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Dielectric Image Line Coupler

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DIELECTRIC IMAGE LINE COUPLER

by

M. Abouzahra and L. Lewin

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ABSTRACT

The transmission line equations for the situation in which a finite length coupler is joined to terminal lengths via curved structure sections are derived. From this analysis the reverse-coupling (directivity) and reflection, as well as corrections to the coupling length, are studied. Numerically, the propagation characteristics and the reflection coefficient due to coupling, as well as the correct 3 dB coupling length are calculated. Second order effects, that determine the bandwidth as well as the coupling, have been considered and found to be very substantial.

In the examples considered the reflection and directivity were both more than 40 dB down, and the 3 dB outputs accurately in quadrature.

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I. Introduction

The formulas of Section III of Report No. 49⁽¹⁾ will be used to analyze the transmission line equations of a finite length coupler joined to curved dielectric slab sections. From this analysis the reflection coefficient due to coupling, the reverse-coupling (directivity) as well as the actual fields in the dielectric slabs are calculated. The end effects of the coupler and the corrected length have been derived. Second order effects (curvature, dielectric constant, and the breadth of the slab) have been found to affect substantially the coupling, reflection coefficient due to coupling, reverse coupling as well as the bandwidth of the coupler. Numerically, a 3 cm coupler has been designed and an error has been found in a January 1979 contract proposal document due to an erroneous formula which seems to be copied and interpreted incorrectly from Marcatili.⁽²⁾ Moreover, this latter formula is correct only when $a \gg 2/h_0$. Our formula, which agrees with earlier results^(3,4,5,6), is found also to agree with Marcatili formula apart from the term in $1/h_0$. All the results in this report are based on the assumption of constant field along the normal axis (x-axis) and on ignoring the reflection coefficient due directly to the curvature discontinuity. We also assume the radii of curvature of the bends to be sufficiently large so that radiation losses are maintained at reasonable levels⁽⁷⁾. These factors are the subject of further study.

The curved section will be considered of parabolic form given by cz^2 , where c is the curvature parameter. The above assumption is adequate in the region where there is significant coupling. In practical couplers the termination is at about one radius distant, but in the present analysis the output will be taken, for reasons of convenience, at $z = \infty$. The additional coupling due to this is completely negligible.

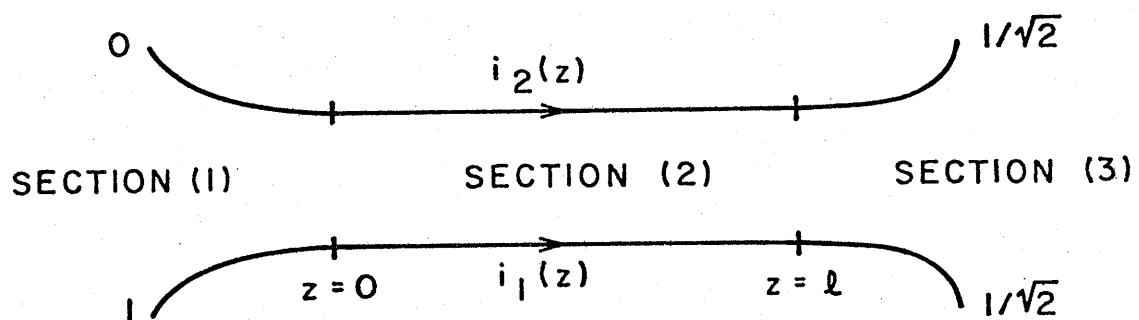


Fig. 1. 3dB Dielectric Directional Coupler

II. Analysis

Figure 1 illustrates the geometry that we use in our description of the 3-dB dielectric image line coupler. We analyze the case where the metal-clad physical arrangement is used. This means that the fine details of the coupling for the case where the dielectric lines are entirely in the air will not be included here, and an alternative simplified structure is analyzed so as to have a good preliminary study of the essential coupling problem. Moreover, small curvature is assumed so that radiation is not significantly important. Mode discontinuity due to the junction will not be investigated here due to its great difficulty and will be the subject of future study.

In our analysis we consider the transmission line equations in which I and V are analogous to the uncoupled field modes of the guides, and where i and v are used to represent the actual coupled fields along each guide. The analysis is undertaken for the three sections. The resulting differential equations for the two curved section does not have an exact closed-form solution because of the variable coupling. Hence an approximate iteration method is used to get an approximate solution which is correct to the first order of the small quantity $\Delta\beta$, the wave number difference due to the coupling.

The full details of this analysis are given in the Appendix of this report. Quoting the results of the analysis, the currents are given by

Section 1:

$$i_1 = e^{-j\beta_0 z} - \psi e^{j\beta_0(z-2\ell)} \sin(2\Delta\beta\ell) \quad (1)$$

$$i_2 = -j \left\{ \Delta\beta e^{-j\beta_0 z} \int_{-\infty}^z e^{-2h_0 cz^2} dz - \Delta\beta e^{j\beta_0 z} \int_0^z e^{-2j\beta_0 z - 2h_0 cz^2} dz + \psi e^{j\beta_0(z-2\ell)} \cos(2\Delta\beta\ell) \right\} \quad (2)$$

Section 2:

$$i_1 = e^{-j\beta_0 z} [\cos(\Delta\beta z) - \phi \sin(\Delta\beta z)] + \psi e^{j\beta_0(z-2\ell)} \sin[\Delta\beta(z-2\ell)] \quad (3)$$

$$i_2 = -j \left\{ e^{-j\beta z} [\sin(\Delta\beta z) + \phi \cos(\Delta\beta z)] + \psi e^{j\beta_0(z-2\ell)} \cos[\Delta\beta(z-2\ell)] \right\} \quad (4)$$

Section 3:

$$i_1 = e^{-j\beta_0 z} [\cos(\Delta\beta\ell) - \phi \sin(\Delta\beta\ell)] - \Delta\beta e^{-j\beta_0 z} \sin(\Delta\beta\ell) \int_0^{z-\ell} e^{-2h_0 cz^2} dz + \Delta\beta e^{j\beta_0(z-2\ell)} \sin(\Delta\beta\ell) \int_{\infty}^{z-\ell} e^{-2j\beta_0 z - 2h_0 cz^2} dz \quad (5)$$

$$i_2 = -j \left\{ e^{-j\beta_0 z} [\sin(\Delta\beta\ell) + \phi \cos(\Delta\beta\ell)] + \Delta\beta e^{-j\beta_0 z} \cos(\Delta\beta\ell) \int_0^{z-\ell} e^{-2h_0 cz^2} dz - \Delta\beta e^{j\beta_0(z-2\ell)} \cos(\Delta\beta\ell) \int_{\infty}^{z-\ell} e^{-2j\beta_0 z - 2h_0 cz^2} dz \right\} \quad (6)$$

By imposing the design requirement for a 3-dB coupler, i.e.,

$|i_1(\infty)| = |i_2(\infty)|$ in Section 3, we can evaluate the length correction due to the end effect which is given by

$$L_{\text{corr}} = \delta\ell + \delta\ell' = \frac{2\phi}{\Delta\beta} = \sqrt{\frac{\pi}{2h_0 c}} \quad (7)$$

where c is the curvature parameter, half the inverse of the radius of curvature of the curved sections at the junction with the straight section.

