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THE USE OF EFFECTIVE APERTURE RELATIONS FOR THE CALCULATION OF THE INPUT CONDUCTANCE OF ELECTRICALLY SMALL ANTENNA

by

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I. Introduction

In determining the input conductance of an electrically short dipole (probe) or any inefficient radiator, one usually encounters difficulties in maintaining the high degree of accuracy with regard to the phase of the current at the feed point, which is essential for an accurate determination of the conductance. In this note an alternative method of computing the conductance via a procedure based upon the effective aperture of the antenna is proposed. This method circumvents the stringent accuracy normally required for the phase of the input current. Using this procedure, a simple expression for the input conductance of a probe is derived and is shown to give results which are in good agreement with those of King [1, Chap. II, Sec 31.]. Our approach, however, provides an explicit functional dependence of the input conductance with regard to the probe length and radius. In addition, we discuss the use of our input conductance formulation to provide an independent check to the numerical solutions for antennas of complicated geometries.

II. Formulation

Consider the system shown in Figure 1a, in which an arbitrary antenna with load $Y_L = G_L - iB_L$ is illuminated by a uniform plane wave incident at the angles $\theta = \theta_i$ and $\phi = \phi_i$. The incident electric field is given by $\vec{E}^{inc}$ and the propagation constant is expressed in terms of the vector, $\vec{k}$. For simplicity we assume the polarizations of the incident plane wave and the antenna to be matched. Shown in Figure 1b, is an equivalent circuit for the antenna system in Figure 1a, which includes an equivalent Norton current source and admittance, $E^{inc} I_{sc}(\theta_i, \phi_i)$ and Y_A respectively, along with the load admittance, Y_L. The Norton equivalent current source is simply the short circuit ($Y_L = \infty$) current at the load point for the antenna in Figure 1a. Note that $I_{sc}(\theta_i, \phi_i)$ is the short circuit current per unit of incident electric field. The source admittance, $Y_A = G_A - iB_A$, may be identified as the input admittance of the antenna in the transmitting mode. Note; throughout this communication, the assumed time variation is $e^{-i\omega t}$.

The maximum power which may be dissipated in the load admittance, Y_L, in Figure 1b, occurs for the conjugate match, $Y_L = Y_A^*$, and is given by

$$P_{max} = \frac{1}{2G_A} \frac{|I_{sc}(\theta_i, \phi_i)|^2}{2} |E^{inc}|^2 \tag{1}$$

On the other hand, the power dissipated in the matched load, Y_L, in Figure 1a may be written in terms of the effective aperture of the antenna, [2, eq.1-2 and 1-8], i.e.,

$$P_{max} = \frac{1}{2\eta} |E^{inc}|^2 \frac{\lambda^2}{4\pi} G(\theta_i, \phi_i) \tag{2}$$

where $G(\theta_i, \phi_i)$ is the gain (power) of the antenna (as referred to an isotropic antenna) as a function of the incident angles θ_i and ϕ_i. $\lambda = 2\pi/k$ is the wavelength in the surrounding medium of the incident plane wave and η is the intrinsic impedance of the surrounding medium.

Since (1) and (2) both predict the maximum possible power dissipated in the load admittance, Y_L, in Figures 1a and 1b, respectively, under the same conditions, they may be equated to yield,

$$G_A\left[\frac{1}{4\pi} G(\theta_i,\phi_i)\right] = \frac{\eta}{4\lambda^2} |I_{sc}(\theta_i,\phi_i)|^2 \quad (3)$$

which upon integration over the solid angle 4π and the subsequent use of the identity,

$$\frac{1}{4\pi}\int_0^{2\pi} d\phi_i \int_0^{\pi} d\theta_i \left[\sin\theta_i\, G(\theta_i,\phi_i)\right] = 1 \quad (4)$$

gives the following equation for the input conductance.

$$G_A = \frac{k^2\eta}{(4\pi)^2}\int_0^{2\pi} d\phi_i \int_0^{\pi} d\theta_i \left[\sin\theta_i |I_{sc}(\theta_i,\phi_i)|^2\right] \quad (5)$$

Thus the input conductance for any antenna can be obtained once $I_{sc}(\theta_i,\phi_i)$ is known for all incident angles of the uniform plane wave (i.e., $0 \le \phi_i \le 2\pi$ and $0 \le \theta_i \le \pi$). Again noting that $I_{sc}(\theta_i,\phi_i)$ is the short circuit current at the antenna input terminals due to a unit ($E^{inc} = 1.0$ V/m) incident uniform plane wave with a polarization matching that of the antenna.

