \times d\vec{S}$$

$$\oint_C \vec{v}(\vec{F} \cdot d\vec{\ell}) = \left\{ \begin{array}{l} \int_S \vec{v}(\nabla \times \vec{F} \cdot d\vec{S}) \\ \int_S \vec{v}[(d\vec{S} \times \nabla) \cdot \vec{F}] \end{array} \right\} - \left\{ \begin{array}{l} \int_S [d\vec{S} \cdot (\vec{F} \times \nabla)] \vec{v} \\ \int_S [(\vec{v} \times \vec{F}) \cdot \nabla] \vec{v} \end{array} \right\}$$

$$\oint_C \bar{F} \times (\bar{v} \times d\bar{\ell}) = \oint_C \bar{v}(\bar{F} \cdot d\bar{\ell}) - \oint_C (\bar{F} \cdot \bar{v})d\bar{\ell}$$

$$= \left\{ \begin{array}{l} \int_S \bar{v}(\nabla \times \bar{F} \cdot d\bar{S}) \\ \int_S \bar{v}[(d\bar{S} \times \nabla) \cdot \bar{F}] \end{array} \right\} - \left\{ \begin{array}{l} \int_S [\bar{dS} \cdot (\bar{F} \times \nabla)]\bar{v} \\ \int_S [(d\bar{S} \times \bar{F}) \cdot \nabla]\bar{v} \end{array} \right\} + \int_S \nabla(\bar{F} \cdot \bar{v}) \times d\bar{S}$$

$$\oint_C F(\bar{v} \times d\bar{\ell}) = \int_S \bar{v} \times (d\bar{S} \times \nabla F) - \int_S F(d\bar{S} \times \nabla) \times \bar{v}$$

$$\oint_C F(\bar{v} \cdot d\bar{\ell}) = \int_S F(\nabla \times \bar{v} \cdot d\bar{S}) + \int_S (\nabla F \times \bar{v}) \cdot d\bar{S}$$

$$= \int_S F[(d\bar{S} \times \nabla) \cdot \bar{v}] + \int_S (\nabla F \times \bar{v}) \cdot d\bar{S}$$

$$\oint_C (\bar{F} \times \bar{v}) \times d\bar{\ell} = - \int_S (d\bar{S} \times \nabla) \times (\bar{F} \times \bar{v})$$

$$= \int_S [\nabla \cdot (\bar{F} \times \bar{v})]d\bar{S} + \int_S [\nabla \times (\bar{F} \times \bar{v})] \times d\bar{S} - \int_S (d\bar{S} \cdot \nabla)(\bar{F} \times \bar{v})$$

$$\left. \begin{array}{l} \oint_C (\bar{F} \times \bar{v}) \cdot d\bar{\ell} \\ \oint_C \bar{F} \cdot (\bar{v} \times d\bar{\ell}) \end{array} \right\} = \left\{ \begin{array}{l} \int_S \nabla \times (\bar{F} \times \bar{v}) \cdot d\bar{S} \\ \int_S (d\bar{S} \times \nabla) \cdot (\bar{F} \times \bar{v}) \end{array} \right.$$

The validity of these is easily demonstrated by methods similar to those used in the proof of Stokes Theorem.

The author has not seen these extensions of Stokes Theorem in the literature; but their simplicity and ease of proof strongly suggest that they are known.

APPENDIX 3

THE DIVERGENCE THEOREM AND RELATED IDENTITIES

The Divergence Theorem is

$$\oint_C \bar{F} \cdot d\bar{S} = \int_V \nabla \cdot \bar{F} \, dv$$

where V is the volume bounded by closed surface C .

