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DIFFERENTIATION OF SURFACE INTEGRALS

by

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Abstract

It is sometimes necessary to differentiate a surface integral with respect to a parameter of the integrand, or of the surface, or of the contour bounding the surface or of any combination of these. Differentiation after the integral has been evaluated presents no difficulties, but occasionally it is desirable, in computation or, more often, in theoretical derivations, to interchange the order of integration and differentiation. This report presents theorems concerning such interchange.

INTRODUCTION

Let Q be defined as a surface integral over surface S bounded by contour C, Q is initially assumed to be of the form

$$Q = \int_S \overline{F} \cdot \overline{dS} \quad ;$$

the results will be extended to other types of surface integrals in a later section. Suppose the integrand, $\overline{F}$, the surface, S, and the contour, C, are functions of a parameter t. They need not all be functions of t, but the procedure to be presented will be based on the assumption that they are, and the results will be applicable whether they are or not.

The symbol, t, can be any parameter but, in this discussion, it will be considered as time. The function $\overline{F}$ may be a function of t, and the surface, S, may be a function of t, that is the surface, S, may be in motion. Such motion may involve motion of the contour C which bounds the surface S.

The motion of the surface can be characterized in terms of a velocity, $\overline{v}$, of each point on the surface. $\overline{v}$ is a function of position on the surface S. The motion, at each point, can be resolved into a component, $\overline{v}_t$, tangent to the surface, and a component, $\overline{v}_n$, normal to the surface. The velocity, $\overline{v}$, can be expressed as

$$\overline{v} = \overline{v}_t + \overline{v}_n$$

The motion of the surface is characterized in terms of a fixed (unmoving) coordinate system. At any instant of time, the integral can be evaluated. Its value is a function of time because, at different times, the integrand, $\overline{F}$, can be different, the surface can be different and the countour bounding the surface can be different.

It is sometimes necessary to differentiate the integral Q. The integration may be performed resulting in an expression for Q as a function of t. This expression may then be differentiated, in a straightforward manner to give $\frac{dQ}{dt}$. It is occasionally desired to interchange the order of differentiation and integration. Such an interchange is sometimes needed in theoretical derivations and sometimes even in computational procedures. The procedure for interchange of integration and differentiation is well known for an integral in a one-dimensional space. It is given by the well-known Leibnitz Theorem (or Leibnitz rule). However, the case of a surface integral, in two or three dimensional space requires an extension of the principle involved in the Leibnitz rule. This report is concerned with that extension.

As mentioned above, the velocity of each point on the surface is resolved into normal and tangential components. The reason for doing so concerns the fact that tangential motion of points within a surface does not influence an integral over that surface unless the boundary is in motion. Such tangential motion is simply a stretching or contraction of the surface, in a fixed coordinate system, without changing its position. Such motion, therefore, does not influence the value of the integral, except along the contour (the boundary of the surface) where the tangential motion will be taken into consideration. On the basis of these observations, it may be expected that the derivative, with respect to time, of a surface integral could be expressed in terms of the normal component of the velocity over the surface and the tangential component of the velocity on the contour. This, in fact, is so as is discussed in the next section.

THEORY

The procedure for interchanging integration and differentiation is given by the following theorem.

Theorem S1

Let Q be the surface integral

$$Q = \int_S \bar{F} \cdot \overline{dS}$$

over the surface S which is bounded by the contour C. Assume that the vector function, $\bar{F}$, is a function of time, t. Also assume the surface S, is in motion with respect to the frame of reference with respect to which $\bar{F}$ is defined. Further assume the contour, C, bounding surface S is in motion. Let the motion, with respect to the frame of reference, be defined by velocity, $\bar{v}$, which is a function of position on surface, S. The derivative of Q with respect to t can be expressed as

$$\frac{dQ}{dt} = \oint_C \bar{F} \times \bar{v}_t \cdot \overline{d\ell} + \int_S \frac{\partial \bar{F}}{\partial t} \cdot \overline{dS} + \int_S (\bar{v}_n \cdot \nabla)\bar{F} \cdot \overline{dS} - \int_S \bar{F} \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n] dS$$

where $\bar{v}_t$ and $\bar{v}_n$ are the tangential and normal components of velocity, $\bar{v}$, with respect to surface S. The operator $\bar{v}_n \cdot \nabla$ is a scalar operator which in a general orthogonal coordinate system, with coordinates u_1, u_2 and u_3 and with metric coefficients h_1, h_2, and h_3 is given by

$$\bar{v}_n \cdot \nabla = (\bar{a}_1 v_{n1} + \bar{a}_2 v_{n2} + \bar{a}_3 v_{n3}) \cdot (\frac{\bar{a}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\bar{a}_2}{h_2} \frac{\partial}{\partial u_1} + \frac{\bar{a}_3}{h_3} \frac{\partial}{\partial u_3})$$

$$= \frac{v_{n1}}{n_1} \frac{\partial}{\partial u_1} + \frac{v_{n2}}{h_2} \frac{\partial}{\partial u_2} + \frac{v_{n3}}{h_3} \frac{\partial}{\partial u_3}$$

where $\bar{a}_1$, $\bar{a}_2$ and $\bar{a}_3$ are unit vectors for the coordinate system u_1, u_2, u_3. The vector $\bar{a}_n$ is a unit vector normal to surface S and is in the direction of $\overline{dS}$. The operator $\bar{a}_n \times \nabla$ is a vector operator which in a general orthogonal coordinate system with coordinates u_1, u_2 and u_3 and with metric coefficients h_1, h_2 and h_3 is given by

$$\bar{a}_n \times \nabla = \bar{a}_n \times \left(\frac{\bar{a}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\bar{a}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\bar{a}_3}{h_3} \frac{\partial}{\partial u_3}\right)$$

$$= \frac{\bar{a}_n \times \bar{a}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\bar{a}_n \times \bar{a}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\bar{a}_n \times \bar{a}_3}{h_3} \frac{\partial}{\partial u_3}$$

The direction of integration in the line integral is related to the direction of $\overline{dS}$ by the right hand rule.

A proof of the above theorem is given in Appendix 1.

If the surface S is a closed surface the line integral vanishes giving the following result.

<u>Corollary S1</u>

If the surface S is closed

$$\frac{dQ}{dt} = \oint_S \frac{\partial \bar{F}}{\partial t} \cdot \overline{dS} + \oint_S (\bar{v}_n \cdot \nabla)\bar{F} \cdot \overline{dS} - \oint_S \bar{F} \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n] dS$$

It may be noted that $\bar{a}_n$ is the unit normal to surface S and it is in the same direction as $\overline{dS}$. Since the product of $\bar{a}_n$ and dS occurs in the last term of the expression for $\frac{dQ}{dt}$, the single factor $\overline{dS} = \bar{a}_n\, dS$ may be used instead. The expression for $\frac{dQ}{dt}$ may thus be written

$$\frac{dQ}{dt} = \oint_C \overline{F} \times \overline{v}_n \cdot \overline{d\ell} + \int_S \frac{\partial \overline{F}}{\partial t} \cdot \overline{dS} + \int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS}$$

$$- \int_S \overline{F} \cdot [(\overline{dS} \times \nabla) \times \overline{v}_n] \ .$$

This notation is slightly more concise, but it may be more confusing than that used in the original statement of the theorem.

EXAMPLE

An example of the use of the differentiation theorem will be given in this section. This example involves the use of the theorem in a theoretical derivation.

Consider the expression

$$\oint_C \overline{E} \cdot \overline{d\ell} = - \frac{d}{dt} \int_S \overline{B} \cdot \overline{dS}$$

which is one of Maxwell's equations. The contour, C, is the boundary of surface, S. This relationship is valid for any choice of the surface S. The direction of integration along contour C is related to the direction of $\overline{dS}$ by the right hand rule.

Suppose that $\overline{B}$ is not a function of time but that the contour C is in motion. Since Maxwell's equation is valid for all choices of the surface, S, the surface can be chosen such that the motion of the contour, C, is tangential to the surface. Then the velocity of the contour is $\overline{v}_t$ in the theorem on differentiation of surface integrals, and $\overline{v}_n$ can be chosen to be zero (because of the arbitrariness of the choice of S). For such a choice of the surface, S, the theorem on differentiation of surface integrals shows that

$$-\frac{d}{dt}\int_S \overline{B}\cdot\overline{dS} = -\oint_C \overline{B}\times\overline{v}_t\cdot\overline{d\ell}$$

The other three integrals on the right hand side vanish because $\overline{v}_n = 0$ or because $\frac{\partial\overline{B}}{\partial t} = 0$. Using this result it is seen that

$$\oint_C \overline{E}\cdot\overline{d\ell} = -\oint_C \overline{B}\times\overline{v}\cdot\overline{d\ell}$$

where $\overline{v}$ is simply the velocity of the contour. An alternative and probably more common form of this relation is obtained by reversing the order of the factors in the vector product and by changing the sign. Thus

$$\oint_C \overline{E}\cdot\overline{d\ell} = \oint_C \overline{v}\times\overline{B}\cdot\overline{d\ell}\ .$$

This well known relation follows in a straightforward manner from Maxwell's equation using the theorem on differentiation of surface integrals. This is not a new result. It has long been known that

$$\oint_C \overline{E}\cdot d\ell = \oint_C \overline{v}\times\overline{B}\cdot\overline{d\ell}$$

can be derived from Maxwell's equation; but, without the use of the theorem on differentiation of surface integrals, the derivation requires clever and intricate argument. (Ref. Engineering Electromagnetic Fields and Waves, Carl T.A. Johnk, John Wiley and Sons 1975, pp. 293-295).

If the same derivation is considered but the vector field, $\overline{B}$, is allowed to be a function of time, then the theorem gives the result

$$\oint_C \overline{E}\cdot\overline{d\ell} = -\int_S \frac{\partial\overline{B}}{\partial t}\cdot dS + \oint_C \overline{v}\times\overline{B}\cdot\overline{d\ell}$$

which is another well known result.

Other examples of the use of the differentiation theorem are given in Appendix 2.

EXTENSIONS TO OTHER TYPES OF SURFACE INTEGRALS

Theorem S1 and Corollary S1 are applicable to a common type of surface integral. There are other types of surface integrals which can be differentiated by procedures similar to those given in Theorem S1. Theorems applicable to several other types of surface integrals are presented in this section. Proofs of these theorems are discussed in Appendix 3 and examples of their use are given in Appendix 4.

<u>Theorem S2</u>

Let a surface integral $\overline{Q}$ be defined by

$$\overline{Q} = \int_S \overline{F} \times \overline{dS} \; .$$

The derivative, $\frac{d\overline{Q}}{dt}$, of $\overline{Q}$ with respect to t can be expressed as

$$\frac{d\overline{Q}}{dt} = \oint_C \overline{F} \times (\overline{v}_t \times \overline{d\ell}) + \int_S \frac{\partial \overline{F}}{\partial t} \times \overline{dS} + \int_S [(\overline{v}_n \cdot \nabla)\overline{F}] \times \overline{dS} - \int_S \overline{F} \times [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS$$

where the symbols have the same meaning as discussed in Theorem S1. The first term, which is a line integral, can be expressed in an alternative form which is sometimes preferable; it is given by

$$\oint_C \overline{F} \times (\overline{v}_t \times \overline{d\ell}) = \oint_C \overline{v}_t (\overline{F} \cdot \overline{d\ell}) - \oint_C (\overline{v}_t \cdot \overline{F}) \overline{d\ell}$$

If the surface S is closed, the theorem simplifies.

Corollary S2

If S is a closed surface

$$\frac{d\overline{Q}}{dt} = \oint_S \frac{\partial \overline{F}}{\partial t} \times \overline{dS} + \oint_S [(\overline{v}_n \cdot \nabla)\overline{F}] \times \overline{dS} - \oint_S \overline{F} \times [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS$$

Another type of surface integral is covered by the next theorem.

Theorem S3

Let a surface integral, $\overline{Q}$, be defined by

$$\overline{Q} = \int_S F \, \overline{dS}$$

where F is a scalar function of position. The derivative, $\frac{d\overline{Q}}{dt}$, with respect to t is expressed by

$$\frac{d\overline{Q}}{dt} = \oint_C F \, \overline{v}_t \times \overline{d\ell} + \int_S \frac{\partial F}{\partial t} \overline{dS} + \int_S (\overline{v}_n \cdot \nabla) F \, \overline{dS} - \int_S F[(\overline{a}_n \times \nabla) \times \overline{v}_n] dS .$$

The case in which the surface, S, is closed is covered by the following:

Corollary S3

If S is a closed surface

$$\frac{d\overline{Q}}{dt} = \oint_S \frac{\partial F}{\partial t} \overline{dS} + \oint_S (v_n \cdot \nabla) F \, \overline{dS} - \oint_S F[(\overline{a}_n \times \nabla) \times \overline{v}_n] dS .$$

Another type of surface integral is that in which the integrand is a vector function of position but the differential of area is scalar. It is treated by the following theorem.

