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Schur--Weyl Duality for Unipotent Upper Triangular Matrices

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by

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Schur–Weyl duality for unipotent upper triangular matrices

Thesis directed by Nathaniel Thiem

Schur–Weyl duality is a fundamental framework in combinatorial representation theory. It intimately relates the irreducible representations of a group to the irreducible representations of its centralizer algebra. We investigate the analogue of Schur–Weyl duality for the group of unipotent upper triangular matrices over a finite field. In this case, the character theory of these upper triangular matrices is "wild" or unattainable. Thus we employ a generalization, known as supercharacter theory, that creates a striking variation on the character theory of the symmetric group with combinatorics built from set partitions. In this thesis, we present a combinatorial formula for calculating a restriction and induction of supercharacters based on statistics of set partitions and seashell inspired diagrams. We use these formulas to create a graph that encodes the decomposition of a tensor space, and develop an analogue of Young tableaux, known as shell tableaux, to index paths in this graph. These paths also help determine a basis for the maps that centralize the action of the group of unipotent upper triangular matrices. We construct a part of this basis by determining copies of certain modules inside a tensor space to construct projection maps onto supermodules that act on a standard basis.

Dedication

To the mathematician that has inspired me the most throughout my life: my mother. If I turn out to be half as smart, half as patient, and half as kind, I would consider myself successful.

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Lastly, I owe many thanks to my wonderful family and friends for supporting me throughout my graduate career. I express my love and thanks to my partner, Caleb, who has stuck by my side during this challenging time.

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Chapter 1

Introduction

Schur–Weyl duality forms an archetypal situation in combinatorial representation theory involving two actions that complement each other. In the basic setup, a G-module M of a finite group G is tensored together k times to form the tensor space

$$M^{\otimes k} = \underbrace{M \otimes \cdots \otimes M}_{k \text{ factors}}.$$

The commuting actions of G and its centralizer algebra $Z_k = \mathrm{End}_G(M^{\otimes k})$ on $M^{\otimes k}$ produce a decomposition

$$M^{\otimes k} \cong \bigoplus_{\lambda} G^{\lambda} \otimes Z_k^{\lambda} \quad \text{as a } (G, Z_k)\text{-bimodule}$$

where the G^{λ} are irreducible G-modules and the Z_k^{λ} are irreducible Z_k-modules. This bimodule decomposition intimately relates the irreducible representations of G with the irreducible representations of Z_k.

In the classical situation, the general linear group $GL_n(\mathbb{C})$ of $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers acts on the tensor space $V^{\otimes k}$ of an n dimensional vector space V, and its centralizer algebra is the symmetric group S_k on the k tensor factors. More recently, the study of new versions of Schur–Weyl duality has led to many remarkable discoveries about algebras of operators on tensor space that are full centralizers of each other. For example,

(1) the Brauer algebra is the centralizer of the symplectic and orthogonal groups acting on tensor space $(\mathbb{C}^n)^{\otimes k}$ [10];

(2) the partition algebra is the centralizer of the symmetric group acting on the tensor space $V^{\otimes k}$ of its permutation representation V [14].

My research focuses on a unipotent analogue of Schur–Weyl duality.

For a positive integer n and a power of a prime $q = p^r$, consider the finite group of unipotent $n \times n$ upper-triangular matrices

$$U_n = \left\{ \begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & 1 \end{bmatrix} \right\}$$

with ones on the diagonal and entries $*$ in the finite field $\mathbb{F}_q$ with q elements. Since U_n is a Sylow p-subgroup of $GL_n(\mathbb{F}_q)$, then every p-group of $GL_n(\mathbb{F}_q)$ is conjugate to a subgroup of U_n. Embedding every finite p-group in $S_n \subseteq GL_n(\mathbb{F}_q)$ as permutation matrices, it follows that every p-group is isomorphic to a subgroup of U_n. This is akin to how every finite group is isomorphic to a subgroup of S_n, so it is not unreasonable to hope that the representation theories of U_n and S_n have comparable structures.

Unlike the combinatorially rich representation theory of S_n [18], the representation theory of U_n is well-known to be intractable or "wild" [13]. Nevertheless, André [2, 3, 4, 5] and Yan [21] constructed a workable approximation that has been useful in studying Fourier analysis [12], random walks [7], and Hopf algebras [1]. In [12] Diaconis and Isaacs generalize this idea to arbitrary finite groups to develop the notion of supercharacter theory. Supercharacter theory approximates the character theory of a finite group by replacing conjugacy classes with certain unions of conjugacy classes called "superclasses" and irreducible characters with certain linear combinations of irreducible characters called "supercharacters".

We study a coarsening of Andre and Yan's traditional super-representation theory on U_n [9]

where there is a one-to-one correspondence between

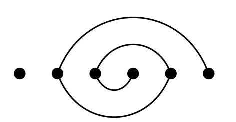
$$\left\{ \begin{array}{c} \text{supercharacters} \\ \\ \text{of } U_n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Set partitions of} \\ \\ \{1, 2, \ldots, n\} \end{array} \right\}.$$

It is becoming ever more apparent that the set partition combinatorics of this super-representation theory is analogous to the classical partition combinatorics of the representation theory of the symmetric group, but with some important differences.

In Chapter 3, we study the decomposition of $V^{\otimes k}$ where $V = \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1}$ as a U_n-supermodule. Much like the partition algebra, we have

$$V^{\otimes k} \cong \underbrace{(\mathrm{Ind}_{U_{n-1}}^{U_n} \mathrm{Res}_{U_{n-1}}^{U_n})^k}_{k \text{ times}}(\mathbb{1})$$

where the trivial supercharacter is restricted and induced k times. We provide a combinatorial formula calculating a restriction of supercharacters from U_n to U_{n-1} where the coefficients of the supercharacters of U_{n-1} are a product of powers of q and $q - 1$ based on statistics of set partitions and seashell inspired diagrams. For example, a shell formed by two set partitions is shown below



Using Frobenius reciprocity, we obtain a corresponding formula for inducing supercharacters. Together these formulas are known as branching rules. As opposed to the representation theory of the symmetric group, they depend on the embedding of U_{n-1} in U_n.

In Chapter 4, we use the branching rules to create a graph that encodes the decomposition of $V^{\otimes k}$ known as the Bratteli diagram. For the symmetric group, paths in the Bratteli diagram are indexed by a set of combinatorial objects called Young tableaux. We create an analogue of Young tableaux, known as shell tableaux, that is built from the combinatorics of the previous chapter using a generalization of shells. Next, we construct a bijection between shell tableaux and paths in the Bratteli diagram. When $q = 2$, we remove a condition on shell tableaux to produce a bijection with weighted paths in the Bratteli diagram. In contrast with the symmetric group, these weights account for the multiplicities in our Bratteli diagram.

In Chapter 5, we apply our results to the theory of Schur–Weyl duality. Since we are approximating by supercharacters, we consider a subalgebra of the centralizer algebra that treats supermodules as irreducibles. We show that the decomposition of $V^{\otimes k}$ as a U_n-supermodule given by the Bratteli diagram produces a decomposition of this subalgebra. We also prove that the dimension of this subalgebra is the product of the weights of pairs in paths in the Bratteli diagram, which is a polynomial in q. Moreover, these paths also help index a basis of the centralizer subalgebra. We construct a part of this basis by determining copies of certain modules inside of $V^{\otimes k}$ to compute projection maps onto supermodules that act on a standard basis of $V^{\otimes k}$. While determining these projections is generally unattainable, it seems to be tractable in this case. We illustrate this when $n = 2$ and $q = 2$ to produce a basis for the full centralizer algebra.

Our construction of a basis is one of the first attempts to carve out a framework for studying a well-defined piece of a centralizer algebra, leading to surprising findings about the super-representation theory of U_n. In particular, we have shown the standard local branching rules of U_n can largely be realized explicitly as modules, which is fairly unexpected in general. The shell combinatorics developed from this may help compute in other algebraic structures related to the supercharacter theory of U_n, such as the Hopf algebra of symmetric functions in noncommuting variables. It also turns out that when the supercharacters coincide with irreducible characters, the dimensions of the centralizer algebras are interesting sequences found in the On-Line Encyclopedia of Integer Sequences (OEIS). However, the dimensions of centralizer subalgebras lead to numerous new combinatorial sequences yet to be explored.

Chapter 2

Preliminaries

We develop background material on character theory, supercharacter theories, and set partition combinatorics with a focus on a supercharacter theory for the group of unipotent upper triangular matrices.

2.1 Character Theory

Let G be a finite group and V be a finite dimensional vector space over the field of complex numbers $\mathbb{C}$. A *representation* over $\mathbb{C}$ is a homomorphism $\rho : G \to GL(V)$ where $GL(V)$ is the group of invertible linear transformations of V. We say V is a left G*-module* with the action

$$gv = \rho(g)v \qquad g \in G,\ v \in V.$$

All groups have a *trivial representation* that sends each $g \in G$ to the identity transformation which maps every vector in V to itself. Furthermore, when G acts on a basis $\mathcal{B}$ for V the *permutation representation* of V is defined by $\rho(g)v = gv$ for all $v \in \mathcal{B}$. In particular, we can form the vector space $\mathbb{C}G$ over $\mathbb{C}$ whose elements are formal linear combinations

$$\sum_{g \in G} a_g g, \qquad a_g \in \mathbb{C}.$$

The permutation representation arising from the action of G on itself is known as the *regular representation.*

In a sense, understanding representations of a group G reduces to studying invariant subspaces of the regular representation. If V is a G-module and W is a subspace such that $\rho(g)(W) = W$

for all $g \in G$, then W is a *submodule* of V. The corresponding representation is known as a *subrepresentation*. We say a G-module is *irreducible* it contains no proper nonzero submodules. We also refer to the representation ρ as irreducible.

It is important to determine when two irreducible representations are isomorphic. A *G-module homomorphism* between two G-modules V and W is a linear transformation $\varphi : V \to W$ such that

$$\varphi(gv) = g\varphi(v) \qquad g \in G, v \in V.$$

A G-module homomorphism from V to itself is known as a *G-module endomorphism*, and a *G-module isomorphism* is a bijective G-module homomorphism. Schur's lemma characterizes G-module homomorphisms.

Proposition 2.1.1 (Schur's Lemma [16, Lemma 1.5])**.** *Let V and W be irreducible G-modules.*

(1) If $\varphi : V \to W$ is a G-module homomorphism then either φ is the zero map or an isomorphism.

(2) If $\varphi : V \to V$ is a G-module endomorphism then φ is a scalar multiple of the identity endomorphism.

Given a representation ρ of G, the *character* of G afforded by ρ (and its corresponding G-module V) is the function

$$\begin{aligned} \chi : G &\longrightarrow \mathbb{C} \\ g &\mapsto \operatorname{tr}(\rho(g)) \end{aligned}$$

where tr denotes the trace of a linear transformation with respect to a basis. If χ is a character of an irreducible representation, we say χ is an *irreducible character*. The set of irreducible characters of G is denoted as $\operatorname{Irr}(G)$.

We refer to the character afforded by the trivial representation as the *trivial character* $\mathbb{1}$. Similarly, the character χ_{reg} corresponding to the regular representation is called the *regular character*. The *degree* of a character χ is the value $\chi(1)$. Characters of degree 1 are called *linear characters*, and are equivalent to homomorphisms from G to the multiplicative group $\mathbb{C}^\times$ of $\mathbb{C}$.

A *class function* of G is a function from G to $\mathbb{C}$ that takes a constant value on each conjugacy class. Let $cl(G)$ denote the set of conjugacy classes of G and define

$$cf(G) = \mathbb{C}\text{-span}\{\varphi : G \to \mathbb{C} \mid \varphi(hgh^{-1}) = \varphi(g) \text{ for all } g, h \in G\}$$

to be the space of class functions. Note that characters are class functions, since if χ is a character of ρ,

$$\chi(hgh^{-1}) = \text{tr}(\rho(hgh^{-1})) = \text{tr}(\rho(h)\rho(g)\rho(h^{-1})) = \text{tr}(\rho(g)) = \chi(g)$$

for all $g, h \in G$.

The space of class functions has an inner product

$$\langle \varphi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\theta(g)},$$

where $\overline{\theta(g)}$ is the complex conjugate of $\theta(g)$. The next proposition shows that with respect to this inner product, the irreducible characters form an orthonormal basis for the space of class functions.

Proposition 2.1.2 ([16, Corollary 2.14, Theorem 2.8])**.** *For irreducible characters χ and ψ of G,*

$$\langle \chi, \psi \rangle = \delta_{\chi,\psi},$$

so that every class function φ of G can be uniquely expressed in the form

$$\varphi = \sum_{\chi \in \text{Irr}(G)} \langle \varphi, \chi \rangle \chi.$$

We say an irreducible character χ in the decomposition of a character ψ with $\langle \psi, \chi \rangle \neq 0$ is a *constituent* of ψ. In addition, the inner product can also be used to determine if a character is irreducible.

Proposition 2.1.3 ([16, Corollary 2.17])**.** *A character χ of G is irreducible if and only if*

$$\langle \chi, \chi \rangle = 1.$$

We now look at ways of building new representations of a group G from existing representations. A G-module V is the *direct sum* $V = U \oplus W$ of two submodules U and W if every element of $v \in V$ can be written uniquely as a sum

$$v = u + w \qquad u \in U, w \in W.$$

If U affords the character χ and W affords the character ψ, then the character of V is $\chi + \psi$.

Proposition 2.1.4 (Maschke's Theorem [16, Theorem 1.9])**.** *Let G be a finite group and V be a non-zero G-module. Then V can be decomposed as a direct sum*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

of irreducible submodules W_i of V.

Thus irreducible modules are the building blocks of all finite dimensional G-modules.