ϕ and ψ are given by

$$\begin{aligned}\phi &= \Delta\beta \int_0^{\infty} e^{-2h_0 cz^2} dz = \Delta\beta \sqrt{\frac{\pi}{8h_0 c}} \quad (8) \\ \psi &= \Delta\beta \int_0^{\infty} e^{-2j\beta_0 z - 2h_0 cz^2} dz = \Delta\beta \sqrt{\frac{\pi}{8h_0 c}} \left\{ e^{-\beta_0^2/2h_0 c} - j \frac{2}{\sqrt{\pi}} \text{Daw}\left(\frac{\beta_0}{\sqrt{2h_0 c}}\right) \right\}\end{aligned}$$

where $\text{Daw}(x)$ is called Dawson's integral, defined as

$$\text{Daw}(x) = e^{-x^2} \int_0^x e^{t^2} dt.$$

By subtracting the length correction, given by eqn. (7), from the uncorrected 3dB length evaluated in the previous report we obtain the corrected length of the 3-dB coupler or simply the length of the straight section, which is found to be

$$\ell = \frac{\pi}{4\Delta\beta} - \sqrt{\frac{\pi}{2h_0 c}} \quad (9)$$

Figure 2 describes the overall structure of the 3dB coupler with the correction sections $\delta\ell$ and $\delta\ell'$ substituting the curved sections. Moreover, and because of the symmetry of the problem, it is easy to see that the corrections on both sides, namely $\delta\ell$ and $\delta\ell'$, are equal.

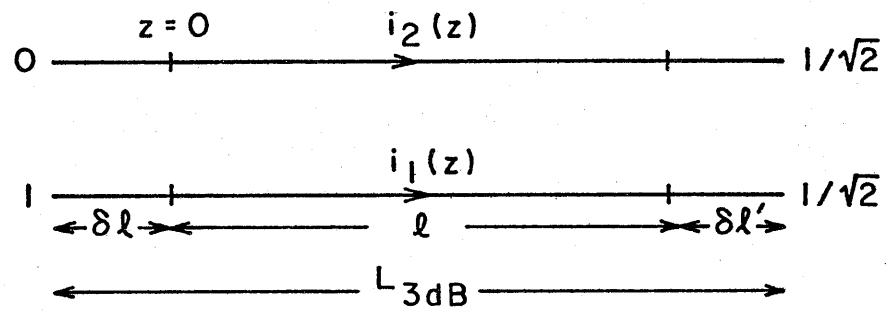


Fig. 2. Equivalent description for the 3dB coupler showing the end effect of the curved sections.

III. Interpretations and Numerical results

The first interesting result that attracts attention in the above mentioned results, eqn. (1) through eqn. (6), is that despite all approximations the coupled outputs or more generally the fields of the two lines are exactly in quadrature all over the three sections. Moreover, the first term on the right-hand side of Eq. (1) represents the unit incident field in line 1 while the second term represents the reflection coefficient due to coupling and the reflection coefficient may be written as

$$R = \psi e^{-j2\beta_0 \ell} \sin(\Delta\beta\ell) \quad (10)$$

On the other hand, i_2 of eqn. (2) does not vanish at $z = -\infty$ and hence zero reverse coupling is not achieved. On the contrary the field of line 2 will retain a finite value, seen in Eq. (2), and given by

$$D(z = -\infty) = j \left\{ \Delta\beta \int_0^{\infty} e^{-2j\beta_0 z - 2h_0 cz^2} dz - \psi e^{-j2\beta_0 \ell} \cos(2\Delta\beta\ell) \right\}$$

which upon carrying the integration reduces to

$$D(z = -\infty) = j\{\psi^* - \psi e^{-j2\beta_0 \ell} \cos(2\Delta\beta\ell)\} \quad (11)$$

where ψ^* is the conjugate of ψ , which is given in Eq. (8). Therefore Eq. (11) gives the value of the reverse coupling or directivity of the coupler. The first two terms on the right-hand side of Eq. (2) describe the effect of internal generation in the initial curved section.

The magnitudes of the actual fields i_1 and i_2 , in the third section and at $z = \infty$, are obtained from Eqs. (5) and (6) and found to be equal to

$$|i_1(\infty)| = \cos(\Delta\beta\ell) - 2\phi \sin(\Delta\beta\ell) \quad (12)$$

and

$$|i_2(\infty)| = \sin(\Delta\beta l) + 2\phi \cos(\Delta\beta l) \quad (13)$$

Numerically, for a 2.54 mm wide Teflon guide with $\epsilon_r = 2.2$ at 94 GHz, the propagation characteristics have been calculated, using the following formulas, derived in a previous report⁽¹⁾

$$\begin{aligned} h_0^2 + p_0^2 &= k_0^2 (\epsilon_r - 1) \\ \tan(p_0 \frac{a}{2}) &= \frac{h_0}{p_0} \\ \Delta\beta &= \frac{p_0^2 h_0 e^{-2h_0 \omega}}{\beta_0 k_0^2 (\epsilon_r - 1) (a/2 + 1/h_0)} \end{aligned} \quad (14)$$

and found to be equal to

$$\begin{aligned} p_0 &= 8.985 \text{ cm}^{-1} \\ h_0 &= 19.608 \text{ cm}^{-1} \\ \beta_0 &= 27.788 \text{ cm}^{-1} \end{aligned}$$

For $c = 0.2$ (corresponding to a 25 mm radius of curvature) and 1 mm spacing, from Eqs. (9) and (14), the 3dB corrected coupling length is found to be equal to 7.5 cm. However this latter result does not agree with the result of a January 1979 contract proposal document. This is due to an error in both copying and interpreting Marcatili's formula. Because we considered this length excessive we reduced the spacing to 0.5 mm which gives a corrected length equal to 2.4 cm, a more suitable value for the coupler length.