The expression for input conductance of an arbitrary antenna in (5) was derived through the use of the effective aperture of the antenna, which may itself be derived from reciprocity considerations. Thus (5) is basically a statement of the reciprocity between an antenna used as a radiating element and a receiving element. However, since this expression involves only the magnitude of the receiving current, we no longer need to maintain the extreme accuracy with regard to the phase, that is required in the direct solution of the transmitting antenna current, in order to obtain a sufficiently accurate result for G_A.

III. Analytical Examples

To enhance the credibility of the formula for input conductance in (5), the following examples are offered.

A. The infinitely long cylinder

The current induced on an infinitely-long, thin (ka <<1) cylinder concentric with the z-axis due to a uniform plane wave of unit amplitude (E^{inc} = 1.0 V/m), polarized in the same plane as the cylinder and incident at an angle θ_i, with the z-axis is given by,

$$I_{s\infty}(\theta_i, z) = -\frac{4}{k\eta} \frac{e^{ikz\cos\theta_i}}{\sin\theta_i \, H_o^{(1)}(ka\sin\theta_i)} \tag{6}$$

where a is the radius of the cylinder and $H_o^{(1)}$ is the zero order Hankel function of the first kind. Using (6) for the short circuit current in (5) and making the substitution $x = ka\sin\theta_i$, the input conductance may be expressed as,

$$G_A = 4\frac{ka}{\pi\eta} \int_o^{ka} \frac{1}{x\sqrt{(ka)^2 - x^2}} \frac{dx}{J_o^2(x) + Y_o^2(x)} \tag{7}$$

where J_o and Y_o are the zero order Bessel and Neumann functions, respectively. Since the cylinder is infinite in length the input conductance is independent of the feed point. Eq. (7) is an expression for the input conductance of an infinitely long cylinder equivalent to that which would be obtained through the more conventional approach of finding the real part of the input current for the driven cylinder [3].

B. The finite length cylinder

A formula for the current on a thin (ka << 1) hollow cylinder of length 2h, illuminated by a uniform plane wave of arbitrary incidence has been given by Chang, Lee, and Rispin in [4] as,

$$I_R(\theta_i,z) = E_\theta^i \{V(\theta_i,z)$$

$$+ [\Delta(\pi-\theta_i,h-z)V(\theta_i,h) + \frac{R(\pi,h-z)}{R(\pi,2h)} C(\pi-\theta_i)] \; U(h-z)$$

$$+ [\Delta(\theta_i,h+z)V(\pi-\theta_i,h) + \frac{R(\pi,h+z)}{R(\pi,2h)} C(\theta_i)] \; U(h+z) \} \tag{8}$$

The reader is directed to the above reference for the definitions of the terms in (8). Using (8) for the short circuit current, $I_{sc}(\theta_i,\phi_i)$ in (5), the input conductance to a finite length cylinder may be found at any z (not approaching either end, see [4]) along the cylinder. Figure 2 shows the input conductance as calculated from (5) and (8) for a center fed finite length thin cylinder (where $\Omega = 2\ \ell n[2h/a] = 10.0$) as a function of the normalized half-length, $kh = 2\pi h/\lambda$. For comparison purposes, the input conductance as predicted by King [1, Chap. II, Sec. 30] is also shown in Figure 2. The agreement between the two curves is quite good. We may view Figure 2 not only as a demonstration of the ability of (5) to calculate antenna conductance, but more importantly in this case as a critical check (necessary, although admittedly not sufficient) for the short circuit current formula in (8).

IV. Input Conductance of a Short Finite Length Thin Cylindrical Antenna

The examples in Section III demonstrated mainly the validity of our formula for calculating antenna input conductance in (5), even though in those examples it provided no particular advantage over the direct approach of solving the transmitting antenna problem. In this section, however, we will use (5) to calculate the input conductance to an electrically very short cylindrical antenna, which to our knowledge has been treated only by a curve fitting procedure on an ad-hoc basis for several values of $\Omega = 2\ \ell n(2h/a)$ by King [1, Chap II, Sec 31].

Briefly, the difficulty encountered in the calculation of the conductance of a short dipole (or any antenna whose dimensions are much smaller than the operating wavelength) is that the input susceptance to the antenna is several orders of magnitude larger than the input conductance. Thus in the direct formulation of the problem, the amplitude as well as the phase of the transmitting current has to be extremely accurate in order to avoid the input conductance being swamped out by the input susceptance.