In addition, we introduce a product operation of two G-modules V and W. Choose bases $\{v_1, \ldots, v_n\}$ for V and $\{w_1, \ldots, w_m\}$ for W. The *tensor product* $V \otimes_{\mathbb{C}} W$ is the set of formal sums of the form

$$\sum a_{ij}(v_i \otimes w_j), \qquad a_{ij} \in \mathbb{C}.$$

When the field is clear we simply write $V \otimes W$. We define an action of G on $V \otimes W$ by setting

$$g(v_i \otimes w_j) = gv_i \otimes gw_j, \qquad g \in G,$$

and extending linearly to all of $V \otimes W$. If V and W afford characters χ and ψ respectively, then $V \otimes W$ affords the character $\chi \odot \psi$ given for $g \in G$ by

$$(\chi \odot \psi)(g) = \chi(g)\psi(g). \tag{2.1}$$

We can also relate representations of a group G to representations of its subgroups. If H is a subgroup of G and V is a G-module, then V is also an H-module denoted by $\mathrm{Res}^G_H(V)$. The afforded character, given by restricting the character χ of the G-module V to H, is called the *restriction* $\mathrm{Res}^G_H(\chi)$ of χ to H. For $h \in H$ and $\varphi \in cf(G)$, this produces a map

$$\mathrm{Res}^G_H : cf(G) \to cf(H)$$

defined by

$$\mathrm{Res}_H^G(\varphi)(h) = \varphi(h).$$

Conversely, we can build up a representation of a group G from a representation of its subgroup H to obtain a map from $cf(H)$ to $cf(G)$. The right action of G on itself makes $\mathbb{C}G$ into a right G-module. Given an H-module W, the *induced module* of W is the G-module

$$\mathrm{Ind}_H^G(W) \cong \mathbb{C}G \otimes_{\mathbb{C}H} W$$

with the action

$$g \cdot (g' \otimes w) = (gg') \otimes w \qquad g, g' \in G,\ w \in W.$$

If ψ is the character of W, the character afforded by $\mathbb{C}G \otimes_{\mathbb{C}H} W$ is known as the *induced character* $\mathrm{Ind}_H^G(\psi)$ of ψ to G, and a map

$$\mathrm{Ind}_H^G : cf(H) \to cf(G)$$

is defined by

$$\mathrm{Ind}_H^G(\theta)(g) = \frac{1}{|H|}\sum_{x \in G} \dot{\chi}(xgx^{-1}) \text{ where } \dot{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H, \\ 0 & \text{if } g \notin H, \end{cases}$$

for $\theta \in ch(H)$ and $g \in G$.

Proposition 2.1.5 (Frobenius reciprocity [16, Lemma 5.2])**.** *Let H be a subgroup of G. Suppose φ is a class function of G and θ is a class function of H. Then*

$$\langle \mathrm{Ind}_H^G(\theta), \varphi \rangle = \langle \theta, \mathrm{Res}_H^G(\varphi) \rangle.$$

When $H \cong G/N$ for a normal subgroup N of G we have an alternative way to use a representation of H to construct a representation of G. In this case $G \cong N \rtimes H$. Let W be an H-module and

$$z_N = \frac{1}{|N|}\sum_{n \in N} n \in Z(\mathbb{C}G),$$

the center of the group algebra $\mathbb{C}G$. Then $\mathbb{C}z_N H$ is a right H-module under right multiplication. The *inflation module*

$$\mathrm{Inf}_H^G(W) \cong \mathbb{C}z_N H \otimes_{\mathbb{C}H} W = \mathbb{C}z_N \otimes_{\mathbb{C}H} W$$

is a G-module with the action for $g = n'h' \in G$ given by

$$g \cdot (z_N h \otimes w) = (n'h'z_N h) \otimes w = z_N h'h \otimes w, \qquad h \in H,\ w \in W.$$

Its character is

$$\mathrm{Inf}_H^G(\psi) = \psi \circ \pi$$

where ψ is the character afforded by W and π is the projection $\pi : G \to G/N$. In this construction irreducible representations of H correspond to irreducible representations of G.

Proposition 2.1.6 ([16, Lemma 2.22])**.** *Let $H \cong G/N$ for a normal subgroup N of G. Then*

$$\mathrm{Inf}_H^G(\psi) \in \mathrm{Irr}(G) \text{ if and only if } \psi \in \mathrm{Irr}(H).$$

Note we can generalize representations to other algebraic structures. Let $\mathbb{F}$ be a field. An $\mathbb{F}$-*algebra* is an $\mathbb{F}$-vector space that is also a ring with 1 such that for all $c \in \mathbb{F}, x, y \in A$, we have

$$(cx)y = c(xy) = x(cy).$$

For example, $M_n(\mathbb{F})$, the set of $n \times n$ matrices over $\mathbb{F}$, is an $\mathbb{F}$-algebra. The vector space $\mathbb{F}G$ with addition and multiplication given by

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$

and

$$\sum_{g \in G} a_g g \times \sum_{g \in G} b_g g = \sum_{g,h \in G} (a_g b_h) gh$$

forms the *group algebra*. In addition $\mathrm{End}(V)$, the set of linear transformations of an $\mathbb{F}$-vector space V, and $\mathrm{End}_A(V)$, the set of A-module linear transformations of V, are $\mathbb{F}$-algebras. The algebra $\mathrm{End}_A(V)$ is known as the *centralizer algebra of* V.

If A and B are $\mathbb{F}$-algebras, then a linear transformation $\varphi : A \to B$ satisfying $\varphi(1) = 1$ and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$ is an *algebra homomorphism.* A *representation* of an $\mathbb{F}$-algebra A on a finite dimensional $\mathbb{F}$-vector space V is an algebra homomorphism $\rho : A \to \mathrm{End}(V)$. We say V together with the action

$$av = \rho(a)v \qquad a \in A,\ v \in V,$$

is a left A-module. If V is a left A-module and a right B-module with compatible actions where

$$(av)b = a(vb) \qquad a \in A, b \in B, v \in V$$

then we say V is an (A, B)-*bimodule.*

There is a natural $(A, \mathrm{End}_A(V))$-bimodule structure on a left A-module V that allows us to study the centralizer algebra $\mathrm{End}_A(V)$.

Theorem 2.1.7 (Double Centralizer [14, Theorem 5.4])**.** *Let $\overline{\mathbb{F}}$ be an algebraically closed field and A be a finite dimensional $\overline{\mathbb{F}}$-algebra. Suppose V is an A-module where*

$$V \cong \bigoplus_{\lambda \in \hat{V}} m_\lambda A^\lambda$$

for an index set $\hat{V}$ of irreducible A-modules A^λ and positive integers m_λ. If $Z = \mathrm{End}_A(V)$, then

(a) $Z \cong \bigoplus_{\lambda \in \hat{V}} M_{m_\lambda}(\overline{\mathbb{F}})$;

(b) as an (A, Z)-bimodule, we have

$$V \cong \bigoplus_{\lambda \in \hat{V}} A^\lambda \otimes Z^\lambda$$

where the Z^λ for $\lambda \in \hat{V}$ are irreducible Z-modules.

2.2 Supercharacter Theories

Supercharacter theory arises as a natural generalization of character theory. Since determining the conjugacy classes and irreducible characters of U_n is impossible or “wild” [13], André [2, 3, 4, 5] and Yan [21] constructed a workable resemblance of character theory by “clumping” together some irreducible characters and some conjugacy classes. This clumping method can be thought of as a way to coarsen the partition of a group into its conjugacy classes that is compatible with certain sums of irreducible characters. In [12] Diaconis and Isaacs generalize this idea to arbitrary finite groups.

A *supercharacter theory* of a group G consists of a set of *superclasses* $\mathcal{K}$ and a set of *supercharacters* $\mathcal{X}$ such that

(a) the set $\mathcal{K}$ is a partition of G into unions of conjugacy classes,

(b) the set $\mathcal{X}$ is a set of characters such that each irreducible character of G is a constituent of exactly one supercharacter,

(c) $|\mathcal{K}| = |\mathcal{X}|$,

(d) the supercharacters are constant on superclasses.

Every group G has two "trivial" supercharacter theories: the usual character theory with $\mathcal{K} = cl(G)$ and $\mathcal{X} = \mathrm{Irr}(G)$, and the supercharacter theory with $\mathcal{K} = \{\{1\}, G - \{1\}\}$ and $\mathcal{X} = \{\mathbb{1}, \chi_{\mathrm{reg}} - \mathbb{1}\}$ where $\mathbb{1}$ is the trivial character of G and χ_{reg} is the regular character. While many finite groups have several supercharacter theories [12], preference is given to supercharacter theories that strike a balance between computability and producing better approximations of the usual character theory.

2.2.1 A supercharacter theory for U_n

We focus on the supercharacter theory on U_n given in [20] that is a slight coarsening of the traditional supercharacter theory of André and Yan.

Let U_n be the subgroup of unipotent upper-triangular matrices of the general linear group $GL_n(\mathbb{F}_q)$ over the finite field $\mathbb{F}_q$ with q elements, B_n be the normalizer of U_n in $GL_n(\mathbb{F}_q)$ consisting of upper triangular matrices, and

$$\mathfrak{u}_n = U_n - 1$$

be the nilpotent $\mathbb{F}_q$-algebra of strictly upper triangular matrices. The subgroup B_n acts by left and right multiplication on $\mathfrak{u}_n$, and the superclasses are given by the two-sided orbits

$$\begin{array}{ccc} B_n\mathfrak{u}_nB_n & \longleftrightarrow & \mathcal{K} \\ B_nxB_n & \mapsto & 1+B_nxB_n. \end{array}$$

Following the construction in [9], fix a nontrivial homomorphism $\vartheta : \mathbb{F}_q^+ \to \mathbb{C}^\times$. The $\mathbb{F}_q$-vector space of $n \times n$ matrices $\mathfrak{gl}_n(\mathbb{F}_q)$ decomposes in terms of upper triangular matrices $\mathfrak{b}_n$ and strictly lower triangular matrices $\mathfrak{l}_n$ as

$$\mathfrak{gl}_n = \mathfrak{b}_n \oplus \mathfrak{l}_n.$$

Identifying $\mathfrak{l}_n$ with $\mathfrak{gl}_n/\mathfrak{b}_n$ makes $\mathfrak{l}_n$ a canonical set of coset representatives in $\mathfrak{gl}_n/\mathfrak{b}_n$. For $v \in \mathfrak{gl}_n$ define

$$\bar{v} = (v + \mathfrak{b}_n) \cap \mathfrak{l}_n.$$

Then for $v \in \mathfrak{l}_n$,

$$\mathbb{C}\text{-span}\{\overline{av} \mid a \in B_n\}$$

is U_n-supermodule with left action

$$uw = \vartheta(\mathrm{tr}((u-1)w)(\overline{uw}) \quad \text{for} \quad u \in U_n, w \in \mathfrak{l}_n$$

and right action

$$wu = \vartheta(\mathrm{tr}(w(u^{-1}-1))(\overline{wu^{-1}}) \quad \text{for} \quad u \in U_n, w \in \mathfrak{l}_n.$$

The two-sided orbits from extending these actions on $\mathfrak{l}_n$ to the normalizer subgroup B_n yields corresponding supercharacters given by,

$$\begin{array}{ccc} B_n\mathfrak{l}_nB_n & \longleftrightarrow & \mathcal{X} \\ B_nvB_n & \mapsto & g \mapsto \dfrac{|B_nv|}{|B_nvB_n|} \displaystyle\sum_{w \in B_nvB_n} \vartheta(\mathrm{tr}((g-1)w)). \end{array}$$

In constructing the supercharacters of U_n it is more common to construct a module structure on the dual $\mathfrak{u}_n^*$, where $\mathfrak{u}_n = U_n - 1$ as in [12]. However, the actions of B_n on $\mathfrak{l}_n$ are a translation of the actions on $\mathfrak{u}_n^*$ that make studying modules more straightforward [9].

By elementary row and column operations we may choose orbit representatives for the two-sided action of B_n on $\mathfrak{u}_n$ and $\mathfrak{l}_n$ so that there is a one to one correspondence between

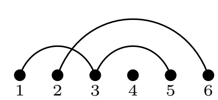
$$\left\{ \begin{array}{c} \text{superclasses} \\ \text{of } U_n \end{array} \right\} \longleftrightarrow \left\{ u \in U_n \;\middle|\; \begin{array}{c} u-1 \text{ has at most one 1} \\ \text{in every row and column} \end{array} \right\}$$

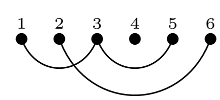
$$\left\{ \begin{array}{c} \text{supercharacters} \\ \text{of } U_n \end{array} \right\} \longleftrightarrow \left\{ v \in \mathfrak{l}_n \;\middle|\; \begin{array}{c} v \text{ has at most one 1} \\ \text{in every row and column} \end{array} \right\}.$$

These representatives are indexed by set partitions.

2.3 Set Partition Combinatorics

Define $[n] = \{1, 2, \ldots, n\}$. A *set partition* λ of $[n]$ is a subset $\{(i,j) \in [n] \times [n] \mid i < j\}$ such that if $(i,k),(j,l) \in \lambda$, then $i = j$ if and only if $k = l$. We represent each set partition $\lambda \vdash [n]$ diagrammatically as a set of arcs on a row of n nodes so that if $(i,j) \in \lambda$, then there is an arc connecting the ith node to the jth node. For example,

$\{1 \frown 3, 3 \frown 5, 2 \frown 6\}$ $\longleftrightarrow$ 

or 

.

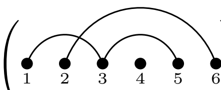
In these diagrams it is natural to draw the arcs above or below the nodes. We will use both orientations to compare set partitions. We typically refer to the pair (i,j) as an *arc* in λ and write $(i,j) = i \frown j$ or $(i,j) = i \smile j$ to specify the arc. For each arc $(i,j) \in \lambda$ we call i the *left endpoint* and j the *right endpoint*. The sets of left and right endpoints of λ are given by

$$le(\lambda) = \{i \in [n] \mid (i,j) \in \lambda, \text{ for some } j \in [n]\}$$

$$re(\lambda) = \{j \in [n] \mid (i,j) \in \lambda, \text{ for some } i \in [n]\}.$$

We say two arcs *conflict* if they have the same the same left or right endpoints. Thus no arcs conflict in a set partition.