Figure 3 illustrates the effect of curvature on the corrected length of the coupler. We found that this effect is small. In Fig. 4 we show

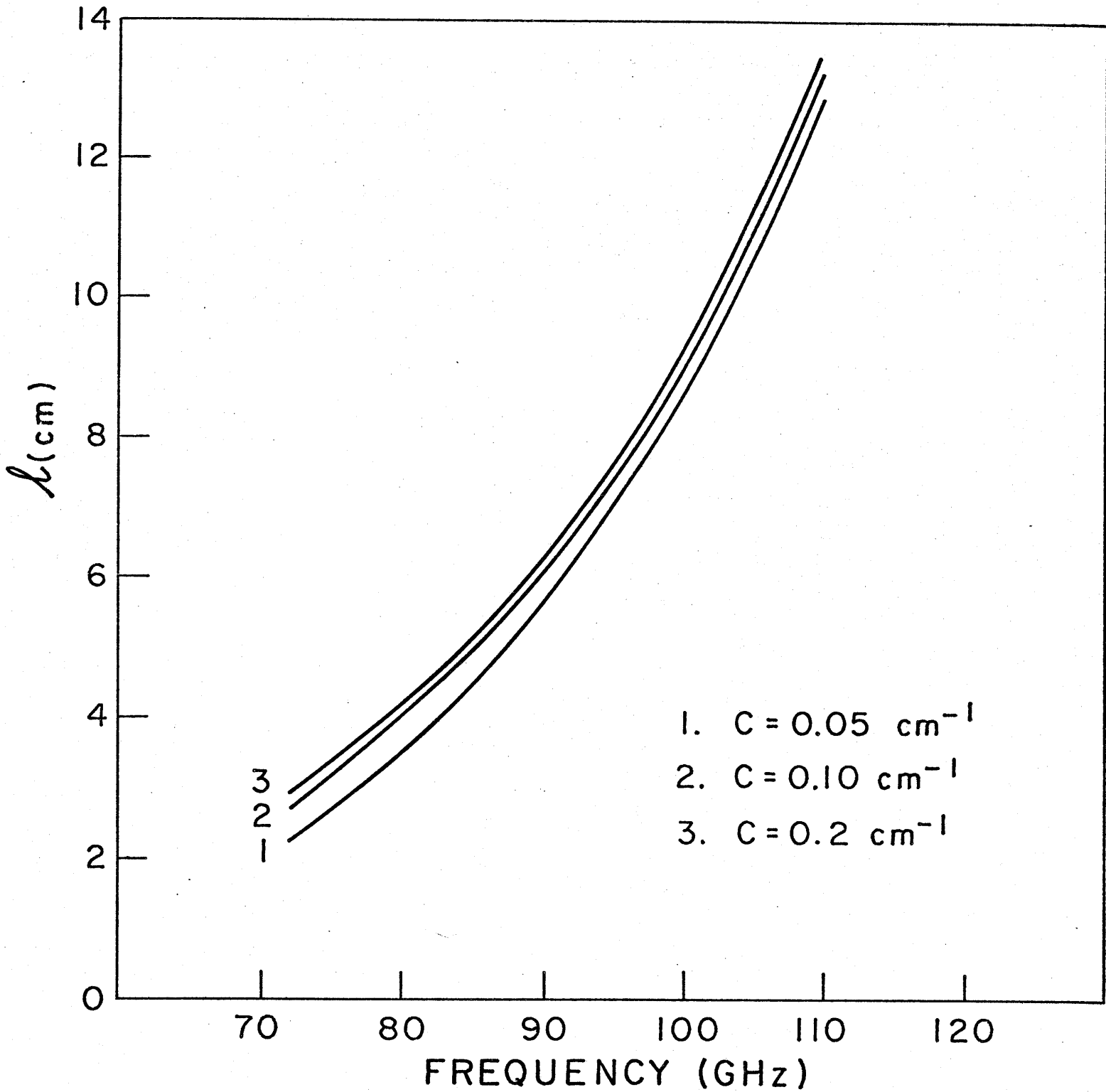


Fig. 3. Corrected length of the coupler vs. frequency.

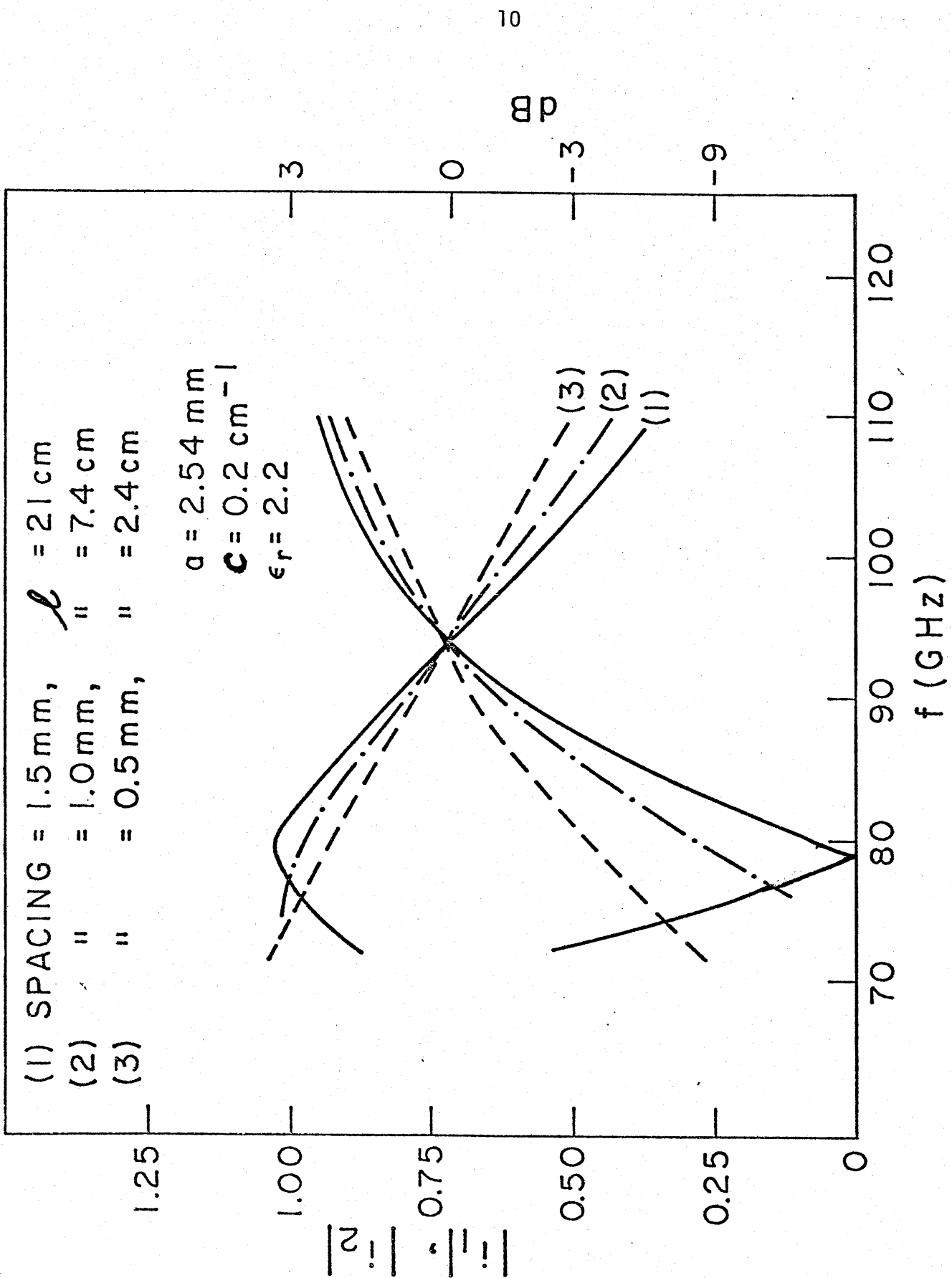


Fig. 4. Frequency dependence of coupling for 3dB at 94 GHz.