Similar identities can be found for other types of integrals; several of these are given below:

$$\oint_S \bar{F} \times d\bar{S} = - \int_V (\nabla \times \bar{F}) \, dv$$

$$\oint_S F \, d\bar{S} = \int_V (\nabla F) \, dv$$

$$\left. \begin{array}{l} \oint_S \bar{F} \times \bar{g} \cdot d\bar{S} \\ \oint_S \bar{F} \cdot \bar{g} \times d\bar{S} \end{array} \right\} = \int_V \nabla \cdot (\bar{F} \times \bar{g}) \, dv$$

$$\oint_S F(\bar{g} \times d\bar{S}) = \int_V (\bar{g} \times \nabla F - F \nabla \times \bar{g}) \, dv$$

$$\oint_S \bar{g}(\bar{F} \cdot d\bar{S}) = \int_V [\bar{g}(\nabla \cdot \bar{F}) + (\bar{F} \cdot \nabla)\bar{g}]dv$$

$$\oint_S \bar{F} \times (\bar{g} \times d\bar{S}) = \int_V [(\nabla \times \bar{g}) \times \bar{F} - (\bar{g} \times \nabla) \times \bar{F}]dv$$

$$\oint_S (\bar{F} \times \bar{g}) \times d\bar{S} = \int_V \nabla \times (\bar{g} \times \bar{F})dv$$

$$\oint_S F(\bar{g} \cdot d\bar{S}) = \int_V [F\nabla \cdot \bar{g} + \nabla F \cdot \bar{g}]dv$$

The validity of these is easily demonstrated by methods similar to those used in the proof of the Divergence Theorem.

The author has not seen these extensions of the Divergence Theorem in the literature; but their simplicity and ease of proof strongly suggest that they are known.

APPENDIX 4

COMPILATION OF RESULTS

Differentiation of Volume Integrals

$$\frac{d}{dt} \int_V F \, dv = \oint_S F(\bar{v} \cdot \bar{dS}) + \int_V \frac{\partial F}{\partial t} \, dv$$

$$\frac{d}{dt} \int_V \bar{F} \, dv = \oint_S \bar{F}(\bar{v} \cdot \bar{dS}) + \int_V \frac{\partial \bar{F}}{\partial t} \, dv$$

Stokes Theorem and Related Identities

$$\begin{aligned} \oint_C \bar{F} \cdot \bar{d\ell} &= \int_S \nabla \times \bar{F} \cdot \bar{dS} \\ &= \int_S (\bar{dS} \times \nabla) \cdot \bar{F} \end{aligned}$$

$$\begin{aligned} \oint_C \bar{F} \times \bar{d\ell} &= - \int_S (\bar{dS} \times \nabla) \times \bar{F} \\ &= \int_S (\nabla \cdot \bar{F}) \bar{dS} + \int_S (\nabla \times \bar{F}) \times \bar{dS} - \int_S (\bar{dS} \cdot \nabla) \bar{F} \end{aligned}$$

$$\oint_C F \, \bar{d\ell} = - \int_S \nabla F \times \bar{dS}$$

$$\oint_C \bar{v}(\bar{F} \cdot d\bar{\ell}) = \left\{ \begin{array}{l} \int_S \bar{v}(\nabla \times \bar{F} \cdot d\bar{S}) \\ \int_S \bar{v}[(d\bar{S} \times \nabla) \cdot \bar{F}] \end{array} \right\} - \left\{ \begin{array}{l} \int_S [d\bar{S} \cdot (\bar{F} \times \nabla)] \bar{v} \\ \int_S [(d\bar{S} \times \bar{F}) \cdot \nabla] \bar{v} \end{array} \right\}$$

$$\begin{aligned} \oint_C \bar{F} \times (\bar{v} \times d\bar{\ell}) &= \oint_C \bar{v}(\bar{F} \cdot d\bar{\ell}) - \oint_C (\bar{F} \cdot \bar{v}) d\bar{\ell} \\ &= \left\{ \begin{array}{l} \int_S \bar{v}(\nabla \times \bar{F} \cdot d\bar{S}) \\ \int_S \bar{v}[(d\bar{S} \times \nabla) \cdot \bar{F}] \end{array} \right\} - \left\{ \begin{array}{l} \int_S [d\bar{S} \cdot (\bar{F} \times \nabla)] \bar{v} \\ \int_S [(d\bar{S} \times \bar{F}) \cdot \nabla] \bar{v} \end{array} \right\} + \int_S \nabla(\bar{F} \cdot \bar{v}) \times d\bar{S} \end{aligned}$$

$$\oint_C \bar{F}(\bar{v} \times d\bar{\ell}) = \int_S \bar{v} \times (d\bar{S} \times \nabla \bar{F}) - \int_S \bar{F}(d\bar{S} \times \nabla) \times \bar{v}$$

