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DIFFERENTIATION OF LINE INTEGRALS

by

S.W. Maley

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Electromagnetics Laboratory
Department of Electrical Engineering
University of Colorado
Boulder, Colorado 80309

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DIFFERENTIATION OF LINE INTEGRALS

S.W. Maley

Abstract

It is sometimes necessary to differentiate a line integral with respect to a parameter of the integrand, or of the contour, or of the end points or of any combination of these. Differentiation after the integral has been evaluated presents no difficulties, but occasionally it is desirable in computation or, more often, in theoretical derivations, to interchange the order of integration and differentiation. This report presents theorems concerning such interchange.

INTRODUCTION

Let Q be defined as a line integral along contour ℓ from point P_1 to point P_2 . More specifically let Q be given by

$$Q = \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell} .$$

Other types of line integrals will be considered in a later section. Suppose the integrand, \bar{F} , the contour C and the end points P_1 and P_2 are functions of a parameter, t . They need not all be functions of t , but the procedure to be presented will be based on the assumption that they are, and the results will be applicable whether they are or not.

The symbol, t , can be any parameter but, in this discussion, it will be interpreted as time. The function \bar{F} may be a function of t , and the contour, C , may be a function of t , that is the contour may be in motion. Such motion may involve movement of the end points P_1 and P_2 . The motion of the contour can be characterized in terms of a velocity, \bar{v} , of each point on the contour. \bar{v} is a function of position along the contour. The motion, at each point, can be resolved into a component, \bar{v}_t , tangent to the contour and a component, \bar{v}_n , normal to the contour.

The velocity, \bar{v} , is of course

$$\bar{v} = \bar{v}_t + \bar{v}_n .$$

The motion of the contour is characterized in terms of a fixed (unmoving) coordinate system. At any instant of time, the integral can be evaluated; its value is a function of time because, at different times the integrand, \bar{F} , can be different, the contour can be different, and the end-points on the contour can be different.

It is sometimes necessary to differentiate the integral Q . The integration may be performed resulting in an expression for Q as a function of t . This expression may then be differentiated, in a straightforward manner, to give $\frac{dQ}{dt}$. It is occasionally desired to interchange the order of differentiation and integration. Such an interchange is sometimes needed in theoretical derivations and sometimes even in computational procedures. The procedure for interchange of integration and differentiation is well known for an integral in a one dimensional space. It is given by the well known Leibnitz Theorem (or Leibnitz rule). However the case of a line integral in two or three dimensional space requires an extension of the principle involved in the Leibnitz rule. This report is concerned with that extension.

As mentioned above, the velocity of each point on the contour is resolved into normal and tangential components. The reason for doing so concerns the fact that tangential motion of points along a contour does not influence an integral along that contour unless the end points are in motion. Such tangential motion is simply a stretching or contraction of the contour, in a fixed coordinate system, without changing its position. Such motion, therefore, does not influence the value of the integral, except at the end points where the tangential motion will be taken into consideration. On the basis of these observations, it may be expected that the derivative, with respect to time, of a line integral could be expressed in terms of the normal component of velocity along the contour and the tangential component of velocity at the end points. This, in fact, is so as is discussed in the next section.

THEORY

The procedure for interchanging integration and differentiation is given by the following theorem.

Theorem L1

Let Q be the line integral

$$Q = \int_{C}^{P_2} \bar{F} \cdot d\bar{\ell}$$

along contour, C , from point P_1 to point P_2 . Assume that the vector function, \bar{F} , is a function of time, t . Also assume the contour, C , is in motion with respect to the frame of reference with respect to which \bar{F} is defined. Further assume the end points P_1 and P_2 are in motion. Let the motion, with respect to the frame of reference, be defined by velocity, \bar{v} , which is a function of position along contour C . The derivative of Q with respect to t can be expressed as

$$\begin{aligned} \frac{dQ}{dt} &= \bar{F}(P_2) \cdot \bar{v}_{t2} - \bar{F}(P_1) \cdot \bar{v}_{t1} \\ &+ \int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} + \int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \cdot d\bar{\ell} + \int_{P_1}^{P_2} \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial \ell} d\ell \end{aligned}$$

where \bar{v}_{t2} is the tangential component of velocity at end-point P_2 and \bar{v}_{t1} is the tangential component of velocity at end-point P_1 . The operator $\bar{v}_n \cdot \nabla$ is a scalar operator which in a general orthogonal coordinate system, with coordinates u_1, u_2 and u_3 and with metric coefficients h_1, h_2 and h_3 is given by

$$\begin{aligned} \bar{v}_n \cdot \nabla &= (\bar{a}_1 v_{n1} + \bar{a}_2 v_{n2} + \bar{a}_3 v_{n3}) \cdot \left(\frac{\bar{a}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\bar{a}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\bar{a}_3}{h_3} \frac{\partial}{\partial u_3} \right) \\ &= \frac{v_{n1}}{h_1} \frac{\partial}{\partial u_1} + \frac{v_{n2}}{h_2} \frac{\partial}{\partial u_2} + \frac{v_{n3}}{h_3} \frac{\partial}{\partial u_3} \end{aligned}$$

where \bar{a}_1, \bar{a}_2 and \bar{a}_3 are unit vectors for the coordinate system u_1, u_2, u_3 .

A proof of the above theorem is given in Appendix 1.

The significance of the various terms in the expression for $\frac{dQ}{dt}$ is apparent from examination in some cases and from the discussion in App. 1 in others. The first term is due to the functional dependence of P_2 on t ; and the second term is due to the functional dependence of P_1 on t . The third term, involving $\frac{\partial \bar{F}}{\partial t}$, is a consequence of the dependence of \bar{F} on t ; more specifically this accounts for the change of \bar{F} with t while all positional coordinates are kept constant. It

therefore does not account for the variation of \bar{F} due to change in positional coordinates which, in turn, is due to motion of the contour. This latter effect is accounted for by the term involving the operator $\bar{v}_t \cdot \nabla$ operating on \bar{F} . Finally the motion of the contour of integration is accounted for by the term involving $\frac{\partial \bar{v}_n}{\partial \ell}$.

In the case of a closed line integral, the first two terms in the expression for $\frac{dQ}{dt}$ cancel giving the following result.

Corollary L1

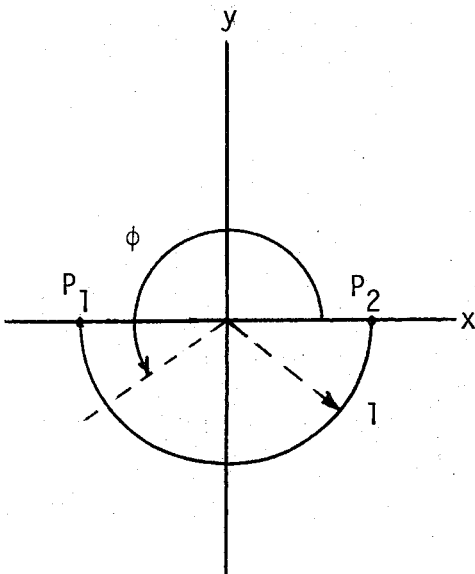
If the contour C is closed,

$$\frac{dQ}{dt} = \oint_C \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} + \oint_C [(\bar{v}_n \cdot \nabla) \bar{F}] \cdot d\bar{\ell} + \oint_C \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial \ell} d\ell.$$

EXAMPLES

As an example of the use of the more general expression for $\frac{dQ}{dt}$ (Theorem L1) consider the contour sketched below. The contour is a semicircle of unit radius. Let \bar{F} be given by

$$\bar{F} = 3xy^2 \bar{a}_y$$



Suppose the contour starts in the position shown and moves upward with velocity $\bar{v} = (1)\bar{a}_y$ starting at $t=0$ without changing shape. (The expression for $\frac{dQ}{dt}$ in Theorem L1 permits changes in the shape of the contour but in this particular example no such change occurs.)

First Q will be evaluated then the result will be differentiated. This then will be compared with the result of evaluation of the right hand side of the expression for $\frac{dQ}{dt}$ given in Theorem LI.

It is convenient to express points on the contour in terms of ϕ (which is defined in terms of the center of the semicircle which is assumed to move up with the contour.) It is easily seen that for $t \geq 0$,

$$x^2 + (y - t)^2 = 1$$

$$\overline{d\ell} = (-\overline{a}_x \sin \phi + \overline{a}_y \cos \phi) d\phi$$

$$x = \cos \phi$$

$$y = \sin \phi + t$$

Q is given by

$$Q = \int_{P_1}^{P_2} \overline{F} \cdot \overline{d\ell} = 3 \int_{\pi}^{2\pi} xy^2 \cos \phi d\phi$$

Expressing x and y in terms of ϕ and evaluating the integral gives the result

$$Q = \frac{3}{2} \pi t^2 - 4t + \frac{3}{8} \pi$$

Differentiation gives the result

$$\frac{dQ}{dt} = 3\pi t - 4.$$

The normal component of the velocity is

$$\overline{v}_n = \overline{a}_x \cos \phi \sin \phi + \overline{a}_y \sin^2 \phi$$

and the operator $\overline{v}_n \cdot \nabla$ is

$$\overline{v}_n \cdot \nabla = \cos \phi \sin \phi \frac{\partial}{\partial x} + \sin^2 \phi \frac{\partial}{\partial y}.$$