the dependence of the output field amplitude, at $z = \infty$ and in both lines of section 3, on frequency. The actual fields are equal at the central frequency f_c and have a value of $\frac{1}{\sqrt{2}}$ so that the sum of the squares is equal to 1. Below and beyond f_c , the squares of the fields, $|i_1(\infty)|^2$ and $|i_2(\infty)|^2$, still add to 1 and this is to be expected because of the conservation of energy. In addition, we can see also the dependence of the flatness of the coupler response on frequency as well as the effect of spacing on the flatness of the coupler response. The bandwidth of the coupler has been found to be inversely proportional to the spacing. This relation is clearly illustrated in Fig. 5 and Fig. 6. In the same way, other second order effects have been examined and were found to affect the bandwidth of the coupler so that the same effect of 0.5 mm spacing and $\epsilon_r = 2.2$ could be achieved by 0.8 mm spacing and $\epsilon_r = 1.8$ with $\ell = 2.5$ cm in both cases. Moreover, it seems to be the case that the same effect could be achieved also by varying the breadth of line as well as the spacing keeping ϵ_r and ℓ at fixed values. These latter facts are clearly described in Fig. 6. Finally, Fig. 7 and Fig. 8 show the dependence of the magnitudes of the reflection coefficient and directivity on frequency.

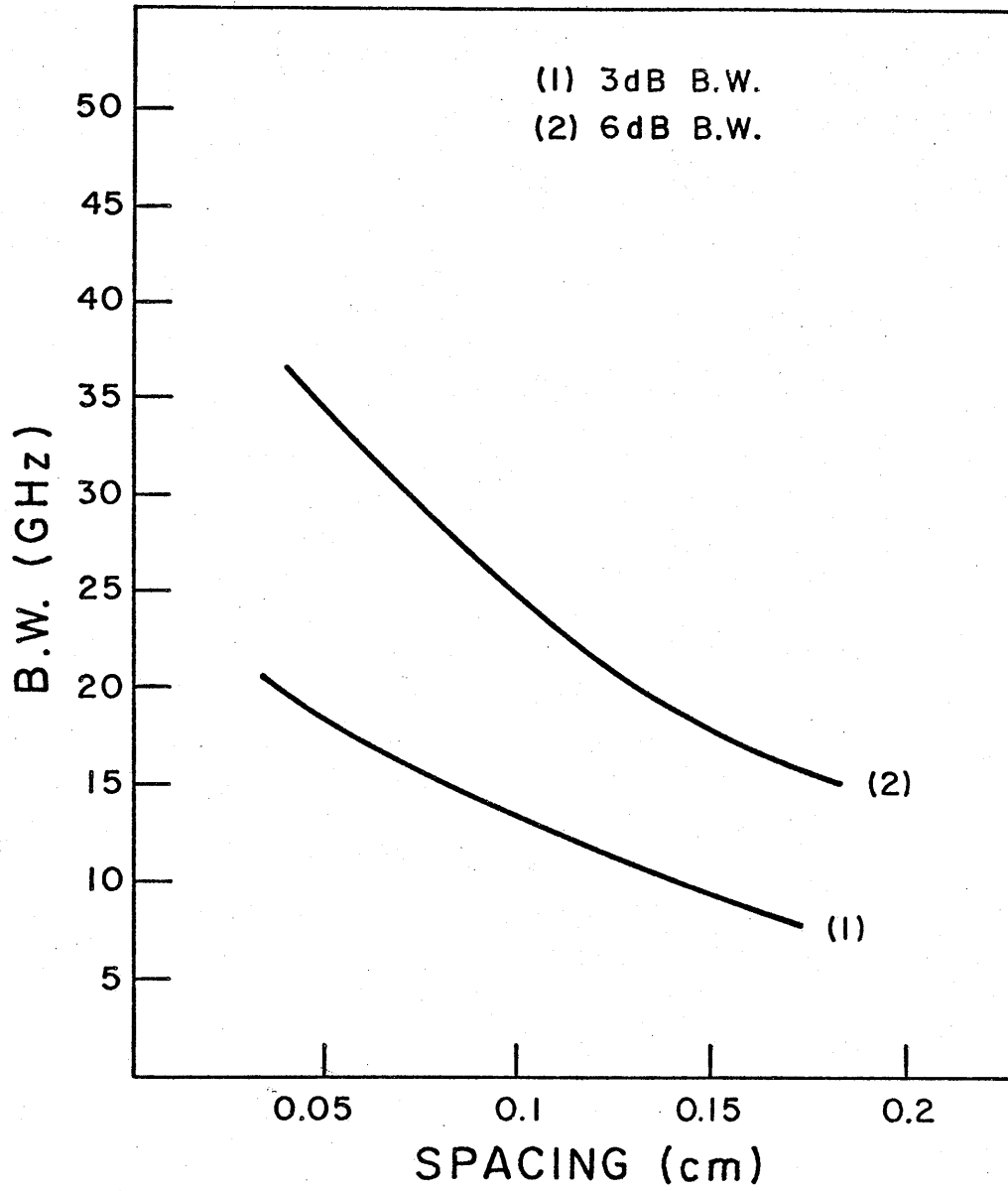


Fig. 5. Bandwidth vs. spacing.