We obtain the more traditional definition of set partitions by taking part(λ) for $\lambda \vdash [n]$ to be the set of equivalence classes on $[n]$ given by the reflexive transitive closure of $i \sim j$ if $(i, j) \in \lambda$. For instance,

part () $= \{\{1, 3, 5\}, \{2, 6\}, \{4\}\}.$

Note the connected components of the diagram are the parts of the set partition and the arcs are the adjacent pairs of elements in each part.

There are some natural statistics on set partitions [11]. For a set partition $\lambda \vdash [n]$ the *dimension* is

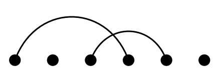
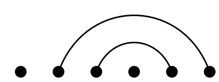
$$\dim(\lambda) = \sum_{i \frown j \in \lambda} j - i - 1.$$

For a pair of set partitions $\lambda, \mu \vdash [n]$ define

$$\mathrm{CRS}(\lambda, \mu) = \{((i, k), (j, l)) \in \lambda \times \mu \mid i < j < k < l\}, \qquad \mathrm{crs}(\lambda, \mu) = |\mathrm{CRS}(\lambda, \mu)|,$$

$$\mathrm{NST}^{\lambda}_{\mu} = \{((i, l), (j, k)) \in \lambda \times \mu \mid i < j < k < l\}, \qquad \mathrm{nst}^{\lambda}_{\mu} = |\mathrm{NST}^{\lambda}_{\mu}|$$

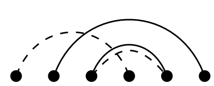
as the *crossing set*, *crossing number*, *nesting set*, and *nesting number* respectively. To illustrate, if

$\lambda =$  and $\mu =$  ,

then we have

$$\dim(\lambda) = 3, \quad \mathrm{crs}(\lambda, \lambda) = 1, \quad \mathrm{nst}^{\lambda}_{\lambda} = 0, \qquad \dim(\mu) = 4, \quad \mathrm{crs}(\mu, \mu) = 0, \quad \mathrm{nst}^{\mu}_{\mu} = 1.$$

Superimposing λ and μ, where the arcs of λ are dashed

$\lambda \cup \mu =$ 

yields

$$\mathrm{CRS}(\lambda, \mu) = \{(1 \frown 4, 2 \frown 6), (1 \frown 4, 3 \frown 5)\}, \qquad \mathrm{NST}^{\lambda}_{\mu} = \emptyset$$

but,

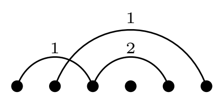
$$\mathrm{CRS}(\mu,\lambda)=\emptyset, \qquad \mathrm{NST}^{\mu}_{\lambda}=\{(2\frown 6, 3\frown 5)\}.$$

While it is not generally true that $\mathrm{CRS}(\lambda,\mu)=\mathrm{CRS}(\mu,\lambda)$, it follows from the definition of a crossing number that for all set partitions $\lambda,\mu,\nu\vdash[n]$,

$$\mathrm{CRS}(\lambda,\mu\cup\nu) = \mathrm{CRS}(\lambda,\mu)+\mathrm{CRS}(\lambda,\nu) \tag{2.2}$$

$$\mathrm{CRS}(\lambda\cup\mu,\nu) = \mathrm{CRS}(\lambda,\nu)+\mathrm{CRS}(\mu,\nu). \tag{2.3}$$

It will also be of interest to consider set partitions where the arcs are labeled or colored by an element of $\mathbb{F}_q^\times$. An $\mathbb{F}_q^\times$-*colored set partition of* $[n]$ is a pair (λ,ϕ), where λ is a set partition of $[n]$ and $\phi:\lambda\to\mathbb{F}_q^\times$ is a coloring of the arcs by elements of $\mathbb{F}_q^\times$. By convention, if $\phi((i,j))=a$ and the orientation of the arc is specified, we write a labeled arc as $i\overset{a}{\frown}j$ or $i\underset{a}{\smile}j$. For example,



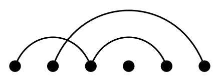
is an $\mathbb{F}_3^\times$-colored set partition of [6].

2.3.1 An uncolored supercharacter theory

We describe the correspondence between set partitions and the superclasses and supercharacters of U_n. Given a set partition $\lambda\vdash[n]$, we construct a representative u_λ of a superclass of U_n by

$$(u_\lambda)_{i,j}=\begin{cases}1 & \text{if } i\frown j\in\lambda \text{ or } i=j\\ 0 & \text{otherwise.}\end{cases}$$

For instance, the correspondence between λ and u_λ is given as follows

$\lambda =$  $\longleftrightarrow$

$$u_\lambda - 1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding superclass $\mathcal{K}_\lambda$ is

$$\mathcal{K}_\lambda = 1 + B_n(u_\lambda - 1)B_n.$$

Similarly, a representative v_λ for the two-sided action of B_n on $\mathfrak{u}_n$ is

$$(v_\lambda)_{k,j} = \begin{cases} 1 & \text{if } j \frown k \in \lambda, \\ 0 & \text{otherwise} \end{cases}$$

so that

$$V^\lambda \cong \mathbb{C}\text{-span}\{\overline{av_\lambda} \mid a \in B_n\}$$

and for $g \in U_n$, the corresponding supercharacter χ^λ is defined as

$$\chi^\lambda(g) = \frac{|B_n v_\lambda|}{|B_n v_\lambda B_n|} \sum_{v \in B_n v_\lambda B_n} \vartheta(\mathrm{tr}((g-1)\ v)).$$

Amazingly, many properties of these supercharacters can be determined using statistics of set partitions.

Proposition 2.3.1 ([9, Bragg, Thiem, Proposition 2.1])**.** *For $\lambda, \mu \vdash [n]$, we have*

$$\chi^\lambda(u_\mu) = \begin{cases} \dfrac{(-1)^{|\lambda\cap\mu|} q^{\dim(\lambda)} (q-1)^{|\lambda-\mu|}}{q^{\mathrm{nst}_\mu^\lambda}} & \textit{if } i < j < k,\ i \frown k \in \lambda \\ & \textit{then } i \frown j, j \frown k \notin \mu, \\ 0 & \textit{otherwise.} \end{cases}$$

In particular the trivial supercharacter $\mathbb{1}$ is the supercharacter $\chi^{\varnothing}$ corresponding to the empty set partition of $[n]$, and the degree of each supercharacter is

$$\chi^{\lambda}(1) = q^{\dim(\lambda)}(q-1)^{|\lambda|}.$$

It also follows from the formula that supercharacters factor as tensor products of arcs

$$\chi^{\lambda} = \bigodot_{i\frown j\in\lambda} \chi^{i\frown j} \qquad \text{where } (\chi \odot \psi)(g) = \chi(g)\psi(g). \tag{2.4}$$

With respect to the inner product the supercharacters form an orthogonal set.

Proposition 2.3.2. *For $\lambda, \mu \vdash [n]$, we have*

$$\langle \chi^{\lambda}, \chi^{\mu}\rangle = \delta_{\lambda\mu}(q-1)^{|\lambda|}q^{\mathrm{crs}(\lambda,\lambda)}.$$

Proposition 2.3.2 can be proved from [20, Thiem, (2.3)]. In light of Proposition 2.1.3, the crossing number $\mathrm{crs}(\lambda, \lambda)$ helps measure how close a supercharacter is to being irreducible.

2.3.2 A colored supercharacter theory

If instead of considering the orbits of the full subgroup B_n, we consider the U_n orbits on the group $\mathfrak{u}_n$ and its dual $\mathfrak{u}_n^*$, then we obtain the traditional supercharacter theory of André and Yan. In this case the combinatorics depends on the finite field $\mathbb{F}_q$ and is based on $\mathbb{F}_q^{\times}$-colored set partitions.

In this traditional supercharacter theory, the superclass $\mathcal{K}_{\lambda,\phi}$ corresponding to an $\mathbb{F}_q^{\times}$-colored set partition (λ, ϕ) of $[n]$ is

$$\mathcal{K}_{\lambda,\phi} = 1 + U_n(u_{\lambda,\phi} - 1)U_n, \qquad \text{where} \qquad (u_{\lambda,\phi})_{i,j} = \begin{cases} \phi(i\frown j) & \text{if } i\frown j \in \lambda, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The supercharacters are given by the following proposition.

Proposition 2.3.3. *Let* $\vartheta : \mathbb{F}_q^+ \to \mathbb{C}^\times$ *be a nontrivial homorphism. For* $\mathbb{F}_q^\times$*-colored set partitions* (λ, ϕ) *and* (μ, ψ) *of* $[n]$*, we have*

$$\chi^{\lambda,\phi}(u_{\mu,\psi}) = \begin{cases} \dfrac{q^{\dim(\lambda)}}{q^{\mathrm{nst}_\mu^\lambda}} \displaystyle\prod_{i \frown l \in \lambda} \vartheta(\phi(i \frown l)\psi(i \frown l)) & \text{if } i < j < k,\ i \frown k \in \lambda \\ & \text{then } i \frown j, j \frown k \notin \mu, \\ 0 & \text{otherwise,} \end{cases}$$

where $\psi(i \frown l) = 0$ *if* $i \frown l \notin \mu$*.*

There is a nice relationship between this colored supercharacter theory and the uncolored supercharacter theory we are considering.

Proposition 2.3.4. *For* $\lambda \vdash [n]$*, we have*

$$\mathcal{K}_\lambda = \bigcup_{\phi \in \mathrm{Col}_{\mathbb{F}_q}(\lambda)} \mathcal{K}_{\lambda,\phi} \qquad \text{and} \qquad \chi^\lambda = \sum_{\phi \in \mathrm{Col}_{\mathbb{F}_q}(\lambda)} \chi^{\lambda,\phi},$$

where $\mathrm{Col}_{\mathbb{F}_q}(\lambda)$ *denotes the set of all* $\mathbb{F}_q^\times$*-colorings of* λ*.*

Proof. Let $\lambda \vdash [n]$. For each $s, t \in T_n$, the group of diagonal matrices, there exists a unique $\phi \in \mathrm{Col}_{\mathbb{F}_q}(\lambda)$ such that

$$s(u_\lambda - 1)t = u_{\lambda,\phi} - 1.$$

Seeing that $B_n = U_n \rtimes T_n$, we obtain

$$\begin{aligned} \mathcal{K}_\lambda &= \{1 + xs(u_\lambda - 1)ty \mid x, y \in U_n, s, t \in T_n\} \\ &= \{1 + x(u_{\lambda,\phi} - 1)y \mid x, y \in U_n, \phi \in \mathrm{Col}_{\mathbb{F}_q}(\lambda)\} \\ &= \bigcup_{\phi \in \mathrm{Col}_{\mathbb{F}_q}(\lambda)} \mathcal{K}_{\lambda,\phi}. \end{aligned}$$

Since χ^λ is constant on the superclasses $\mathcal{K}_\lambda$, it follows that χ^λ takes a constant value each superclass $\mathcal{K}_{\lambda,\phi}$ for $\phi \in \mathrm{Col}_{\mathbb{F}_q}(\lambda)$. Thus it suffices to show that

$$\chi^\lambda(u_{\mu,\psi}) = \sum_{\phi \in \mathrm{Col}_{\mathbb{F}_q}(\lambda)} \chi^{\lambda,\phi}(u_{\mu,\psi})$$

for an $\mathbb{F}_q^\times$-colored set partition (μ,ψ). By Propositions 2.3.3 and 2.3.1 if $i \frown l \in \lambda$, and $i \frown j \in \mu$ or $j \frown k \in \mu$ for $i<j<k$ then

$$\sum_{\phi \in \operatorname{Col} \mathbb{F}_q(\lambda)} \chi^{\lambda,\phi}(u_{\mu,\psi}) = 0 = \chi^\lambda(u_{\mu,\psi}).$$

Otherwise, we obtain

$$\sum_{\phi \in \operatorname{Col} \mathbb{F}_q(\lambda)} \chi^{\lambda,\phi}(u_{\mu,\psi}) = \frac{q^{\dim(\lambda)}}{q^{\operatorname{nst}_\mu^\lambda}} \sum_{\phi \in \operatorname{Col} \mathbb{F}_q(\lambda)} \prod_{i \frown l \in \lambda} \vartheta(\phi(i \frown l)\psi(i \frown l)).$$

If

$$\lambda - \mu = \{i_1 \frown l_1, \ldots, i_t \frown l_t\} \qquad \text{and} \qquad \lambda \cap \mu = \{j_1 \frown k_1, \ldots, j_s \frown k_s\},$$

then we can identify any $\mathbb{F}_q^\times$-coloring ϕ with a tuple

$$(a_1, \ldots, a_t, b_1, \ldots, b_s) \in (\mathbb{F}_q^\times)^{t+s}.$$

As a result, we obtain

$$\begin{aligned} &\sum_{\phi \in \operatorname{Col} \mathbb{F}_q(\lambda)} \prod_{i \frown l \in \lambda} \vartheta(\phi(i \frown l)\psi(i \frown l)) \\ &\quad = \sum_{a_1,\ldots,a_t \in \mathbb{F}_q^\times} \prod_{r=1}^{t} \vartheta(\phi(i_r \frown l_r)\psi(i_r \frown l_r)) \cdot \sum_{b_1,\ldots,b_s \in \mathbb{F}_q^\times} \prod_{r=1}^{s} \vartheta(\phi(j_r \frown k_r)\psi(j_r \frown k_r)). \end{aligned} \tag{2.5}$$