$$\begin{aligned} \oint_C \bar{F}(\bar{v} \cdot d\bar{\ell}) &= \int_S \bar{F}(\nabla \times \bar{v} \cdot d\bar{S}) + \int_S (\nabla \bar{F} \times \bar{v}) \cdot d\bar{S} \\ &= \int_S \bar{F}[(d\bar{S} \times \nabla) \cdot \bar{v}] + \int_S (\nabla \bar{F} \times \bar{v}) \cdot d\bar{S} \end{aligned}$$

$$\oint_C (\bar{F} \times \bar{v}) \times d\bar{\ell} = - \int_S (\bar{dS} \times \nabla) \times (\bar{F} \times \bar{v})$$

$$= \int_S [\nabla \cdot (\bar{F} \times \bar{v})] \bar{dS} + \int_S [\nabla \times (\bar{F} \times \bar{v})] \times \bar{dS} - \int_S (\bar{dS} \cdot \nabla) (\bar{F} \times \bar{v})$$

$$\left. \begin{aligned} \oint_C (\bar{F} \times \bar{v}) \cdot d\bar{\ell} \\ \oint_C \bar{F} \cdot (\bar{v} \times d\bar{\ell}) \end{aligned} \right\} = \left. \begin{aligned} \int_S \nabla \times (\bar{F} \times \bar{v}) \cdot \bar{dS} \\ \int_S (\bar{dS} \times \nabla) \cdot (\bar{F} \times \bar{v}) \end{aligned} \right\}$$

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Divergence Theorem and Related Identities

$$\oint_C \bar{F} \cdot d\bar{S} = \int_V \nabla \cdot \bar{F} \, dv$$

$$\oint_S \bar{F} \times d\bar{S} = - \int_V (\nabla \times \bar{F}) \, dv$$

$$\oint_S F \, d\bar{S} = \int_V (\nabla F) \, dv$$

$$\left. \begin{aligned} \oint_S \bar{F} \times \bar{g} \cdot d\bar{S} \\ \oint_S \bar{F} \cdot \bar{g} \times d\bar{S} \end{aligned} \right\} = \int_V \nabla \cdot (\bar{F} \times \bar{g}) \, dv$$

$$\oint_S F(\bar{g} \times d\bar{S}) = \int_V (\bar{g} \times \nabla F - F \nabla \times \bar{g}) \, dv$$

$$\oint_S \bar{g}(\bar{F} \cdot d\bar{S}) = \int_V [\bar{g}(\nabla \cdot \bar{F}) + (\bar{F} \cdot \nabla)\bar{g}] dv$$

$$\oint_S \bar{F} \times (\bar{g} \times d\bar{S}) = \int_V [(\nabla \times \bar{g}) \times \bar{F} - (\bar{g} \times \nabla) \times \bar{F}] dv$$

$$\oint_S (\bar{F} \times \bar{g}) \times d\bar{S} = \int_V \nabla \times (\bar{g} \times \bar{F}) dv$$

$$\oint_S F(\bar{g} \cdot d\bar{S}) = \int_V [F\nabla \cdot \bar{g} + \nabla F \cdot \bar{g}] dv$$

* * * * *

Differentiation of Line Integrals

In terms of tangential and normal components \bar{v}_t and \bar{v}_n of velocity of C ,

$$\begin{aligned} \frac{d}{dt} \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell} &= \bar{F}(P_2) \cdot \bar{v}_{t2} - \bar{F}(P_1) \cdot \bar{v}_{t1} \\ &+ \int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} + \int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla)\bar{F}] \cdot d\bar{\ell} + \int_{P_1}^{P_2} \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial t} d\ell \end{aligned}$$

In terms of total velocity \bar{v} of C ,

$$\frac{d}{dt} \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell} = \bar{F}(P_2) \cdot \bar{v}_2 - \bar{F}(P_1) \cdot \bar{v}_1 + \int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} + \int_{P_1}^{P_2} (\nabla \times \bar{F}) \times \bar{v} \cdot d\bar{\ell}.$$

Using an integral rather than the terms involving endpoints.