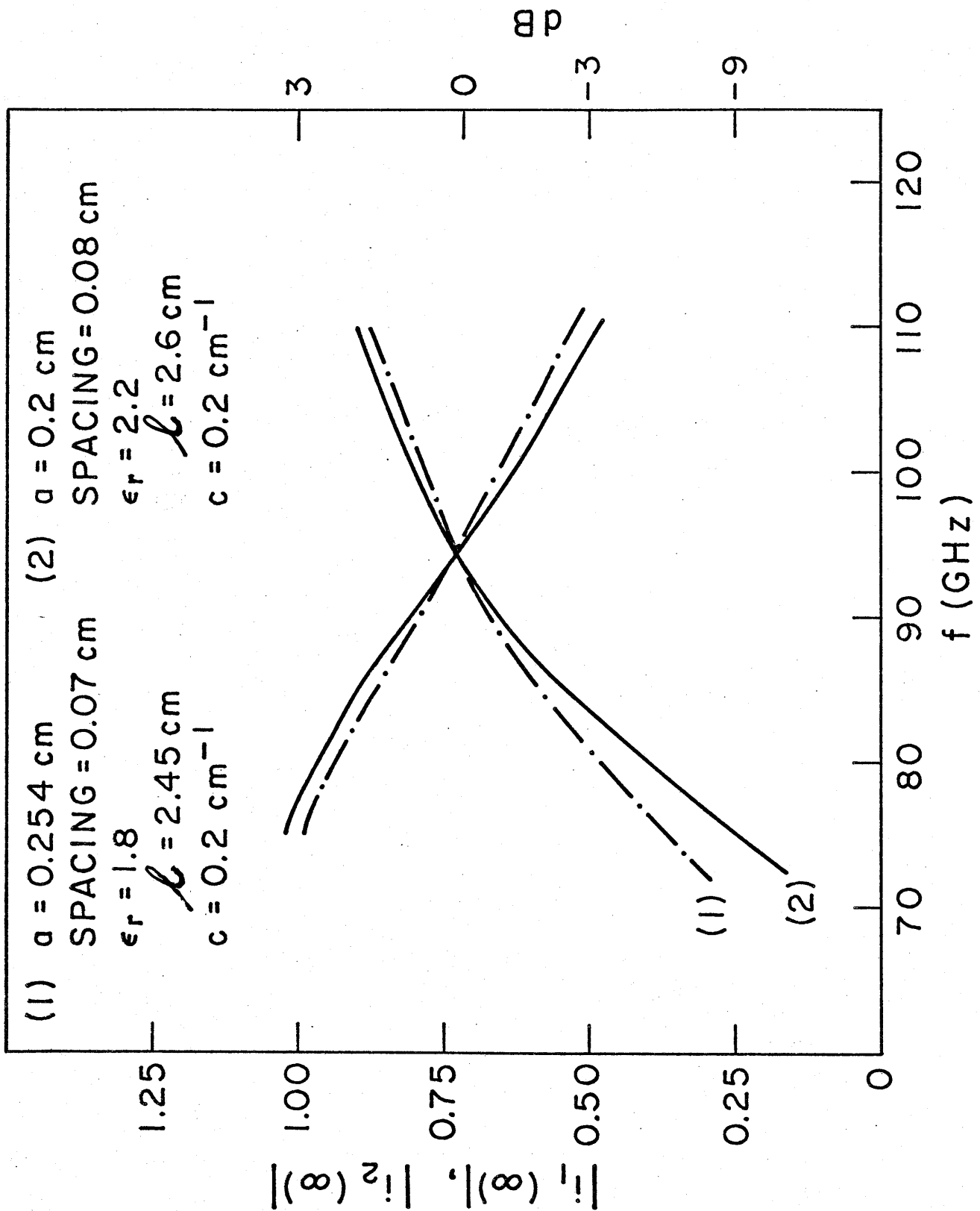


Fig. 6. Effect of line width and dielectric constant on coupling.

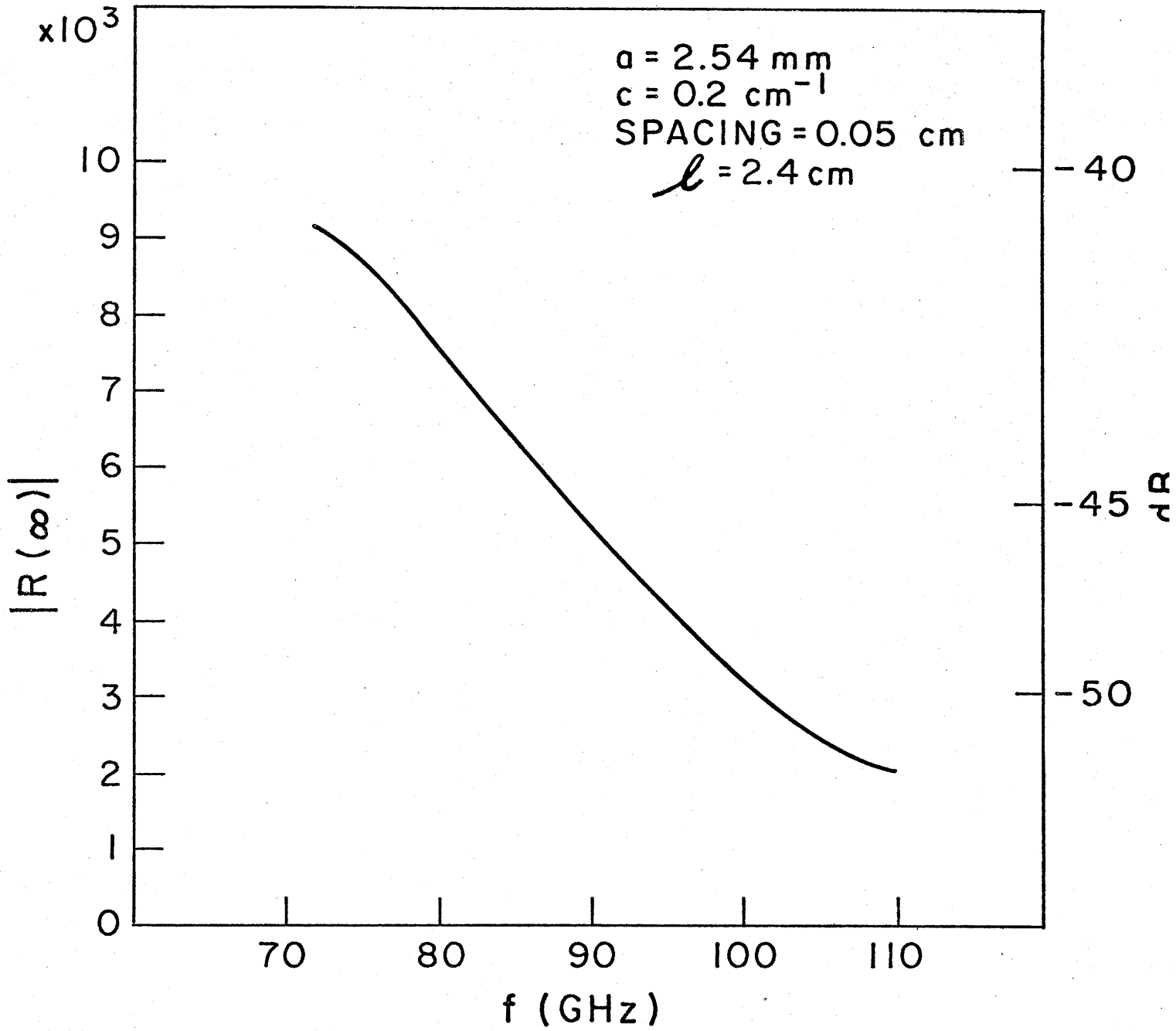


Fig. 7. Reflection coefficient vs. frequency.