Since $i_r \frown l_r \notin \mu$, it follows that $\psi(i_r \frown l_r) = 0$, which implies

$$\sum_{a_1,\ldots,a_t \in \mathbb{F}_q^\times} \prod_{r=1}^{t} \vartheta(\phi(i_r \frown l_r)\psi(i_r \frown l_r)) = \sum_{a_1,\ldots,a_t \in \mathbb{F}_q^\times} \prod_{r=1}^{t} \vartheta(0) = \sum_{a_1,\ldots,a_t \in \mathbb{F}_q^\times} 1 = (q-1)^t. \tag{2.6}$$

Because ϑ is a nontrivial linear character of $\mathbb{F}_q^+$, we have

$$\begin{aligned} \sum_{b_1,\ldots,b_s \in \mathbb{F}_q^\times} \prod_{r=1}^{s} \vartheta(\phi(j_r \frown k_r)\psi(j_r \frown k_r) &= \sum_{b_1,\ldots,b_s \in \mathbb{F}_q^\times} \prod_{r=1}^{s} \vartheta(b_r\psi(j_r \frown k_r)) \\ &= \sum_{b_1,\ldots,b_s \in \mathbb{F}_q^\times} \prod_{r=1}^{s} \vartheta(b_r) \\ &= \left(\sum_{b \in \mathbb{F}_q^\times} \vartheta(b) \right)^s \\ &= (-1)^s. \end{aligned} \tag{2.7}$$

Substituting (2.6) and (2.7) into Equation (2.5) yields

$$\sum_{\phi \in \mathrm{Col}\,\mathbb{F}_q(\lambda)} \prod_{i \frown l \in \lambda} \vartheta(\phi(i \frown l)\psi(i \frown l)) = (-1)^{|\lambda \cap \mu|}(q-1)^{|\lambda - \mu|}$$

as $t = |\lambda - \mu|$ and $s = |\lambda \cap \mu|$. Therefore, we obtain

$$\sum_{\phi \in \mathrm{Col}\,\mathbb{F}_q(\lambda)} \chi^{\lambda,\phi}(u_{\mu,\psi}) = \frac{(-1)^{|\lambda \cap \mu|} q^{\dim(\lambda)} (q-1)^{|\lambda - \mu|}}{q^{\mathrm{nst}_\mu^\lambda}} = \chi^{\lambda}(u_{\mu,\psi})$$

by Proposition 2.3.3. □

This proposition allows us to translate many results on the colored supercharacter theory to the uncolored one.

Chapter 3

Branching Rules

An important property of the supercharacters of U_n is that their restriction to any subgroup is a linear combination of supercharacters with nonnegative integer coefficients [12]. However, the coefficients in the restriction decompositions are not well understood [20]. We provide a combinatorial formula for calculating the restriction of supercharacters of U_n to U_{n-1}. Using Frobenius reciprocity, we obtain a corresponding formula for inducing supercharacters. Since these formulas depend on the number of nonzero elements in the field $\mathbb{F}_q$, fix

$$t = q - 1$$

for this chapter.

3.1 Restriction

We consider the restriction of supercharacters from U_n to U_{n-1} by embedding $U_{n-1} \subseteq U_n$ as

$$U_{n-1} = \{u \in U_n \mid (u-1)_{ij} \neq 0 \text{ implies } i < j < n\}.$$

Since supercharacters decompose into tensor products of arcs (2.4), for $\lambda \vdash [n]$, we have

$$\chi^\lambda = \bigodot_{i\frown l \in \lambda} \chi^{i\frown l} \quad \text{and} \quad \mathrm{Res}^{U_n}_{U_{n-1}}(\chi^\lambda) = \bigodot_{i\frown l \in \lambda} \mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{i\frown l}).$$

Consequently we compute restrictions for each $\chi^{i\frown l}$ and use the tensor product to glue together the resulting restrictions.

The restriction of the supercharacter $\chi^{i\frown l}$ is given using the formulas in [20] for computing restrictions in the colored supercharacter theory [cf. Section 2.3.2].

Proposition 3.1.1. *For $1 \leq i < l \leq n$, the restriction $\mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{i\frown l})$ is given by*

$$\mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{i\frown l}) = \begin{cases} \chi^{i\frown l} & \text{if } l \neq n, \\ t\Big(\mathbb{1} + \sum\limits_{i<k<l} \chi^{i\frown k}\Big) & \text{if } l = n. \end{cases}$$

Proof. By the formulas for restriction of colored arcs [20, Theorem 4.5], for $l \neq n$, we have

$$\mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{i\frown l}) = \sum_{a\in\mathbb{F}_q^\times} \mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{i\overset{a}{\frown} l}) = \sum_{a\in\mathbb{F}_q^\times} \chi^{i\overset{a}{\frown} l} = \chi^{i\frown l},$$

and for $l = n$, we have

$$\mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{i\frown l}) = \sum_{a\in\mathbb{F}_q^\times} \mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{i\overset{a}{\frown} l}) = \sum_{a\in\mathbb{F}_q^\times}\Bigg(\mathbb{1} + \sum_{\substack{i<k<l \\ b\in\mathbb{F}_q^\times}} \chi^{i\overset{b}{\frown} k}\Bigg) = t\Big(\mathbb{1} + \sum_{i<k<l} \chi^{i\frown k}\Big).$$

□

Intuitively, restricting an arc corresponds to removing the last node and reattaching the arc in all possible ways.

We now use the tensor product to glue together the resulting restrictions. For $1 \leq i < l$, define

$$\chi^{i\frown\!\times l} = t\Big(\mathbb{1} + \sum_{i<k<l} \chi^{i\frown k}\Big) \quad \text{and} \quad \chi^{i\times\!\frown l} = t\Big(\mathbb{1} + \sum_{i<j<l} \chi^{j\frown l}\Big).$$

Using the formulas in [20] for the colored supercharacter theory yields the following proposition.

Proposition 3.1.2. *For $1 \leq i < l \leq n$ and $1 \leq j < k \leq n$ such that $(i, l) \neq (j, k)$,*

$$\chi^{i\frown l} \odot \chi^{j\frown k} = \begin{cases} \chi^{\{i\frown l, j\frown k\}} & \text{if } k \neq l, i \neq j, \\ \chi^{i\frown l} \odot \chi^{j\frown\!\times k} & \text{if } i < j < k = l, \\ \chi^{i\frown l} \odot \chi^{j\times\!\frown k} & \text{if } i = j < k < l. \end{cases}$$

Proof. Let $1 \leq i < l \leq n$ and $1 \leq j < k \leq n$ such that $(i, l) \neq (j, k)$. For $k \neq l$ and $i \neq j$, the tensor product $\chi^{i\frown l} \odot \chi^{j\frown k}$ is given by

$$\chi^{i\frown l} \odot \chi^{j\frown k} = \sum_{a\in\mathbb{F}_q^\times} \sum_{b\in\mathbb{F}_q^\times} \chi^{i\overset{a}{\frown} l} \odot \chi^{j\overset{b}{\frown} k} = \sum_{a\in\mathbb{F}_q^\times} \sum_{b\in\mathbb{F}_q^\times} \chi^{\{i\overset{a}{\frown} l, j\overset{b}{\frown} k\}} = \chi^{\{i\frown l, j\frown k\}},$$

for $i < j < k = l$, we have

$$\chi^{i\frown l} \odot \chi^{j\frown l} = \sum_{a\in\mathbb{F}_q^\times}\sum_{b\in\mathbb{F}_q^\times} \chi^{i\overset{a}{\frown} l} \odot \chi^{j\overset{b}{\frown} l} = \sum_{a\in\mathbb{F}_q^\times}\sum_{b\in\mathbb{F}_q^\times} \chi^{i\overset{a}{\frown} l} \odot \Bigg(\mathbb{1} + \sum_{\substack{j<k<l\\ c\in\mathbb{F}_q^\times}} \chi^{j\overset{c}{\frown} k}\Bigg) = \chi^{i\frown l} \odot \chi^{j\underset{\times}{\frown} l},$$

and for $i = j < k < l$, we obtain

$$\chi^{i\frown l} \odot \chi^{i\frown k} = \sum_{a\in\mathbb{F}_q^\times}\sum_{b\in\mathbb{F}_q^\times} \chi^{i\overset{a}{\frown} l} \odot \chi^{i\overset{b}{\frown} k} = \sum_{a\in\mathbb{F}_q^\times}\sum_{b\in\mathbb{F}_q^\times} \chi^{i\overset{a}{\frown} l} \odot \Bigg(\mathbb{1} + \sum_{\substack{i<j<k\\ c\in\mathbb{F}_q^\times}} \chi^{j\overset{c}{\frown} k}\Bigg) = \chi^{i\frown l} \odot \chi^{i\underset{\times}{\frown} k}$$

by the tensor formulas for colored arcs [20, Lemma 4.6]. □

Thus the tensor product provides a rule for resolving conflicting arcs that have the same right endpoint by removing the smaller arc and reattaching it in all possible ways.

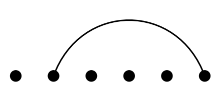
Next we work toward providing a combinatorial description of the coefficients in the tensor product based on statistics of set partitions and seashell inspired diagrams.

Definition 3.1.3. Let $s' \in \{s, s+1\}$ for $s \in \mathbb{Z}_{\geq 1}$ and $1 \leq i \leq l \leq n$. A *shell* of *size* n and *width* $l - i$ is a set of arcs on n nodes of the form

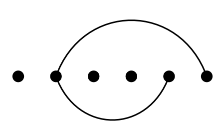
$$\bigcup_{r=1}^{s}\{i_r \frown l_r\} \cup \bigcup_{r=1}^{s'-1}\{i_r \smile l_{r+1}\}$$

where $i = i_1 < \cdots < i_s \leq l_{s'} < \cdots < l_1 = l$.

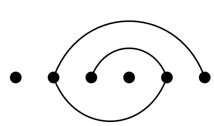
For example, some shells of size 6 and width $6 - 2$ are



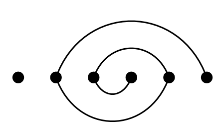
$\{2 \frown 6\}$,



$\{2 \frown 6\} \cup \{2 \smile 5\}$



$\{2 \frown 6, 3 \frown 5\} \cup \{2 \smile 5\}$,

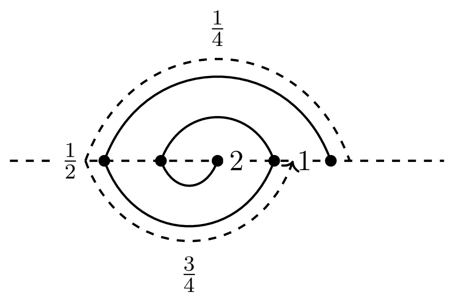


$\{2 \frown 6, 3 \frown 5\} \cup \{2 \smile 5, 3 \smile 4\}$.

A *whorl* is pair of consecutive arcs $(i \frown l, i \smile j)$ in a shell corresponding to a 360° rotation in the spiral configuration. Following the notation of Definition 3.1.3, the number of whorls of a shell is

$$\left\lceil \frac{s+s'-1}{2} \right\rceil$$

as each arc is half a whorl. We use the convention that whorls are counted from the right endpoint l spiraling inward. For instance, in the shell below we count the two whorls $(1 \frown 5, 1 \smile 4)$ and $(2 \frown 4, 2 \smile 3)$ as follows



If the whorls of a shell are given by $(i_1 \frown l_1, i_1 \smile l_2), \ldots, (i_s \frown l_s, i_s \smile l_{s+1})$ we say the pair $(i_1 \frown l_1, i_1 \smile l_2)$ is the *outer whorl* and the other whorls are *inner whorls*.

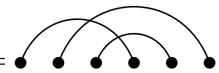
We can use shells to determine the partitions that appear in the restriction of a supercharacter. More precisely, drawing the arcs of a partition $\mu \vdash [n-1]$ below the nodes, and identifying the nodes with the leftmost $n-1$ nodes of a partition $\lambda \vdash [n]$ allows us to characterize the partitions with nonzero coefficients in the restriction of λ as the partitions $\mu \vdash [n-1]$ such that the symmetric difference between λ and μ form a shell.

Definition 3.1.4. For $\lambda \vdash [n]$ and $1 \leq i < l \leq n$ with $i \notin le(\lambda)$, the *shell set* $C^{\lambda, i \frown l}$ of $\lambda \cup \{i \frown l\}$ is

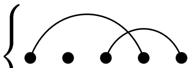
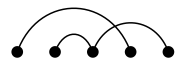
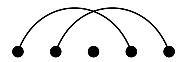
$$C^{\lambda,i\frown l} = \big\{\mu \vdash [n-1] \mid \big((\lambda \cup \{i \frown l\}) - \mu\big) \cup \big(\mu - (\lambda \cup \{i \frown l\})\big) \text{ is a shell of width } l-i\big\}.$$

This corresponds to all the ways to reattach the arc $i \frown l$ and "straighten" the resulting diagram by resolving all the conflicting arcs that share the same right endpoint.

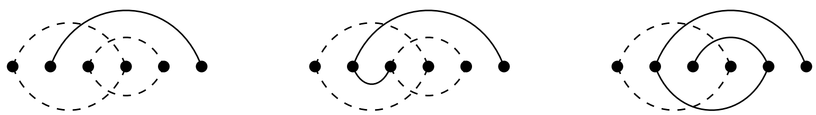
Example 3.1.5. Suppose $\lambda = \{1 \frown 4, 3 \frown 5\} \vdash [6]$. Consequently, we have

$\lambda \cup \{2 \frown 6\} =$ 

and

$C^{\lambda, 2\frown 6} = \Big\{$  ,  ,  $\Big\}$.

The seashells created by the symmetric differences between $\lambda \cup \{2 \frown 6\}$ and $\mu \in C^{\lambda,2\frown 6}$ are shown as solid lines



while the arcs in $\lambda \cap \mu$ are dashed.

It will be of interest to examine the shell sets $C^{\lambda,i\frown l}$ by considering the right endpoints $re(\lambda)$.