Theorem S4

Let a surface integral, $\overline{Q}$, be defined by

$$\overline{Q} = \int_S \overline{F}\, dS$$

The derivative of $\overline{Q}$ with respect to t is given by

$$\frac{d\overline{Q}}{dt} = \oint_C \overline{F}\, [\overline{a}_n \cdot (\overline{v}_t \times \overline{a}_t)] d\ell + \int_S \frac{\partial \overline{F}}{\partial t}\, dS + \int_S (\overline{v}_n \cdot \nabla)\overline{F}\, dS$$

$$- \int_S \overline{F}\{\overline{a}_n \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n]\}\, dS$$

where $\overline{a}_t$ is the unit vector, tangent to contour C, in the direction of integration.

The case in which the surface, S, is closed is covered by the following:

Corollary S4

If S is a closed surface

$$\frac{d\overline{Q}}{dt} = \oint_S \frac{\partial \overline{F}}{\partial t}\, dS + \oint_S (\overline{v}_n \cdot \nabla)\overline{F}\, dS$$

$$- \oint_S \overline{F}\, \{\overline{a}_n \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n]\} dS\ .$$

The final type of surface integral to be considered is that in which both the integrand and the differential of area are scalars. The following theorem is applicable to that case.

Theorem S5

Let a surface integral, Q, be defined by

$$Q = \int_S F\, dS\ .$$

The derivative of Q with respect to t is given by

$$\frac{dQ}{dt} = \oint_C F\,[\bar{a}_n \cdot (\bar{v}_t \times \bar{a}_t)]d\ell + \int_S \frac{\partial F}{\partial t}\,dS + \int_S [(\bar{v}_n \cdot \nabla)F]dS$$
$$- \int_S F\{\bar{a}_n \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n]\}dS$$

where $\bar{a}_t$ is a unit vector tangent to contour C in the direction of integration.

The case in which the surface, S, is closed is covered by the following Corollary.

<u>Corollary S5</u>

If S is a closed surface

$$\frac{dQ}{dt} = \oint_S \frac{\partial F}{\partial t}\,dS + \oint_S (\bar{v}_n \cdot \nabla)F\,dS$$
$$- \oint_S F\{\bar{a}_n \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n]\}.$$

LIMITATIONS

Nothing has been said concerning restrictions on the functions involved in the integrals treated by the foregoing theorems. Sufficient conditions for the validity of the theorems will be given in this section.

It is assumed that the surface of integration, S, is a two-sided surface having a unit normal which is a continuous function of position on the surface. Surfaces having an edge, that is a line along which the unit normal is discontinuous, can be handled by considering the surface as two surfaces which are joined along the edge. The theorems can then be applied to each individual surface. This procedure of placing edges along the bounding contours of surfaces can be extended to any number of edges.

It is assumed that the integrand, denoted by F or $\overline{F}$ in the statements of the theorems, is continuous and has continuous derivatives with respect to t and spacial coordinates at all points on the surface S. Integrands not meeting these conditions can still be treated if it is possible to express the integral as a sum of surface integrals over subdivisions of the surface, the integrand meeting the continuity requirements on each of the subdivisions. The theorems can then be applied to each of the integrals in the sum. This procedure can be described as placing discontinuities on the bounding contours of surfaces where they do not prevent the application of the theorems.

It is assumed that the normal component, $\overline{v}_n$, of the velocity of surface S is continuous and has continuous derivatives in directions tangent to the surface. If these conditions are not met, the theorems can still be applied if it is possible to express the integral as a sum of integrals over subdivisions of the surface, S, the continuity conditions on $\overline{v}_n$ being satisfied on each of the subdivisions. This procedure also amounts to placing the discontinuities on the bounding contours of surfaces such that the theorems can be applied.

COMPILATION OF RESULTS

The five theorems and five corollaries given in this report are listed below in concise notation. This involves the following replacements where appropriate:

$$\bar{a}_t d\ell = \overline{d\ell}$$

$$\bar{a}_n dS = \overline{dS}$$

Theorem S1

$$\text{If} \quad Q = \int_S \bar{F} \cdot \overline{dS}$$

$$\text{then } \frac{dQ}{dt} = \oint_C \bar{F} \times \bar{v}_t \cdot \overline{d\ell} + \int_S \frac{\partial \bar{F}}{\partial t} \cdot \overline{dS}$$

$$+ \int_S [(\bar{v}_n \cdot \nabla)\bar{F}] \cdot \overline{dS} + \int_S \bar{F} \cdot [(-\overline{dS} \times \nabla) \times \bar{v}_n]$$

Corollary S1

If S is a closed surface

$$\text{then } \frac{dQ}{dt} = \oint_S \frac{\partial \bar{F}}{\partial t} \cdot \overline{dS} + \oint_S [(\bar{v}_n \cdot \nabla)\bar{F}] \cdot \overline{dS}$$

$$+ \oint_S \bar{F} \cdot [(-\overline{dS} \times \nabla) \times \bar{v}_n]$$

Theorem S2

$$\text{If} \quad \bar{Q} = \int_S \bar{F} \times \overline{dS}$$

$$\text{then } \frac{d\bar{Q}}{dt} = \oint_C \bar{F} \times (\bar{v}_t \times \overline{d\ell}) + \int_S \frac{\partial \bar{F}}{\partial t} \times \overline{dS}$$

$$+ \int_S [(\bar{v}_n \cdot \nabla)\bar{F}] \times \overline{dS} + \int_S \bar{F} \times [(-\overline{dS} \times \nabla) \times \bar{v}_n]$$

Corollary S2

If S is a closed surface

$$\text{then } \frac{d\overline{Q}}{dt} = \oint_S \frac{\partial \overline{F}}{\partial t} \times \overline{dS} + \oint_S [(\overline{v}_n \cdot \nabla)\overline{F}] \times \overline{dS}$$

$$+ \oint_S \overline{F} \times [(-\overline{dS} \times \nabla) \times \overline{v}_n]$$

Theorem S3

$$\text{If } \overline{Q} = \int_S F\, \overline{dS}$$

$$\text{then } \frac{d\overline{Q}}{dt} = \oint_C F\, \overline{v}_t \times \overline{d\ell} + \int_S \frac{\partial F}{\partial t} \overline{dS}$$

$$+ \int_S [\overline{v}_n \cdot \nabla)F]\overline{dS} + \int_S F[(-\overline{dS} \times \nabla) \times \overline{v}_n]$$

Corollary S3

If S is a closed surface

$$\text{then } \frac{d\overline{Q}}{dt} = \oint_S \frac{\partial F}{\partial t} \overline{dS} + \oint [(\overline{v}_n \cdot \nabla)F]\overline{dS}$$

$$+ \oint_S F[(-\overline{dS} \times \nabla) \times \overline{v}_n]$$

Theorem S4

$$\text{If}\quad \overline{Q} = \int_S \overline{F}\, dS$$

$$\text{then}\quad \frac{d\overline{Q}}{dt} = \oint_C \overline{F}[\overline{a}_n \cdot (\overline{v}_t \times \overline{d\ell})] + \int_S \frac{\partial \overline{F}}{\partial t}\, dS$$

$$+ \int_S [(\overline{v}_n \cdot \nabla)\overline{F}]dS + \int_S \overline{F}\{\overline{a}_n \cdot [(-\overline{dS} \times \nabla) \times \overline{v}_n]\}$$

Corollary S4

If S is a closed surface

$$\text{then}\quad \frac{d\overline{Q}}{dt} = \oint_S \frac{\partial F}{\partial t}\, dS + \oint_S [(\overline{v}_n \cdot \nabla)\overline{F}]dS$$

$$+ \oint_S \overline{F}\{\overline{a}_n \cdot [(-\overline{dS} \times \nabla) \times \overline{v}_n]\}$$

Theorem S5

$$\text{If}\quad Q = \int_S F\, dS$$

$$\text{then}\quad \frac{dQ}{dt} = \oint_C F[\overline{a}_n \cdot (\overline{v}_t \times \overline{d\ell})] + \int_S \frac{\partial F}{\partial t}\, dS$$

$$+ \int_S [(\overline{v}_n \cdot \nabla)F]dS + \int_S F\{\overline{a}_n \cdot [(-\overline{dS} \times \nabla) \times \overline{v}_n]\}$$

Corollary S5

If S is a closed surface

$$\text{then}\quad \frac{dQ}{dt} = \oint_S \frac{\partial F}{\partial t}\, dS + \oint_S [(\overline{v}_n \cdot \nabla)F]dS$$

$$+ \oint_S F\{\overline{a}_n \cdot [(-\overline{dS} \times \nabla) \times \overline{v}_n]\}$$

APPENDIX 1

Proof of Theorem S1

Q is defined by

$$Q = \int_S \overline{F} \cdot \overline{dS}$$

It is assumed that surface S is in motion and that the motion is described in terms of its velocity, $\overline{v}$, which is a function of position on surface S. It is further assumed that the integrand $\overline{F}$ is a function of time, t, and of position in space. Q is therefore a function of time. The derivative, $\frac{dQ}{dt}$ of Q with respect to time will have contributions due to the motion of the contour C which bounds surface S, due to the variation of $\overline{F}$ with time, due to the variation of $\overline{F}$ caused by the motion of surface S and due to the variation of $\overline{dS}$ caused by the motion of surface S. An expression for $\frac{dQ}{dt}$ must take all of these contributions into consideration.

The velocity, $\overline{v}$, can be expressed in terms of components tangential, $\overline{v}_t$, and normal, $\overline{v}_n$, to the surface. Thus

$$\overline{v} = \overline{v}_t + \overline{v}_n \ .$$

The tangential component at an interior point of the surface (i.e. any point other than one on the bounding contour C) causes relative motion of the points within the surface but causes no motion of the surface itself; and, therefore, has no effect upon the surface integral. For this reason only the normal component $\bar{v}_n$ of velocity need be considered at an interior point of the surface. However, the tangential component, $\bar{v}_t$, on the bounding contour, C, does influence the surface integral because it influences the area of surface S. It is apparent that the influence of the motion of surface S, on Q is the sum of contributions due to the normal component, $\bar{v}_n$ of velocity over the surface and due to the tangential component on the bounding contour C.

To evaluate the contribution due to the tangential component, $\bar{v}_t$, on the bounding contour C, assume that $\bar{v}_n = 0$ and $\frac{\partial \bar{F}}{\partial t} = 0$. Next assume the integral is formulated in terms of a set of orthogonal coordinates u_1, u_2 on surface S. The coordinates are selected such that $ds = du_1 du_2$ and such that the bounding countour C coincides with a line defined by u_1 = constant. Next a third coordinate, u_3, is introduced in a direction normal to surface S such that the differential of volume is $dv = du_1 du_2 du_3$. Let $\bar{a}_1$, $\bar{a}_2$ and $\bar{a}_3$ be the unit vectors in this right handed orthogonal coordinate system. $\bar{a}_3$ may also be denoted by $\bar{a}_n$, the unit normal to surface S. Expanding the integrand in the coordinate system u_1, u_2, u_3 gives

$$Q = \int_S F_3 du_1 du_2$$

At t = 0, Q may be expressed as

$$Q(0) = \int_{S(0)} F_3 du_1 du_2$$

where $S(0)$ simply means surface S at $t = 0$. At $t = 0 + dt$, Q may be expressed as

$$Q(0 + dt) = \int_{S(0)} F_3 du_1 du_2 + \oint_C (v_t \cos\theta \, dt) F_3 du_2$$

where v_t is the magnitude of $\bar{v}_t$, and θ is the angle between $\bar{v}_t$ and $\bar{a}_1$. Let $\bar{v}_t = v_t \bar{a}_t$ where $\bar{a}_t$ is a unit vector. It is tangent to surface S along contour C. It can be expressed as $\bar{a}_t = \bar{a}_1 \cos\theta + \bar{a}_2 \sin\theta$. Now consider the expression $\bar{F} \times \bar{a}_t \cdot \bar{a}_2$; it can be manipulated as follows:

$$\bar{F} \times \bar{a}_t \cdot \bar{a}_2 = (a_1 \bar{F}_1 + \bar{a}_2 F_2 + \bar{a}_3 F_3) \times (\bar{a}_1 \cos\theta + \bar{a}_2 \sin\theta) \cdot \bar{a}_2$$

$$= \cos\theta \; F_3 \; .$$

Substitution of this result into the expression for $Q(0 + dt)$ gives

$$Q(0 + dt) = \int_{S(0)} F_3 \, du_1 du_2 + dt \oint_C v_t (\bar{F} \times \bar{a}_t \cdot \bar{a}_2) du_2$$

or

$$Q(0 + dt) = \int_{S(0)} F_3 du_1 du_2 + dt \oint_C \bar{F} \times (v_t \bar{a}_t) \cdot \bar{a}_2 du_2$$

$$\int_{S(0)} F_3 du_1 du_2 + dt \oint_C \bar{F} \times \bar{v}_t \cdot \overline{d\ell}.$$

where $\overline{d\ell} = \bar{a}_2 du_2$ is the vector differential length along contour C. The first integral in this expression is independent of time; therefore the derivative, $\frac{dQ}{dt}$, can be expressed as

$$\frac{dQ}{dt} = \oint_C \bar{F} \times \bar{v}_t \cdot \overline{d\ell}$$

This is the contribution to $\frac{dQ}{dt}$ due to the tangential component of the motion of the bounding contour C.