This operator applied to \bar{F} gives

$$(\bar{v}_n \cdot \nabla) \bar{F} = (\cos \phi \sin \phi 3y^2 + \sin^2 \phi 6xy) \bar{a}_y$$

and substituting for x and y

$$\begin{aligned} \int_{P_1}^{P_2} (\bar{v}_n \cdot \nabla) \bar{F} \cdot d\bar{\ell} &= \int_{\pi}^{2\pi} (\cos \phi \sin \phi 3(t + \sin \phi)^2 + \sin^2 \phi 6 \cos \phi (t + \sin \phi)) \cos \phi d\phi \\ &= -2t^2 + \frac{3}{2} \pi t - \frac{12}{5} . \end{aligned}$$

Distance, ℓ , along the contour is given by

$$\ell = \phi - \pi$$

so

$$\frac{\partial \bar{v}_n}{\partial \ell} = \frac{\partial \bar{v}_n}{\partial \phi} = \bar{a}_x (\cos^2 \phi - \sin^2 \phi) + \bar{a}_y 2 \sin \phi \cos \phi .$$

The integral involving $\frac{\partial \bar{v}_n}{\partial \ell}$ can now be evaluated

$$\begin{aligned} \int_{P_1}^{P_2} \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial \ell} d\ell &= 6 \int_{\pi}^{2\pi} xy^2 \sin \phi \cos \phi d\phi \\ &= 6 \int_{\pi}^{2\pi} \cos \phi (t + \sin \phi)^2 \sin \phi \cos \phi d\phi \\ &= -4t^2 + \frac{3}{2} \pi t - \frac{8}{5} . \end{aligned}$$

The expressions involving the end points are

$$\begin{aligned} &\bar{F}(P_2) \cdot \bar{v}_{t2} - \bar{F}(P_1) \cdot \bar{v}_{t1} \\ &= (3(1)t^2 \bar{a}_y \cdot \bar{a}_y)(1) - (3(-1)t^2 \bar{a}_y \cdot (\bar{a}_y)) = 6t^2 . \end{aligned}$$

The integral involving $\frac{\partial \bar{F}}{\partial t}$ is zero because \bar{F} does not explicitly involve t . The expression for $\frac{dQ}{dt}$ in Theorem L1 now gives

$$\begin{aligned}\frac{dQ}{dt} &= 6t^2 + (-2t^2 + \frac{3}{2}\pi t - \frac{12}{5}) + (-4t^2 + \frac{3}{2}\pi t - \frac{8}{5}) \\ &= 3\pi t - 4\end{aligned}$$

The expression for $\frac{dQ}{dt}$ thus gives the correct result in this example. Another example is given in App. 2.

EXTENSIONS TO OTHER TYPES OF LINE INTEGRALS

Theorem L1 and Corollary L1 are applicable to a common type of line integral. There are other types of line integrals which can be differentiated by procedures similar to those given in Theorem L1. Theorems applicable to several other types of line integrals are presented in this section. Proofs of these theorems are discussed in Appendix 3 and examples of their use are given in Appendix 4.

Theorem L2

Let a line integral \bar{Q} be defined by

$$\bar{Q} = \int_{C^{P_1}}^{P_2} \bar{F} \times d\bar{\ell} .$$

The derivative, $\frac{d\bar{Q}}{dt}$, of \bar{Q} with respect to t can be expressed as

$$\begin{aligned} \frac{d\bar{Q}}{dt} = & \bar{F}(P_2) \times \bar{v}_{t2} - \bar{F}(P_1) \times \bar{v}_{t1} + \int_{C^{P_1}}^{P_2} \frac{\partial \bar{F}}{\partial t} \times d\bar{\ell} + \int_{C^{P_1}}^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \times d\bar{\ell} \\ & + \int_{C^{P_1}}^{P_2} \bar{F} \times \frac{\partial \bar{v}_n}{\partial \ell} d\ell \end{aligned}$$

where the symbols have the same meaning as discussed in Theorem L1.

If the contour is closed, the theorem simplifies.

Corollary L2

If C is a closed contour

$$\frac{d\bar{Q}}{dt} = \oint_C \frac{\partial \bar{F}}{\partial t} \times d\bar{\ell} + \oint_C [(\bar{v}_n \cdot \nabla) \bar{F}] \times d\bar{\ell} + \oint_C \bar{F} \times \frac{\partial \bar{v}_n}{\partial \ell} d\ell$$

Another type of line integral is covered by the next theorem.

Theorem L3

Let a line integral, \bar{Q} , be defined by

$$\bar{Q} = \int_{C}^{P_2} F d\ell$$

where F is a scalar function of position. The derivative, $\frac{d\bar{Q}}{dt}$, of \bar{Q} with respect to t can be expressed by

$$\begin{aligned} \frac{d\bar{Q}}{dt} = & F(P_2)\bar{v}_{t2} - F(P_1)\bar{v}_{t1} + \int_{C}^{P_2} \frac{\partial F}{\partial t} d\ell \\ & + \int_{C}^{P_2} [(\bar{v}_n \cdot \nabla)F] d\ell + \int_{C}^{P_2} F \frac{\partial \bar{v}_n}{\partial \ell} d\ell \end{aligned}$$

The case in which the contour, C , is closed is covered by the following:

Corollary L3

If C is a closed contour

$$\frac{d\bar{Q}}{dt} = \oint_C \frac{\partial F}{\partial t} d\ell + \oint_C [(\bar{v}_n \cdot \nabla)F] d\ell + \oint_C F \frac{\partial \bar{v}_n}{\partial \ell} d\ell$$

Another type of line integral is that in which the integrand is a vector function of position but the differential of length is scalar. It is treated by the following theorem.

Theorem L4.

Let a line integral, \bar{Q} , be defined by

$$\bar{Q} = \int_{C}^{P_2} \bar{F} d\ell$$

The derivative of \bar{Q} with respect to t is given by

$$\frac{d\bar{Q}}{dt} = \bar{F}(P_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - \bar{F}(P_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1})$$

$$+ \int_{C^{P_1}}^{P_2} \frac{\partial \bar{F}}{\partial t} d\ell + \int_{C^{P_1}}^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] d\ell + \int_{C^{P_1}}^{P_2} \bar{F} [\bar{a}_\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}] d\ell$$

where \bar{a}_ℓ is a unit vector tangent to contour C in the direction of integration. The case in which the contour, C , is closed is treated by the following corollary.

Corollary L4

If contour, C , is closed

$$\frac{d\bar{Q}}{dt} = \oint_C \frac{\partial \bar{F}}{\partial t} d\ell + \oint_C [(\bar{v}_n \cdot \nabla) \bar{F}] d\ell + \oint_C \bar{F} [\bar{a}_\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}] d\ell$$

The final type of line integral to be considered is that in which both the integrand and the differential of length are scalars. The following theorem is applicable to that case.

Theorem L5

Let a line integral, Q , be defined by

$$Q = \int_{C^{P_1}}^{P_2} F d\ell$$

The derivative of Q with respect to t is given by

$$\frac{dQ}{dt} = F(P_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - F(P_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1}) + \int_{C^{P_1}}^{P_2} \frac{\partial F}{\partial t} d\ell$$

$$+ \int_{C^{P_1}}^{P_2} [(\bar{v}_n \cdot \nabla) F] d\ell + \int_{C^{P_1}}^{P_2} F [\bar{a}_\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}] d\ell$$

where \bar{a}_ℓ is a unit vector tangent to contour C in the direction of integration.

The case in which contour, C , is closed is treated by the following corollary.

Corollary L5

If contour, C , is closed

$$\frac{dQ}{dt} = \oint_C \frac{\partial F}{\partial t} d\ell + \oint_C [(\bar{v}_n \cdot \nabla)F] d\ell + \oint_C F \left[\bar{a}_\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell} \right] d\ell$$

LIMITATIONS

Nothing has been said concerning restrictions on the functions involved in the integrals treated by the foregoing theorems. Sufficient conditions for the validity of the theorems will be given in this section.

It is assumed that the contour, C , of integration, is a continuous, smooth curve. A continuous curve that is not smooth but has points at which the direction abruptly changes cannot be handled directly by the above theorems; however, it can be handled by a very simple extension. It is only necessary to express the line integral, along the contour in question, as a sum of line integrals along segments of the contour, each segment being smooth. Then the theorems can be applied to each of the integrals.

It is assumed that the integrand, denoted by F or \bar{F} in the statements of the theorems, is continuous and has continuous derivatives with respect to t and spacial coordinates at all points along contour C . Integrands not meeting these conditions can still be treated if it is possible to express the integral as a sum of line integrals along segments of the contour, the integrand meeting the continuity requirements along each segment. The theorems can then be applied to each of the integrals in the sum. This procedure can be described as placing discontinuities at the end points of integrals where they do not prevent the application of the theorems.

It is assumed that the normal component, \bar{v}_n , of the velocity, is continuous and has a continuous derivative, $\frac{\partial \bar{v}_n}{\partial \lambda}$, along the contour. If these conditions are not met, the theorems can still be applied if it is possible to express the integral as a sum of integrals long segments of the contour, the continuity conditions on \bar{v}_n being met on each of the segments of the contour. This, also, amounts to placing the discontinuities at the end points of the integrals so the theorems can be applied to each of the individual integrals.