$$\frac{d}{dt} \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell} = \int_C \left[\nabla(\bar{F} \cdot \bar{v}) + (\nabla \times \bar{F}) \times \bar{v} + \frac{\partial \bar{F}}{\partial t} \right] \cdot d\bar{\ell} .$$

In terms of tangential and normal components, \bar{v}_t and \bar{v}_n , of velocity of C ,

$$\begin{aligned} \frac{d}{dt} \int_{P_1}^{P_2} \bar{F} \times d\bar{\ell} &= \bar{F}(P_2) \times \bar{v}_{t2} - \bar{F}(P_1) \times \bar{v}_{t1} + \int_C \frac{\partial \bar{F}}{\partial t} \times d\bar{\ell} \\ &+ \int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \times d\bar{\ell} + \int_C \bar{F} \times \frac{\partial \bar{v}_n}{\partial t} d\ell \end{aligned}$$

In terms of total velocity \bar{v} of C ,

$$\frac{d}{dt} \int_{P_1}^{P_2} \bar{F} \times d\bar{\ell} = \bar{F}(P_2) \times \bar{v}_2 - \bar{F}(P_1) \times \bar{v}_1 + \int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} \times d\bar{\ell} + \int_{P_1}^{P_2} [(\bar{d}\bar{\ell} \times \bar{v}) \times \nabla] \times \bar{F}$$

Using an integral rather than the terms involving endpoints,

$$\frac{d}{dt} \int_{P_1}^{P_2} \bar{F} \times d\bar{\ell} = \int_C (\bar{d}\bar{\ell} \cdot \nabla) (\bar{F} \times \bar{v}) + \int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} \times d\bar{\ell} + \int_C [(\bar{d}\bar{\ell} \times \bar{v}) \times \nabla] \times \bar{F}$$

In terms of tangential and normal components, \bar{v}_t and \bar{v}_n , of velocity of C ,

$$\frac{d}{dt} \int_C^{P_2} F \, d\bar{\ell} = F(P_2)\bar{v}_{t2} - F(P_1)\bar{v}_{t1} + \int_C^{P_2} \frac{\partial F}{\partial t} \, d\bar{\ell} + \int_C^{P_2} [(\bar{v}_n \cdot \nabla)F] \, d\bar{\ell} + \int_C^{P_2} F \frac{\partial \bar{v}_n}{\partial \ell} \, d\ell$$

In terms of total velocity \bar{v} of C ,

$$\frac{d}{dt} \int_C^{P_2} F \, d\bar{\ell} = F(P_2)\bar{v}_2 - F(P_1)\bar{v}_1 + \int_C^{P_2} \frac{\partial F}{\partial t} \, d\bar{\ell} - \int_C^{P_2} \nabla F \times (\bar{v} \times d\bar{\ell})$$

Using an integral rather than the terms involving endpoints,

$$\frac{d}{dt} \int_C^{P_2} F \, d\bar{\ell} = \int_C^{P_2} (\bar{d\ell} \cdot \nabla) (F\bar{v}) + \int_C^{P_2} \frac{\partial F}{\partial t} \, d\bar{\ell} - \int_C^{P_2} \nabla F \times (\bar{v} \times d\bar{\ell})$$

In terms of tangential and normal components, \bar{v}_t and v_n , of velocity C , ($\bar{v}_t = \bar{a}_t v_t$ where \bar{a}_t is the unit vector tangent to C in the direction of integration).

$$\frac{d}{dt} \int_C^{P_2} \bar{F} \, d\ell = \bar{F}(P_2)v_{t2} - \bar{F}(P_1)v_{t1} + \int_C^{P_2} \frac{\partial \bar{F}}{\partial t} \, d\ell + \int_C^{P_2} [(\bar{v}_n \cdot \nabla)\bar{F}] \, d\ell + \int_C^{P_2} \bar{F} [d\bar{\ell} \cdot \frac{\partial \bar{v}_n}{\partial \ell}]$$

Using an integral rather than the terms involving the endpoints,

$$\frac{d}{dt} \int_C^{P_2} \bar{F} d\ell = \int_C^{P_2} (\bar{d\ell} \cdot \nabla) F v_t + \int_C^{P_2} \frac{\partial \bar{F}}{\partial t} d\ell + \int_C^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] d\ell + \int_C^{P_2} \bar{F} [\bar{d\ell} \cdot \frac{\partial \bar{v}_n}{\partial \ell}]$$