The contribution to $\frac{dQ}{dt}$ due to the explicit dependence of $\bar{F}$ on t is of the same form as for other types of integrals. It will simply be stated without further discussion; it is

$$\int_S \frac{\partial \bar{F}}{\partial t} \cdot \overline{dS} \ .$$

To complete the proof of Theorem S1, it is necessary to add terms resulting from non-zero $\bar{v}_n$.

The normal component, $\bar{v}_n$, of the velocity is a measure of the transverse movement of each point on the surface. As a point on the surface moves, the function, $\bar{F}$, evaluated at that point, changes. The rate of change of $\bar{F}$ can be expressed as the product of the magnitude, v_n, of the normal component of the velocity and the directional derivative of $\bar{F}$ in the direction of $\bar{v}_n$. Let α be distance in the direction of $\bar{v}_n$ (that is in the direction normal to the surface); then the directional derivative needed is $\frac{\partial \bar{F}}{\partial \alpha}$. Letting $\bar{a}_n$ be a unit normal to surface S, the differentiation operator $\frac{\partial}{\partial \alpha}$ can be expressed as

$$\frac{\partial}{\partial \alpha} = \bar{a}_n \cdot \nabla$$

Therefore

$$v_n \frac{\partial \overline{F}}{\partial \alpha} = v_n(\overline{a}_n \cdot \nabla)\overline{F}$$

$$= (\overline{v}_n \cdot \nabla)\overline{F}$$

and the contribution to $\frac{dQ}{dt}$ resulting from the variation of $\overline{F}$, caused by the transverse motion of surface S, is

$$\int_S [(\overline{v}_n \cdot \nabla)\overline{F}] \cdot \overline{dS} \quad .$$

All contributions to $\frac{dQ}{dt}$ have now been evaluated except that due to the change of $\overline{dS}$ with time; so $\frac{dQ}{dt}$ can be written

$$\frac{dQ}{dt} = \oint_C \overline{F} \times \overline{v}_t \cdot \overline{d\ell} + \int_S \frac{\partial \overline{F}}{\partial t} \cdot \overline{dS} + \int_S [(\overline{v}_n \cdot \nabla)\overline{F}] \cdot \overline{dS} + \int_S \overline{F} \cdot \frac{d(\overline{dS})}{dt}$$

The last term in this expression, at first glance, seems surprising since differentials, in simple integrals, are usually not functions of parameters. However, surface integrals are frequently formulated such that $\overline{dS}$ is a function of position; then, if the surface if integration is in motion, $\frac{d(\overline{dS})}{dt}$ will be non-zero.

The last term of the above relation is in a form that is somewhat awkward to use. A much more satisfactory result can be obtained by formulating it in terms of the normal component, $\overline{v_n}$, of velocity. To do this, first express $\overline{dS}$ in the form

$$\overline{dS} = \overline{a}_n \, du_1 du_2$$

using the coordinate system u_1, u_2 and u_3 introduced earlier in this section. Since $\overline{a}_n = \overline{a}_3$, it can be expressed as

$$\overline{a}_n = \overline{a}_n \times \overline{a}_2 .$$

Thus $\overline{dS}$ now becomes

$$\overline{dS} = \overline{a}_1 \times \overline{a}_2 \, du_1 du_2 .$$

Next note that $\overline{a}_1$ and $\overline{a}_2$ can be expressed in terms of the position vector, $\overline{R}$.

$$\overline{a}_1 = \frac{\partial \overline{R}}{\partial u_1}$$

$$\overline{a}_2 = \frac{\partial \overline{R}}{\partial u_2} .$$

These expressions are valid regardless of the location of the origin of the coordinate system in which the position vector, $\overline{R}$, is defined.

The differential of area, $\overline{dS}$, can now be written

$$\overline{dS} = \frac{\partial \overline{R}}{\partial u_1} \times \frac{\partial \overline{R}}{\partial u_2} \, du_1 du_2 ,$$

and

$$\frac{d(\overline{dS})}{dt} = \left[\frac{\partial(\frac{\partial \overline{R}}{\partial t})}{\partial u_1} \times \frac{\partial \overline{R}}{\partial u_2} + \frac{\partial \overline{R}}{\partial u_1} \times \frac{\partial(\frac{\partial \overline{R}}{\partial t})}{\partial u_2}\right] du_1 du_2 .$$

It is not necessary to consider the variation of du_1 or du_2 with time because any such variation of $du_1 du_2$ in the numerator will cancel the variation of ∂u_1 and ∂u_2 in the denominator.

Since, in this analysis, only the normal component, $\overline{v}_n$, of velocity is being considered, $\frac{\partial \overline{R}}{\partial t}$ may be replaced by $\overline{v}_n$;

$$\frac{\partial \overline{R}}{\partial t} = \overline{v}_n .$$

Using this, it is seen that

$$\frac{d(\overline{dS})}{dt} = [\frac{\partial \overline{v}_n}{\partial u_1} \times \frac{\partial \overline{R}}{\partial u_2} + \frac{\partial \overline{R}}{\partial u_1} \times \frac{\partial \overline{v}_n}{\partial u_2}] \, du_1 du_2$$

$$= [\frac{\partial \overline{v}_n}{\partial u_1} \times \overline{a}_2 + \overline{a}_1 \frac{\partial \overline{v}_n}{\partial u_2}] \, du_1 du_2 \; .$$

Next it is observed that

$$\overline{a}_n \times \nabla = \overline{a}_3 \times (\overline{a}_1 \frac{\partial}{\partial u_1} + \overline{a}_2 \frac{\partial}{\partial u_2} + \overline{a}_3 \frac{\partial}{\partial u_3})$$

$$= -\overline{a}_1 \frac{\partial}{\partial u_2} + \overline{a}_2 \frac{\partial}{\partial u_1}$$

and

$$(\overline{a}_n \times \nabla) \times \overline{v}_n = -\overline{a}_1 \times \frac{\partial \overline{v}_n}{\partial u_2} + \overline{a}_2 \times \frac{\partial \overline{v}_n}{\partial u_1}$$

$$= - (\frac{\partial \overline{v}_n}{\partial u_1} \times \overline{a}_2 + \overline{a}_1 \times \frac{\partial \overline{v}_n}{\partial u_2}) \; ;$$

so

$$\frac{d(\overline{dS})}{dt} = - (\overline{a}_n \times \nabla) \times \overline{v}_n dS \; ,$$

where $dS = du_1 du_2$ is the scalar differential of area. Finally $\frac{dQ}{dt}$ can be written

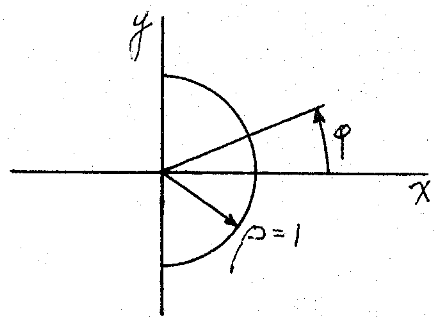
$$\frac{dQ}{dt} = \oint_C \overline{F} \times \overline{v}_t \cdot \overline{d\ell} + \int_S \frac{\partial \overline{F}}{\partial t} \cdot \overline{dS} + \int_S [(\overline{v}_n \cdot \nabla)\overline{F}] \cdot \overline{dS} - \int_S \overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS$$

This completes the proof of Theorem S1.

APPENDIX 2

Examples of the Use of Theorem S1

As an example of the use of the more general expression for $\frac{dQ}{dt}$ consider the surface sketched below. It is a semicylindrical surface



with elements parallel to the z-axis, defined by $\rho = 1$, and extending from $z = 0$ to $z = 1$. Assume it is in motion with velocity $\overline{v} = \overline{a}_x$. Let $\overline{F}$ be given by

$$\overline{F} = \overline{a}_x \, 4xy^2z \ .$$

Since the surface is in motion parallel to the x-axis at a velocity of unity, the relations among x, y, ϕ and t are

$$x - t = \cos \phi$$

and

$$y = \sin \phi$$

The expression for Q can be evaluated to give

$$Q = \int_S \overline{F} \cdot d\overline{S} = \int_S \overline{a}_x 4xy^2z \cdot \rho(\overline{a}_x \cos \phi + \overline{a}_y \sin \phi) d\phi \, dz$$

where the differential of area has been taken to be

$$\overline{dS} = \rho(\overline{a}_x \cos\phi + \overline{a}_y \sin\phi)d\phi\, dz$$

and ρ , in this example, is unity. Thus

$$Q = 4 \int_o^1 z\, dz \int_{-\pi/2}^{\pi/2} xy^2 \cos\phi\; d\phi.$$

Evaluating the integral over z and substituting for x and y in terms of ϕ gives

$$Q = 2 \int_{-\pi/2}^{\pi/2} (t + \cos\phi)\sin^2\phi \cos\phi\, d\phi$$

$$= 2t \int_{-\pi/2}^{\pi/2} \sin^2\phi \cos\phi\, d\phi + 2 \int_{-\pi/2}^{\pi/2} \sin^2\phi \cos^2\phi\, d\phi .$$

These integrals may be evaluated to give

$$Q = 2t\, \frac{\sin^3\phi}{3} \Big|_{-\pi/2}^{\pi/2} + 2(\tfrac{\pi}{8}) = 2t\, \frac{2}{3} + 2\, \frac{\pi}{8}$$

or

$$Q = \frac{4}{3}t + \frac{\pi}{4}$$

From this it is seen that

$$\frac{dQ}{dt} = \frac{4}{3}$$

Now $\frac{dQ}{dt}$ will be found from

$$\frac{dQ}{dt} = \oint_C \overline{F} \times \overline{v}_t \cdot \overline{d\ell} + \int_S \frac{\partial \overline{F}}{\partial t} \cdot \overline{dS} + \int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS} - \int_S \overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS$$

to verify that the same result is obtained. The normal component of the velocity, $\overline{v}$, is

$$\overline{v}_n = v_n\ (\overline{a}_x \cos\phi + \overline{a}_y \sin\phi)$$

where

$$v_n = \cos\phi.$$

The operator $\overline{v}_n \cdot \nabla$ is given by

$$\overline{v}_n \cdot \nabla = \cos^2\phi \frac{\partial}{\partial x} + \cos\phi\ \sin\phi \frac{\partial}{\partial y}$$

and

$$(\overline{v}_n \cdot \nabla)\overline{F} = (4y^2 z \cos^2\phi + 8xyz \cos\phi\ \sin\phi)\overline{a}_x\ .$$

From this, the following integral can be evaluated

$$\int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS} = 4\int_0^1 z\,dz \left[\int_{-\pi/2}^{\pi/2} y^2 \cos^2\phi \cos\phi\ d\phi + 2\int_{-\pi/2}^{\pi/2} xy \cos\phi \sin\phi \cos\phi\ d\phi\right]$$

$$= 2\int_{-\pi/2}^{\pi/2} \sin^2\phi \cos^3\phi\ d\phi + 4\int_{-\pi/2}^{\pi/2} (t + \cos\phi)\sin^2\phi \cos^2\phi\ d\phi.$$

This simplifies to give

$$\int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS} = 2\int_{-\pi/2}^{\pi/2} \sin^2\phi \cos^3\phi\ d\phi + 4t\int_{-\pi/2}^{\pi/2} \sin^2\phi \cos^2\phi\ d\phi + 4\int_{-\pi/2}^{\pi/2} \sin^2\phi \cos^3\phi\ d\phi.$$

The integrals are easily evaluated to give

$$\int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS} = \frac{\pi}{2}t + \frac{8}{5}\ .$$

The tangential component of velocity on the surface edges parallel to the z-axis is $\overline{v}_t = \overline{a}_x$; it is zero on the ends of the cylindrical surface. Therefore

$$\oint \overline{F} \times \overline{v}_t \cdot \overline{d\ell} = 0$$

since $\overline{F}$ and $\overline{v}_t$ are in the same direction at all points at which $\overline{v}_t$ is nonzero.