COMPILATION OF RESULTS

The five theorems and five corollaries given in this report are listed below in concise notation. This involves the following replacement where appropriate:

$$\bar{a}_\ell d\ell = d\bar{\ell}$$

Theorem L1

$$\begin{aligned} \text{If } Q &= \int_{C, p_1}^{p_2} \bar{F} \cdot d\bar{\ell} \\ \text{then } \frac{dQ}{dt} &= \bar{F}(p_2) \cdot \bar{v}_{t2} - \bar{F}(p_1) \cdot \bar{v}_{t1} + \int_{C, p_1}^{p_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} \\ &+ \int_{C, p_1}^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \cdot d\bar{\ell} + \int_{C, p_1}^{p_2} \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial \ell} d\ell \end{aligned}$$

Corollary L1

If C is a closed contour

$$\begin{aligned} \text{then } \frac{dQ}{dt} &= \oint_C \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} + \oint_C [(\bar{v}_n \cdot \nabla) \bar{F}] \cdot d\bar{\ell} \\ &+ \oint_C \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial \ell} d\ell \end{aligned}$$

Theorem L2

$$\begin{aligned} \text{If } \bar{Q} &= \int_{C, p_1}^{p_2} \bar{F} \times d\bar{\ell} \\ \text{then } \frac{d\bar{Q}}{dt} &= \bar{F}(p_2) \times \bar{v}_{t2} - \bar{F}(p_1) \times \bar{v}_{t1} + \int_{C, p_1}^{p_2} \frac{\partial \bar{F}}{\partial t} \times d\bar{\ell} \\ &+ \int_{C, p_1}^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \times d\bar{\ell} + \int_{C, p_1}^{p_2} \bar{F} \times \frac{\partial \bar{v}_n}{\partial \ell} d\ell \end{aligned}$$

Corollary L2

If C is a closed contour

$$\text{then } \frac{d\bar{Q}}{dt} = \oint_C \frac{\partial \bar{F}}{\partial t} \times \bar{d\ell} + \oint_C [(\bar{v}_n \cdot \nabla) \bar{F}] \times \bar{d\ell} \\ + \oint_C \bar{F} \times \frac{\partial \bar{v}_n}{\partial \ell} d\ell$$

Theorem L3

$$\text{If } \bar{Q} = \int_{p_1}^{p_2} \bar{F} d\bar{\ell}$$

$$\text{then } \frac{d\bar{Q}}{dt} = F(p_2) \bar{v}_{t2} - F(p_1) \bar{v}_{t1} + \int_{p_1}^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \bar{d\ell} \\ + \int_{p_1}^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \bar{d\ell} + \int_{p_1}^{p_2} \bar{F} \frac{\partial \bar{v}_n}{\partial \ell} d\ell$$

Corollary L3

If C is a closed contour

$$\text{then } \frac{d\bar{Q}}{dt} = \oint_C \frac{\partial \bar{F}}{\partial t} \bar{d\ell} + \oint_C [(\bar{v}_n \cdot \nabla) \bar{F}] \bar{d\ell} \\ + \oint_C \bar{F} \frac{\partial \bar{v}_n}{\partial \ell} d\ell$$

Theorem L4

$$\text{If } \bar{Q} = \int_{C}^{p_2} \bar{F} d\ell$$

$$\text{then } \frac{d\bar{Q}}{dt} = \bar{F}(p_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - \bar{F}(p_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1})$$

$$+ \int_{C}^{p_2} \frac{\partial \bar{F}}{\partial t} d\ell + \int_{C}^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] d\ell + \int_{C}^{p_2} \bar{F} [d\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}]$$

Corollary L4

If C is a closed contour

$$\text{then } \frac{d\bar{Q}}{dt} = \oint_C \frac{\partial \bar{F}}{\partial t} d\ell + \oint_C [(\bar{v}_n \cdot \nabla) \bar{F}] d\ell + \oint_C \bar{F} [d\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}]$$

Theorem L5

$$\text{If } Q = \int_{C}^{p_2} F d\ell$$

$$\text{then } \frac{dQ}{dt} = F(p_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - F(p_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1})$$

$$+ \int_{C}^{p_2} \frac{\partial F}{\partial t} d\ell + \int_{C}^{p_2} [(\bar{v}_n \cdot \nabla) F] d\ell + \int_{C}^{p_2} F [d\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}]$$

Corollary L5

If C is a closed contour

$$\text{then } \frac{dQ}{dt} = \oint_C \frac{\partial F}{\partial t} d\ell + \oint_C [(\bar{v}_n \cdot \nabla) F] d\ell + \oint_C F [d\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}]$$

APPENDIX 1

Proof of Theorem L1

Q is defined by

$$Q = \int_{C^{p_1}}^{p_2} \bar{F} \cdot d\bar{\ell}$$

It is assumed that contour C is in motion and that the motion is described in terms of its velocity, \bar{v} , which is a function of position along contour C. It is further assumed that the integrand \bar{F} is a function of time, t, and of position in space. Q is therefore a function of time. The derivative, $\frac{dQ}{dt}$, of Q with respect to time will have contribution due to the motion of the end points P_1 and P_2 , due to the variation of \bar{F} with time, due to the variation of \bar{F} caused by the motion of contour C and due to the variation of $d\bar{\ell}$ caused by the motion of contour C. An expression for $\frac{dQ}{dt}$ must take all of these contributions into consideration.

The velocity, \bar{v} , can be expressed in terms of components tangential, \bar{v}_t , and normal, \bar{v}_n , to the contour. Thus

$$\bar{v} = \bar{v}_t + \bar{v}_n .$$

The tangential component at an interior point of the contour (i.e. any point other than an endpoint) causes stretching or compression of the sequence of points along the contour; it does not result in any movement of the contour in space, and, therefore, has no effect upon the line integral. For this reason, only the normal component, \bar{v}_n , of velocity

need be considered at interior points. However, the tangential component of velocity at the end points, P_1 and P_2 influences the integral in the same manner as for an integral of a function of one spatial variable. For that case it may be recalled that if I is defined by

$$I = \int_{a_1}^{a_2} D(t,x) dx$$

where a_1 and a_2 are functions of t and D is a function of t and x . Then I is a function of t and the derivative, $\frac{dI}{dt}$, of I with respect to t is given by

$$\frac{dI}{dt} = D(t,a_2) \frac{da_2}{dt} - D(t,a_1) \frac{da_1}{dt} + \int_{a_1}^{a_2} \frac{\partial D(t,x)}{\partial t} dx$$

This result is applicable to the problem under consideration for the special case in which $\bar{v}_n = 0$. For this case, it is apparent that

$$\frac{dQ}{dt} = \bar{F}(p_2) \cdot \bar{v}_{t2} - \bar{F}(p_1) \cdot \bar{v}_{t1} + \int_{p_1}^{p_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell}$$

To complete the proof of Theorem L1, it is necessary to add terms resulting from non-zero \bar{v}_n to the above expression.

The normal component, \bar{v}_n , of the velocity is a measure of the transverse movement of each point on the contour. As a point on the contour moves, the function, \bar{F} , evaluated at that point, changes. The rate of change of \bar{F} can be expressed as the product of the magnitude, v_n , of the normal component of the velocity and the directional derivative of \bar{F} in the direction of \bar{v}_n . Let α be distance in the direction of \bar{v}_n (that is in the direction normal to the surface); then the directional derivative needed is $\frac{\partial \bar{F}}{\partial \alpha}$.

Let \bar{a}_{vn} be a unit vector in the direction of \bar{v}_n ; then the operator $\bar{a}_{vn} \cdot \nabla$ is a scalar operator that performs differentiation in the direction of \bar{a}_{vn} . Thus

$$\frac{\partial}{\partial \alpha} = \bar{a}_{vn} \cdot \nabla$$

and

$$v_n \frac{\partial \bar{F}}{\partial \alpha} = v_n (\bar{a}_{vn} \cdot \nabla) \bar{F} .$$

Since $v_n \bar{a}_{vn} = \bar{v}_n$ this can be written

$$\frac{\partial \bar{F}}{\partial \alpha} = (\bar{v}_n \cdot \nabla) \bar{F}$$

and the contribution to $\frac{dQ}{dt}$ resulting from the variation of \bar{F} , caused by the transverse motion of contour C , is

$$\int_{p_1}^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \cdot d\bar{\ell}$$

All contributions to $\frac{dQ}{dt}$ have now been evaluated except that due to the change of $d\bar{\ell}$ with time; so $\frac{dQ}{dt}$ can be written

$$\frac{dQ}{dt} = \bar{F}(p_2) \cdot \bar{v}_{t2} - \bar{F}(p_1) \cdot \bar{v}_{t1} + \int_{p_1}^{p_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} + \int_{p_1}^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \cdot d\bar{\ell} + \int_{p_1}^{p_2} \bar{F} \frac{d(d\bar{\ell})}{dt}$$

The last term of this expression, at first glance, seems surprising since differentials, in simple integrals, are usually not functions of parameters. However, line integrals are frequently formulated such that $d\bar{\ell}$ is a function of position; then if the contour of integration is in motion, $\frac{d(d\bar{\ell})}{dt}$ will be nonzero.

The last term of the above relation is in a form that is somewhat awkward to use. A much more satisfactory result can be obtained by formulating it in terms of the normal component \bar{v}_n , of velocity. To do this, first express $\bar{d\ell}$ in terms of the position vector, \bar{R} , of a point on the contour of integration. This can be done by writing

$$\bar{d\ell} = \bar{a}_\ell d\ell$$

where \bar{a}_ℓ is a unit vector tangent to the contour. \bar{a}_ℓ can be expressed as

$$\bar{a}_\ell = \frac{\partial \bar{R}}{\partial \ell} .$$

Thus

$$\bar{d\ell} = \frac{\partial \bar{R}}{\partial \ell} d\ell .$$

The derivative of this with respect to time is given by

$$\frac{d(\bar{d\ell})}{dt} = \frac{\partial \left(\frac{d\bar{R}}{dt} \right)}{\partial \ell} d\ell .$$

It is not necessary to consider the variation of $d\ell$ or $\partial \ell$ with respect to time because any such variation in $d\ell$, in the numerator, cancels the variation in $\partial \ell$ in the denominator.