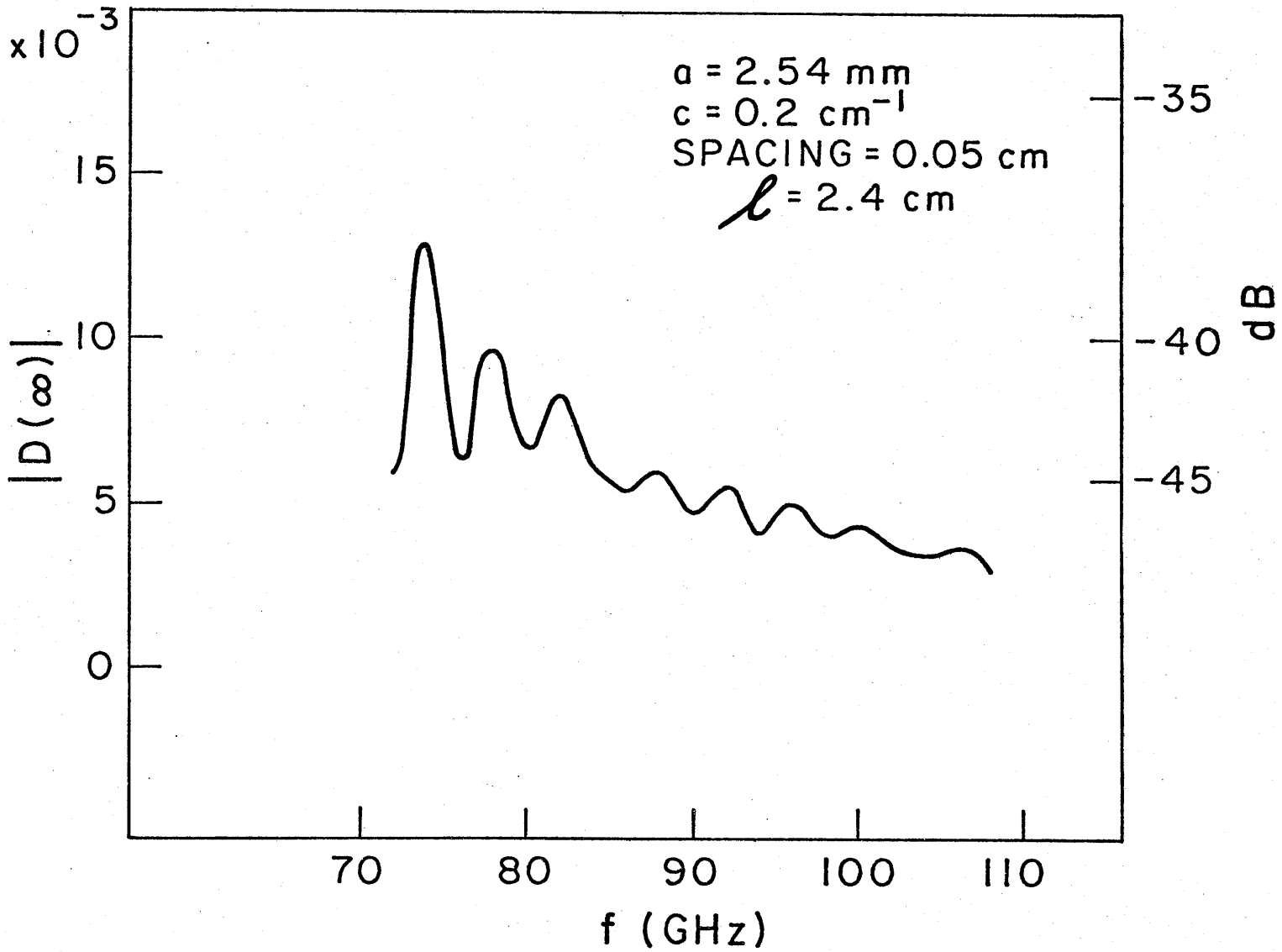


Fig. 8. Directivity vs. frequency.

IV. Conclusions

1. As long as the curvature is small enough, it has negligible effect on the reflection coefficient and the directivity of the coupler.
2. Despite all approximations in the calculations the fields on both lines are always in quadrature.
3. There are substantial second order effects which affect the bandwidth of the coupler, and, at least in part, increasing the coupling seems to increase the bandwidth.
4. It seems to be the case that the same effect on the bandwidth of tight coupling at 0.5mm spacing can also be achieved by decreasing the value of ϵ_r and increasing the spacing.

All the above mentioned conclusions are applicable to the metal covered case and are not necessarily valid for the case of dielectric lines in air.

V. List of Topics for Future Work

As time allows, the following problems will be tackled:

1. Treating the case where the dielectric lines are in air.
2. Solving the radiation problem due to curvature.
3. Calculating the reflection due to the discontinuity of the curvature.

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APPENDIX

The transmission line equations will be used in order to solve for the fields and the coupling coefficients for the linear as well as the curved guide sections. In this analogy each line is to correspond to a slab. The propagation constant is β_+ for the symmetrical modes and β_- for the antisymmetrical modes. The actual fields along each line will be represented by i and v as shown in Fig. 9. Because of the symmetry of the problem the mutual coupling coefficients L_{12} , L_{21} will be equal, as will C_{12} , C_{21} and hence the coupled Eqns. of this symmetrical system become:

$$\frac{\partial i_1}{\partial z} = -C \frac{\partial v_1}{\partial t} - \frac{1}{2} C_{12} \frac{\partial}{\partial t} (v_1 - v_2)$$

$$\frac{\partial i_2}{\partial z} = -C \frac{\partial v_2}{\partial t} - \frac{1}{2} C_{12} \frac{\partial}{\partial t} (v_2 - v_1)$$

(A-1a)

$$\frac{\partial v_1}{\partial z} = -L \frac{\partial i_1}{\partial t} - L_{12} \frac{\partial i_2}{\partial t}$$

$$\frac{\partial v_2}{\partial z} = -L \frac{\partial i_2}{\partial t} - L_{12} \frac{\partial i_1}{\partial t}$$

Here, L and C represent the self coupling coefficients and L_{12} and C_{12} represent the mutual coupling coefficients.