The operator $\overline{a}_n \times \nabla$ is given by

$$\overline{a}_n \times \nabla = (\overline{a}_x \cos\phi + \overline{a}_y \sin\phi) \times (\overline{a}_x \frac{\partial}{\partial x} + \overline{a}_y \frac{\partial}{\partial y} + \overline{a}_z \frac{\partial}{\partial z})$$

$$= \overline{a}_x \sin\phi \frac{\partial}{\partial z} - \overline{a}_y \cos\phi \frac{\partial}{\partial z} + \overline{a}_z (\cos\phi \frac{\partial}{\partial y} - \sin\phi \frac{\partial}{\partial x}).$$

Operating on $\overline{v}_n$ gives

$$(\overline{a}_n \times \nabla) \times \overline{v}_n = (\overline{a}_n \times \nabla) \times (\overline{a}_x \cos^2\phi + \overline{a}_y \cos\phi \sin\phi)$$

$$= (\overline{a}_n \times \nabla) \times (\overline{a}_n (x-t)^2 + \overline{a}_y y(x-t))$$

$$= \overline{a}_x [-\cos\phi (x-t) + \sin\phi y] + \overline{a}_y [-\sin\phi 2(x-t)]$$

$$= \overline{a}_x [\sin^2\phi - \cos^2\phi] + \overline{a}_y [-2 \sin\phi \cos\phi] \cdot$$

From this result it is seen that

$$\overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] = 4xy^2 z(\sin^2\phi - \cos^2\phi)$$

and

$$\int_S \overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS = 4 \int_0^1 z\, dz \int_{-\pi/2}^{\pi/2} (t + \cos\phi) \sin^2\phi (\sin^2\phi - \cos^2\phi) d\phi$$

$$= 2t \int_{-\pi/2}^{\pi/2} \sin^4\phi\, d\phi - 2t \int_{-\pi/2}^{\pi/2} \sin^2\phi \cos^2\phi\, d\phi + 2 \int_{-\pi/2}^{\pi/2} \cos\phi \sin^4\phi\, d\phi - 2 \int_{-\pi/2}^{\pi/2} \cos^3\phi \sin^2\phi\, d\phi .$$

These integrals can be evaluated to give

$$\int_S \overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS = 2t(\tfrac{3\pi}{8}) - 2t(\tfrac{\pi}{8}) + 2(\tfrac{2}{5}) - 2(\tfrac{4}{15})$$

$$= \frac{\pi}{2} t + \frac{4}{15} .$$

Finally, since $\frac{\partial \overline{F}}{\partial t} = 0$

$$\oint_C \overline{F} \times \overline{v}_n \cdot \overline{d\ell} + \int_S \frac{\partial \overline{F}}{\partial t} \cdot \overline{dS} + \int_S (\overline{v}_n \cdot \nabla) \overline{F} \cdot \overline{dS}$$

$$- \int_S \overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n]\, dS$$

$$= 0 + 0 + (\tfrac{\pi}{2} t + \tfrac{8}{5}) - (\tfrac{\pi}{2} t + \tfrac{4}{15}) = \tfrac{4}{3} .$$

This is the same result as obtained by integration for Q followed by differentiation with respect to t.

The above example suggests that the differentiation formulae herein presented involve more time consuming analytical evaluation than the more straightforward evaluation of the integral followed by differentiation. This, in fact, is true for most integrals; indeed the primary utility of the theorem on differentiation of surface integrals presented here is in theoretical derivations rather than in evaluation of derivatives.

To gain greater understanding of the use of the theorem on differentiation of surface integrals, another example will be presented.

Consider the same cylindrical surface having the same motion as in the previous example but let the vector field, $\overline{F}$, be given by

$$\overline{F} = \overline{a_y}\, 4xyz \cos \omega t.$$

The following relations are the same as in the previous problem.

$$x - t = \cos \phi \, ,$$

$$y = \sin \phi \, ,$$

$$v_n = \cos \phi \, ,$$

$$\overline{v}_n = v_n \, (\overline{a}_x \cos \phi + \overline{a}_y \sin \phi \,),$$

$$\overline{v}_t = \overline{a}_x \text{ (on the parallel edges of the cylinder)},$$

$$\overline{v}_t = 0 \text{ (on the ends of the cylinder)},$$

$$\overline{v}_n \cdot \nabla = \cos^2\phi \frac{\partial}{\partial x} + \sin \phi \cos \phi \frac{\partial}{\partial y} \, ,$$

$$\overline{dS} = \rho(\overline{a}_x \cos \phi + \overline{a}_y \sin \phi) d\phi \, dz \text{ (with } \rho = 1),$$

$$\frac{\partial \overline{dS}}{\partial t} = \cos \phi (\overline{a}_x \cos \phi + \overline{a}_y \sin \phi) d\phi \, dz \text{ (since } \frac{\partial \rho}{\partial t} = \cos \phi).$$

First $\frac{dQ}{dt}$ will be evaluated by integration to obtain Q followed by differentiation with respect to t.

$$Q = \int_S \overline{F} \cdot \overline{dS} = 4 \cos \omega t \int_0^1 z \, dz \int_{-\pi/2}^{\pi/2} xy \sin \phi \, d\phi$$

$$= 2 \cos \omega t \int_{-\pi/2}^{\pi/2} (t + \cos \phi) \sin^2 \phi \, d\phi$$

$$= 2(\cos \omega t) t \int_{-\pi/2}^{\pi/2} \sin^2 \phi \, d\phi + 2 \cos \omega t \int_{-\pi/2}^{\pi/2} \cos \phi \sin^2 \phi \, d\phi$$

$$= 2(\cos \omega t) t \frac{\pi}{2} + 2(\cos \pi t) \frac{2}{3}$$

$$= \pi t \cos \omega t + \frac{4}{3} \cos \omega t.$$

The derivative with respect to t is

$$\frac{dQ}{dt} = \pi \cos \omega t - \pi \omega t \sin \omega t - \frac{4}{3} \omega \sin \omega t.$$

To demonstrate the use of the theorem five integrals must be evaluated. First consider

$$\oint_C \overline{F} \times \overline{v}_t \cdot \overline{d\ell} = \int_1^0 \overline{F}(t,-1,z) \times \overline{a}_x \cdot \overline{a}_z dz + \int_0^1 \overline{F}(t,1,z) \times \overline{a}_x \cdot \overline{a}_z dz$$

$$= 4t \cos \omega t \int_1^0 z \, dz - 4t \cos \omega t \int_0^1 z \, dz = -4t \cos \omega t.$$

Next consider

$$\int_S (\overline{v}_n \cdot \nabla) \overline{F} \cdot \overline{dS} = 4 \cos \omega t \int_S [yz \cos^2 \phi + xz \sin \phi \cos \phi] \sin \phi \, d\phi \, dz$$

$$= 4 \cos \omega t \int_0^1 z \, dz \int_{-\pi/2}^{\pi/2} [\sin^2 \phi \cos^2 \phi + (t + \cos \phi) \sin^2 \phi \cos \phi] d\phi$$

$$= 2 \cos \omega t \left[\frac{\pi}{8} + t \frac{2}{3} + \frac{\pi}{8}\right]$$

$$= \frac{\pi}{2} \cos \omega t + \frac{4}{3} t \cos \omega t.$$

Just as in the previous example

$$(\bar{a}_n \times \nabla) \times \bar{v}_n = \bar{a}_x(\sin^2\phi - \cos^2\phi) + \bar{a}_y[-2 \sin\phi \cos\phi] .$$

From this it is seen that

$$\bar{F} \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n] = -2 \sin\phi \cos\phi \, (4xyz \cos \omega t)$$

$$= -8 \, (t + \cos\phi) \sin^2\phi \cos\phi \, z \cos \omega t$$

and

$$\int_S \bar{F} \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n] dS = -8 \int_0^1 z \, dz \int_{-\pi/2}^{\pi/2} (t + \cos\phi) \sin^2\phi \cos\phi \cos \omega t \, d\phi$$

$$= -4(\cos \omega t) t \int_{-\pi/2}^{\pi/2} \sin^2\phi \cos\phi \, d\phi - 4 \cos \omega t \int_{-\pi/2}^{\pi/2} \sin^2\phi \cos^2\phi \, d\phi$$

$$= -4(\cos \omega t) \, t \left(\frac{2}{3}\right) - 4 \cos \omega t \left(\frac{\pi}{8}\right).$$

The next integral to be evaluated is

$$\int_C \frac{\partial \bar{F}}{\partial t} \cdot \overline{dS} = -4 \, \omega \sin \omega t \int_0^1 z \, dz \int_{-\pi/2}^{\pi/2} (t + \cos\phi) \sin\phi \sin\phi \, d\phi$$

$$= -2\omega(\sin \omega t) t \left(\frac{\pi}{2}\right) - 2\omega(\sin \omega t)\left(\frac{2}{3}\right)$$

$$= -\pi\omega t \sin \omega t - \frac{4}{3} \omega \sin \omega t$$

Collecting these results it is seen that

$$\oint \overline{F} \times \overline{v_n} \cdot \overline{d\ell} + \int_S \frac{\partial \overline{F}}{\partial t} \cdot \overline{dS} + \int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS}$$

$$- \int_S \overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS$$

$$= (-4t \cos \omega t) + (-\pi\omega t \sin \omega t - \frac{4}{3}\omega \sin \omega t)$$

$$+ (\frac{\pi}{2} \cos \omega t + \frac{4}{3} t \cos \omega t) - (-\frac{8}{3} t \cos \omega t - \frac{\pi}{2} \cos \omega t)$$

$$= \pi \cos \pi t - \pi\omega t \sin \omega t - \frac{4}{3}\omega \sin \omega t \ .$$

This result agrees with that obtained by integration to find Q followed by differentiation with respect to t.

It seems desirable to present one example involving a surface having curvature in two dimensions. Consider

$$Q = \int_S \overline{F} \cdot \overline{dS}$$

where

$$\overline{F} = 4x^2y^2z \ \overline{a}_z$$

and let S be a hemispherical surface of unit radius. More specifically, at $t = 0$, S is the hemisphere, for which $z > 0$, having unit radius, centered at the origin and moving, undistorted, in the direction of $\overline{a}_z$ with unit velocity. Thus

$$\overline{v} = \overline{a}_z \ ,$$

and on surface S

$$z = t + \cos\theta$$

$$x = \sin\theta \cos\phi$$

$$y = \sin\theta \sin\phi$$

where θ and ϕ are coordinates in a spherical coordinate system having an origin at the center of the spherical surface moving at the same velocity as the surface.