Next it is observed that $\frac{d\bar{R}}{dt}$ is simply the normal component, \bar{v}_n , of the velocity of the contour. (As explained above the tangential component of velocity is ignored, at interior points of the contour, because it does not influence Q.) The above relation may now be written

$$\frac{d(\bar{d\ell})}{dt} = \frac{\partial \bar{v}_n}{\partial \ell} d\ell$$

and the expression for $\frac{dQ}{dt}$ can be written

$$\frac{dQ}{dt} = \bar{F}(p_2) \cdot \bar{v}_{t2} - \bar{F}(p_1) \cdot \bar{v}_{t1} + \int_{p_1}^{p_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} + \int_{p_1}^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \cdot d\bar{\ell} + \int_{p_1}^{p_2} \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial \bar{\ell}} d\bar{\ell}.$$

This completes the proof of Theorem L1.

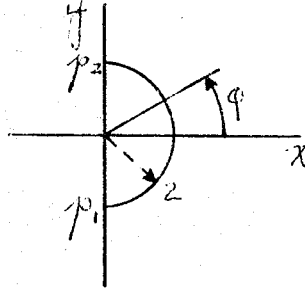
APPENDIX 2

Example of the Use of Theorem L1

Integral: $Q = \int_C^{p_2} \bar{F} \cdot d\bar{\ell}$

Integrand: $\bar{F} = \bar{a}_y 4xy^2 \sin \omega t$

Contour:



Motion: $\bar{v} = \bar{a}_x$

Definitions: $x = 2 \cos \phi$

$$y = 2 \sin \phi$$

$$d\bar{\ell} = (-\bar{a}_x \sin \phi + \bar{a}_y \cos \phi) 2 d\phi$$

$$d\ell = 2d\phi$$

$$\bar{v}_n = (\bar{a}_x \cos \phi + \bar{a}_y \sin \phi) \cos \phi$$

$$\bar{v}_{t2} = \bar{a}_x$$

$$\bar{v}_{t1} = \bar{a}_x$$

$$\bar{v}_n \cdot \nabla = \cos^2 \phi \frac{\partial}{\partial x} + \sin \phi \cos \phi \frac{\partial}{\partial y}$$

$$(\bar{v}_n \cdot \nabla) \bar{F} = \bar{a}_y (4y^2 \sin \omega t \cos^2 \phi + 8xy \sin \omega t \sin \phi \cos \phi)$$

Definitions:
(continued)

$$\frac{\partial \bar{v}_n}{\partial \ell} = \frac{\partial \bar{v}_n}{\partial \phi} \frac{\partial \phi}{\partial \ell} = \frac{1}{2} [\bar{a}_x (-2 \cos \phi \sin \phi) + \bar{a}_y (\cos^2 \phi - \sin^2 \phi)]$$

$$\frac{\partial \bar{F}}{\partial t} = \bar{a}_y 4xy^2 \omega \cos \omega t$$

Evaluation of Q:

$$Q = \int_{p_1}^{p_2} (\bar{a}_y 4xy^2 \sin \omega t) \cdot (-\bar{a}_x \sin \phi + \bar{a}_y \cos \phi) 2d\phi$$

$$= 8 \sin \omega t \int_{-\pi/2}^{\pi/2} xy^2 \cos \phi \, d\phi$$

$$= 8 \sin \omega t \int_{-\pi/2}^{\pi/2} (t + 2 \cos \phi) 4 \cos \phi \sin^2 \phi \, d\phi$$

$$= 8 \sin \omega t \left[4t \left(\frac{2}{3} \right) + 8 \left(\frac{\pi}{8} \right) \right]$$

$$= \frac{64}{3} t \sin \omega t + 8\pi \sin \omega t$$

Evaluation of $\frac{dQ}{dt}$:

$$\frac{dQ}{dt} = \frac{64}{3} \sin \omega t + \frac{64}{3} \omega t \cos \omega t + 8\pi \omega \cos \omega t$$

Evaluation of $\bar{F}(p_2) \cdot \bar{v}_{t2} - \bar{F}(p_1) \cdot \bar{v}_{t1}$:

$$\bar{F}(p_2) \cdot \bar{v}_{t2} - \bar{F}(p_1) \cdot \bar{v}_{t1} = 4t(2)^2 \bar{a}_y \cdot \bar{a}_x - 4t(-2)^2 \bar{a}_y \cdot \bar{a}_x$$

$$= 0$$

Evaluation of $\int_C^{p_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell}$:

$$\int_C^{p_2} \frac{\partial \bar{F}}{\partial t} \cdot d\bar{\ell} = 4\omega \cos \omega t \int_{-\pi/2}^{\pi/2} xy^2 \bar{a}_y \cdot (-\bar{a}_x \sin \phi + \bar{a}_y \cos \phi) 2d\phi$$

$$= 32\omega \cos \omega t \int_{-\pi/2}^{\pi/2} (t + 2 \cos \phi) \cos \phi \sin^2 \phi d\phi$$

$$= 32\omega \cos \omega t \left[t \left(\frac{2}{3} \right) + 2 \left(\frac{\pi}{8} \right) \right]$$

$$= \frac{64}{3} \omega t \cos \omega t + 8\pi \omega \cos \omega t$$

Evaluation of $\int_C^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \cdot d\bar{\ell}$:

$$\int_C^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] d\ell = 4 \sin \omega t \int_{-\pi/2}^{\pi/2} (y^2 \cos^2 \phi + 2xy \sin \phi \cos \phi) \cos \phi 2d\phi$$

$$= 8 \sin \omega t \int_{-\pi/2}^{\pi/2} (4 \sin^2 \phi \cos^3 \phi + 4(t+2 \cos \phi) \cos^2 \phi \sin^2 \phi) d\phi$$

$$= 8 \sin \omega t \left[4 \left(\frac{4}{15} \right) + 4t \left(\frac{\pi}{8} \right) + 8 \left(\frac{4}{15} \right) \right]$$

$$= \frac{384}{15} \sin \omega t + 4\pi t \sin \omega t$$

Evaluation of $\int_{C^{p_1}}^{p_2} \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial \ell} d\ell$:

$$\begin{aligned}
 \int_{C^{p_1}}^{p_2} \bar{F} \cdot \frac{\partial \bar{v}_n}{\partial \ell} d\ell &= 4 \sin \omega t \int_{-\pi/2}^{\pi/2} \frac{1}{2} xy^2 (\cos^2 \phi - \sin^2 \phi) 2 d\phi \\
 &= 16 \sin \omega t \int_{-\pi/2}^{\pi/2} (t + 2 \cos \phi) \sin^2 \phi (\cos^2 \phi - \sin^2 \phi) d\phi \\
 &= 16 \sin \omega t \left[t \left(\frac{\pi}{8} \right) - t \left(\frac{3\pi}{8} \right) + 2 \left(\frac{4}{15} \right) - 2 \left(\frac{2}{5} \right) \right] \\
 &= -4\pi t \sin \omega t - \frac{64}{15} \sin \omega t
 \end{aligned}$$

Calculation of $\frac{dQ}{dt}$ using Theorem L1:

$$\begin{aligned}
 \frac{dQ}{dt} &= [0] + \left[\frac{64}{3} \omega t \cos \omega t + 8\pi \omega \cos \omega t \right] \\
 &+ \left[\frac{384}{15} \sin \omega t + 4\pi t \sin \omega t \right] + \left[-4\pi t \sin \omega t - \frac{64}{15} \sin \omega t \right] \\
 &= \frac{64}{3} \sin \omega t + \frac{64}{3} \omega t \cos \omega t + 8\pi \omega \cos \omega t
 \end{aligned}$$

APPENDIX 3

Proofs of Theorems L2, L3, L4 and L5

Theorem L1 has been proved in Appendix 1. The proofs of Theorems L2, L3, L4 and L5 are merely slight modifications of that proof. Only the modifications will be discussed in this Appendix. Two of these theorems involve an expression for $\frac{d(d\ell)}{dt}$. This can be evaluated by first writing $d\ell$ in terms of the position vector \bar{R} . Thus

$$d\ell = \left| \frac{\partial \bar{R}}{\partial \ell} \right| d\ell .$$

This is true since $\frac{\partial \bar{R}}{\partial \ell} = \bar{a}_\ell$ where \bar{a}_ℓ is the unit tangent vector to contour C. This seems a surprising starting point since the term involving the magnitude brackets is unity. Nevertheless this will lead to a useful formulation. \bar{R} can be expressed as

$$\bar{R} = \bar{a}_x x + \bar{a}_y y + \bar{a}_z z$$

where x , y and z are the coordinates of points on contour C. They are functions of distance, ℓ , along the contour. From this it is seen that

$$\frac{\partial \bar{R}}{\partial \ell} = \bar{a}_x \frac{\partial x}{\partial \ell} + \bar{a}_y \frac{\partial y}{\partial \ell} + \bar{a}_z \frac{\partial z}{\partial \ell}$$

and

$$\left| \frac{\partial \bar{R}}{\partial \ell} \right| = \sqrt{\left(\frac{\partial x}{\partial \ell}\right)^2 + \left(\frac{\partial y}{\partial \ell}\right)^2 + \left(\frac{\partial z}{\partial \ell}\right)^2} .$$