Definition 3.1.6. For each $j \frown k \in \lambda$ with $i < j < k < l$ define $\lambda|_{j\mapsto i}$ as the set partition obtained by replacing $j \frown k$ with $i \frown k$ and leaving everything else in λ the same. That is,

$$\lambda|_{j\mapsto i} = \lambda \cup \{i \frown k\} - \{j \frown k\}.$$

With this notation we can describe the shell set $C^{\lambda,i\frown l}$ as a union of shells with half a whorl, shells with one whorl, and shells with greater than one whorl.

Lemma 3.1.7. *For $\lambda \vdash [n]$, and $i \notin le(\lambda)$, the shell set is given by*

$$C^{\lambda,i\frown l} = \{\lambda\} \cup \{\lambda \cup \{i \frown k\} \mid i < k < l, k \notin re(\lambda)\} \cup \{\mu \in C^{\lambda|_{j\mapsto i}, j\frown k} \mid i < j < k < l, j \frown k \in \lambda\}.$$

Proof. By definition $\{\lambda, \lambda \cup \{i \frown k\} \mid i < k < l, k \notin re(\lambda)\} \subseteq C^{\lambda,i\frown l}$, so it suffices to show that

$$C^{\lambda,i\frown l} \backslash \{\lambda, \lambda \cup \{i \frown k\} \mid i < k < l, k \notin re(\lambda)\} = \{\mu \in C^{\lambda|_{j\mapsto i}, j\frown k} \mid i < j < k < l, j \frown k \in \lambda\}.$$

There exist $j = j_1 < \cdots < j_s < k_{s'} < \cdots < k_1 = k$ with $s' \in \{s, s+1\}$ such that

$$(\lambda|_{j\mapsto i} \cup \{j \frown k\}) - \mu = \{j_1 \frown k_1, j_2 \frown k_2, \ldots, j_s \frown k_s\}, \text{ and}$$

$$\mu - (\lambda|_{j\mapsto i} \cup \{j \frown k\}) = \{j_1 \frown k_2, j_2 \frown k_3, \ldots, j_{s'-1} \frown k_{s'}\}$$

if and only if there exist $i < j = j_1 < \cdots < j_s < k_{s'} < \cdots < k_1 = k < l$ such that

$$(\lambda \cup \{i \frown l\}) - \mu = \{i \frown l, j_1 \frown k_1, \ldots, j_s \frown k_s\}, \text{ and}$$

$$\mu - (\lambda \cup \{i \frown l\}) = \{i \frown k, j_1 \frown k_2, \ldots, j_{s'-1} \frown k_{s'}\}.$$

Thus $\mu \in C^{\lambda|_{j\mapsto i}, j\frown k}$ for some $i < j < k < l, j \frown k \in \lambda$ if and only if $\mu \in C^{\lambda, i\frown l} \backslash \{\lambda, \lambda \cup \{i \frown k\} \mid i < k < l, k \notin re(\lambda)\}$ as desired. □

Definition 3.1.8. For each $\mu \in C^{\lambda, i\frown l}$ define the *shell coefficient* of $\lambda \cup \{i \frown l\}$ and μ as

$$c_\mu^{\lambda, i\frown l} = \frac{t^{|(\lambda\cup\{i\frown l\})-\mu|} q^{\mathrm{crs}((\lambda\cup\{i\frown l\})\cap\mu, (\lambda\cup\{i\frown l\})-\mu)}}{q^{\mathrm{crs}((\lambda\cup\{i\frown l\})\cap\mu, \mu-(\lambda\cup\{i\frown l\}))}}$$

where $t = q - 1$ and $\mathrm{crs}(\cdot, \cdot)$ is the crossing number of two set partitions given in Section 2.3.

We can associate each shell coefficient $c_\mu^{\lambda, i\frown l}$ to the shell created by the symmetric difference of $\lambda \cup \{i \frown l\}$ and μ. The next lemma shows the shell coefficient is the product of the shell coefficient of the outer whorl with the shell coefficient of the inner whorls.

Lemma 3.1.9. *Let $\lambda \vdash [n]$, $i \notin le(\lambda)$, and $h \frown l, j \frown k \in \lambda$ with $1 \leq h < i < j < k < l \leq n$. If $\mu \in C^{\lambda|_{j\mapsto i}, j\frown k}$ then*

$$c_\mu^{\lambda, i\frown l} = c_{\lambda\cup\{i\frown k\}}^{\lambda, i\frown l} c_\mu^{\lambda|_{j\mapsto i}, j\frown k}.$$

Proof. Let $\mu \in C^{\lambda|_{j\mapsto i}, j\frown k}$. By construction $i \notin le(\lambda|_{j\mapsto i})$, so $i \frown l \notin \mu$. Thus we have

$$(\lambda \cup \{i \frown l\}) - \mu = \{i \frown l\} \cup (\lambda - \mu),$$

hence

$$(\lambda \cup \{i \frown l\}) \cap \mu = \lambda \cap \mu.$$

Substituting this and applying the crossing number equation (2.2), it follows that

$$\begin{aligned} c_\mu^{\lambda, i\frown l} &= \frac{t^{|(\lambda\cup\{i\frown l\})-\mu|} q^{\mathrm{crs}((\lambda\cup\{i\frown l\})\cap\mu, (\lambda\cup\{i\frown l\})-\mu)}}{q^{\mathrm{crs}((\lambda\cup\{i\frown l\})\cap\mu, \mu-(\lambda\cup\{i\frown l\}))}} \\ &= \frac{t^{|\{i\frown l\}\cup(\lambda-\mu)|} q^{\mathrm{crs}(\lambda\cap\mu, \{i\frown l\}\cup(\lambda-\mu))}}{q^{\mathrm{crs}(\lambda\cap\mu, \mu-(\lambda\cup\{i\frown l\}))}} \\ &= \frac{t^{|i\frown l|} q^{\mathrm{crs}(\lambda\cap\mu, i\frown l)} t^{|\lambda-\mu|} q^{\mathrm{crs}(\lambda\cap\mu, \lambda-\mu)}}{q^{\mathrm{crs}(\lambda\cap\mu, \mu-(\lambda\cup\{i\frown l\}))}}. \end{aligned}$$

Similarly since $j \frown k \in \lambda$ and $i \frown k \in \mu - \lambda$, we have

$$\mu - (\lambda \cup \{i \frown l\}) = \{i \frown k\} \cup (\mu - (\lambda|_{j\mapsto i} \cup \{j \frown k\}))$$

and thus

$$\lambda - \mu = \lambda|_{j\mapsto i} \cup \{j \frown k\} - \mu.$$

By the crossing number equation (2.2),

$$\begin{aligned} c_\mu^{\lambda,i\frown l} &= \frac{t^{|i\frown l|}q^{\mathrm{crs}(\lambda\cap\mu,i\frown l)}t^{|(\lambda|_{j\mapsto i}\cup j\frown k)-\mu|}q^{\mathrm{crs}(\lambda\cap\mu,(\lambda|_{j\mapsto i}\cup j\frown k)-\mu)}}{q^{\mathrm{crs}(\lambda\cap\mu,\{i\frown k\}\cup(\mu-(\lambda|_{j\mapsto i}\cup j\frown k)))}} \\ &= \frac{t^{|i\frown l|}q^{\mathrm{crs}(\lambda\cap\mu,i\frown l)}}{q^{\mathrm{crs}(\lambda\cap\mu,i\frown k)}}\cdot\frac{t^{|(\lambda|_{j\mapsto i}\cup j\frown k)-\mu|}q^{\mathrm{crs}(\lambda\cap\mu,(\lambda|_{j\mapsto i}\cup j\frown k)-\mu)}}{q^{\mathrm{crs}(\lambda\cap\mu,\mu-(\lambda|_{j\mapsto i}\cup j\frown k))}}. \end{aligned}$$

Moreover any arc in λ that crosses with $i \frown k$ or $i \frown l$ must be in μ, implying

$$\begin{aligned} c_\mu^{\lambda,i\frown l} &= \frac{t^{|i\frown l|}q^{\mathrm{crs}(\lambda,i\frown l)}}{q^{\mathrm{crs}(\lambda,i\frown k)}}\cdot\frac{t^{|(\lambda|_{j\mapsto i}\cup j\frown k)-\mu|}q^{\mathrm{crs}((\lambda|_{j\mapsto i}\cup j\frown k)\cap\mu,(\lambda|_{j\mapsto i}\cup j\frown k)-\mu)}}{q^{\mathrm{crs}((\lambda|_{j\mapsto i}\cup j\frown k)\cap\mu,\mu-(\lambda|_{j\mapsto i}\cup j\frown k))}} \\ &= c_{\lambda\cup\{i\frown k\}}^{\lambda,i\frown l}c_\mu^{\lambda|_{j\mapsto i},j\frown k}. \end{aligned}$$

□

Theorem 3.1.10. *For $\lambda \vdash [n]$, $i \notin le(\lambda)$, and $1 \leq i < l \leq n$, we have*

$$\chi^\lambda \odot \chi^{i\frown l} = \sum_{\mu\in C^{\lambda,i\frown l}} c_\mu^{\lambda,i\frown l}\chi^\mu$$

where $C^{\lambda,i\frown l}$ is the shell set of $\lambda \cup \{i \frown l\}$ and $c_\mu^{\lambda,i\frown l}$ is the shell coefficient of $\lambda \cup \{i \frown l\}$ and μ.

Before proving the theorem we state a lemma about the q-analogue of a crossing number. In general, the q-analogue of a nonnegative integer n is

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Lemma 3.1.11. *For $\lambda \vdash [n]$, and $1 \leq j < l \leq n$ where $j \notin le(\lambda)$, we have*

$$\sum_{\substack{i\frown k\in\lambda \\ i<j<k<l}} q^{\mathrm{crs}(\lambda,j\frown k)} = \frac{q^{\mathrm{crs}(\lambda,j\frown l)} - 1}{q - 1} = [\mathrm{crs}(\lambda, j \frown k)]_q,$$

the q-analogue of $\mathrm{crs}(\lambda, j \frown k)$.

Proof. Let $\lambda \vdash [n]$, $1 \leq j < l \leq n$, and $j \notin le(\lambda)$. If the set of arcs in λ that cross with $j \frown k$ is given by

$$\{i \frown k \in \lambda \ \mid i < j < k < l\} = \{i_1 \frown k_1, i_2 \frown k_2, \ldots, i_r \frown k_r\},$$

then for $1 \leq s \leq r$

$$\{i \frown k \in \lambda \ \mid i < j < k < k_s\} = \{i_1 \frown k_1, i_2 \frown k_2, \ldots, i_{s-1} \frown k_{s-1}\}.$$

By the definition of the crossing number

$$\sum_{\substack{i<j<k<l \\ i\frown k \in \lambda}} q^{\mathrm{crs}(\lambda, j\frown k)} = \sum_{s=1}^{r} q^{\#\{i\frown k\in\lambda | i<j<k<k_s\}} = \sum_{s=1}^{r} q^{s-1} = \frac{q^r - 1}{q-1} = \frac{q^{\mathrm{crs}(\lambda, j\frown l)} - 1}{q-1}.$$

□

We are now ready to prove Theorem 3.1.10.

Proof. We induct on $l - i$. For the base case assume $l - i = 1$. Then $C^{\lambda, i\frown l} = \{\lambda\}$, and we obtain

$$\chi^\lambda \odot \chi^{i \frown\!\!\!\times l} = \chi^\lambda \odot t\mathbb{1} = t\chi^\lambda = c_\lambda^{\lambda, i\frown l}\chi^\lambda$$

as desired.

Assume the formula holds for all $k - j < l - i$. Then, this yields

$$\begin{aligned}
\chi^\lambda \odot \chi^{i \frown\!\!\!\times l} &= \chi^\lambda \odot t\Big(\mathbb{1} + \sum_{i<k<l} \chi^{i\frown k}\Big) \\
&= t\chi^\lambda + t\sum_{i<k<l} \chi^\lambda \odot \chi^{i\frown k} \\
&= t\chi^\lambda + t\Big(\sum_{\substack{i<k<l \\ k\notin re(\lambda)}} \chi^\lambda \odot \chi^{i\frown k} + \sum_{\substack{h<i<k<l \\ h\frown k\in\lambda}} \chi^\lambda \odot \chi^{i\frown k} + \sum_{\substack{i<j<k<l \\ j\frown k\in\lambda}} \chi^\lambda \odot \chi^{i\frown k}\Big),
\end{aligned}$$

which by Proposition 3.1.2 is equal to

$$= \quad t\chi^\lambda + t\sum_{\substack{i<k<l \\ k\notin re(\lambda)}} \chi^{\lambda\cup\{i\frown k\}} + t\sum_{\substack{h<i<k<l \\ h\frown k\in\lambda}} \chi^\lambda \odot \chi^{i \frown\!\!\!\times k} + t\sum_{\substack{i<j<k<l \\ j\frown k\in\lambda}} \chi^{\lambda\cup\{i\frown k\}-\{j\frown k\}} \odot \chi^{j \frown\!\!\!\times k}.$$

Recall from Definition 3.1.6 that $\lambda|_{j\mapsto i} = \lambda \cup \{i \frown k\} - \{j \frown k\}$ for each $j \frown k \in \lambda$ such that $i < j < k < l$. By the induction hypothesis the tensor product $\chi^\lambda \odot \chi^{i \frown\!\!\times l}$ is

$$\begin{aligned}
&= t\chi^\lambda + t\sum_{\substack{i<k<l\\ k\notin re(\lambda)}} \chi^{\lambda\cup\{i\frown k\}} + t\sum_{\substack{h<i<k<l\\ h\frown k\in\lambda}}\Bigg(\sum_{\mu\in C^{\lambda,i\frown k}} c_\mu^{\lambda,i\frown k}\chi^\mu\Bigg) + t\sum_{\substack{i<j<k<l\\ j\frown k\in\lambda}}\Bigg(\sum_{\mu\in C^{\lambda|_{j\mapsto i},j\frown k}} c_\mu^{\lambda|_{j\mapsto i},j\frown k}\chi^\mu\Bigg)\\
&= t\chi^\lambda + t\sum_{\substack{i<k<l\\ k\notin re(\lambda)}} \chi^{\lambda\cup\{i\frown k\}} + t\sum_{\mu\vdash[n]}\Bigg(\sum_{\substack{h<i<k<l\\ h\frown k\in\lambda\\ \mu\in C^{\lambda,i\frown k}}} c_\mu^{\lambda,i\frown k}\chi^\mu\Bigg) + t\sum_{\mu\vdash[n]}\Bigg(\sum_{\substack{i<j<k<l\\ j\frown k\in\lambda\\ \mu\in C^{\lambda|_{j\mapsto i},j\frown k}}} c_\mu^{\lambda|_{j\mapsto i},j\frown k}\chi^\mu\Bigg).
\end{aligned}$$