Q can be evaluated as follows:

$$\overline{dS} = \bar{a}_r \, d\theta \sin\theta \, d\phi ,$$

where

$$\bar{a}_r = \bar{a}_x \sin\theta\cos\phi + \bar{a}_y \sin\theta\sin\phi + \bar{a}_z\cos\theta .$$

So Q can be written

$$Q = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} 4x^2y^2z \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= 4 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (t + \cos\theta)\sin^2\theta \cos^2\phi \sin^2\theta \sin^2\phi \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= 4 \int_{0}^{2\pi} \cos^2\phi \sin^2\phi \, d\phi \left[t \int_{0}^{\pi/2} \sin^5\theta \cos\theta \, d\theta + \int_{0}^{\pi/2} \sin^5\theta \cos^2\theta \, d\theta \right]$$

$$= 4 \left(\frac{\pi}{4}\right) \left[t \frac{1}{6} + \frac{8}{105}\right] = \frac{\pi}{6} t + \frac{8\pi}{105} .$$

From this, it is seen that

$$\frac{dQ}{dt} = \frac{\pi}{6} .$$

Next $\frac{dQ}{dt}$ will be evaluated from

$$\frac{dQ}{dt} = \oint_C \overline{F} \times \overline{v}_t \cdot \overline{d\ell} + \int_S \frac{\partial \overline{F}}{\partial t} \cdot \overline{dS} + \int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS}$$

$$- \int_S \overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS \ .$$

It is noted that $\overline{F} \times \overline{v}_t = 0$ at all points on the contour bounding S; so the first integral on the right hand side vanishes. The second integral vanishes because $\frac{\partial \overline{F}}{\partial t} = 0$. The normal component of the velocity is

$$\overline{v}_n = \overline{a}_r \cos\theta = \overline{a}_x \sin\theta \cos\theta \cos\phi + \overline{a}_y \sin\theta \cos\theta \sin\phi + \overline{a}_z \cos^2\theta,$$

so

$$\overline{v}_n \cdot \nabla = \cos\theta \sin\theta \cos\phi \frac{\partial}{\partial x} + \cos\theta \sin\theta \sin\phi \frac{\partial}{\partial y} + \cos^2\phi \frac{\partial}{\partial z}$$

and

$$(\overline{v}_n \cdot \nabla)\overline{F} = (\sin\theta \cos\theta \cos\phi \, 8xy^2z + \sin\theta \cos\theta \sin\phi \, 8x^2yz + \cos^2\theta \, 4x^2y^2)\overline{a}_z$$

so

$$\int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS} = 8 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin\theta \cos\theta \cos\phi \sin\theta \cos\phi \sin^2\theta \sin^2\phi (t+\cos\theta) \cos\theta \sin\theta \, d\theta \, d\phi$$

$$+ 8 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin\theta \cos\theta \sin\phi \sin^2\theta \cos^2\phi \sin\theta \sin\phi \, (t + \cos\theta) \cos\theta \sin\theta \, d\theta \, d\phi$$

$$+ 4 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \cos^2\theta \sin^2\theta \cos^2\phi \sin^2\theta \sin^2\phi \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= 16t \int_0^{2\pi} \cos^2\phi \, \sin^2\phi \, d\phi \int_0^{\pi/2} \sin^5\theta \, \cos^2\theta \, d\theta$$

$$+ 20 \int_0^{2\pi} \cos^2\phi \, \sin^2\phi \, d\phi \int_0^{\pi/2} \sin^5\theta \, \cos^2\theta \, d\theta$$

$$= \frac{32}{105} \pi t + \frac{5\pi}{24} .$$

The operator $\bar{a}_n \times \nabla$ is easily expanded in terms of coordinates θ and ϕ.

$$\bar{a}_n \times \nabla = \bar{a}_r \times \nabla = - \frac{\bar{a}_\theta}{\sin\theta} \frac{\partial}{\partial\phi} + \bar{a}_\phi \frac{\partial}{\partial\theta}$$

and

$$(\bar{a}_n \times \nabla) \times \bar{v}_n = \left(- \frac{\bar{a}_\theta}{\sin\theta} \frac{\partial}{\partial\phi} + \bar{a}_\phi \frac{\partial}{\partial\theta}\right) \times \bar{a}_r \cos\theta$$

$$= - 2\bar{a}_r \cos\theta - \bar{a}_\theta \sin\theta$$

where the relations

$$\frac{\partial\bar{a}_r}{\partial\theta} = \bar{a}_\theta$$

and

$$\frac{\partial\bar{a}_r}{\partial\phi} = \sin\theta \, \bar{a}_\phi$$

have been used. In rectangular coordinates this is

$$(\bar{a}_n \times \nabla) \times \bar{v}_n = -\bar{a}_x \, 3 \sin\theta \cos\theta \cos\phi - \bar{a}_y \sin\theta \cos\theta \sin\phi + \bar{a}_z(\sin^2\theta - 2\cos^2\theta) ,$$

$$\int_S \overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS = \int_S 4x^2y^2z(\sin^2\theta - 2\cos^2\theta)dS$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} 4 \sin^2\theta \cos^2\phi \sin^2\theta \sin^2\phi (t+\cos\theta)(\sin^2\theta - 2\cos^2\theta)\sin\theta \, d\phi \, d\theta$$

$$= 4 \int_0^{2\pi} \cos^2\phi \sin^2\phi \, d\phi \int_0^{\pi/2} \sin^7\theta (t + \cos\theta) d\theta$$

$$- 8 \int_0^{2\pi} \cos^2\phi \sin^2\phi \, d\phi \int_0^{\pi/2} \sin^5\theta \cos^2\theta (t + \cos\theta) d\theta$$

$$= 4(\tfrac{\pi}{4})[t(\tfrac{16}{35}) + \tfrac{1}{8}] - 8(\tfrac{\pi}{4})[t(\tfrac{8}{105}) + \tfrac{1}{24}] = \pi t \, \frac{32}{105} + \frac{\pi}{24} \; .$$

Finally

$$\oint_C \overline{F} \times \overline{v}_t \cdot \overline{d\ell} + \int_S \frac{\partial \overline{F}}{\partial t} \cdot \overline{dS} + \int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS}$$

$$- \int_S \overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS$$

$$= 0 + 0 + (\frac{32}{105} \pi t + \frac{5\pi}{24}) - (\pi t \, \frac{32}{105} + \frac{\pi}{24}) = \frac{\pi}{6} \; .$$

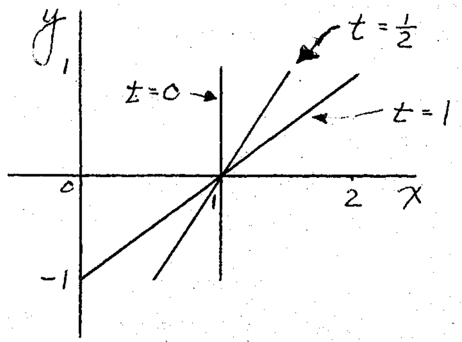
This result agrees with that obtained by differentiation of the expression for Q.

Consider next an example for which

$$\overline{F} = 4xy^2z \, \overline{a}_x$$

and the surface is a plane extending from $z = 0$ to $z = 1$. At time $t = 0$ the plane extends from the line $x = 1, y = -1$ to the line $x = 1, y = 1$. Thus the surface is a rectangle of 2 units on one side and one on the other.

It is assumed the rectangular, plane surface rotates about the axis $x = 1$, $y = 0$ such that the upper edge has a velocity of $\bar{a}_x$ and the lower edge has velocity $-\bar{a}_x$. The surface remains a planar rectangle, of unit length in the z direction, as it rotates, but its width increases as it rotates. The surface is sketched below. The sketch shows the position of the surface at $t = 0$, $t = \frac{1}{2}$ and $t = 1$. The relation between x and y on the surface can be expressed as



$$x = 1 + yt.$$

If ϕ denotes the angle through which the planar surface has been rotated, the differential of area can be written

$$\overline{dS} = (\bar{a}_x \cos\phi - \bar{a}_y \sin\phi)\,\frac{dy}{\cos\phi}\;dz$$

and the surface integral can be written

$$Q = \int_S \bar{F}\cdot\overline{dS} = 4\int_{z=0}^{1}\int_{y=-1}^{1} xy^2z\;dy\;dz\;.$$

Substituting for x in terms of y this becomes

$$Q = 4\int_0^1 z\,dz\int_{-1}^{1}(1+yt)y^2dy$$

$$= 2\left[\frac{y^3}{3}\Big|_{-1}^{1} + t\,\frac{y^4}{4}\Big|_{-1}^{1}\right] = \frac{4}{3}$$

From this

$$\frac{dQ}{dt} = 0\;.$$

Now the derivative $\frac{dQ}{dt}$ will be found by evaluating the expression

$$\frac{dQ}{dt} = \oint_C \bar{F} \times \bar{v}_t \cdot \overline{d\ell} + \int_S \frac{\partial \bar{F}}{\partial t} \cdot \overline{dS} + \int_S (\bar{v}_n \cdot \nabla)\bar{F} \cdot \overline{dS}$$

$$- \int_S \bar{F} \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n]\, dS \; .$$

The tangential and normal components of the velocity can be expressed as

$$v_n = (\bar{a}_x \cos\phi - \bar{a}_y \sin\phi) y \cos\phi$$

$$\bar{v}_t = (\bar{a}_x \sin\phi + \bar{a}_y \cos\phi) \sin\phi \quad \text{(on upper edge)}$$

$$\bar{v}_t = - (\bar{a}_x \sin\phi + \bar{a}_y \cos\phi) \sin\phi \quad \text{(on lower edge)}$$

From these expressions

$$\bar{F} \times \bar{v}_t = 4(1 + t) z \sin\phi \cos\phi \, \bar{a}_z$$

on the upper edge and

$$\bar{F} \times \bar{v}_t = -4(1-t) z \sin\phi \cos\phi \, \bar{a}_z$$

on the lower edge.

Using these expressions

$$\oint_C \bar{F} \times \bar{v}_t \cdot \overline{d\ell} = -4 \int_1^0 (1-t) z \sin\phi \cos\phi \, dz + 4 \int_0^1 (1+t) z \sin\phi \cos\phi \, dz$$

$$= 4 \sin\phi \cos\phi \, .$$

Next it is seen that

$$\overline{v}_n \cdot \nabla = y \cos^2\phi \frac{\partial}{\partial x} - y \sin\phi \cos\phi \frac{\partial}{\partial y}$$

and

$$(\overline{v}_n \cdot \nabla)\overline{F} = (y \cos^2\phi \; 4y^2 z - y \sin\phi \cos\phi \, 8xyz)\overline{a}_z \; .$$

Therefore

$$\int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS} = 4 \int_o^1 z \, dz \int_{-1}^1 y \cos^2\phi \; y^2 dy - 8 \int_o^1 z \, dz \int_{-1}^1 \sin\phi \cos\phi \, (1+yt) y^2 dy$$

$$= - 4 \sin\phi \cos\phi \frac{y^3}{3} \Big|_{-1}^{1} = - \frac{8}{3} \sin\phi \cos\phi \; .$$

The operator $\overline{a}_n \times \nabla$ is given by

$$\overline{a}_n \times \nabla = -\overline{a}_x \sin\phi \frac{\partial}{\partial z} - \overline{a}_y \cos\phi \frac{\partial}{\partial z} + \overline{a}_z (\cos\phi \frac{\partial}{\partial y} + \sin\phi \frac{\partial}{\partial x})$$

and

$$(\overline{a}_n \times \nabla) \times \overline{v}_n = \overline{a}_x \cos^2\phi \sin\phi + \overline{a}_y \cos^3\phi \; .$$

Finally the last of the required integrals is

$$\int_S \overline{F} \cdot [\overline{a}_n \times \nabla) \times \overline{v}_n] \, dS = 4 \int_o^1 z \, dz \int_{-1}^1 xy^2 \cos^2\phi \sin\phi \frac{dy}{\cos\phi}$$

$$= 2 \int_{-1}^1 (1 + yt) y^2 \cos\phi \sin\phi \, dy$$

$$= 2 \cos\phi \sin\phi \frac{y^3}{3} \Big|_{-1}^{1} = \frac{4}{3} \cos\phi \sin\phi \; .$$

Collecting these results it is seen that

$$\oint_C \overline{F} \times \overline{v_t} \cdot \overline{d\ell} + \int_S \frac{\partial \overline{F}}{\partial t} \cdot \overline{dS} + \int_S (\overline{v}_n \cdot \nabla)\overline{F} \cdot \overline{dS}$$

$$- \int_S \overline{F} \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS$$

$$= (4 \sin\phi \cos\phi) + 0 + (- \frac{8}{3} \sin\phi \cos\phi) - (\frac{4}{3} \cos\phi \sin\phi)$$

$$= 0 .$$

This is the same result as obtained by integration for Q followed by differentiation with respect to t.