In a recent paper by Rispin and Chang [5], it is shown that the Wiener-Hopf approach to linear antenna problems is not restricted to long ($kh > 1$) and thin ($ka << 1$) antennas, but instead it is primarily dependent upon the condition $\Omega = 2\ \ell n(2h/a) >> \ell n(2kh)$. The parameter, Ω, being the same as that used by King in his classical investigation into linear antennas [1]. Thus with the above restriction on Ω satisfied, we may confidently use the receiving current formula of Chang, et al. [4] given again in (8) for the short circuit current on a short dipole. In Figure 3, the input conductance of a short cylindrical antenna as calculated from (5) and (8) for a value of $\Omega = 10.0$ is shown as a function of the normalized half length $kh = 2\pi(h/\lambda)$. For comparison purposes the input conductance for the same parameters as determined by King

[1, Chap. II, Sec 31.] through the interpolation of graphical data is also shown in Figure 3. It is apparent the agreement between our conductance and the conductance predicted by King is excellent. Unlike King's formula, however, our result need not be modified for different values of Ω. Furthermore, since the receiving current, $I_R(\theta_i, z)$, in (8) exhibits the angular dependence of approximately $\sin\theta_i$, we may write the receiving current formula in (8) for short dipoles as,

$$I_R(\theta_i, z) \simeq I_R(\pi/2, z)\ \sin\theta_i \quad ; \qquad kh \lesssim 1 \tag{9}$$

Using (9) for the short circuit current in (5), the integrations may be carried out analytically, yielding the expression for the conductance of a short ($kh \lesssim 1$) thin ($\Omega = 2\ln[2h/a] >> \ln[2kh]$) cylindrical antenna given by,

$$G_A \simeq \eta \left(\frac{k}{4}\right)^2 \left|I_R\ (\pi/2, z)\right|^2 \tag{10}$$

Conductance values calculated from (10) for a short cylindrical antenna in which $\Omega = 2\ln(2h/a) = 10.0$ as a function of the normalized half length, $kh = 2\pi(h/\lambda)$ is given in Figure 3. The correspondence between the conductance values obtained through the numerical integration of (5) with (8) and those from the simple expression in (10) is still very good. These results clearly demonstrate the fact that, in contrast with the direct approach, one needs not maintain the same order of accuracy in our method to obtain a relatively accurate input conductance.

V. Conclusion

Relationships concerning the effective aperture of an antenna have to be utilized as a mathematical artifice to yield an expression for the input conductance of the antenna. This formula, given in (5), is particularly powerful when applied to electrically short antennas and probes. Even when the antenna is not electrically short, we have shown that our conductance formula can be used as an independent check between the receiving short circuit current and the antenna input conductance. This check may be applied to wholly numerical solutions, such as the moment method applied to antennas of complicated geometries. Although the computational procedures for the transmitting and receiving antenna are similar, one can show without difficulty that an error in the impedance or moment matrix will yield a different value of input conductance with the application of our conductance formula.

References

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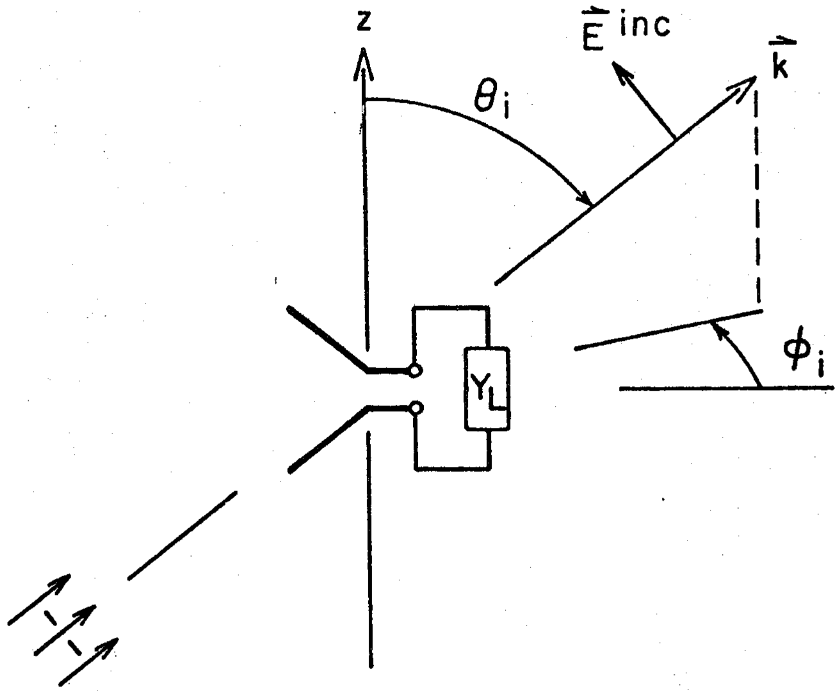
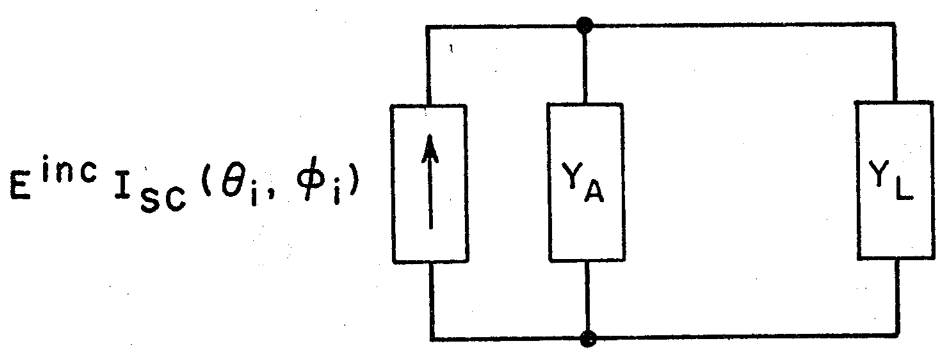