By Lemma 3.1.11, the coefficient of χ^λ will be

$$t + t\sum_{\substack{h<i<k<l\\ h\frown k\in\lambda}} c_\lambda^{\lambda,i\frown k} = t\Bigg(1+\sum_{\substack{h<i<k<l\\ h\frown k\in\lambda}} tq^{\mathrm{crs}(\lambda,i\frown k)}\Bigg) = t\Bigg(1+t\cdot\frac{q^{\mathrm{crs}(\lambda,i\frown l)}-1}{t}\Bigg) = tq^{\mathrm{crs}(\lambda,i\frown l)} = c_\lambda^{\lambda,i\frown l}.$$

Similarly for $\lambda\cup\{i\frown k'\}$ where $i<k'<l$ and $k'\notin re(\lambda)$, the coefficient of $\chi^{\lambda\cup\{i\frown k'\}}$ is

$$t + t\sum_{\substack{h<i<k'<k<l\\ h\frown k\in\lambda}} c_\mu^{\lambda,i\frown k} = t\Bigg(1+\sum_{\substack{h<i<k'<k<l\\ h\frown k\in\lambda}} t\frac{q^{\mathrm{crs}(\lambda,i\frown k)}}{q^{\mathrm{crs}(\lambda,i\frown k')}}\Bigg) = t\Bigg(1+\sum_{\substack{h<i<k<l\\ h\frown k\in\lambda}} tq^{\mathrm{crs}(\lambda-\nu,i\frown k)}\Bigg),$$

where $\nu = \{h\frown k\in\lambda \mid (h\frown k, i\frown k')\in \mathrm{CRS}(\lambda, i\frown k')\}$. This is equivalent to

$$t\Bigg(1+t\cdot\frac{q^{\mathrm{crs}(\lambda-\nu,i\frown l)}-1}{t}\Bigg) = t\Bigg(1+t\cdot\frac{q^{\mathrm{crs}(\lambda,i\frown l)-\mathrm{crs}(\lambda,i\frown k')}-1}{t}\Bigg) = \frac{tq^{\mathrm{crs}(\lambda,i\frown l)}}{q^{\mathrm{crs}(\lambda,i\frown k')}} = c_{\lambda\cup\{i\frown k\}}^{\lambda,i\frown l}$$

by Lemma 3.1.11. If $j\frown k'\in\lambda$ is such that $i<j<k'<l$ then we have $\lambda|_{j\mapsto i} = \lambda\cup\{i\frown k'\} - \{j\frown k'\}$. Let $\nu = \{h\frown k\in\lambda \mid (h\frown k, i\frown k')\in \mathrm{CRS}(\lambda, i\frown k')\}$. Using Lemma 3.1.9, the coefficient of χ^μ for each $\mu\in C^{\lambda|_{j\mapsto i},j\frown k'}$ is

$$\begin{aligned}
tc_\mu^{\lambda|_{j\mapsto i},j\frown k'} + t\sum_{\substack{h<i<k'<k<l\\ h\frown k\in\lambda}} c_\mu^{\lambda,i\frown k} &= tc_\mu^{\lambda|_{j\mapsto i},j\frown k'} + t\sum_{\substack{h<i<k'<k<l\\ h\frown k\in\lambda}} c_{\lambda\cup i\frown k'}^{\lambda,i\frown k}c_\mu^{\lambda|_{j\mapsto i},j\frown k'}\\
&= tc_\mu^{\lambda|_{j\mapsto i},j\frown k'}\Bigg(1+\sum_{\substack{h<i<k'<k<l\\ h\frown k\in\lambda}} t\frac{q^{\mathrm{crs}(\lambda,i\frown k)}}{q^{\mathrm{crs}(\lambda,i\frown k')}}\Bigg)\\
&= tc^{\lambda|_{j\mapsto i},j\frown k'}\Bigg(1+t\cdot\frac{q^{\mathrm{crs}(\lambda-\nu,i\frown l)}-1}{t}\Bigg).
\end{aligned}$$

Applying Lemmas 3.1.11 and 3.1.9 yields

$$\begin{aligned} tc^{\lambda|_{j\mapsto i,j\frown k'}}\left(1+t\cdot\frac{q^{\mathrm{crs}(\lambda-\nu,i\frown l)}-1}{t}\right) &= tc^{\lambda|_{j\mapsto i,j\frown k'}}\left(1+t\cdot\frac{q^{\mathrm{crs}(\lambda,i\frown l)-\mathrm{crs}(\lambda,i\frown k')}-1}{t}\right)\\ &= \frac{tq^{\mathrm{crs}(\lambda,i\frown l)}}{q^{\mathrm{crs}(\lambda,i\frown k')}}c_{\mu}^{\lambda|_{j\mapsto i,j\frown k'}}\\ &= c^{\lambda,i\frown l}_{\lambda\cup\{i\frown k\}'}c_{\mu}^{\lambda|_{j\mapsto i,j\frown k'}}\\ &= c_{\mu}^{\lambda,i\frown l}. \end{aligned}$$

Substituting this into the equation for $\chi^{\lambda}\odot\chi^{i\frown l}$ and applying Lemma 3.1.7 we obtain

$$\begin{aligned} \chi^{\lambda}\odot\chi^{i\frown l} &= c_{\lambda}^{\lambda,i\frown l}\chi^{\lambda}+\sum_{\substack{i<k<l\\ k\notin re(\lambda)}}c^{\lambda,i\frown l}_{\lambda\cup\{i\frown k\}}\chi^{\lambda\cup\{i\frown k\}}+\sum_{\mu\vdash[n]}\sum_{\substack{i<j<k<l\\ j\frown k\in\lambda\\ \mu\in C^{\lambda|_{j\mapsto i,j\frown k}}}}c_{\mu}^{\lambda,i\frown l}\chi^{\mu}\\ &= \sum_{\mu\in C^{\lambda,i\frown l}}c_{\mu}^{\lambda,i\frown l}\chi^{\mu}. \end{aligned}$$

□

This combinatorial description of the coefficients in the tensor product leads to a combinatorial description of the coefficients in the restriction to U_{n-1}.

Corollary 3.1.12. *For $\lambda\vdash[n]$, the restriction $\mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{\lambda})$ is given by*

$$\mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{\lambda})=\sum_{\mu\vdash[n-1]}c_{\mu}^{\lambda}\chi^{\mu}$$

where

$$c_{\mu}^{\lambda}=\begin{cases}\delta_{\lambda\mu} & \text{if } n\notin re(\lambda),\\ \dfrac{t^{|\lambda-\mu|}q^{\mathrm{crs}(\lambda\cap\mu,\lambda-\mu)}}{q^{\mathrm{crs}(\lambda\cap\mu,\mu-\lambda)}} & \text{if } \mu\in C^{\lambda-\{i\frown n\},i\frown n},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Applying Propositions 3.1.1, 3.1.2, and Theorem 3.1.10 respectively,

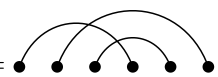
$$\begin{aligned} \mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{\lambda}) &= \bigodot_{i\frown l\in\lambda}\mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{i\frown l})=\bigodot_{\substack{j\frown l\in\lambda\\ l\neq n}}\chi^{j\frown l}\odot\chi^{i\frown n}\\ &= \chi^{\lambda-\{i\frown n\}}\odot\chi^{i\frown n}=\sum_{\mu\in C^{\lambda-\{i\frown n\},i\frown n}}c_{\mu}^{\lambda}\chi^{\mu} \end{aligned}$$

where

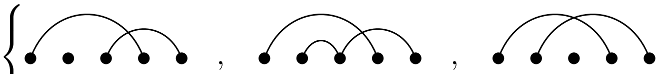
$$c_\mu^\lambda = c_\mu^{\lambda - \{i \frown n\}, i \frown n} = \frac{t^{|\lambda-\mu|} q^{\mathrm{crs}(\lambda\cap\mu, \lambda-\mu)}}{q^{\mathrm{crs}(\lambda\cap\mu, \mu-\lambda)}}.$$

□

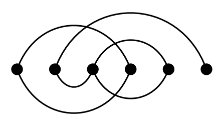
Example 3.1.13. Similar to Example 3.1.5, let

$\lambda =$ 

so that

$C^{\lambda-\{2\frown 6\}, 2\frown 6} = \Big\{$  $\Big\}$.

Drawing the arcs of $\mu = \{1 \frown 4, 2 \frown 3 \frown 5\}$ below the nodes of λ as shown below



illustrates that

$$c_\mu^\lambda = \frac{t^1 \cdot q^1}{q^0} = tq$$

since

$$\lambda - \mu = \{2 \frown 6\}, \quad \mathrm{CRS}(\lambda\cap\mu, \lambda-\mu) = \{(1 \frown 4, 2 \frown 6)\}, \quad \mathrm{crs}(\lambda\cap\mu, \lambda-\mu) = 1, \quad \mathrm{crs}(\lambda\cap\mu, \mu-\lambda) = 0.$$

We can calculate the other coefficients in the same manner to obtain

$$\mathrm{Res}_{U_5}^{U_6}(\chi^\lambda) = tq\chi^{\{1\frown 4, 3\frown 5\}} + tq\chi^{\{2\frown 3\frown 5, 1\frown 4\}} + t^2 q\chi^{\{1\frown 4, 2\frown 5\}}.$$

3.2 Induction and Superinduction

While the restriction of a supercharacter of U_n is a nonnegative integer linear combination of supercharacters, an induced supercharacter may not be a sum of supercharacters. In fact, the induced character may not even be a superclass function; for an example see [12, Section 6]. If instead we generalize to superinduction by averaging over superclasses in the same way that induction averages over conjugacy classes, then the constructed function will be a linear combination of supercharacters with rational coefficients [12, Lemma 6.7].

Suppose $H \subseteq G$ and χ is a superclass function of H. If $\mathcal{K}_g$ is the superclass containing $g \in G$, then the *superinduction* $\mathrm{SInd}_H^G(\chi)$ is

$$\mathrm{SInd}_H^G(\chi)(g) = |G:H| \frac{1}{|\mathcal{K}_g|} \sum_{x \in \mathcal{K}_g} \dot{\chi}(x) \text{ where } \dot{\chi}(x) = \begin{cases} \chi(x) & \text{if } x \in H \\ 0 & \text{if } x \notin H. \end{cases}$$

A nice property of superinduction is that the analog of Frobenius reciprocity holds.

Proposition 3.2.1 (Frobenius Reciprocity [15, Lemma 5.2]). *Let H be a subgroup of G. Suppose φ is a superclass function of G and θ is a class function of H. Then*

$$\langle \mathrm{SInd}_H^G(\theta), \varphi \rangle_G = \langle \theta, \mathrm{Res}_H^G(\varphi) \rangle_H.$$

However, superinduced characters are not necessarily characters so it is useful to know when superinduction is equivalent to induction.

[19, Section 3.2] examines some cases when this occurs for a larger class of p-groups known as algebra groups. If J is a finite dimensional nilpotent associative algebra over $\mathbb{F}_q$, then the algebra group based on J is $G = \{1 + x \mid x \in J\}$ under the multiplication $(1+x)(1+y) = 1 + x + y + xy$. In particular, Marberg and Thiem show if we embed U_{n-1} into U_n by

$$U_{n-1} = \{u \in U_n \mid (u-1)_{ij} \neq 0 \text{ implies } i < j < n\}$$

then for any superclass function χ of U_{n-1},

$$\mathrm{SInd}_{U_{n-1}}^{U_n}(\chi) = \mathrm{Ind}_{U_{n-1}}^{U_n}(\chi).$$

They also provide some conditions when superinduction is the same as induction.

Proposition 3.2.2 ([19, Theorem 3.1]). *Let H be a subalgebra group of an algebra group G, and suppose*

(1) no two superclasses of H are in the same superclass of G, and

(2) $x(h-1)+1 \in H$ *for all* $x \in G, h \in H$.

Then the superinduction of any superclass function χ *of* H *is*

$$\mathrm{SInd}_H^G(\chi) = \mathrm{Ind}_H^G(\chi).$$

If we embed U_{n-1} into U_n by

$$U_{n-1} = \{u \in U_n \mid u_{n-1,n} = 0 \text{ and } u_{i,n-1} = 0 \text{ for } i < n-1\}$$

then we have the following corollary.

Corollary 3.2.3. *Let* $U_{n-1} = \{u \in U_n \mid u_{n-1,n} = 0$ *and* $u_{i,n-1} = 0$ *for* $i < n-1\}$. *Then the superinduction any superclass function* χ *of* U_{n-1} *is*

$$\mathrm{SInd}_{U_{n-1}}^{U_n}(\chi) = \mathrm{Ind}_{U_{n-1}}^{U_n}(\chi).$$

Proof. It suffices to show the hypotheses of the previous theorem hold. Because there is an injective function from superclasses of U_{n-1} to U_n then no two superclasses of U_{n-1} are in the same superclass of U_n.