APPENDIX 3

PROOFS OF THEOREMS S2, S3, S4 and S5

Theorem S1 has been proved in Appendix 1. The proof of theorems S2, S3, S4 and S5 are merely slight modifications of that proof. Only the modifications will be discussed in this Appendix. Two of these theorems involve an expression for $\frac{d(dS)}{dt}$. This can be evaluated by first writing dS in terms of the position vector $\overline{R}$. Thus

$$dS = \left|\frac{\partial \overline{R}}{\partial u_1}\right| \left|\frac{\partial \overline{R}}{\partial u_2}\right| du_1 du_2$$

where u_1 and u_2 are the orthogonal, rectangular coordinates introduced in Appendix 1 and $\frac{\partial \overline{R}}{\partial u_2} = \overline{a}_1$ and $\frac{\partial \overline{R}}{\partial u_2} = \overline{a}_2$. This seems a surprising starting point since both of the terms involving magnitude brackets are unity and, of course, $du_1 du_2 = dS$. Nevertheless, this will lead to a useful formulation. In this relation, $\overline{R}$ can be expressed as

$$\overline{R} = \overline{a}_x x + \overline{a}_y y + \overline{a}_z z$$

where x, y and z are the coordinates of points on surface S. They are functions of the position coordinates u_1 and u_2 on S. From this it is seen that

$$\frac{\partial \overline{R}}{\partial u_1} = \overline{a}_x \frac{\partial x}{\partial u_1} + \overline{a}_y \frac{\partial y}{\partial u_1} + \overline{a}_z \frac{\partial z}{\partial u_1}$$

and

$$\frac{\partial \bar{R}}{\partial u_2} = \bar{a}_x \frac{\partial x}{\partial u_2} + \bar{a}_y \frac{\partial y}{\partial u_2} + \bar{a}_z \frac{\partial z}{\partial u_2} .$$

From these expressions it is clear that

$$\left| \frac{\partial \bar{R}}{\partial u_1} \right| = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}$$

and

$$\left| \frac{\partial \bar{R}}{\partial u_2} \right| = \sqrt{\left(\frac{\partial x}{\partial u_2}\right)^2 + \left(\frac{\partial y}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2} .$$

Using these results dS can be written

$$dS = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2} \sqrt{\left(\frac{\partial x}{\partial u_2}\right)^2 + \left(\frac{\partial y}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2} \; du_1 du_2 .$$

From this

$$\frac{d(dS)}{dt} = \left[\frac{\frac{\partial x}{\partial u_1} \frac{\partial\left(\frac{\partial x}{t}\right)}{\partial u_1} + \frac{\partial y}{\partial u_1} \frac{\partial\left(\frac{\partial y}{\partial t}\right)}{\partial u_1} + \frac{\partial z}{\partial u_1} \frac{\partial\left(\frac{\partial z}{\partial t}\right)}{\partial u_1}}{\sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}} \sqrt{\left(\frac{\partial x}{\partial u_2}\right)^2 + \left(\frac{\partial y}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2} \right.$$

$$\left. + \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2} \; \frac{\frac{\partial x}{\partial u_2} \frac{\partial\left(\frac{\partial x}{\partial t}\right)}{\partial u_2} + \frac{\partial y}{\partial u_2} \frac{\partial\left(\frac{\partial y}{\partial t}\right)}{\partial u_2} + \frac{\partial z}{\partial u_2} \frac{\partial\left(\frac{\partial z}{\partial t}\right)}{\partial u_2}}{\sqrt{\left(\frac{\partial x}{\partial u_2}\right)^2 + \left(\frac{\partial y}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2}} \right] du_1 du_2$$

In obtaining this result the order of differentiation, with respect to time and with respect to spacial coordinates has been interchanged.

The motion under consideration in this expression is motion in the direction of the normal to surface S having velocity, $\overline{v_n}$, which can be written

$$\overline{v}_n = \overline{a}_x \frac{\partial x}{\partial t} + \overline{a}_y \frac{\partial y}{\partial t} + \overline{a}_z \frac{\partial z}{\partial t} .$$

Using this, the above expression can be written

$$\frac{d(dS)}{dt} = \left[\frac{\frac{\partial \overline{R}}{\partial u_1} \cdot \frac{\partial \overline{v}_n}{\partial u_1}}{\left| \frac{\partial \overline{R}}{\partial u_1} \right|} \left| \frac{\partial \overline{R}}{\partial u_2} \right| + \left| \frac{\partial \overline{R}}{\partial u_1} \right| \frac{\frac{\partial \overline{R}}{\partial u_2} \cdot \frac{\partial \overline{v}_n}{\partial u_2}}{\left| \frac{\partial \overline{R}}{\partial u_2} \right|} \right] du_1 du_2 .$$

Next, using the relations (see Appendix 1)

$$\frac{\partial \overline{R}}{\partial u_1} = \overline{a}_1 ,$$

$$\frac{\partial \overline{R}}{\partial u_2} = \overline{a}_2 ,$$

$$\left| \frac{\partial \overline{R}}{\partial u_1} \right| = 1 ,$$

and

$$\left| \frac{\partial \overline{R}}{\partial u_2} \right| = 1 .$$

It follows that

$$\frac{d(dS)}{dt} = \left[\overline{a}_1 \cdot \frac{\partial \overline{v}_n}{\partial u_1} + \overline{a}_2 \cdot \frac{\partial \overline{v}_n}{\partial u_2} \right] du_1 du_2 .$$

In order to express this result so that it does not involve a coordinate system, note that

$$\bar{a}_n \times \nabla = \bar{a}_3 \times (\bar{a}_1 \frac{\partial}{\partial u_1} + \bar{a}_2 \frac{\partial}{\partial u_2} + \bar{a}_3 \frac{\partial}{\partial u_3})$$

$$= \bar{a}_2 \frac{\partial}{\partial u_1} - \bar{a}_1 \frac{\partial}{\partial u_2} ,$$

and

$$(\bar{a}_n \times \nabla) \times \overline{v_n} = \bar{a}_2 \times \frac{\partial \overline{v_n}}{\partial u_1} - \bar{a}_1 \times \frac{\partial \overline{v_n}}{\partial u_2} .$$

From this, it is seen that

$$\bar{a}_n \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n] = \bar{a}_3 \cdot \bar{a}_2 \times \frac{\partial \bar{v}_n}{\partial u_1} - \bar{a}_3 \cdot \bar{a}_1 \times \frac{\partial \bar{v}_n}{\partial u_2}$$

$$= \bar{a}_3 \times \bar{a}_2 \cdot \frac{\partial \bar{v}_n}{\partial u_1} - \bar{a}_3 \times \bar{a}_1 \cdot \frac{\partial \bar{v}_n}{\partial u_2}$$

$$= - \bar{a}_1 \cdot \frac{\partial \bar{v}_n}{\partial u_1} - \bar{a}_2 \cdot \frac{\partial \bar{v}_n}{\partial u_2} .$$

Using this result and using $du_1 du_2 = dS$, the desired expression for $\frac{d(dS)}{dt}$ can be written

$$\frac{d(dS)}{dt} = - \bar{a}_n \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n] dS$$

Now theorem S2 will be considered. The integral in this theorem differs from that in theorem S1 only in the presence of a vector product in place of the scalar product in the integral of theorem S1. Theorem S2 can, therefore, be obtained from theorem S1 by replacing scalar products by vector products; the result is

$$\frac{d\overline{Q}}{dt} = \oint_C \overline{F} \times (\overline{v}_t \times \overline{d\ell}) + \int_S \frac{\partial \overline{F}}{\partial t} \times \overline{dS} + \int_S [(\overline{v}_n \cdot \nabla)\overline{F}] \times \overline{dS} - \int_S \overline{F} \times [(\overline{a}_n \times \nabla) \times \overline{v}_n] dS \ .$$

The integral in theorem S3 has a scalar integrand, F, but a vector differential of area, $\overline{dS}$. The theorem can be obtained from theorem S1 by replacing scalar products by ordinary products and by replacing $\overline{F}$ by F. The result is

$$\frac{d\overline{Q}}{dt} = \oint_C F \ \ \overline{v}_t \times \overline{d\ell} + \int_S \frac{\partial F}{\partial t} \overline{dS} + \int_S [(\overline{v}_n \cdot \nabla)F]\overline{dS} - \int_S F[(\overline{a}_n \times \nabla) \times \overline{v}_n] dS \ .$$

The integral in theorem S4 has a scalar differential of area, dS, for which the derivative, with respect to time, is as given above. Using this result and replacing scalar multiplication by ordinary multiplication gives the desired result.

$$\frac{d\overline{Q}}{dt} = \oint_C \overline{F}[\overline{a}_n \cdot (\overline{v}_t \times \overline{a}_t)] d\ell + \int_S \frac{\partial \overline{F}}{\partial t} dS + \int_S [(\overline{v}_n \cdot \nabla)\overline{F} dS$$
$$- \int_S \overline{F}\{\overline{a}_n \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n]\} dS \ .$$

The first term on the right hand side in the above expression is seen, from a review of Appendix 1 to be a line integral along the closed contour C of $\overline{F}v_t \sin\phi \, d\ell$ where ϕ is the angle between $\overline{v}_t$ and the unit vector tangent to C. This can be written as $\overline{F}|(\overline{v}_t \times \overline{a}_t)| d\ell$; but $\overline{v}_t \times \overline{a}_t$ is normal to the surface; so $|(\overline{v}_t \times \overline{a}_t)| = \overline{a}_n \cdot (\overline{v}_t \times \overline{a}_t)$. This leads to the above expression.

The integral in theorem S5 involves no vector functions. The modifications needed, in this case, are a combination of those used in theorems S3 and S4; the result is

$$\frac{dQ}{dt} = \oint_C F[\bar{a}_n \cdot (\bar{v}_t \times \bar{a}_t)]d\ell + \int_S \frac{\partial F}{\partial t}\, dS + \int_S [(\bar{v}_n \cdot \nabla)F]dS$$

$$- \int_S F\{\bar{a}_n \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n]\}dS.$$

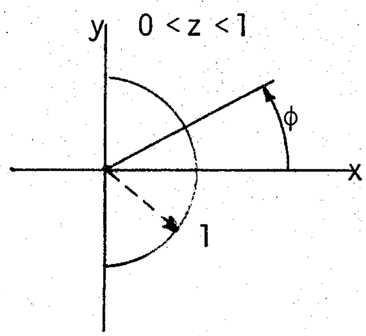
This completes the proofs of theorems S2, S3, S4 and S5.

APPENDIX 4

EXAMPLES OF THE USE OF THEOREMS S2, S3, S4 and S5

Example involving Theorem S2

Consider an example in which the surface S is semicylindrical of unit radius and length as sketched below. Let the surface move, without changing shape, with velocity $\overline{v} = \overline{a}_x$. Let $\overline{F} = 4xyz\,\overline{a}_x$ be a vector field and define $\overline{Q}$ by



$$\overline{Q} = \int_S \overline{F} \times \overline{dS}.$$

Coordinates x and y can be expressed in terms of ϕ and t.

$$x = t + \cos\phi$$

$$y = \sin\phi$$

$\overline{Q}$ can be evaluated to give

$$\overline{Q} = 4\int_0^1 zdz \int_{-\pi/2}^{\pi/2} xy\;\overline{a}_x \times (\overline{a}_x\cos\phi + \overline{a}_y\sin\phi)d\phi$$

$$= 2\overline{a}_z \int_{-\pi/2}^{\pi/2} (t + \cos\phi)\sin^2\phi\;d\phi$$

$$= 2\overline{a}_z[t(\tfrac{\pi}{2}) + \tfrac{2}{3}] = \overline{a}_z[\pi t + \tfrac{4}{3}]$$

From this it is seen that

$$\frac{d\overline{Q}}{dt} = \overline{a}_z \pi .$$

Now $\frac{d\overline{Q}}{dt}$ will be evaluated using Theorem S2.

$$\oint \overline{F} \times (\overline{v}_t \times \overline{d\ell}) = 4\overline{a}_x \times (\overline{a}_x \times \overline{a}_y) \int_o^1 t(1)zdz$$

$$+ 4\overline{a}_x \times (\overline{a}_x \times (-\overline{a}_z)) \int_1^o t(-1)z(-dz)$$

$$= -4\overline{a}_z t(\tfrac{1}{2}) + 4\overline{a}_z t(-\tfrac{1}{2}) = -4t\overline{a}_z .$$