Thus

$$d\ell = \sqrt{\left(\frac{\partial x}{\partial \ell}\right)^2 + \left(\frac{\partial y}{\partial \ell}\right)^2 + \left(\frac{\partial z}{\partial \ell}\right)^2} d\ell.$$

Using this relation it is seen that

$$\frac{d(d\ell)}{dt} = \frac{\frac{\partial x}{\partial \ell} \frac{\partial(\frac{\partial x}{\partial t})}{\partial \ell} + \frac{\partial y}{\partial \ell} \frac{\partial(\frac{\partial y}{\partial t})}{\partial \ell} + \frac{\partial z}{\partial \ell} \frac{\partial(\frac{\partial z}{\partial t})}{\partial \ell}}{\sqrt{\left(\frac{\partial x}{\partial \ell}\right)^2 + \left(\frac{\partial y}{\partial \ell}\right)^2 + \left(\frac{\partial z}{\partial \ell}\right)^2}} d\ell$$

In obtaining this result, the order of differentiation with respect to time and with respect to ℓ has been interchanged. The motion under consideration in this expression is motion in the direction normal to contour C with velocity, \bar{v}_n , which can be written

$$\bar{v}_n = \bar{a}_x \frac{\partial x}{\partial t} + \bar{a}_y \frac{\partial y}{\partial t} + \bar{a}_z \frac{\partial z}{\partial t}.$$

Using this, the above expression can be written

$$\frac{d(d\ell)}{dt} = \frac{\frac{\partial \bar{R}}{\partial \ell} \cdot \frac{\partial \bar{v}_n}{\partial \ell}}{\sqrt{\left(\frac{\partial x}{\partial \ell}\right)^2 + \left(\frac{\partial y}{\partial \ell}\right)^2 + \left(\frac{\partial z}{\partial \ell}\right)^2}} d\ell$$

Finally using the relations

$$\frac{\partial \bar{R}}{\partial \ell} = \bar{a}_\ell$$

and

$$\sqrt{\left(\frac{\partial x}{\partial \ell}\right)^2 + \left(\frac{\partial y}{\partial \ell}\right)^2 + \left(\frac{\partial z}{\partial \ell}\right)^2} = \left| \frac{\partial \bar{R}}{\partial \ell} \right| = \left| \bar{a}_\ell \right| = 1.$$

The desired result is obtained

$$\frac{d(\overline{d\ell})}{dt} = \overline{a}_\ell \cdot \frac{\partial \overline{v}_n}{\partial \ell} d\ell .$$

Now Theorem L2 will be considered. The integral in this theorem differs from that in Theorem L1 only in the presence of a vector product in place of the scalar product in the integral of Theorem L1. Theorem L2 can, therefore, be obtained from Theorem L1 by replacing scalar products by vector products; the result is

$$\begin{aligned} \frac{d\overline{Q}}{dt} = & \overline{F}(p_2) \times \overline{v}_{t_2} - \overline{F}(p_1) \times \overline{v}_{t_1} + \int_C^{p_2} \frac{\partial \overline{F}}{\partial t} \times \overline{d\ell} + \int_C^{p_2} [(\overline{v}_n \cdot \nabla) \overline{F}] \times \overline{d\ell} \\ & + \int_C^{p_2} \overline{F} \times \frac{\partial \overline{v}_n}{\partial \ell} d\ell . \end{aligned}$$

The integral in Theorem L3 has a scalar integrand, F , but a vector differential of length, $\overline{d\ell}$. The theorem can be obtained from Theorem L1 by replacing scalar products by ordinary products and by replacing \overline{F} by F . The result is

$$\begin{aligned} \frac{d\overline{Q}}{dt} = & F(p_2) \overline{v}_{t_2} - F(p_1) \overline{v}_{t_1} + \int_C^{p_2} \frac{\partial F}{\partial t} \overline{d\ell} + \int_C^{p_2} [(\overline{v}_n \cdot \nabla) F] \overline{d\ell} \\ & + \int_C^{p_2} F \frac{\partial \overline{v}_n}{\partial \ell} d\ell . \end{aligned}$$

The integral in Theorem L4 has a scalar differential of length, $d\ell$, for which the derivative, with respect to time, is as given above. Using this result and replacing scalar multiplication by ordinary multiplication gives the desired result.

$$\begin{aligned} \frac{d\bar{Q}}{dt} = & \bar{F}(p_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - \bar{F}(p_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1}) + \int_C^{p_2} \frac{\partial \bar{F}}{\partial t} d\ell + \int_C^{p_2} [(\bar{v}_n \cdot \nabla) \bar{F}] d\ell \\ & + \int_C^{p_2} \bar{F} [\bar{a}_\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}] d\ell. \end{aligned}$$

One additional modification is necessary in achieving this result. The first two terms involve the magnitude of the component of \bar{v}_t in the direction of integration so \bar{v}_{t2} and \bar{v}_{t1} in Theorem L1 are replaced by $\bar{v}_{t2} \cdot \bar{a}_{\ell 2}$ and $\bar{v}_{t1} \cdot \bar{a}_{\ell 1}$ respectively.

The integral in Theorem L5 involves no vector functions. The modifications needed, in this case, are a combination of those used in Theorems L3 and L4; the result is

$$\begin{aligned} \frac{dQ}{dt} = & F(p_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - F(p_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1}) + \int_C^{p_2} \frac{\partial F}{\partial t} d\ell \\ & + \int_C^{p_2} [(\bar{v}_n \cdot \nabla) F] d\ell + \int_C^{p_2} F [\bar{a}_\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}] d\ell \end{aligned}$$

This completes proofs of Theorems L2, L3, L4 and L5.

APPENDIX 4

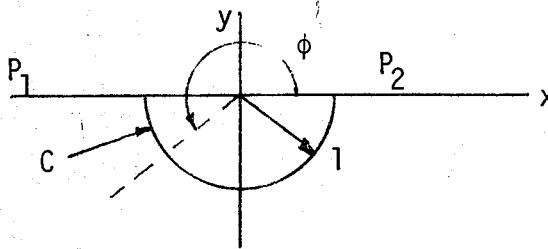
EXAMPLES OF THE USE OF THEOREMS L2, L3, L4 and L5

Example involving Theorem L2.

Integral:
$$\bar{Q} = \int_{P_1}^{P_2} \bar{F} \times d\bar{\ell}$$

Integrand:
$$\bar{F} = 3xy^2 \bar{a}_x$$

Contour:



Motion:

$$\bar{v} = \bar{a}_y$$

Definitions:

$$x = \cos \phi$$

$$y = \sin \phi$$

$$d\bar{\ell} = (-\bar{a}_x \sin \phi + \bar{a}_y \cos \phi) d\phi$$

$$\bar{v}_n = (\bar{a}_x \cos \phi + \bar{a}_y \sin \phi) \sin \phi$$

$$\bar{v}_{t1} = \bar{a}_y$$

$$\bar{v}_{t2} = \bar{a}_y$$

$$\bar{v}_n \cdot \nabla = \cos \phi \sin \phi \frac{\partial}{\partial x} + \sin^2 \phi \frac{\partial}{\partial y}$$

$$(\bar{v}_n \cdot \nabla) \bar{F} = \bar{a}_x (\cos \phi \sin \phi 3y^2 + \sin^2 \phi 6xy)$$

$$\frac{\partial \bar{v}_n}{\partial \ell} = \frac{\partial \bar{v}_n}{\partial \phi} = \bar{a}_x (\cos^2 \phi - \sin^2 \phi) + \bar{a}_y 2 \sin \phi \cos \phi$$

Evaluation of \bar{Q} :

$$\begin{aligned} \bar{Q} &= 3 \int_{P_1}^{P_2} xy^2 \bar{a}_x \times (-\bar{a}_x \sin \phi + \bar{a}_y \cos \phi) d\phi \\ &= 3\bar{a}_z \int_{\pi}^{2\pi} \cos \phi (t + \sin \phi)^2 \cos \phi d\phi \\ &= 3\bar{a}_z \int_{\pi}^{2\pi} \cos \phi (t^2 + 2t \sin \phi + \sin^2 \phi) \cos \phi d\phi \\ &= 3\bar{a}_z [t^2(\frac{\pi}{2}) + 2t(\frac{-2}{3}) + \frac{\pi}{8}] \\ &= \bar{a}_z [\frac{3}{2} \pi t^2 - 4t + \frac{3}{8} \pi] \end{aligned}$$

Evaluation of $\frac{d\bar{Q}}{dt}$: $\frac{d\bar{Q}}{dt} = \bar{a}_z [3\pi t - 4]$

Evaluation of $\bar{F}(P_2) \times \bar{v}_{t2} - \bar{F}(P_1) \times \bar{v}_{t1}$:

$$\bar{F}(P_2) \times \bar{v}_{t2} - \bar{F}(P_1) \times \bar{v}_{t1} = 3t^2 \bar{a}_x \times \bar{a}_y - 3(-1)t^2 \bar{a}_x \times \bar{a}_y = \bar{a}_z 6t^2$$

Evaluation of $\int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} \times d\bar{\ell}$:

$$\int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} \times d\bar{\ell} = 0 .$$