In terms of tangential and normal components, \bar{v}_t and \bar{v}_n , of velocity of C , ($\bar{v}_t = \bar{a}_t v_t$ where \bar{a}_t is the unit vector tangent to C in the direction of integration).

$$\frac{d}{dt} \int_C^{P_2} F d\ell = F(P_2) v_{t2} - F(P_1) v_{t1} + \int_C^{P_2} \frac{\partial F}{\partial t} d\ell + \int_C^{P_2} [(\bar{v}_n \cdot \nabla) F] d\ell + \int_C^{P_2} F [\bar{d\ell} \cdot \frac{\partial \bar{v}_n}{\partial \ell}]$$

Using an integral rather than terms involving the end points,

$$\frac{d}{dt} \int_C^{P_2} F d\ell = \int_C^{P_2} (\bar{d\ell} \cdot \nabla) (F v_t) + \int_C^{P_2} \frac{\partial F}{\partial t} d\ell + \int_C^{P_2} [(\bar{v}_n \cdot \nabla) F] d\ell + \int_C^{P_2} F [\bar{d\ell} \cdot \frac{\partial \bar{v}_n}{\partial \ell}]$$

For each of the above formulas on line integrals, the terms involving the end points, P_1 and P_2 , vanish if contour C is closed.

Differentiation of Surface Integrals

In terms of tangential and normal components, \bar{v}_t and \bar{v}_n , of velocity of S,

$$\begin{aligned} \frac{d}{dt} \int_S \bar{F} \cdot d\bar{S} &= \oint_C \bar{F} \times \bar{v}_t \cdot d\bar{\ell} + \int_S \frac{\partial \bar{F}}{\partial t} \cdot d\bar{S} \\ &+ \int_S [(\bar{v}_n \cdot \nabla) \bar{F}] \cdot d\bar{S} + \int_S \bar{F} \cdot [(-d\bar{S} \times \nabla) \times \bar{v}_n] \end{aligned}$$

In terms of total velocity, \bar{v} , of S

$$\frac{d}{dt} \int_S \bar{F} \cdot d\bar{S} = \oint_C \bar{F} \times \bar{v} \cdot \overbrace{d\bar{\ell}}^{\cancel{d\bar{\ell}}} + \int_S \frac{\partial \bar{F}}{\partial t} \cdot d\bar{S} + \int_S (\nabla \cdot \bar{F}) \bar{v} \cdot d\bar{S}$$

Using a surface integral rather than the closed contour integral,

$$\frac{d}{dt} \int_S \bar{F} \cdot d\bar{S} = \int_S [\nabla \times (\bar{F} \times \bar{v}) + (\nabla \cdot \bar{F}) \bar{v} + \frac{\partial \bar{F}}{\partial t}] \cdot d\bar{S}$$

In terms of tangential and normal components, \bar{v}_t and \bar{v}_n , of velocity of S,

$$\begin{aligned} \frac{d}{dt} \int_S \bar{F} \times d\bar{S} &= \oint_C \bar{F} \times (\bar{v}_t \times d\bar{\ell}) + \int_S \frac{\partial \bar{F}}{\partial t} \times d\bar{S} \\ &+ \int_S [(\bar{v}_n \cdot \nabla) \bar{F}] \times d\bar{S} + \int_S \bar{F} \times [(-d\bar{S} \times \nabla) \times \bar{v}_n] \end{aligned}$$

In terms of total velocity, \bar{v} , of S

$$\frac{d}{dt} \int_S \bar{F} \times d\bar{S} = \oint_C \bar{F} \times (\bar{v} \times d\bar{\ell}) + \int_S \frac{\partial \bar{F}}{\partial t} \times d\bar{S} - \int_S (\nabla \times \bar{F}) (\bar{v} \cdot d\bar{S})$$

The closed contour integral could be replaced by a surface integral using one of the extensions of Stokes Theorem (Ref. App. 2).