The operator $\overline{v}_n \cdot \nabla$ is given by

$$\overline{v}_n \cdot \nabla = [(\overline{a}_x \cos\phi + \overline{a}_y \sin\phi)\cos\phi] \cdot \nabla$$

$$= \cos^2\phi \frac{\partial}{\partial x} + \cos\phi \sin\phi \frac{\partial}{\partial y}$$

so

$$\int_S [(\overline{v}_n \cdot \nabla)\overline{F}] \times \overline{dS} = 4 \int_o^1 z\,dz \int_{-\pi/2}^{\pi/2} (\cos^2\phi y + \cos\phi \sin\phi x)\overline{a}_x \times (\overline{a}_x \cos\phi + \overline{a}_y \sin\phi)d\phi$$

$$= 2\overline{a}_z \int_{-\pi/2}^{\pi/2} (\cos^2\phi \sin^2\phi + \cos\phi \sin^2\phi(t + \cos\phi))d\phi$$

$$= 2\overline{a}_z[\tfrac{\pi}{8} + t(\tfrac{2}{3}) + \tfrac{\pi}{8}] = \overline{a}_z[\tfrac{\pi}{2} + \tfrac{4}{3}t] .$$

Next, it is noted that

$$(\bar{a}_n \times \nabla) \times \bar{v}_n = \bar{a}_x[\sin^2\phi - \cos^2\phi] + \bar{a}_y[-2 \cos\phi \sin\phi]$$

from this

$$\int_S \bar{F} \times [(\bar{a}_n \times \nabla) \times \bar{v}_n] dS$$

$$= 4 \int_0^1 z dz \int_{-\pi/2}^{\pi/2} xy \; \bar{a}_x \times [\bar{a}_x(\sin^2\phi - \cos^2\phi) + \bar{a}_y(-2\cos\phi \sin\phi)] d\phi$$

$$= -4\bar{a}_z \int_{\pi/2}^{\pi/2} (t + \cos\phi)\sin^2\phi \cos\phi \; d\phi$$

$$= -4\bar{a}_z[t(\tfrac{2}{3}) + \tfrac{\pi}{8}] = - \bar{a}_z[\tfrac{8}{3}t + \tfrac{\pi}{2}] .$$

Observing that $\frac{\partial \bar{F}}{\partial t} = 0$ and collecting the above results, it is seen that

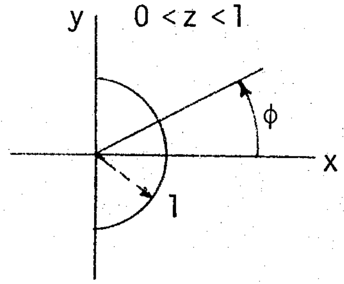
$$\oint_C \bar{F} \times (\bar{v}_t \times \overline{d\ell}) + \int_S \frac{\partial \bar{F}}{\partial t} \times \overline{dS} + \int_S [(\bar{v}_n \cdot \nabla)\bar{F}] \times \overline{dS} - \int_S \bar{F} \times [(\bar{a}_n \times \nabla) \times \bar{v}_n] dS$$

$$= -4t\bar{a}_z + 0 + \bar{a}_z[\tfrac{\pi}{2} + \tfrac{4}{3}t] - [-\bar{a}_z(\tfrac{8}{3}t + \tfrac{\pi}{2})] = \bar{a}_z\pi .$$

This result verifies the correctness of Theorem S2 for this particular problem.

Example involving Theorem S3.

Consider an example in which the surface S is semicylindrical, of unit radius and length one as sketched below. Let the surface move, without changing shape, with velocity $\overline{v} = \overline{a}_x$.



Let $F = 4xy^2z$ be a scalar field and define $\overline{Q}$ as

$$\overline{Q} = \int_S F\,\overline{dS}$$

Coordinates x and y can be expressed in terms of ϕ and t.

$$x = t + \cos\phi$$

$$y = \sin\phi$$

$\overline{Q}$ can be evaluated to give

$$\overline{Q} = 4\int_0^1 z dz \int_{-\pi/2}^{\pi/2} xy^2(\overline{a}_x\cos\phi + \overline{a}_y\sin\phi)d\phi$$

$$= 2\int_{-\pi/2}^{\pi/2} (t + \cos\phi)\sin^2\phi(\overline{a}_x\cos\phi + \overline{a}_y\sin\phi)d\phi$$

$$= 2\overline{a}_x(t(\tfrac{2}{3}) + \tfrac{\pi}{8}) + 2\overline{a}_y(t(0) + (0))$$

$$= \overline{a}_x[(\tfrac{4}{3})t + \tfrac{\pi}{4}]$$

It follows from this that

$$\frac{d\overline{Q}}{dt} = \overline{a}_x(\tfrac{4}{3}) .$$

The derivative will now be evaluated using Theorem S3. The tangential component of the velocity is $\bar{v}_t = \bar{a}_x$ on the edges of the surface at $x = 0$ and $y = \pm 1$. $\bar{v}_t = 0$ on the remainder of the boundary. Thus

$$\oint_C F\, \bar{v}_t \times d\ell = 4 \int_0^1 t(2)^2 z(-\bar{a}_y)dz + 4 \int_1^0 t(2)^2 z(\bar{a}_y)(-dz)$$

$$= 0 \ .$$

The operator $\bar{v}_n \cdot \nabla$ is given by

$$\bar{v}_n \cdot \nabla = [(\bar{a}_x \cos\phi + \bar{a}_y \sin\phi)\cos\phi] \cdot \nabla$$

$$= \cos^2\phi \frac{\partial}{\partial x} + \sin\phi \cos\phi \frac{\partial}{\partial y}$$

so

$$\int_S (\bar{v}_n \cdot \nabla) F\, \overline{dS} = 4 \int_0^1 z\, dz \int_{-\pi/2}^{\pi/2} (\cos^2\phi\, y^2 + 2 \sin\phi \cos\phi\, xy)(\bar{a}_x \cos\phi + \bar{a}_y \sin\phi) d\phi$$

$$= 2\bar{a}_x \int_{-\pi/2}^{\pi/2} (\cos^3\phi \sin^2\phi + 2 \sin^2\phi(t + \cos\phi \cos^2\phi) d\phi$$

$$+ 2\bar{a}_y \int_{-\pi/2}^{\pi/2} (\cos^2\phi \sin^3\phi + 2 \sin^3\phi \cos\phi (t + \cos\phi) d\phi$$

$$= 2\bar{a}_x [(3)\frac{(2)^2}{5(3)} + 2t(\frac{\pi}{8})] + 2\bar{a}_y [0 + 2t(0) + 2(0)]$$

$$= \bar{a}_x [\frac{8}{5} + \frac{\pi t}{2}] \ .$$

Next, it is seen that

$$(\bar{a}_n \times \nabla) \times \bar{v}_n = \bar{a}_x (\sin^2\phi - \cos^2\phi) + \bar{a}_y (-2 \sin\phi \cos\phi)$$

so

$$\int_S F[(\bar{a}_n \times \nabla) \times \bar{v}_n]dS = 4\int_0^1 z\,dz \int_{-\pi/2}^{\pi/2} xy^2[\bar{a}_x(\sin^2\phi - \cos^2\phi) + \bar{a}_y(-2\sin\phi\cos\phi)]d\phi$$

$$= 2\bar{a}_x \int_{-\pi/2}^{\pi/2} (t+\cos\phi)\sin^2\phi(\sin^2\phi - \cos^2\phi)d\phi - 8\bar{a}_y \int_{-\pi/2}^{\pi/2} (t+\cos\phi)\sin^3\phi\cos\phi\, d\phi$$

$$= 2\bar{a}_x[t(\frac{3\pi}{8} - \frac{\pi}{8}) + (\frac{2}{5} - \frac{(2)^2}{3(5)})] - 8\bar{a}_y[t(0) + (0)]$$

$$= \bar{a}_x[\frac{\pi t}{2} + \frac{4}{15}] .$$

Assembling these results and observing that $\frac{\partial F}{\partial t} = 0$, it is seen that

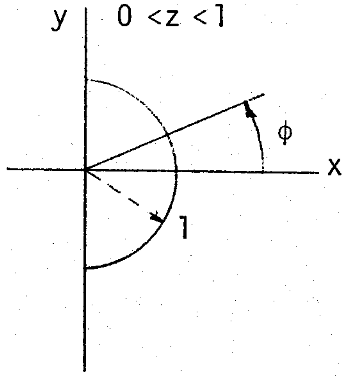
$$\oint_C F\,\bar{v}_t \times \overline{d\ell} + \int_S \frac{\partial F}{\partial t}\overline{dS} + \int_S (\bar{v}_n \cdot \nabla)F\,\overline{dS} - \int_S F[\bar{a}_n \times \nabla) \times \bar{v}_n]dS$$

$$= \bar{a}_x[\frac{8}{5} + \frac{\pi t}{2}] - \bar{a}_x[\frac{\pi t}{2} + \frac{4}{15}] = \bar{a}_x(\frac{4}{3}) .$$

This verifies the correctness of Theorem S3 for this particular example.

Example involving Theorem S3.

Consider an example in which the surface S is semicylindrical, of radius one, and length one as sketched below. Let the surface move, without changing shape, with velocity, $\bar{v} = \bar{a}_x$. Let $F = 4xyz$ be a scalar field and define $\bar{Q}$ as



$$\overline{Q} = \int_S F\,\overline{dS}$$

Coordinates x and y can be expressed in terms of ϕ and t.

$$x = t + \cos\phi$$

$$y = \sin\phi$$

$\overline{Q}$ can be evaluated to give

$$\overline{Q} = 4\int_0^1 z\,dz \int_{-\pi/2}^{\pi/2} xy(\overline{a}_x\cos\phi + \overline{a}_y\sin\phi)d\phi$$

$$= 2\overline{a}_x\int_{-\pi/2}^{\pi/2}(t+\cos\phi)\sin\phi\,\cos\phi\,d\phi + 2\overline{a}_y\int_{-\pi/2}^{\pi/2}(t+\cos\phi)\sin^2\phi\,d\phi$$

$$= 2\overline{a}_x[0 + 0] + 2\overline{a}_y[t(\tfrac{\pi}{2}) + \tfrac{2}{3}] = \overline{a}_y(\pi t + \tfrac{4}{3}) .$$

From this, it is seen that

$$\frac{d\overline{Q}}{dt} = \overline{a}_y\pi .$$

Now the derivative, $\frac{d\overline{Q}}{dt}$, will be evaluated using Theorem S3.

$$\oint_C F\,\overline{v}_t \times d\ell = 4\int_0^1 t(1)z(-\overline{a}_y)dz + 4\int_1^0 t(-1)(\overline{a}_y)(-dz)$$

$$= -2t\,\overline{a}_y - 2t\,\overline{a}_y = -4t\,\overline{a}_y .$$

The operator $\overline{v}_n \cdot \nabla$ is given by

$$\overline{v}_n \cdot \nabla = [(\overline{a}_x \cos\phi + \overline{a}_y \sin\phi)\cos\phi] \cdot \nabla$$

$$= \cos^2\phi \frac{\partial}{\partial x} + \sin\phi \cos\phi \frac{\partial}{\partial y}$$

so

$$\int_S (\overline{v}_n \cdot \nabla) F \, \overline{dS} = 4 \int_o^1 z \, dz \int_{-\pi/2}^{\pi/2} (y \cos^2\phi + x \sin\phi\cos\phi)(\overline{a}_x \cos\phi + \overline{a}_y \sin\phi) d\phi$$

$$= 2\overline{a}_x \int_{-\pi/2}^{\pi/2} (\sin\phi \cos^3\phi + (t + \cos\phi)\sin\phi \cos^2\phi) d\phi$$

$$+ 2\overline{a}_y \int_{-\pi/2}^{\pi/2} (\sin^2\phi \cos^2\phi + (t + \cos\phi)\sin^2\phi \cos\phi) d\phi$$

$$= 2\overline{a}_x [0 + t(0) + (0)] + 2\overline{a}_y [\tfrac{\pi}{8} + t(\tfrac{2}{3}) + \tfrac{\pi}{8}]$$

$$= 2\overline{a}_y [t(\tfrac{2}{3}) + \tfrac{\pi}{4}] = \overline{a}_y [\tfrac{4}{3} t + \tfrac{\pi}{2}] \ .$$

Next, since

$$(\overline{a}_n \times \nabla) \times \overline{v}_n = \overline{a}_x (\sin^2\phi - \cos^2\phi) + \overline{a}_y (-2 \sin\phi \cos\phi)$$

$$\int_S F[(\bar{a}_n + \nabla) \times \overline{v}_n] dS = 4 \int_o^1 z \, dz \int_{-\pi/2}^{\pi/2} xy[\overline{a}_x (\sin^2\phi - \cos^2\phi) + \overline{a}_y (-2 \sin\phi \cos\phi)] d\phi$$

$$= 2\overline{a}_x \int_{-\pi/2}^{\pi/2} (t + \cos\phi)(\sin\phi)(\sin^2\phi - \cos^2\phi) d\phi$$

$$-4\bar{a}_y \int_{-\pi/2}^{\pi/2} (t + \cos\phi)\sin^2\phi \; \cos\phi \; d\phi$$

$$= 2\bar{a}_x[t(0) + t(0) + (0) + (0)] - 4\bar{a}_y[t(\tfrac{2}{3}) + \tfrac{\pi}{8}]$$

$$= -\bar{a}_y[\tfrac{8}{3}\, t + \tfrac{\pi}{2}] \; .$$

Observing that $\frac{\partial F}{\partial t} = 0$ and collecting these results it is seen that

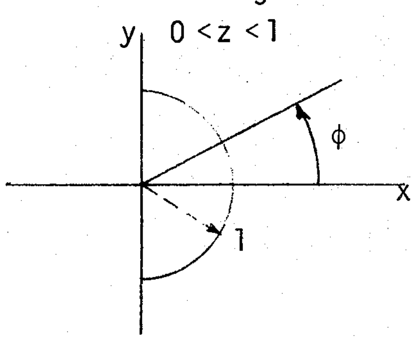
$$\oint_C F \; \bar{v}_t \times d\bar{\ell} + \int_S \frac{\partial F}{\partial t} \, \overline{dS} + \int_S (\bar{v}_n \cdot \nabla)F \; \overline{dS} - \int_S F[(\bar{a}_n \times \nabla) \times \bar{v}_n] dS$$

$$= -4t\bar{a}_y + \bar{a}_y[\tfrac{4}{3}t + \tfrac{\pi}{2}] - \{- \bar{a}_y[\tfrac{8}{3}t + \tfrac{\pi}{2}]\} = \bar{a}_y \pi$$

which verifies the theorem (S3) for this example.