(a)



(b)

Figure 1. (a) An arbitrary antenna illuminated by a uniform plane wave (polarizations are matched).

(b) Equivalent circuit representation.

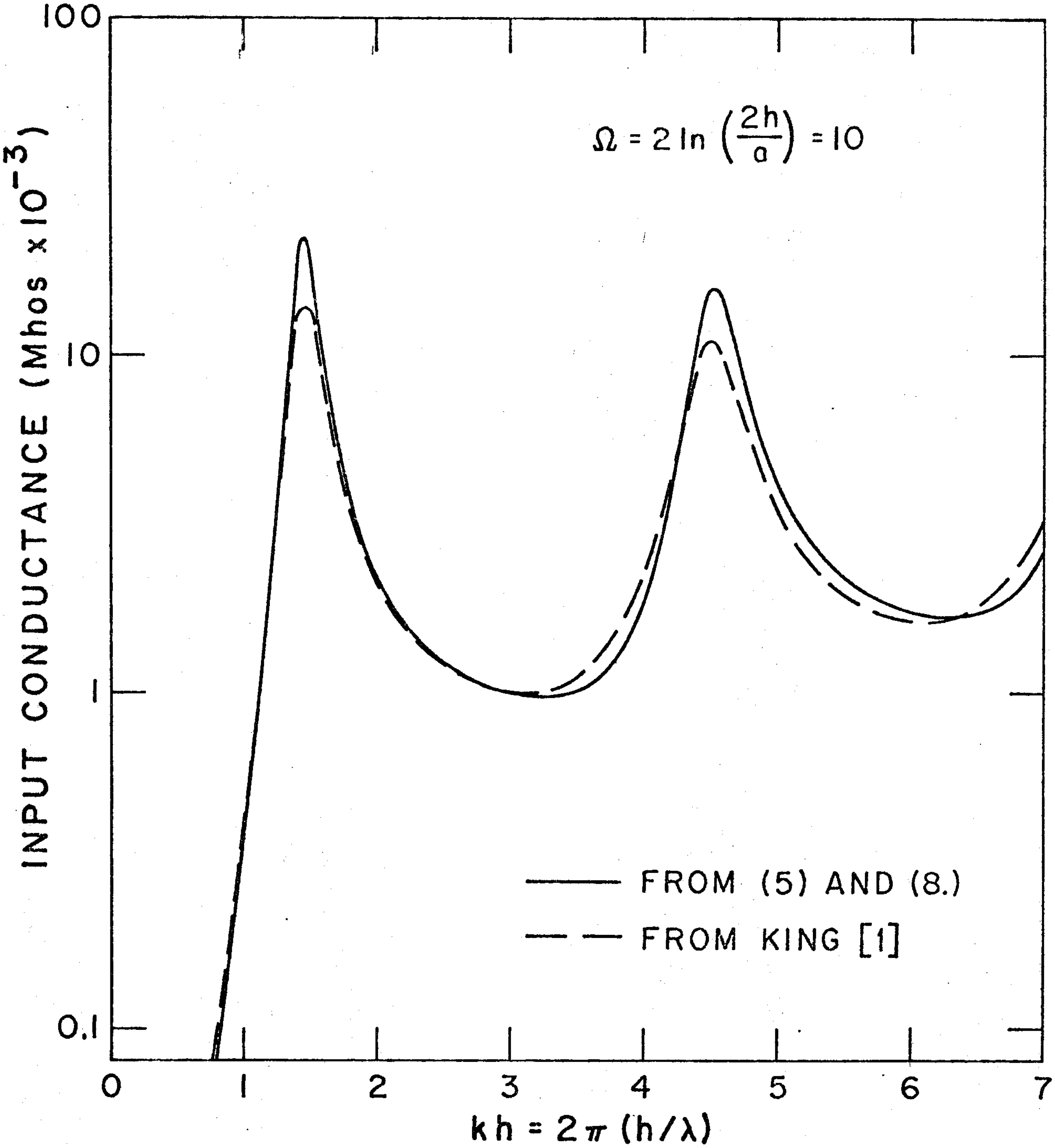