In terms of tangential and normal components, \bar{v}_t and \bar{v}_n , of velocity of S,

$$\begin{aligned} \frac{d}{dt} \int_S F \bar{dS} &= \oint_C F \bar{v}_t \times \bar{d\ell} + \int_S \frac{\partial F}{\partial t} \bar{dS} \\ &+ \int_S [(\bar{v}_n \cdot \nabla) F] \bar{dS} + \int_S F [(-\bar{dS} \times \nabla) \times \bar{v}_n] \end{aligned}$$

In terms of total velocity, \bar{v} , of S,

$$\frac{d}{dt} \int_S F \bar{dS} = \oint_C F(\bar{v} \times \bar{d\ell}) + \int_S \frac{\partial F}{\partial t} \bar{dS} + \int_S \nabla F(\bar{v} \cdot \bar{dS}) .$$

The closed contour integral could be replaced by a surface integral using one of the extensions of Stokes Theorem (Ref. App. 2).

In terms of tangential and normal components, \bar{v}_t and \bar{v}_n , of velocity of S,

$$\begin{aligned} \frac{d}{dt} \int_S \bar{F} dS &= \oint_C \bar{F} [\bar{a}_n \cdot (\bar{v}_t \times \bar{d\ell})] + \int_S \frac{\partial \bar{F}}{\partial t} dS \\ &+ \int_S [(\bar{v}_n \cdot \nabla) \bar{F}] dS + \int_S \bar{F} [\bar{a}_n \cdot [(-\bar{dS} \times \nabla) \times \bar{v}_n]] \end{aligned}$$

where \bar{a}_n is the unit normal vector to S in the direction of \bar{dS} .

In terms of tangential and normal components, \bar{v}_t and \bar{v}_n , of velocity of S,

$$\begin{aligned} \frac{d}{dt} \int_S F dS &= \oint_C F[\bar{a}_n \cdot (\bar{v}_t \times d\bar{\ell})] + \int_S \frac{\partial F}{\partial t} dS \\ &+ \int_S [(\bar{v}_n \cdot \nabla)F] dS + \int_S F\{\bar{a}_n \cdot [(-d\bar{S} \times \nabla) \times \bar{v}_n]\} \end{aligned}$$

where \bar{a}_n is the unit normal vector to S in the direction of $d\bar{S}$.

APPENDIX 5

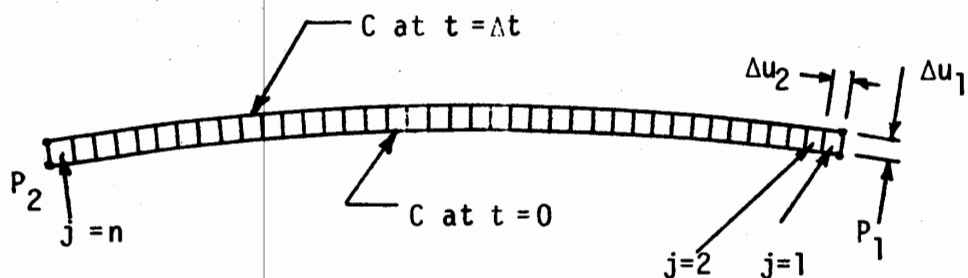
ALTERNATIVE PROOFS FOR THEOREMS 1 AND 2

Consider a line integral

$$Q = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{\ell}$$

from point P_1 to point P_2 on contour C . Let $\frac{\partial \vec{F}}{\partial t} = 0$ and let C be in motion but let its velocity \vec{v} at every point be normal to C ; thus $\vec{v} = \vec{v}_n$. For these assumptions the derivative of Q with respect to time is the quantity denoted $\left(\frac{dQ}{dt}\right)_{\vec{v}_n}$ in the section on line integrals.

An orthogonal coordinate system u_1, u_2, u_3 with unit vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$, where $\vec{a}_3 = \vec{a}_1 \times \vec{a}_2$, will be used in the derivation. It will be oriented such that contour C is described by $u_1 = \text{constant}$ and $u_3 = \text{constant}$, u_2 is distance along C , and \vec{a}_1 is in the direction of \vec{v}_n ; so $\vec{v}_n = \vec{a}_1 v_n$.