Let $x \in U_n$, $h \in U_{n-1}$ and $u = x(h-1)+1$. Since $h_{i,n-1} - 1 = 0$ we have $u_{i,n-1} = 0$ for $i < n-1$. Similarly, $x_{n-1,j} = 0$ for $j < n-1$ and $h_{n-1,j} - 1 = 0$ for $j \geq n-1$ implies $u_{n-1,n} = 0$. This shows $u \in U_{n-1}$. Therefore, $\mathrm{SInd}_{U_{n-1}}^{U_n}(\chi) = \mathrm{Ind}_{U_{n-1}}^{U_n}(\chi)$ for any superclass function χ of U_{n-1} by Proposition 3.2.2.

□

Unlike in the representation theory of the symmetric group, the decomposition of induced characters depends on the embedding of U_{n-1} into U_n. If we instead consider right modules then superinduction is equivalent to induction for the following embeddings

$$U_{n-1} = \{u \in U_n \mid (u-1)_{ij} \neq 0 \text{ implies } 1 < i < j\}$$

and

$$U_{n-1} = \{u \in U_n \mid u_{1,2} = 0 \text{ and } u_{2,j} = 0 \text{ for } 2 < j\}$$

[19, Section 3.1]. However, it is not known if superinduction is the same as induction for other embeddings. In our case we use the embedding of $U_{n-1} \subseteq U_n$ obtained by removing the last column so that superinduction is in fact induction.

We now derive a corresponding formula for induction from restriction.

Corollary 3.2.4. *For $\mu \vdash [n-1]$, the induction $\mathrm{Ind}_{U_{n-1}}^{U_n}(\chi^\mu)$ is given by*

$$\mathrm{Ind}_{U_{n-1}}^{U_n}(\chi^\mu) = \sum_{\lambda \vdash [n]} d_\mu^\lambda \chi^\lambda,$$

where

$$d_\mu^\lambda = \begin{cases} \delta_{\lambda\mu} & \text{if } n \notin re(\lambda) \\ \dfrac{t^{|\mu-\lambda|} q^{\mathrm{crs}(\mu-\lambda,\lambda\cap\mu)}}{q^{\mathrm{crs}(\lambda-\mu,\lambda\cap\mu)}} & \text{if } \mu \in C^{\lambda-\{i\frown n\},i\frown n} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda \vdash [n]$ and $\mu \vdash [n-1]$. Frobenius reciprocity, Proposition 3.2.1, shows

$$\langle \chi^\lambda, \mathrm{SInd}_{U_{n-1}}^{U_n}(\chi^\mu)\rangle_{U_n} = \langle \mathrm{Res}_{U_{n-1}}^{U_n}(\chi^\lambda), \chi^\mu\rangle_{U_{n-1}}.$$

Thus if

$$\mathrm{Ind}_{U_{n-1}}^{U_n}(\chi^\mu) = \sum_\gamma d_\mu^\gamma \chi^\gamma \quad \text{and} \quad \mathrm{Res}_{U_{n-1}}^{U_n}(\chi^\lambda) = \sum_\nu c_\nu^\lambda \chi^\nu$$

then the inner product, Proposition 2.3.2, yields

$$q^{\mathrm{crs}(\lambda,\lambda)} t^{|\lambda|} d_\mu^\lambda = q^{\mathrm{crs}(\mu,\mu)} t^{|\mu|} c_\mu^\lambda.$$

Therefore, the coefficient d_μ^λ is

$$\begin{aligned} d_\mu^\lambda &= \frac{t^{|\mu|-|\lambda|} q^{\mathrm{crs}(\mu,\mu)}}{q^{\mathrm{crs}(\lambda,\lambda)}} c_\mu^\lambda \\ &= \frac{t^{|\mu|-|\lambda|} q^{\mathrm{crs}(\mu,\mu)}}{q^{\mathrm{crs}(\lambda,\lambda)}} \cdot \frac{t^{|\lambda-\mu|} q^{\mathrm{crs}(\lambda\cap\mu,\lambda-\mu)}}{q^{\mathrm{crs}(\lambda\cap\mu,\mu-\lambda)}} \\ &= \frac{t^{|\mu-\lambda|} q^{\mathrm{crs}(\mu,\mu)-\mathrm{crs}(\lambda\cap\mu,\mu-\lambda)}}{q^{\mathrm{crs}(\lambda,\lambda)-\mathrm{crs}(\lambda\cap\mu,\lambda-\mu)}} \end{aligned}$$

since $|\mu-\lambda| = |\mu| - |\lambda| + |\lambda-\mu|$. From the crossing number equation (2.2) we obtain

$$\begin{aligned} d^{\lambda}_{\mu} &= \frac{t^{|\mu-\lambda|} q^{\mathrm{crs}(\mu-(\lambda\cap\mu),\mu-(\mu-\lambda))}}{q^{\mathrm{crs}(\lambda-(\lambda\cap\mu),\lambda-(\lambda-\mu))}} \\ &= \frac{t^{|\mu-\lambda|} q^{\mathrm{crs}(\mu-\lambda,\lambda\cap\mu)}}{q^{\mathrm{crs}(\lambda-\mu,\lambda\cap\mu)}}. \end{aligned}$$

□

Together Corollaries 3.1.12 and 3.2.4 for decomposing restricted and induced supercharacters are known as *branching rules*, which we restate due to their importance.

Theorem 3.2.5 (Branching Rules). *For $\lambda \vdash [n]$, the restriction $\mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{\lambda})$ is given by*

$$\mathrm{Res}^{U_n}_{U_{n-1}}(\chi^{\lambda}) = \sum_{\mu \vdash [n-1]} c^{\lambda}_{\mu} \chi^{\mu}$$

where

$$c^{\lambda}_{\mu} = \begin{cases} \delta_{\lambda\mu} & \text{if } n \notin re(\lambda), \\ \dfrac{t^{|\lambda-\mu|} q^{\mathrm{crs}(\lambda\cap\mu,\lambda-\mu)}}{q^{\mathrm{crs}(\lambda\cap\mu,\mu-\lambda)}} & \text{if } \mu \in C^{\lambda-\{i\frown n\},i\frown n}, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mu \vdash [n-1]$, the induction $\mathrm{Ind}^{U_n}_{U_{n-1}}(\chi^{\mu})$ is given by

$$\mathrm{Ind}^{U_n}_{U_{n-1}}(\chi^{\mu}) = \sum_{\lambda \vdash [n]} d^{\lambda}_{\mu} \chi^{\lambda},$$

where

$$d^{\lambda}_{\mu} = \begin{cases} \delta_{\lambda\mu} & \text{if } n \notin re(\lambda) \\ \dfrac{t^{|\mu-\lambda|} q^{\mathrm{crs}(\mu-\lambda,\lambda\cap\mu)}}{q^{\mathrm{crs}(\lambda-\mu,\lambda\cap\mu)}} & \text{if } \mu \in C^{\lambda-\{i\frown n\},i\frown n} \\ 0 & \text{otherwise.} \end{cases}$$

Since the branching rules are simple and easily computable, we include code in Appendix A for a program in SAGE that takes a supercharacter as input and outputs these restriction and induction decompositions. This enables us to quickly compute meaningful examples of restricting and inducing a supercharacter multiple times. While these formulas allow us to better understand restriction and induction, they are also useful for Schur–Weyl duality.

Chapter 4

Shell Tableaux

We use the branching rules to create a graph known as the Bratteli diagram. For the symmetric group, paths in the Bratteli diagram are indexed by a set of combinatorial objects called Young tableaux [18]. Building from the combinatorics of the previous chapter, we create an analogue of Young tableaux known as shell tableaux and construct a bijection between shell tableaux and paths in the Bratteli diagram.

For $k \in \mathbb{Z}_{\geq 1}$, consider

$$V^k = \underbrace{(\mathrm{Ind}_{U_{n-1}}^{U_n} \mathrm{Res}_{U_{n-1}}^{U_n})^k}_{k \text{ times}}(\mathbb{1})$$

where $\mathbb{1}$ is the trivial supercharacter of U_n that is restricted and induced k times. Let

$$\begin{aligned} \hat{Z}_k &= \big\{\lambda \vdash [n] \text{ corresponding to supercharacters of } V^k\big\} \\ \hat{Z}_{k+\frac{1}{2}} &= \big\{\mu \vdash [n-1] \text{ corresponding to supercharacters of } \mathrm{Res}_{U_{n-1}}^{U_n}(V^k)\big\}. \end{aligned}$$

The *Bratteli diagram* $\Lambda(n)$ is the graph with

(a) vertices $\{(\lambda, k) \mid k \in \mathbb{Z}_{\geq 0}, \lambda \in \hat{Z}_k\} \cup \{(\mu, k+\frac{1}{2}) \mid k \in \mathbb{Z}_{\geq 0}, \mu \in \hat{Z}_{k+\frac{1}{2}}\}$,

(b) an edge $(\lambda, k) \to (\mu, k+\frac{1}{2})$ if $\langle \mathrm{Res}_{U_{n-1}}^{U_n}(\chi^\lambda), \chi^\mu \rangle \neq 0$,

(c) an edge $(\mu, k+\frac{1}{2}) \to (\lambda, k+1)$ if $\langle \chi^\lambda, \mathrm{Ind}_{U_{n-1}}^{U_n}(\chi^\mu) \rangle \neq 0$,

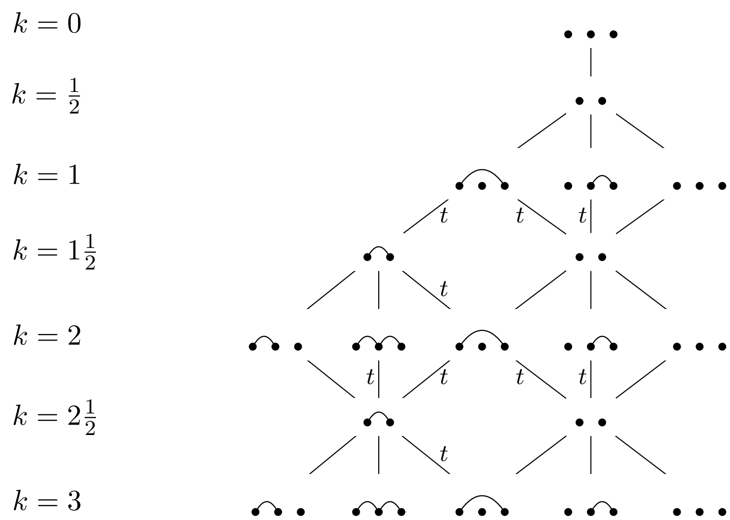
(d) an edge labeling $m : E \to \mathbb{Z}_{\geq 1}$ on the set of edges E defined by

$$m((\lambda, k) \to (\mu, k+\tfrac{1}{2})) = \frac{(q-1)^{|\lambda-\mu|} q^{\mathrm{crs}(\lambda\cap\mu, \lambda-\mu)}}{q^{\mathrm{crs}(\lambda\cap\mu, \mu-\lambda)}}$$

$$m((\mu, k+\tfrac{1}{2}) \to (\lambda, k+1)) = \frac{(q-1)^{|\mu-\lambda|} q^{\mathrm{crs}(\mu-\lambda, \lambda\cap\mu)}}{q^{\mathrm{crs}(\lambda-\mu, \lambda\cap\mu)}}.$$

Recall from the branching rules, Theorem 3.2.5, that the edge labeling $m((\lambda, k) \to (\mu, k+\frac{1}{2}))$ is the restriction coefficient which specifies the multiplicity that χ^μ appears in $\mathrm{Res}^{U_n}_{U_{n-1}}(\chi^\lambda)$. Similarly, the edge labeling $m((\lambda, k+\frac{1}{2}) \to (\mu, k+1))$ is the induction coefficient which specifies the multiplicity that χ^λ appears in $\mathrm{Ind}^{U_n}_{U_{n-1}}(\chi^\mu)$.

When drawing the Bratteli diagram, we place all the vertices (λ, l) in the lth row and simply write λ. For example, the Bratteli diagram for $\Lambda(3)$ up to row 3 is



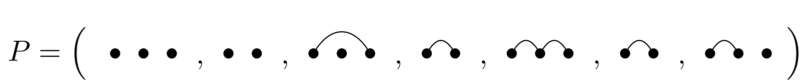
where $t = q - 1$.

A *path* P in the Bratteli diagram $\Lambda(n)$ to $\lambda \in \hat{Z}_k$ is a sequence $P = (\lambda^0, \lambda^{\frac{1}{2}}, \ldots, \lambda^{k-\frac{1}{2}}, \lambda^k = \lambda)$ such that for $0 \leq r \leq k-1$,

(a) (λ^r, r) and $(\lambda^{r+\frac{1}{2}}, r+\frac{1}{2})$ are vertices in $\Lambda(n)$

(b) $(\lambda^r, r) \to (\lambda^{r+\frac{1}{2}}, r+\frac{1}{2})$ and $(\lambda^{r+\frac{1}{2}}, r+\frac{1}{2}) \to (\lambda^{r+1}, r+1)$ are edges in $\Lambda(n)$.

For instance,



is a path in $\Lambda(3)$.

Taking the edge labeling into account, we say the *weight* $\mathrm{wt}(P)$ of a path P is the product

$$\prod_{r=1}^{k-1} m((\lambda^r, r) \to (\lambda^{r+\frac{1}{2}}, r+\tfrac{1}{2}))m((\lambda^{r+\frac{1}{2}}, r+\tfrac{1}{2}) \to (\lambda^{r+1}, r+1))$$

of its edge labels. The sum of the weights of the paths to $\lambda \in \hat{Z}_k$ is the multiplicity that χ^λ appears in V^k. The path given above has weight t^2 since $m((\lambda^1, 1) \to (\lambda^{1\frac{1}{2}}, \frac{1}{2})) = t$ and $m((\lambda^2, 2) \to (\lambda^{2\frac{1}{2}}, \frac{1}{2})) = t$.

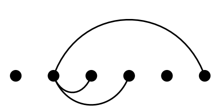
Let $\mathcal{P}_k(\lambda)$ be the set of paths in $\Lambda(n)$ to $\lambda \in \hat{Z}_k$. There is a combinatorial way to encode paths in $\mathcal{P}_k(\lambda)$ using a generalization of shells.