Assuming $e^{j\omega t}$ time dependence and setting

$$V = v_1 + v_2 ; \quad v = v_1 - v_2$$

$$I = i_1 + i_2 ; \quad i = i_1 - i_2$$

(A-1b)

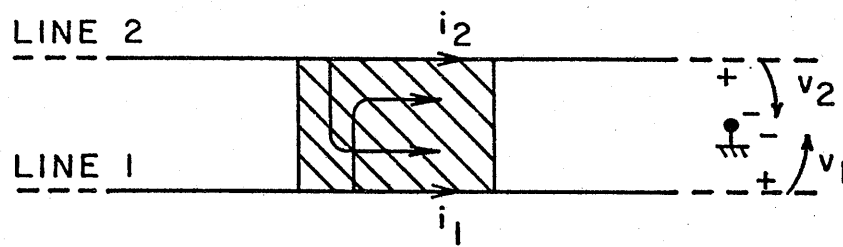


Fig. 9. Representation of two coupled lines.

where I and V stand for the symmetrical modes, i and v stand for the antisymmetrical modes, the coupled equations decompose to

$$\frac{dI}{dz} = -j\omega CV$$

$$\frac{di}{dz} = -j\omega(C + C_{12})v$$

$$\frac{dV}{dz} = -j\omega(L + L_{12})I \quad (\text{A-2})$$

$$\frac{dv}{dz} = -j\omega(L - L_{12})i$$

which can be written as

$$\frac{d^2 I}{dz^2} = -\omega^2 LC \left(1 + \frac{L_{12}}{L}\right) I \quad (\text{A-3.a})$$

$$\frac{d^2 i}{dz^2} = -\omega^2 LC \left(1 + \frac{C_{12}}{C}\right) \left(1 - \frac{L_{12}}{L}\right) i \quad (\text{A-3.b})$$

If the single line, in isolation, supports a single mode with propagation constant β_0 given by

$$\beta_0^2 = \omega^2 LC \quad (\text{A-4})$$

then the coupled system will support two types of modes, namely symmetrical and antisymmetrical, with two different propagation constants β_+ and β_- written respectively as

$$\beta_+^2 = \beta_0^2 \left(1 + \frac{L_{12}}{L}\right) \quad (\text{A-5})$$

$$\beta_-^2 = \beta_0^2 \left(1 + \frac{C_{12}}{C}\right) \left(1 - \frac{L_{12}}{L}\right) \quad (\text{A-6})$$

Knowing that β_+ and β_- are also given by

$$\beta_+ = \beta_0 + \Delta\beta$$

$$\beta_- = \beta_0 - \Delta\beta$$

and by means of Eqs. (A-5) and (A-6) the mutual coupling coefficients can be evaluated:

$$\frac{L_{12}}{L} \approx 2 \frac{\Delta\beta}{\beta_0} \quad (\text{A-7a})$$

$$C_{12} \approx 0 \quad (\text{A-7b})$$

where $\Delta\beta$ is the shift in the propagation constant due to coupling and is equal to

$$\Delta\beta = \frac{p_0^2 h_0^2 e^{-2h_0 w}}{\beta_0 k_0^2 (\epsilon_r - 1) \left(\frac{a}{2} + \frac{1}{h_0} \right)} \quad (\text{A-8})$$

for the linear section shown in Fig. 9. For the curved sections, sections (1) and (3), $\Delta\beta$ becomes a function of z and Eqn. (A-8) becomes

$$\Delta\beta = \frac{p_0^2 h_0^2 e^{-2h_0 (w + cz^2)}}{\beta_0 k_0^2 (\epsilon_r - 1) \left(\frac{a}{2} + \frac{1}{h_0} \right)} \quad (\text{A-9})$$

where cz^2 , in the exponential term, stands for the parabolic form of the curved sections.

Substituting (A-4), (A-7), (A-8) and (A-9) in (A-3) we obtain the field differential equations over the three sections of Fig. 10. Using the perturbation technique to get an approximate solution correct to the order of $\Delta\beta$ for the symmetric and antisymmetric modes of the curved

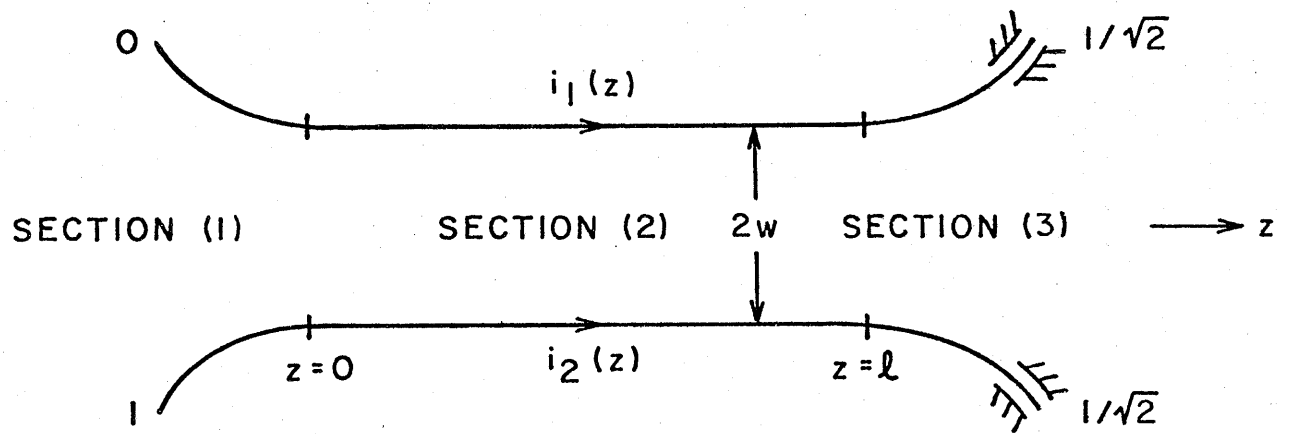


Fig. 10. Dielectric Directional Coupler.

sections and imposing the boundary conditions on the junctions $z = 0$ and $z = \ell$ we obtain:

Section 1:

Field amplitude differential equations

$$\frac{d^2 I}{dz^2} + \beta_0^2 I = -M e^{-2h_0 cz^2} I$$

$$\frac{d^2 i}{dz^2} + \beta_0^2 i = M e^{-2h_0 cz^2} i$$

$$\frac{d^2 V}{dz^2} + \beta_0^2 V = -M e^{-2h_0 cz^2} V$$

$$\frac{d^2 v}{dz^2} + \beta_0^2 v = M e^{-2h_0 cz^2} v$$

where

$$M = 2\beta_0 \Delta\beta = \frac{2p_0^2 h_0 e^{-2h_0 \omega}}{k_0^2 (\epsilon_r - 1) \left(\frac{a}{2} + \frac{1}{h_0}\right)}$$

Solutions

$$I = e^{-j\beta_0 z} + \frac{M}{2j\beta_0} e^{-j\beta_0 z} \int_{-\infty}^z e^{-2h_0 cz^2} dz - \frac{M}{2j\beta_0} e^{j\beta_0 z} \int_0^z e^{2j\beta_0 z - 2h_0 cz^2} dz \quad (\text{A-10})$$

$$i = e^{-j\beta_0 z} - \frac{M}{2j\beta_0} e^{-j\beta_0 z} \int_{-\infty}^z e^{-2h_0 cz^2} dz + \frac{M}{2j\beta_0} \int_0^z e^{-2j\beta_0 z - 2h_0 cz^2} dz$$

$$+ j\psi e^{-2j\beta_0 \ell} e^{j\beta_0 z}$$

where ψ is given in Eq. (A-16).