Evaluation of $\int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \times d\bar{\ell} :$

$$\begin{aligned}
 \int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] \times d\bar{\ell} &= 3 \int_{\pi}^{2\pi} (\cos \phi \sin \phi y^2 + \sin^2 \phi 2xy) \bar{a}_x \times (-\bar{a}_x \sin \phi + \bar{a}_y \cos \phi) d\phi \\
 &= 3\bar{a}_z \int_{\pi}^{2\pi} [\cos \phi \sin \phi (t + \sin \phi)^2 + 2 \sin^2 \phi \cos \phi (t + \sin \phi)] \cos \phi d\phi \\
 &= 3\bar{a}_z \int_{\pi}^{2\pi} [\cos^2 \phi \sin \phi (t^2 + 2t \sin \phi + \sin^2 \phi) + 2 \sin^2 \phi \cos^2 \phi (t + \sin \phi)] d\phi \\
 &= 3\bar{a}_z [t^2 \left(\frac{-2}{3}\right) + 2t \left(\frac{\pi}{8}\right) + \left(-\frac{4}{15}\right) + 2t \left(\frac{\pi}{8}\right) + 2 \left(-\frac{4}{15}\right)] \\
 &= \bar{a}_z [-2t^2 + \frac{3}{2}\pi t - \frac{12}{5}] .
 \end{aligned}$$

Evaluation of $\int_{P_1}^{P_2} \bar{F} \times \frac{\partial \bar{v}_n}{\partial \ell} d\ell :$

$$\begin{aligned}
 \int_{P_1}^{P_2} \bar{F} \times \frac{\partial \bar{v}_n}{\partial \ell} d\ell &= 3 \int_{\pi}^{2\pi} xy^2 \bar{a}_x \times [\bar{a}_x (\cos^2 \phi - \sin^2 \phi) + \bar{a}_y 2 \sin \phi \cos \phi] d\phi \\
 &= 6\bar{a}_z \int_{\pi}^{2\pi} xy^2 \sin \phi \cos \phi d\phi \\
 &= 6\bar{a}_z \int_{\pi}^{2\pi} \cos^2 \phi (t + \sin \phi)^2 \sin \phi d\phi
 \end{aligned}$$

$$\begin{aligned}
&= 6\bar{a}_z \int_{\pi}^{2\pi} \cos^2 \phi \sin \phi (t^2 + 2t \sin \phi + \sin^2 \phi) d\phi \\
&= 6\bar{a}_z \left[t^2 \left(-\frac{2}{3}\right) + 2t \left(\frac{\pi}{8}\right) + \left(-\frac{4}{15}\right) \right] \\
&= \bar{a}_z \left[-4t^2 + \frac{3}{2}\pi t - \frac{8}{5} \right]
\end{aligned}$$

Evaluation of $\frac{d\bar{Q}}{dt}$ using Theorem L2:

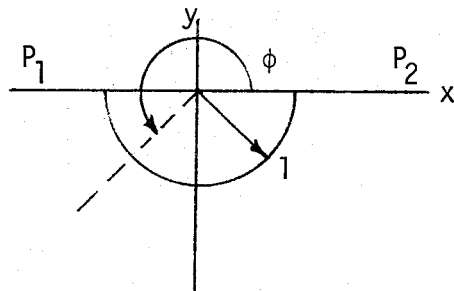
$$\begin{aligned}
\frac{d\bar{Q}}{dt} &= [\bar{a}_z 6t^2] + 0 + \left[\bar{a}_z \left(-2t^2 + \frac{3}{2}\pi t - \frac{12}{5}\right) \right] + \left[\bar{a}_z \left(-4t^2 + \frac{3}{2}\pi t - \frac{8}{5}\right) \right] \\
&= \bar{a}_z [3\pi t - 4]
\end{aligned}$$

Example involving Theorem L3

Integral:
$$\bar{Q} = \int_{P_1}^{P_2} F \overline{d\ell}$$

Integrand:
$$F = 3x^2y^2$$

Contour:



Motion:
$$\bar{v} = \bar{a}_y$$

Definitions: $x = \cos\phi$

$$y = t + \sin\phi$$

$$d\ell = (-\bar{a}_x \sin\phi + \bar{a}_y \cos\phi)d\phi$$

$$\bar{v}_n = (\bar{a}_x \cos\phi + \bar{a}_y \sin\phi)\sin\phi$$

$$\bar{v}_{t1} = \bar{a}_y$$

$$\bar{v}_{t2} = \bar{a}_y$$

$$\bar{v}_n \cdot \nabla = \cos\phi \sin\phi \frac{\partial}{\partial x} + \sin^2\phi \frac{\partial}{\partial y}$$

$$(\bar{v}_n \cdot \nabla)F = \cos\phi \sin\phi 6xy^2 + \sin^2\phi 6x^2y$$

$$\frac{\partial v_n}{\partial \ell} = \bar{a}_x(\cos^2\phi - \sin^2\phi) + \bar{a}_y 2\sin\phi \cos\phi$$

Evaluation of \bar{Q} :

$$\begin{aligned} \bar{Q} &= 3 \int_{\pi}^{2\pi} x^2 y^2 (-\bar{a}_x \sin\phi + \bar{a}_y \cos\phi) d\phi \\ &= 3 \int_{\pi}^{2\pi} \cos^2\phi (t + \sin\phi)^2 (-\bar{a}_x \sin\phi + \bar{a}_y \cos\phi) d\phi \\ &= 3 \int_{\pi}^{2\pi} [-\bar{a}_x \cos^2\phi \sin\phi (t^2 + 2t \sin\phi + \sin^2\phi) \\ &\quad + \bar{a}_y \cos^3\phi (t^2 + 2t \sin\phi + \sin^2\phi)] d\phi \\ &= 3[-\bar{a}_x (t^2(\frac{-2}{3}) + 2t(\frac{\pi}{8}) - \frac{4}{15}) + \bar{a}_y (t^2(0) + 2t(0) + (0))] \\ &= \bar{a}_x (2t^2 - \frac{3}{4}\pi t + \frac{4}{5}) . \end{aligned}$$

Evaluation of $\frac{d\bar{Q}}{dt}$:

$$\frac{d\bar{Q}}{dt} = \bar{a}_x (4t - \frac{3}{4}\pi)$$

Evaluation of $F(p_2)\bar{v}_{t2} - F(p_1)\bar{v}_{t1}$:

$$F(p_2)\bar{v}_{t2} - F(p_1)\bar{v}_{t1} = 3(1)^2 t^2 \bar{a}_y - 3(-1)^2 t^2 \bar{a}_y = 0$$

Evaluation of $\int_{P_1}^{P_2} \frac{\partial F}{\partial t} d\bar{\ell}$: $\int_{P_1}^{P_2} \frac{\partial F}{\partial t} d\bar{\ell} = 0$

Evaluation of $\int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla)F] d\bar{\ell}$:

$$\int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla)F] d\bar{\ell} = 6 \int_{\pi}^{2\pi} (\cos\phi \sin\phi x y^2 + \sin^2\phi x^2 y)(-\bar{a}_x \sin\phi + \bar{a}_y \cos\phi) d\phi$$

$$= -6 \int_{\pi}^{2\pi} [\cos^2\phi \sin^2\phi (t + \sin\phi)^2 + \sin^3\phi \cos^2\phi (t + \sin\phi)] \bar{a}_x d\phi$$

$$+ 6 \int_{\pi}^{2\pi} [\cos^3\phi \sin\phi (t + \sin\phi)^2 + \sin^2\phi \cos^3\phi (t + \sin\phi)] \bar{a}_y d\phi$$

$$= -6\bar{a}_x \int_{\pi}^{2\pi} [\cos^2\phi \sin^2\phi (t^2 + 2t \sin\phi + \sin^2\phi) + \sin^3\phi \cos^2\phi (t + \sin\phi)] d\phi$$

$$+ 6\bar{a}_y \int_{\pi}^{2\pi} [\cos^3\phi \sin\phi (t^2 + 2t \sin\phi + \sin^2\phi) + \sin^2\phi \cos^3\phi (t + \sin\phi)] d\phi$$

$$= -6\bar{a}_x [t^2(\frac{\pi}{8}) + 2t(-\frac{4}{15}) + \frac{\pi}{16} + t(-\frac{4}{15}) + \frac{\pi}{16}]$$

$$+ 6\bar{a}_y [0 + 0 + 0 + 0 + 0]$$

$$= \bar{a}_x [-\frac{3}{4} \pi t^2 + \frac{24}{5} t - \frac{3}{4} \pi]$$

Evaluation of $\int_{P_1}^{P_2} F \frac{\partial \bar{v}_n}{\partial \ell} d\ell$:

$$\int_{P_1}^{P_2} F \frac{\partial \bar{v}_n}{\partial \ell} d\ell = 3 \int_{P_1}^{P_2} x^2 y^2 [\bar{a}_x (\cos^2 \phi - \sin^2 \phi) + \bar{a}_y 2 \sin \phi \cos \phi] d\phi$$

$$= 3 \int_{\pi}^{2\pi} [\cos^4 \phi (t + \sin \phi)^2 - \cos^2 \phi \sin^2 \phi (t + \sin \phi)^2] \bar{a}_x d\phi$$

$$+ 6 \int_{\pi}^{2\pi} [\cos^3 \phi \sin \phi (t + \sin \phi)^2] \bar{a}_y d\phi$$

$$= 3 \int_{\pi}^{2\pi} [\cos^4 \phi (t^2 + 2t \sin \phi + \sin^2 \phi) - \cos^2 \phi \sin^2 \phi (t^2 + 2t \sin \phi + \sin^2 \phi)] \bar{a}_x d\phi$$

$$+ 6 \int_{\pi}^{2\pi} [\cos^3 \phi \sin \phi (t^2 + 2t \sin \phi + \sin^2 \phi)] \bar{a}_y d\phi$$

$$= 3\bar{a}_x [t^2 (\frac{3}{8} \pi) + 2t (-\frac{2}{5}) + \frac{\pi}{16} - t^2 (\frac{\pi}{8}) - 2t (-\frac{4}{15}) - \frac{\pi}{16}]$$

$$+ 6\bar{a}_y [t^2 (0) + 2t(0) + 0]$$

$$= \bar{a}_x [\frac{3}{4} \pi t^2 - \frac{4}{5} t]$$

Evaluation of $\frac{d\bar{Q}}{dt}$ from Theorem L3:

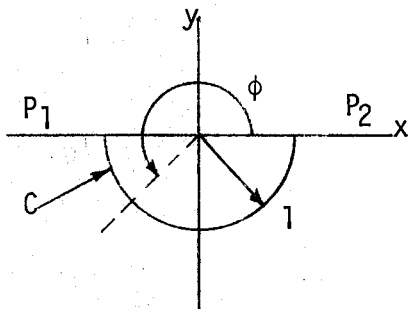
$$\begin{aligned}\frac{d\bar{Q}}{dt} &= [0] + [0] + [\bar{a}_x(-\frac{3}{4}\pi t^2 + \frac{24}{5}t - \frac{3}{4}\pi)] + [\bar{a}_x(\frac{3}{4}\pi t^2 - \frac{4}{5}t)] \\ &= \bar{a}_x[4t - \frac{3}{4}\pi]\end{aligned}$$

Example involving Theorem L4:

Integral:
$$\bar{Q} = \int_{P_1}^{P_2} \bar{F} \cdot d\bar{\ell}$$

Integrand:
$$\bar{F} = 3xy^2\bar{a}_y$$

Contour:



Motion:

$$\bar{v} = \bar{a}_y$$

Definitions:

$$x = \cos \phi$$

$$y = t + \sin \phi$$

$$d\ell = d\phi$$

$$\bar{v}_n = (\bar{a}_x \cos \phi + \bar{a}_y \sin \phi) \sin \phi$$

$$\bar{v}_{t1} = \bar{a}_y$$

$$\bar{v}_{t2} = \bar{a}_y$$

$$\bar{v}_n \cdot \nabla = \cos \phi \sin \phi \frac{\partial}{\partial x} + \sin^2 \phi \frac{\partial}{\partial y}$$

$$(\bar{v}_n \cdot \nabla) \bar{F} = \bar{a}_y (\cos \phi \sin \phi 3y^2 + \sin^2 \phi 6xy)$$

$$\frac{\partial \bar{v}_n}{\partial \ell} = \bar{a}_x (\cos^2 \phi - \sin^2 \phi) + \bar{a}_y 2 \sin \phi \cos \phi$$

$$\bar{a}_\ell = -\bar{a}_x \sin \phi + \bar{a}_y \cos \phi$$

Evaluation of \bar{Q} :

$$\bar{Q} = 3\bar{a}_y \int_{P_1}^{P_2} xy^2 d\phi$$

$$= 3\bar{a}_y \int_{\pi}^{2\pi} \cos \phi (t + \sin \phi)^2 d\phi$$

$$= 3\bar{a}_y \int_{\pi}^{2\pi} \cos \phi (t^2 + 2t \sin \phi + \sin^2 \phi) d\phi$$

$$= 3\bar{a}_y [t^2(0) + 2t(0) + 0]$$

$$= 0.$$

Evaluation of $\frac{d\bar{Q}}{dt}$:

$$\frac{d\bar{Q}}{dt} = 0.$$

Evaluation of $\bar{F}(p_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - \bar{F}(p_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1})$:

$$\bar{F}(p_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - \bar{F}(p_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1}) = 3(1)t^2 \bar{a}_y (\bar{a}_y \cdot \bar{a}_y) - 3(-1)t^2 \bar{a}_y (\bar{a}_y \cdot (-\bar{a}_y)) = 0.$$

Evaluation of $\int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} d\ell$:

$$\int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} d\ell = 0$$

Evaluation of $\int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] d\ell$:

$$\begin{aligned} \int_{P_1}^{P_2} (\bar{v}_n \cdot \nabla) \bar{F} d\ell &= 3\bar{a}_y \int_{\pi}^{2\pi} (\cos\phi \sin\phi y^2 + 2 \sin^2\phi xy) d\phi \\ &= 3\bar{a}_y \int_{\pi}^{2\pi} [\cos\phi \sin\phi (t + \sin\phi)^2 + 2 \sin^2\phi \cos\phi (t + \sin\phi)] d\phi \\ &= 3\bar{a}_y \int_{\pi}^{2\pi} [\cos\phi \sin\phi (t^2 + 2t\sin\phi + \sin^2\phi) + 2\sin^2\phi \cos\phi (t + \sin\phi)] d\phi \\ &= 3\bar{a}_y [t^2(0) + 2t(0) + 0 + 2t(0) + 2(0)] \\ &= 0. \end{aligned}$$

Evaluation of $\int_{P_1}^{P_2} \bar{F} [\bar{a}_\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}] d\ell$:

$$\begin{aligned} \int_{P_1}^{P_2} \bar{F} [\bar{a}_\ell \cdot \frac{\partial \bar{v}_n}{\partial \ell}] d\ell &= 3\bar{a}_y \int_{\pi}^{2\pi} xy^2 (-\bar{a}_x \sin\phi + \bar{a}_y \cos\phi) \cdot [\bar{a}_x (\cos^2\phi - \sin^2\phi) + \bar{a}_y 2 \sin\phi \cos\phi] d\phi \\ &= 3\bar{a}_y \int_{\pi}^{2\pi} xy^2 (-\cos^2\phi \sin\phi + \sin^3\phi + 2\sin\phi \cos^2\phi) d\phi \end{aligned}$$

$$\begin{aligned}
&= 3\bar{a}_y \int_{\pi}^{2\pi} xy^2 \sin\phi \, d\phi \\
&= 3\bar{a}_y \int_{\pi}^{2\pi} \cos\phi (t + \sin\phi)^2 \sin\phi \, d\phi \\
&= 3\bar{a}_y \int_{\pi}^{2\pi} \cos\phi (t^2 + 2t \sin\phi + \sin^2\phi) \sin\phi \, d\phi \\
&= 3\bar{a}_y [t^2(0) + 2t(0) + 0] \\
&= 0.
\end{aligned}$$

Evaluation of $\frac{d\bar{Q}}{dt}$ from Theorem L4:

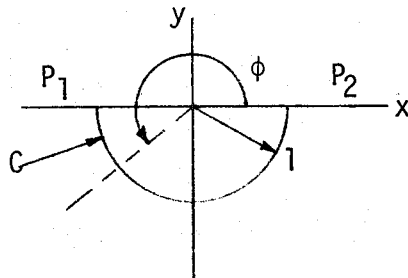
$$\frac{d\bar{Q}}{dt} = 0 + 0 + 0 + 0 = 0.$$

Example involving Theorem L4.

Integral: $\bar{Q} = \int_{P_1}^{P_2} \bar{F} \, d\ell$

Integrand: $\bar{F} = 3x^2y^2\bar{a}_y$

Contour:



Motion: $\bar{v} = \bar{a}_y$

Definitions:

$$x = \cos \phi$$

$$y = t + \sin \phi$$

$$d\ell = d\phi$$

$$\bar{v}_n = (\bar{a}_x \cos \phi + \bar{a}_y \sin \phi) \sin \phi$$

$$\bar{v}_{t1} = \bar{a}_y$$

$$\bar{v}_{t2} = \bar{a}_y$$

$$\bar{v}_n \cdot \nabla = \cos \phi \sin \phi \frac{\partial}{\partial x} + \sin^2 \phi \frac{\partial}{\partial y}$$

$$(\bar{v}_n \cdot \nabla) \bar{F} = (\cos \phi \sin \phi 6xy^2 + \sin^2 \phi 6x^2y) \bar{a}_y$$

$$\frac{\partial \bar{v}_n}{\partial \ell} = \bar{a}_x (\cos^2 \phi - \sin^2 \phi) + \bar{a}_y 2 \sin \phi \cos \phi$$

$$\bar{a}_\ell = -\bar{a}_x \sin \phi + \bar{a}_y \cos \phi$$

Evaluation of \bar{Q} :

$$\bar{Q} = 3\bar{a}_y \int_{P_1}^{P_2} x^2 y^2 d\ell$$

$$= 3\bar{a}_y \int_{\pi}^{2\pi} \cos^2 \phi (t + \sin \phi)^2 d\phi$$

$$= 3\bar{a}_y \int_{\pi}^{2\pi} \cos^2 \phi (t^2 + 2t \sin \phi + \sin^2 \phi) d\phi$$

$$= 3\bar{a}_y \left[t^2 \left(\frac{\pi}{2} \right) + 2t \left(\frac{-2}{3} \right) + \frac{\pi}{8} \right]$$

$$= \bar{a}_y \left[\frac{3}{2} \pi t^2 - 4t + \frac{3}{8} \pi \right]$$

Evaluation of $\frac{d\bar{Q}}{dt}$:

$$\frac{d\bar{Q}}{dt} = (3\pi t - 4) \bar{a}_y$$

Evaluation of $\bar{F}(p_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - \bar{F}(p_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1})$:

$$\begin{aligned}\bar{F}(p_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - \bar{F}(p_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1}) &= 3(1)^2 t^2 \bar{a}_y (\bar{a}_y \cdot \bar{a}_y) - 3(-1)^2 t^2 \bar{a}_y (\bar{a}_y \cdot (-\bar{a}_y)) \\ &= 6t^2 \bar{a}_y\end{aligned}$$