Two consecutive positions of C , at $t = 0$ and at $t = \Delta t$, are sketched above. The distance between them is $\Delta u_1 = v_n \Delta t$, which is a function of position along C . The region between the two positions of C is divided into n elementary areas, S_j , for $j = 1$ to n and having dimensions Δu_1 and Δu_2 . Consider the sum of the line integrals around the boundaries of the elementary areas. In the sum, the contributions in the \bar{a}_1 direction at interior boundaries cancel; thus

$$\sum_{j=1}^n \oint_{C_j} \bar{F} \cdot d\bar{\ell} = \bar{F}(P_1) \cdot \bar{a}_1 \Delta u_1(P_1) + \bar{F}(P_2) \cdot (-\bar{a}_1) \Delta u_1(P_2) \\ + \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell} + \int_{P_2}^{P_1} \bar{F} \cdot d\bar{\ell}$$

$C(\Delta t) \qquad C(0)$

where $C(0)$ denotes C at $t = 0$ and $C(\Delta t)$ denotes C at $t = \Delta t$. Note that the last term has limits reversed with respect to the previous term. It is convenient to make the limits the same and place a negative sign on the last term.

The desired result is obtained by using $\Delta u_1 = v_n \Delta t$ and $\bar{v}_n = \bar{a}_1 v_n$, division by Δt and taking the limit as $n \rightarrow \infty$ and $\Delta t \rightarrow 0$. The last two terms on the right become

$$\frac{d}{dt} \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell} = \left(\frac{dQ}{dt} \right) \bar{v}_n$$

and the two terms involving the end points become

$$\bar{F}(P_1) \cdot \bar{v}_n - \bar{F}(P_2) \cdot \bar{v}_n$$

On the left hand side, the basic definition of the curl operator,

$$(\nabla \times \bar{F}) \cdot \bar{a}_3 = \lim_{\Delta u_1 \Delta u_2 \rightarrow 0} \frac{\oint_{C_j} \bar{F} \cdot d\bar{\ell}}{\Delta u_1 \Delta u_2}$$

is used. The expression to be considered is thus

$$\sum_{j=1}^n (\nabla \times \bar{F}) \cdot \bar{a}_3 \Delta u_1 \Delta u_2 .$$

Since $\Delta u_1 = \frac{v_n}{\Delta t} \Delta t$, $\bar{v}_n = \bar{a}_1 v_n$, $\bar{a}_3 = \bar{a}_1 \times \bar{a}_2$ and $d\bar{\ell} = \bar{a}_2 \Delta u_2$, this becomes

$$\sum_{j=1}^n (\nabla \times \bar{F}) \cdot (\bar{v}_n \times d\bar{\ell}) \Delta t$$

or

$$\sum_{j=1}^n (\nabla \times \bar{F}) \times \bar{v}_n \cdot d\bar{\ell} \Delta t$$

Finally, in the limit, this becomes

$$\int_C^{P_2} (\nabla \times \bar{F}) \times \bar{v}_n \cdot d\bar{\ell} ,$$

and the end result is

$$\int_C^{P_2} (\nabla \times \bar{F}) \times \bar{v}_n \cdot d\bar{\ell} = \bar{F}(P_1) \cdot \bar{v}_n \Big|_{P_1}^{P_2} - \bar{F}(P_2) \cdot \bar{v}_n \Big|_{P_1}^{P_2} + \left(\frac{dQ}{dt} \right) \bar{v}_n .$$

This is the same result for $\left(\frac{dQ}{dt} \right) \bar{v}_n$ as obtained in the section

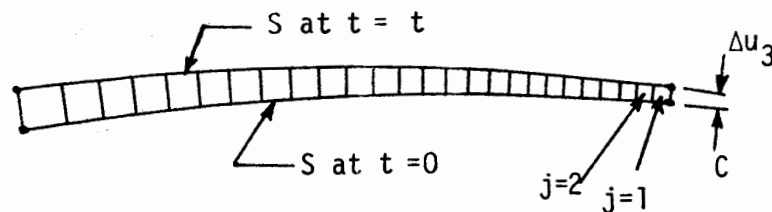
on line integrals (p. 11); it, therefore provides an alternative derivation for Theorem 1.