Example Involving Theorem S4.

Consider an example in which the surface S is semicylindrical of unit radius and length as sketched below. Suppose the surface is moving



with velocity $\bar{r} = \bar{a}_x$. Let a vector field, $\bar{F}$, be defined by

$$\bar{F} = 4xy^2 z \; \bar{a}_x$$

Let the vector surface integral $\bar{Q}$ be defined by

$$\bar{Q} = \int_S \bar{F} \; dS$$

To evaluate $\overline{Q}$, note that x and y can be expressed in terms of ϕ .

$$x = t + \cos\phi$$

$$y = \sin\phi \; .$$

The normal component of the velocity is

$$\overline{v}_n = (\overline{a}_x\cos\phi + \overline{a}_y\sin\phi)\cos\phi \; .$$

The tangential component of the velocity is non-zero only on the parts of the contour parallel to the z-axis. Along the line $x = 0$, $y = 1$, it is $\overline{v}_t = \overline{a}_x$, and it has the same value along the line $x = 0$, $y = -1$.

First the integral for $\overline{Q}$ will be evaluated.

$$\overline{Q} = \overline{a}_x 4 \int_{z=0}^{1} z\,dz \int_{\phi=-\frac{\pi}{2}}^{\pi/2} xy^2 d\phi = \overline{a}_x 4 \int_0^1 z\,dz \int_{-\pi/2}^{\pi/2} (t + \cos\phi)\sin^2\phi\, d\phi$$

$$= \overline{a}_x (4)(\tfrac{1}{2})[t \int_{-\pi/2}^{\pi/2} \sin^2\phi\, d\phi + \int_{-\pi/2}^{\pi/2} \sin^2\phi \cos\phi\, d\phi]$$

$$= 2\overline{a}_x [t\,\frac{\pi}{2} + \frac{2}{3}]$$

$$= \overline{a}_x [\pi t + \frac{4}{3}] \; .$$

From this it follows that

$$\frac{d\overline{Q}}{dt} = \overline{a}_x \pi \; .$$

Theorem S4 will now be applied to this integral.

First the line integral around the contour bounding the surface will be evaluated.

$$\int_C \overline{F}[\overline{a}_n \cdot \overline{v}_t \times \overline{a}_t]d\ell = [-\int_0^1 4(t)(1)^2 z\,dz - \int_1^0 4(t)(1)^2 z(-dz)]\overline{a}_x$$

$$= [-4t(\tfrac{1}{2}) - 4t(\tfrac{1}{2})]\overline{a}_x = (-4t)\overline{a}_x \ .$$

Next evaluate the operator $\overline{v}_n \cdot \nabla$.

$$\overline{v}_n \cdot \nabla = (\overline{a}_x \cos^2\phi + \overline{a}_y \sin\phi\ \cos\phi) \cdot (\overline{a}_x \frac{\partial}{\partial x} + \overline{a}_y \frac{\partial}{\partial y} + \overline{a}_z \frac{\partial}{\partial z})$$

$$= \cos^2\phi \frac{\partial}{\partial x} + \sin\phi\ \cos\phi \frac{\partial}{\partial y} \ .$$

From this it is seen that

$$(\overline{v}_n \cdot \nabla)\overline{F} = (\cos^2\phi\ 4y^2 z + \sin\phi\ \cos\phi\ 8xyz)\overline{a}_x$$

and

$$\int_S (\overline{v}_n \cdot \nabla)\overline{F}\,dS = 4\overline{a}_x \int_0^1 z\,dz \int_{-\pi/2}^{\pi/2} (\cos^2\phi\ y^2 + \cos\phi\ \sin\phi\ 2xy)d\phi$$

$$= 2\overline{a}_x\ [\int_{-\pi/2}^{\pi/2} \cos^2\phi\ \sin^2\phi\ d\phi + 2\int_{-\pi/2}^{\pi/2} \sin^2\phi(t + \cos\phi)\cos\phi\ d\phi]$$

$$= 2\overline{a}_x[\tfrac{\pi}{8} + 2(\tfrac{2}{3}t + \tfrac{\pi}{8})] = \overline{a}_x(\tfrac{8}{3}t + \tfrac{3\pi}{4}) \ .$$

Next note that

$$(\overline{a}_n \times \nabla) \times \overline{v}_n = \overline{a}_x(\sin^2\phi - \cos^2\phi) + \overline{a}_y[-2\ \sin\phi\cos\phi]$$

and

$$\bar{a}_n \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n] = \cos\phi \sin^2\phi - \cos^3\phi - 2\sin^2\phi \cos\phi$$

$$= -\cos^3\phi - \sin^2\phi \cos\phi = -\cos\phi;$$

so

$$\int_S \bar{F}\{\bar{a}_n \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n]\} dS = -4\bar{a}_x \int_0^1 z\, dz \int_{-\pi/2}^{\pi/2} xy^2 \cos\phi\, d\phi$$

$$= -2\bar{a}_x \int_{-\pi/2}^{\pi/2} (t + \cos\phi)\sin^2\phi \cos\phi\, d\phi$$

$$= -2\bar{a}_x[t(\tfrac{2}{3}) + \tfrac{\pi}{8}] = -\bar{a}_x[\tfrac{4}{3}t + \tfrac{\pi}{4}] .$$

Assembling these results, it is seen that

$$\int_C \bar{F}[\bar{a}_n \cdot \bar{v}_t \times \bar{a}_t] d\ell + \int_S \frac{\partial \bar{F}}{\partial t} dS + \int_S (\bar{v}_n \cdot \nabla)\bar{F}\, dS$$

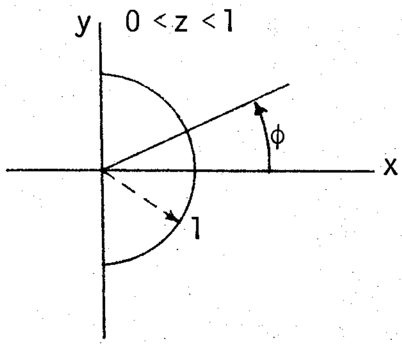
$$-\int_S F\{\bar{a}_n \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n]\} dS = -4t\bar{a}_x + \bar{a}_x(\tfrac{8}{3}t + \tfrac{3\pi}{4})$$

$$+ \bar{a}_x(\tfrac{4}{3}t + \tfrac{\pi}{4}) = \bar{a}_x \pi$$

which agrees with the result obtained by integrating first then differentiating. This verifies Theorem S4 for this particular case.

Example involving Theorem S5

Consider an example in which the surface S is semicylindrical, of unit radius and length as sketched below. Let the surface move, without



changing shape, with velocity $\overline{v} = \overline{a}_x$.

Let $F = 4xy^2z$ be a scalar field and define Q by

$$Q = \int_S F\,dS$$

x and y can be expressed in terms of ϕ .

$$x = t + \cos\phi$$

$$y = \sin\phi$$

The normal component of velocity is

$$\overline{v}_n = (\overline{a}_x\cos\phi + \overline{a}_y\sin\phi)\cos\phi$$

Q as a function of t is easily evaluated.

$$Q = 4\int_0^1 z\,dz \int_{-\pi/2}^{\pi/2} (t + \cos\phi)\sin^2\phi\,d\phi$$

$$= 2[t(\tfrac{\pi}{2}) + \tfrac{2}{3}] = \pi t + \tfrac{4}{3} .$$

From this

$$\frac{dQ}{dt} = \pi .$$

Theorem S5 will now be applied to this problem

$$\oint F[\overline{a}_n \cdot (\overline{v}_t \times \overline{a}_t)d\ell = 4\int_0^1 t(1)^2z(-1)dz + 4\int_1^0 t(-1)^2z(-1)(-dz)$$

$$= -2t - 2t = -4t .$$

The operator $\overline{v}_n \cdot \nabla$ is given by

$$\overline{v}_n \cdot \nabla = [(\overline{a}_x \cos\phi + \overline{a}_y \sin\phi)\cos\phi] \cdot \nabla$$

$$= \cos^2\phi \frac{\partial}{\partial x} + \sin\phi \cos\phi \frac{\partial}{\partial y}$$

so

$$\int_S (\overline{v}_n \cdot \nabla) F \, dS = 4 \int_0^1 z \, dz \int_{-\pi/2}^{\pi/2} (\cos^2\phi \; y^2 + \sin\phi \cos\phi \; 2xy) d\phi$$

$$= 2 \int_{-\pi/2}^{\pi/2} (\cos^2\phi \sin^2\phi + 2 \sin^2\phi \cos\phi(t + \cos\phi)) d\phi$$

$$= 2[\frac{\pi}{8} + 2t(\frac{2}{3}) + 2(\frac{\pi}{8})] = \frac{8}{3} t + \frac{3}{4}\pi$$

Next note that

$$(\overline{a}_n \times \nabla) \times \overline{v}_n = \overline{a}_x(\sin^2\phi - \cos^2\phi) + \overline{a}_y(-2 \sin\phi \cos\phi)$$

and

$$\overline{a}_n \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n] = (\overline{a}_x \cos\phi + \overline{a}_y \sin\phi) \cdot [\overline{a}_x(\sin^2\phi - \cos^2\phi) + \overline{a}_y(-2 \sin\phi \cos\phi)]$$

$$= \cos\phi \sin^2\phi - \cos^3\phi - 2 \sin^2\phi \cos\phi$$

$$= - \cos\phi \; .$$

Therefore

$$\int_S F\{\overline{a}_n \cdot [(\overline{a}_n \times \nabla) \times \overline{v}_n]\} dS$$

$$= -4 \int_0^1 z \, dz \int_{-\pi/2}^{\pi/2} xy^2 \cos\phi \, d\phi$$

$$= -2 \int_{-\pi/2}^{\pi/2} (t + \cos\phi)\sin^2\phi \cos\phi \, d\phi$$

$$= -2[t(\tfrac{2}{3}) + \tfrac{\pi}{8}] = -\tfrac{4}{3}t - \tfrac{\pi}{4}$$

Noting that $\frac{\partial F}{\partial t} = 0$ and collecting the above results, it is seen that

$$\oint_C F[\bar{a}_n \cdot (\bar{v}_n \times \bar{a}_t)]d\ell + \int_S \frac{\partial F}{\partial t} dS + \int_S (\bar{v}_n \cdot \nabla)F\, dS - \int_S F\{\bar{a}_n \cdot [(\bar{a}_n \times \nabla) \times \bar{v}_n]\}dS$$

$$= -4t + 0 + (\tfrac{8}{3}t + \tfrac{3}{4}\pi) - (-\tfrac{4}{3}t - \tfrac{\pi}{4}) = \pi$$

which verifies the correctness of Theorem S5 for this particular example.