Evaluation of $\int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} d\ell$:

$$\int_{P_1}^{P_2} \frac{\partial \bar{F}}{\partial t} d\ell = 0$$

Evaluation of $\int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] d\ell$:

$$\begin{aligned}\int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla) \bar{F}] d\ell &= 6\bar{a}_y \int_{\pi}^{2\pi} (\cos\phi \sin\phi xy^2 + \sin^2\phi x^2 y) d\phi \\ &= 6\bar{a}_y \int_{\pi}^{2\pi} (\cos^2\phi \sin\phi (t + \sin\phi)^2 + \sin^2\phi \cos^2\phi (t + \sin\phi)) d\phi \\ &= 6\bar{a}_y \int_{\pi}^{2\pi} (\cos^2\phi \sin\phi (t^2 + 2t\sin\phi + \sin^2\phi) + \sin^2\phi \cos^2\phi (t + \sin\phi)) d\phi \\ &= 6\bar{a}_y [t^2(-\frac{2}{3}) + 2t(\frac{\pi}{8}) - \frac{4}{15} + t(\frac{\pi}{8}) - \frac{4}{15}] \\ &= \bar{a}_y [-4t^2 + \frac{9}{4}\pi t - \frac{16}{5}]\end{aligned}$$

Evaluation of $\int_{P_1}^{P_2} \bar{F}[\bar{a}_\ell \cdot \frac{\partial \bar{v}}{\partial \ell}] d\ell$:

$$\begin{aligned}
\int_{P_1}^{P_2} \bar{F}[\bar{a}_\ell \cdot \frac{\partial \bar{v}}{\partial \ell}] d\ell &= 3\bar{a}_y \int_{\pi}^{2\pi} x^2 y^2 [(-\bar{a}_x \sin\phi + \bar{a}_y \cos\phi) \cdot (\bar{a}_x (\cos^2\phi - \sin^2\phi) \\
&\quad + \bar{a}_y 2 \sin\phi \cos\phi)] d\phi \\
&= 3\bar{a}_y \int_{\pi}^{2\pi} \cos^2\phi (t + \sin\phi)^2 (-\cos^2\phi \sin\phi + \sin^3\phi + 2\sin\phi \cos^2\phi) d\phi \\
&= 3\bar{a}_y \int_{\pi}^{2\pi} \cos^2\phi (t^2 + 2t \sin\phi + \sin^2\phi) \sin\phi d\phi \\
&= 3\bar{a}_y [t^2(-\frac{2}{3}) + 2t(\frac{\pi}{8}) - \frac{4}{15}] \\
&= \bar{a}_y [-2t^2 + \frac{3}{4}\pi t - \frac{4}{5}]
\end{aligned}$$

Evaluation of $\frac{d\bar{Q}}{dt}$ from Theorem L4:

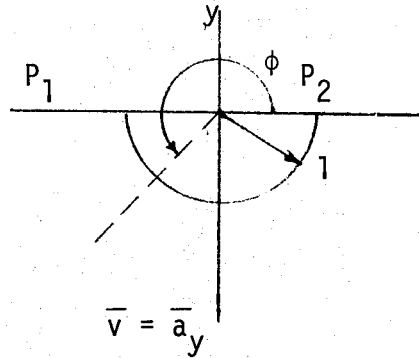
$$\begin{aligned}
\frac{d\bar{Q}}{dt} &= [6t^2\bar{a}_y] + [0] + [\bar{a}_y(-4t^2 + \frac{9}{4}\pi t - \frac{16}{5})] + [\bar{a}_y(-2t^2 + \frac{3}{4}\pi t - \frac{4}{5})] \\
&= \bar{a}_y[3\pi t - 4]
\end{aligned}$$

Example involving Theorem L5.

Integral: $Q = \int_{P_1}^{P_2} F d\ell$

Integrand: $F = 3x^2y^2$

Contour:



Motion:

$$\bar{v} = \bar{a}_y$$

Definitions:

$$x = \cos\phi$$

$$y = t + \sin\phi$$

$$d\ell = d\phi$$

$$\bar{v}_n = (\bar{a}_x \cos\phi + \bar{a}_y \sin\phi) \sin\phi$$

$$\bar{v}_{t1} = \bar{a}_y$$

$$\bar{v}_{t2} = \bar{a}_y$$

$$\bar{v}_n \cdot \nabla = \cos\phi \sin\phi \frac{\partial}{\partial x} + \sin^2\phi \frac{\partial}{\partial y}$$

$$(\bar{v}_n \cdot \nabla)F = \cos\phi \sin\phi 6xy^2 + \sin^2\phi 6x^2y$$

$$\frac{\partial \bar{v}_n}{\partial \ell} = \bar{a}_x (\cos^2\phi - \sin^2\phi) + \bar{a}_y 2\sin\phi \cos\phi$$

$$\bar{a}_\ell = -\bar{a}_x \sin\phi + \bar{a}_y \cos\phi$$

Evaluation of Q:

$$Q = 3 \int_{P_1}^{P_2} x^2 y^2 d\ell$$

$$= 3 \int_{\pi}^{2\pi} \cos^2\phi (t + \sin\phi)^2 d\phi$$

$$\begin{aligned}
&= 3 \int_{\pi}^{2\pi} \cos^2 \phi (t^2 + 2t \sin \phi + \sin^2 \phi) d\phi \\
&= 3 \left[t^2 \left(\frac{\pi}{2} \right) + 2t \left(-\frac{2}{3} \right) + \frac{\pi}{8} \right] \\
&= \frac{3}{2} \pi t^2 - 4t + \frac{3}{8} \pi
\end{aligned}$$

Evaluation of $\frac{dQ}{dt}$: $\frac{dQ}{dt} = 3\pi t - 4$

Evaluation of $F(p_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - F(p_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1})$

$$\begin{aligned}
F(p_2)(\bar{v}_{t2} \cdot \bar{a}_{\ell 2}) - F(p_1)(\bar{v}_{t1} \cdot \bar{a}_{\ell 1}) &= 3(1)^2 t^2 (\bar{a}_y \cdot \bar{a}_y) - 3(-1)^2 t^2 (\bar{a}_y \cdot (-\bar{a}_y)) \\
&= 6t^2 .
\end{aligned}$$

Evaluation of $\int_{P_1}^{P_2} \frac{\partial F}{\partial t} d\ell$: $\int_{P_1}^{P_2} \frac{\partial F}{\partial t} d\ell = 0$

Evaluation of $\int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla)F] d\ell$:

$$\begin{aligned}
\int_{P_1}^{P_2} [(\bar{v}_n \cdot \nabla)F] d\ell &= 6 \int_{\pi}^{2\pi} (\cos \phi \sin \phi xy^2 + \sin^2 \phi x^2 y) d\phi \\
&= 6 \int_{\pi}^{2\pi} [\cos^2 \phi \sin \phi (t + \sin \phi)^2 + \sin^2 \phi \cos^2 \phi (t + \sin \phi)] d\phi \\
&= 6 \int_{\pi}^{2\pi} [\cos^2 \phi \sin \phi (t^2 + 2t \sin \phi + \sin^2 \phi) + \sin^2 \phi \cos^2 \phi (t + \sin \phi)] d\phi
\end{aligned}$$

$$\begin{aligned}
&= 6\left[t^2\left(-\frac{2}{3}\right) + 2t\left(\frac{\pi}{8}\right) - \frac{4}{15} + t\left(\frac{\pi}{8}\right) - \frac{4}{15}\right] \\
&= -4t^2 + \frac{9}{4}\pi t - \frac{16}{5}.
\end{aligned}$$

Evaluation of $\int_{P_1}^{P_2} F[\bar{a}_\ell \cdot \frac{\partial \bar{v}}{\partial \ell} \bar{n}] d\ell$:

$$\begin{aligned}
\int_{P_1}^{P_2} F[\bar{a}_\ell \cdot \frac{\partial \bar{v}}{\partial \ell} \bar{n}] d\ell &= 3 \int_{\pi}^{2\pi} x^2 y^2 [(-\bar{a}_x \sin \phi + \bar{a}_y \cos \phi) \cdot (\bar{a}_x (\cos^2 \phi - \sin^2 \phi) + \bar{a}_y 2 \sin \phi \cos \phi)] d\phi \\
&= 3 \int_{\pi}^{2\pi} x^2 y^2 (-\cos^2 \phi \sin \phi + \sin^3 \phi + 2 \cos^2 \phi \sin \phi) d\phi \\
&= 3 \int_{\pi}^{2\pi} \cos^2 \phi (t + \sin \phi)^2 (\sin \phi) d\phi \\
&= 3 \int_{\pi}^{2\pi} \cos^2 \phi \sin \phi (t^2 + 2t \sin \phi + \sin^2 \phi) d\phi \\
&= 3\left[t^2\left(-\frac{2}{3}\right) + 2t\left(\frac{\pi}{8}\right) - \frac{4}{15}\right] \\
&= -2t^2 + \frac{3}{4}\pi t - \frac{4}{5}.
\end{aligned}$$

Evaluation of $\frac{dQ}{dt}$ from Theorem L5:

$$\frac{dQ}{dt} = [6t^2] + [0] + [-4t^2 + \frac{9}{4}\pi t - \frac{16}{5}] + [-2t^2 + \frac{3}{4}\pi t - \frac{4}{5}] = 3\pi t - 4.$$