In the case of a surface integral, consider

$$Q = \int_S \vec{F} \cdot \vec{dS}$$

over surface S . Let $\frac{\partial \vec{F}}{\partial t} = 0$ and let S be in motion but let its velocity at every point be normal to S ; thus $\vec{v} = \vec{v}_n$. For these assumptions the derivative of Q with respect to time is the quantity denoted $\left(\frac{dQ}{dt}\right)_{\vec{v}_n}$ in the section on surface integrals.

An orthogonal coordinate system similar to that used above will be employed. Again the bounding contour C is defined by $u_1 = \text{constant}$ and $u_3 = \text{constant}$, u_2 is distance along C and \vec{a}_3 is normal to S in the direction of \vec{dS} and \vec{v}_n . Thus $\vec{dS} = \vec{a}_3 dS$ and $\vec{v}_n = \vec{a}_3 v_n$.



Two consecutive positions of surface S at $t = 0$ and at $t = \Delta t$, are sketched above. The distance between them is $\Delta u_3 = v_n \Delta t$ which is a function of position on S . The volume between the two consecutive positions of C is divided into n elementary volumes, v_j , for $j = 1$ to n , having volume $v_j = \Delta u_1 \Delta u_2 \Delta u_3$. Consider the sum of the surface integrals over the boundaries of the elementary volumes. In the sum, the contributions over interior surfaces cancel; thus

$$\sum_{j=1}^n \oint_{S_j} \vec{F} \cdot \vec{dS} = \oint_C \vec{F} \cdot (\vec{d\ell} \times \vec{a}_3 \Delta u_3) + \int_{S(0)} \vec{F} \cdot \vec{dS} - \int_{S(\Delta t)} \vec{F} \cdot \vec{dS}$$

where $S(0)$ and $S(\Delta t)$ denote surface S at times $t = 0$ and $t = \Delta t$ respectively. The 1st term has a negative sign because the \overline{dS} is taken to be in the same direction as in the previous term.

The desired result is obtained by using $\Delta u_3 = v_n \Delta t$ and $\overline{v}_n = \overline{a}_3 v_n$, division by Δt and taking the limit as $n \rightarrow \infty$ and $\Delta t \rightarrow 0$. The last two terms on the right become

$$\frac{d}{dt} \int_S \overline{F} \cdot \overline{dS} = \left(\frac{dQ}{dt} \right) \overline{v}_n$$

and the other term on the right becomes

$$\oint_C \overline{F} \cdot (\overline{d\ell} \times \overline{v}_n)$$

This however can be written

$$- \oint_C \overline{F} \times \overline{v}_n \cdot \overline{d\ell}$$

On the left hand side, the basic definition of the divergence operator,

$$\nabla \cdot \overline{F} = \lim_{\Delta u_1 \Delta u_2 \Delta u_3 \rightarrow 0} \frac{\oint_{S_j} \overline{F} \cdot \overline{dS}}{\Delta u_1 \Delta u_2 \Delta u_3},$$

is used. The expression to be considered is

$$\sum_{j=1}^n \Delta u_1 \Delta u_2 \Delta u_3 \nabla \cdot \overline{F}$$

Since $\Delta u_3 = v_n \Delta t$ this can be written

$$\sum_{j=1}^n \Delta S \mathbf{v}_n \cdot \nabla \cdot \bar{\mathbf{F}} \Delta t$$

where $\Delta S = \Delta u_1 \Delta u_2$. In the limit as $n \rightarrow \infty$ and $\Delta t \rightarrow 0$, this can be written

$$\int_S (\nabla \cdot \bar{\mathbf{F}}) \bar{\mathbf{v}} \cdot d\bar{\mathbf{S}} .$$

Finally collecting all of the terms results in

$$\int_S (\nabla \cdot \bar{\mathbf{F}}) \bar{\mathbf{v}} \cdot d\bar{\mathbf{S}} = - \oint_C \bar{\mathbf{F}} \times \bar{\mathbf{v}} \cdot d\bar{\boldsymbol{\ell}} + \left(\frac{dQ}{dt} \right)_{\bar{\mathbf{v}}_n} .$$

This is the same result for $\left(\frac{dQ}{dt} \right)_{\bar{\mathbf{v}}_n}$ as obtained in the section on surface integrals (p. 16); it, therefore provides an alternative derivation for Theorem 2.