Figure 2. Input conductance of a finite length thin cylindrical antenna.

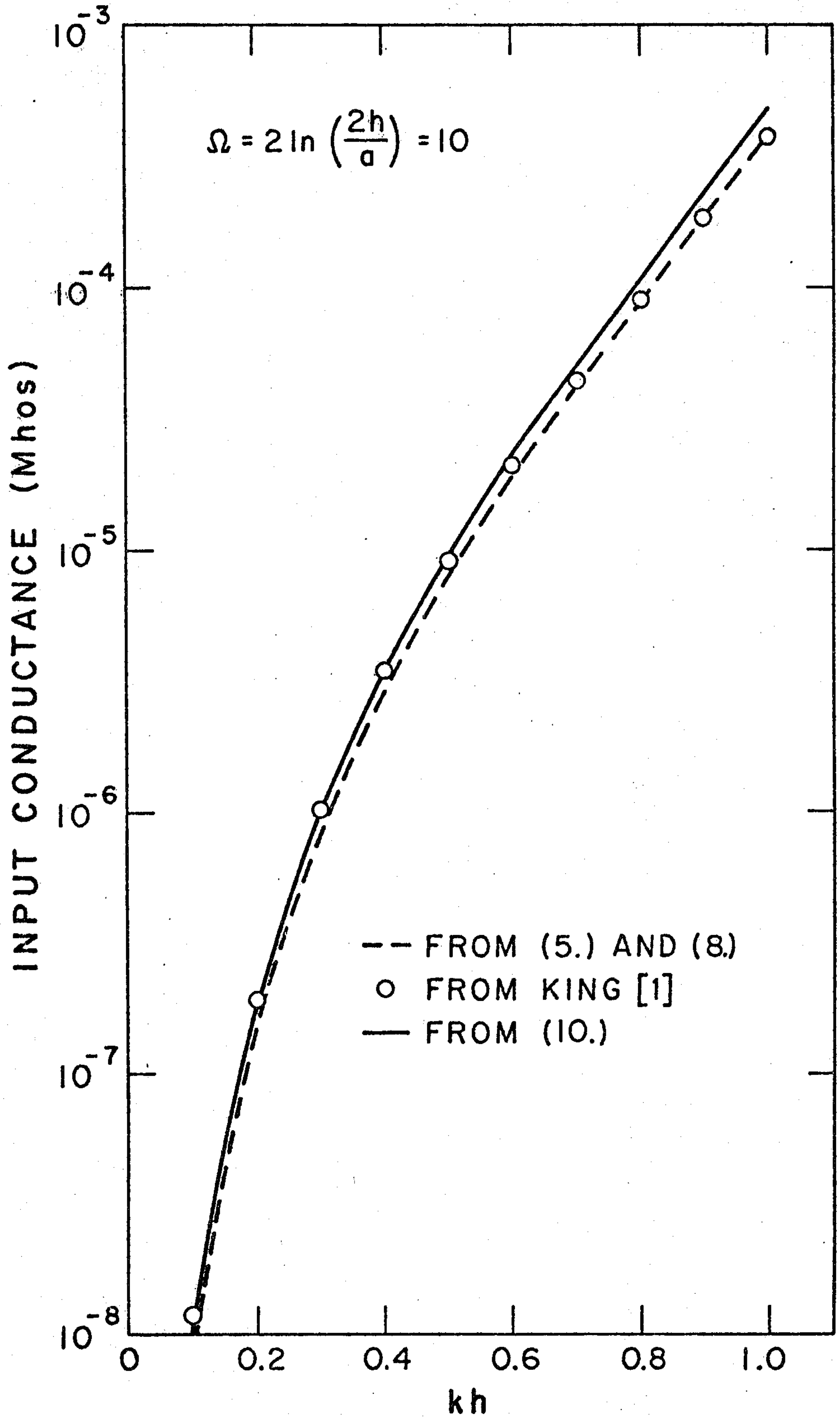


Figure 3. Input conductance of a short cylindrical antenna.