By means of Eqn. (A-1b), that is by addition and subtraction of the previous results we obtain

$$\begin{aligned}
 i_1 &= e^{-j\beta_0 z} - \psi e^{j\beta_0(z-2\ell)} \sin(2\Delta\beta\ell) \quad (A-11) \\
 i_2 &= -j\Delta\beta e^{-j\beta_0 z} \int_{-\infty}^z e^{-2h_0 cz^2} dz + j\Delta\beta e^{j\beta_0 z} \int_0^z e^{-2j\beta_0 z - 2h_0 cz^2} dz \\
 &\quad - j\psi e^{j\beta_0(z-2\ell)} \cos(2\Delta\beta\ell)
 \end{aligned}$$

Section 2:

Field amplitude diff. eqns.

$$\frac{d^2 I}{dz^2} + \beta_+^2 I = 0$$

$$\frac{d^2 V}{dz^2} + \beta_+^2 V = 0$$

$$\frac{d^2 i}{dz^2} + \beta_-^2 i = 0$$

$$\frac{d^2 v}{dz^2} + \beta_-^2 v = 0$$

Solutions

$$I = i_1 + i_2 = (1-j\phi)e^{-j\beta_+ z} - j\psi e^{-2j\beta_+ \ell} e^{j\beta_+ z} \quad (A-12)$$

$$i = i_1 - i_2 = (1+j\phi)e^{-j\beta_- z} + j\psi e^{-2j\beta_- \ell} e^{j\beta_- z}$$

hence

$$i_1 = e^{-j\beta_0 z} [\cos(\Delta\beta z) - \phi \sin(\Delta\beta z)] + \psi e^{j\beta_0(z-2\ell)} \sin[\Delta\beta(z-2\ell)] \quad (\text{A-13})$$

$$i_2 = -je^{-j\beta_0 z} [\sin(\Delta\beta z) + \phi \cos(\Delta\beta z)] - j\psi e^{j\beta_0(z-2\ell)} \cos[\Delta\beta(z-2\ell)]$$

Section 3:

Filed amplitude diff. equations, same as in Section I.

Solutions

$$I = i_1 + i_2 = (1 - j\phi) e^{-j\beta_0(z-\ell)} e^{-j\beta_0 \ell} \left[1 + \frac{M}{2j\beta_0} \int_0^{z-\ell} e^{-2h_0 cz^2} dz \right] \\ - \frac{M}{2j\beta_0} e^{-j\beta_0 \ell} e^{j\beta_0(z-\ell)} \int_{\infty}^{z-\ell} e^{-2j\beta_0 z - 2h_0 cz^2} dz \quad (\text{A-14})$$

$$i = i_1 - i_2 = (1 + j\phi) e^{-j\beta_0(z-\ell)} e^{-j\beta_0 \ell} \left[1 - \frac{M}{2j\beta_0} \int_0^{z-\ell} e^{-2h_0 cz^2} dz \right] \\ + \frac{M}{2j\beta_0} e^{-j\beta_0 \ell} e^{j\beta_0(z-\ell)} \int_{\infty}^{z-\ell} e^{-2j\beta_0 z - 2h_0 cz^2} dz ,$$

from which the actual fields can be evaluated and found to be

$$i_1 = e^{-j\beta_0 z} [\cos(\Delta\beta \ell) - \phi \sin(\Delta\beta \ell)] - e^{-j\beta_0 z} \sin(\Delta\beta \ell) \Delta\beta \int_0^{z-\ell} e^{-2h_0 cz^2} dz \\ + e^{j\beta_0(z-2\ell)} \sin(\Delta\beta \ell) \cdot \Delta\beta \int_{\infty}^{z-\ell} e^{-2j\beta_0 z - 2h_0 cz^2} dz \quad (\text{A-15})$$

$$i_2 = -je^{-j\beta_0 z} [\sin(\Delta\beta \ell) + \phi \cos(\Delta\beta \ell)] - je^{-j\beta_0 z} \cos(\Delta\beta \ell) \Delta\beta \int_0^{z-\ell} e^{-2h_0 cz^2} dz \\ + je^{j\beta_0(z-2\ell)} \cos(\Delta\beta \ell) \cdot \Delta\beta \int_{\infty}^{z-\ell} e^{-2j\beta_0 z - 2h_0 cz^2} dz$$

where

$$\phi = \Delta\beta \int_0^{\infty} e^{-2h_0 cz^2} dz = \Delta\beta \frac{\pi}{\sqrt{8h_0 c}}$$

and

$$\begin{aligned}\psi &= \Delta\beta \int_0^{\infty} e^{-2j\beta_0 z - 2h_0 cz^2} dz \\ &= \Delta\beta \sqrt{\frac{\pi}{8h_0 c}} \left[e^{-\beta_0^2/2h_0 c} - j \frac{2}{\sqrt{\pi}} \text{Daw}\left(\frac{\beta_0}{\sqrt{2h_0 c}}\right) \right]\end{aligned}\quad (\text{A-16})$$

$\text{Daw}(x)$ is called Dawson's integral and it is defined as

$$\text{Daw}(x) = e^{-x^2} \int_0^x e^{t^2} dt .$$

By now we are ready to impose the design requirement on $i_1(z)$ and $i_2(z)$ of Section 3, at $z = \infty$, which is

$$|i_1(z=\infty)| = |i_2(z=\infty)|$$

By satisfying the above condition, the length corrections due to the end effect of the curved sections will be obtained as follows:

$$\cos(\Delta\beta\ell) - 2\phi \sin(\Delta\beta\ell) = \sin(\Delta\beta\ell) + 2\phi \cos(\Delta\beta\ell)$$

which reduces to

$$\tan(\Delta\beta\ell) \approx 1 - 4\phi \quad (\text{A-17})$$

substituting for ℓ and $L_{3\text{dB}}$ in Eqn. (A-17) by

$$\ell = L_{3\text{dB}} - L_c$$

where $L_{3\text{dB}}$ and L_c represent the uncorrected 3dB length of the coupler and the length correction respectively, with

$$L_{3\text{dB}} = \frac{\pi}{4\Delta\beta} , \quad (\text{A-18})$$

we obtain

$$L_c = \delta l + \delta l' = 2 \frac{\phi}{\Delta\beta} = \frac{\pi}{\sqrt{2h_0 c}} \quad (\text{A-19})$$

δl and $\delta l'$ represent the correction at the two ends of the coupler.

Because of symmetry δl and $\delta l'$ are equal.

The reflection coefficient and directivity can be readily extracted from Eqn. (11) by taking $z = -\infty$.