Definition 4.1.1. Let $s' \in \{s, s+1\}$ for $s \in \mathbb{Z}_{\geq 1}$ and $1 \leq i \leq l \leq n$. A *generalized shell* of *width* $l - i$ is a set of arcs on n nodes of the form

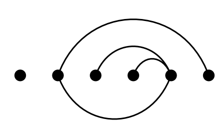
$$\bigcup_{r=1}^{s} \{j \frown \min L_r \mid j \in I_r\} \cup \bigcup_{r=1}^{s'-1} \{\max I_r \smile m \mid m \in L_{r+1}\}$$

where $I_r, L_r \subseteq [n]$ with $\{i\} = I_1 < \cdots < I_s \leq L_{s'} < \cdots < L_1 = \{l\}$.

For subsets $I, L \subseteq [n]$ we say $I < L$ if $i < l$ for each $i \in I$ and $l \in L$. If $\max I = \min L$, we say $I \leq L$. It follows that a generalized shell with $|I_r| = 1$ and $|L_r| = 1$ for all r is simply a shell in the sense of Definition 3.1.3. Some generalized shells of size 6 and width $6-2$ are

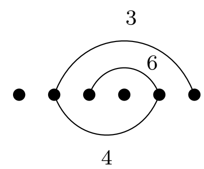


$\{2 \frown 6\} \cup \{2 \smile m \mid m \in \{3,4\}\}$,



$\{2 \frown 6\} \cup \{j \frown 5 \mid j \in \{3,4\}\} \cup \{2 \smile 5\}$.

A *labeled shell* is a pair (ς, τ) for a generalized shell ς and a map $\tau : \varsigma \to \mathbb{Z}_{\geq 0}$. We say the labeling τ is *strict* if every pair of arcs $(i,l), (j,m) \in \varsigma$ with $\dim(i,j) > \dim(j,m)$ satisfies $\tau(i,j) < \tau(j,m)$, and $\tau(j,m) \neq \tau(i,l) + 1$ if $i = j$ or $l = m$. If $\tau(i,j) = a$, we write the labeled arc as $(i,j;a)$. When the orientation of the arc is specified we write $(i \frown j; a)$ or $(i \smile j; a)$. For example, in the case of the shell



$$\{(2 \frown 6;3),(3 \frown 5;6),(2 \smile 5;4)\}.$$

From strictly labeled shells, we define the key notion shell tableaux.

Definition 4.1.2. A *shell tableau* $T = (\varsigma^1, \dots, \varsigma^k)$ of *length* k is a sequence of strictly labeled shells ς^r of size n and width $n - i_r$ such that

(1) for $1 \le r < k, \varsigma^r = \{(n \frown n; a)\}$ or $|\varsigma^r| \ge 2$, and $\varsigma^k = \{(i_k \frown n; a)\}$;

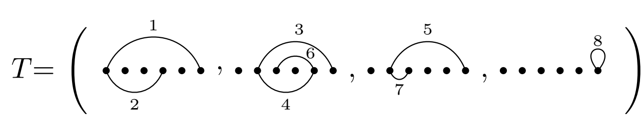
(2) each arc has a distinct label in $\{1, 2, \dots, \sum_{r=1}^{k} |\varsigma^r|\}$;

(3) the two smallest labels of each labeled shell ς^r are less than the smallest label in ς^{r+1};

(4) for $l \neq m$ and $i < j \le \min\{l, m\}$, if $(i, l; a) \in \varsigma^{r_l}$ then there exists a minimal $b > a$ such that $(i, m; b) \in \varsigma^{r_m}$ if and only if $(j, \min\{l, m\}; b + 1) \in \varsigma^{r_{\min\{l,m\}}}$;

(5) for $i \neq j$ and $\max\{i, j\} \le l < m$, if $(j, m; a) \in \varsigma^{r_j}$ then there exists a minimal $b > a$ such that $(i, m; b) \in \varsigma^{r_i}$ if and only if $(\max\{i, j\}, l; b + 1) \in \varsigma^{r_{\max\{i,j\}}}$.

Conditions 1–3 provide the basic set up of the shells and labeling that generalizes the condition of increasing entires along the rows and columns in standard Young tableaux. Intuitively conditions 4 and 5 say a strictly labeled shell in a shell tableau has inner whorls if and only if its outer whorl conflicts with the outer whorl of another shell. As an example consider the tableau



of length 4. By condition 4, the inner half whorl $(3 \frown 5; 6)$ lies in ς^2 since $(2 \smile 5; 4) \in \varsigma^2$ conflicts with $(2 \frown 6; 5) \in \varsigma^3$.

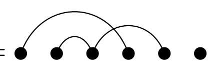
Let $\mathcal{ST}_k$ denote the set of shell tableaux of length k.

Definition 4.1.3. Define the map

$$\begin{aligned}\mathrm{sh} : (\mathbb{Z}_{\geq 0}, \mathcal{ST}_k) &\longrightarrow \text{Set of Arcs}\\ (a,T) &\longmapsto \bigcup_{r=1}^{k} \left\{ (i,l) \;\middle|\; \begin{array}{c} i \neq l \text{ and } \tau((i,l)) \text{ is maximal} \\ \text{among all labels } b \in \varsigma^r \text{ with } b \leq a \end{array} \right\},\end{aligned}$$

and $\mathrm{sh}(T) = \mathrm{sh}(|T|, T)$ be the *shape* of a shell tableau T.

For T in the example above, we have

sh$(T) =$ 

because $\tau(1,4) = 2$ is the maximal label ς^1, $\tau(3,5) = 6$ is the maximal label in ς^2 and $\tau(2,3)$ is the maximal label in ς^3. For $\lambda \vdash [n]$, let $\mathcal{ST}_k(\lambda)$ denote the set of shell tableaux of shape λ.

Theorem 4.1.4. *Let $\lambda \in \hat{Z}_k$. There is a bijection between $\mathcal{P}_k(\lambda)$ and $\mathcal{ST}_k(\lambda)$.*

Proof. Let $\lambda \in \hat{Z}_k$. Given a path $P = (\lambda^0, \lambda^{\frac{1}{2}}, \lambda^1, \ldots, \lambda^{k-\frac{1}{2}}, \lambda^k) \in \mathcal{P}_k(\lambda)$ we will recursively define a sequence

$$T_0, T_{\frac{1}{2}}, T_1, \ldots, T_{k-\frac{1}{2}}, T_k$$

where T_j is a shell tableau of length j and shape λ^j, and $T_{j+\frac{1}{2}}$ is a shell tableau of length $j+1$ and shape $\lambda^{j+\frac{1}{2}}$. Let T_0 be the empty shell tableau of length 0.

(1) If $\lambda^{j+\frac{1}{2}} = \lambda^j$, we define

$$T_{j+\frac{1}{2}} = (\varsigma^1_{j+\frac{1}{2}}, \varsigma^2_{j+\frac{1}{2}}, \ldots, \varsigma^{j+1}_{j+\frac{1}{2}}),$$

where

$$\varsigma^r_{j+\frac{1}{2}} = \begin{cases} \varsigma^r_j & \text{if } r < j+1, \\ (n, n; |T_j| + 1) & \text{if } r = j+1. \end{cases}$$

(2) If $\lambda^{j+\frac{1}{2}} \neq \lambda^j$, suppose

$$\bigcup_{s=1}^{t} \{i_s \frown l_s\} \cup \bigcup_{s=1}^{t} \{i_s \smile l_{s+1}\}$$

is the shell created by the symmetric difference of λ^r and $\lambda^{r+\frac{1}{2}}$. Since $\mathrm{sh}(T_j) = \lambda^j$, for $1 \le s \le t$, $(i_s, l_s; a_s)$ is an arc with maximal label $a_s \le |T_j|$ in a diagram $\varsigma_j^{r_s}$ of T_j. Let

$$T_{j+\frac{1}{2}} = (\varsigma_{j+\frac{1}{2}}^1, \varsigma_{j+\frac{1}{2}}^2, \ldots, \varsigma_{j+\frac{1}{2}}^{j+1}),$$

where

$$\varsigma_{j+\frac{1}{2}}^r = \begin{cases} \varsigma_j^r & \text{if } r \ne r_s \text{ for any } s, \\ \varsigma_j^r + (i_s, l_{s+1}; |T_j| + s) & \text{if } r = r_s \text{ for some } s, \\ (n, n; |T_j| + t + 1) & \text{if } r = j+1. \end{cases}$$

(3) If $\lambda^{j+1} = \lambda^{j+\frac{1}{2}}$, define $T_{j+1} = T_{j+\frac{1}{2}}$.

(4) If $\lambda^{j+1} \ne \lambda^{j+\frac{1}{2}}$, suppose

$$\bigcup_{s=1}^{t} \{i_s \frown l_s\} \cup \bigcup_{s=1}^{t-1} \{i_s \smile l_{s+1}\}$$

is the shell created by the symmetric difference of λ^r and $\lambda^{r+\frac{1}{2}}$. Since $\mathrm{sh}(T_{j+\frac{1}{2}}) = \lambda^{j+\frac{1}{2}}$ then for $1 \le s \le t-1$, $(i_s, l_{s+1}; a_s)$ is an arc with maximal label $a_s \le |T_{j+\frac{1}{2}}|$ in a distinct diagram $\varsigma_{j+\frac{1}{2}}^{r_s}$ in $T_{j+\frac{1}{2}}$. We define

$$T_{j+1} = (\varsigma_{j+1}^1, \varsigma_{j+1}^2, \ldots, \varsigma_{j+1}^{j+1}),$$

where

$$\varsigma_{j+1}^r = \begin{cases} \varsigma_{j+\frac{1}{2}}^r & \text{if } r \ne r_s \text{ for any } s, \\ \varsigma_{j+\frac{1}{2}}^r + (i_{s+1}, l_{s+1}; |T_{j+\frac{1}{2}}| + s) & \text{if } r = r_s \text{ for some } s, \\ (i_1, l_1, |T_{j+\frac{1}{2}}|) & \text{if } r = j+1. \end{cases}$$

In the above construction, we have $T_{\frac{1}{2}} = (\varsigma_{\frac{1}{2}}^1)$ where $\varsigma_{\frac{1}{2}}^1 = \{(n \frown n; 1)\}$ is a shell tableau of length 1 and shape $\emptyset$. If T_j is a shell tableau of length j and shape λ^j, then $T_{j+\frac{1}{2}}$ has length $j+1$ and

$$\mathrm{sh}(T_{j+\frac{1}{2}}) = (\lambda^j \cap \lambda^{j+\frac{1}{2}}) \cup (\lambda^{j+\frac{1}{2}} - \lambda^j) = \lambda^{j+\frac{1}{2}}.$$

It is straightforward to check that $T_{j+\frac{1}{2}}$ satisfies conditions 1–4. Since T_j is a shell tableau, it suffices to prove condition 5 for the arcs $(i_s, l_{s+1}; |T_j| + s)$. For $s > 1$, consider $(i_s, l_s; a_s) \in \varsigma_{j+\frac{1}{2}}^{r_s}$. Then

$(i_{s-1}, l_s; |T_j| + s - 1)$ lies in $\varsigma^{r_{s-1}}_{j+\frac{1}{2}}$ where $|T_j| + s - 1 > a_s$ is minimal, and $(i_s, l_{s+1}; |T_j| + s) \in \varsigma^{r_s}_{j+\frac{1}{2}}$. Thus condition 5 holds, so $T_{j+\frac{1}{2}}$ is in fact a shell tableau. A similar argument can be used to verify each T_j is a shell tableau of length j and shape λ^j.

For $\lambda \in \hat{Z}_k$, define

$$\begin{aligned} \varphi : \mathcal{P}_k(\lambda) &\longrightarrow \mathcal{ST}_k(\lambda) \\ P &\mapsto T_k. \end{aligned}$$

The map φ is bijective since the construction of the sequence of shell tableaux can be reversed as follows. Given a shell tableau $T = (\varsigma^1, \varsigma^2, \ldots, \varsigma^k)$ of shape λ, let $T_k = T$.

(1) If $\varsigma^j = \{(n \frown n; a)\}$, define $T_{j-\frac{1}{2}} = T_j$.

(2) If $(i \frown n; a) \in \varsigma^j$ for $i < n$, let $(i_1, l_1; a_1), (i_2, l_2; a_2), \ldots, (i_t, l_t; a_t)$ be the arcs in T_j with $i_s \frown l_s \in \varsigma^{r_s}_j$ and $a_s \geq a$. We define

$$T_{j-\frac{1}{2}} = (\varsigma^1_{j-\frac{1}{2}}, \varsigma^2_{j-\frac{1}{2}}, \ldots, \varsigma^j_{j-\frac{1}{2}}),$$

where

$$\varsigma^r_{j-\frac{1}{2}} = \begin{cases} \varsigma^r_j & \text{if } r \neq r_s \text{ for any } s, \\ \varsigma^r_j - (i_s, l_s; a_s) & \text{if } r = r_s \text{ for some } s, \\ (n, n, |T_j| - t + 1) & \text{if } r = j. \end{cases}$$

(3) If $\varsigma^j = \{(n \frown n; a)\}$, define $T_j = (\varsigma^1_j, \varsigma^2_j, \ldots, \varsigma^{j-1}_j)$ where $\varsigma^r_j = \varsigma^r_{j+\frac{1}{2}}$.

(4) If $(i \frown n; a) \in \varsigma^j$ for $i < n$, let $(i_1, l_1; a_1), (i_2, l_2; a_2), \ldots, (i_t, l_t; a_t)$ be the arcs in T_j with $i_s \frown l_s \in \varsigma^{r_s}_j$ and $a_s > a$. We define

$$T_j = (\varsigma^1_j, \varsigma^2_j, \ldots, \varsigma^{j-1}_j),$$

where

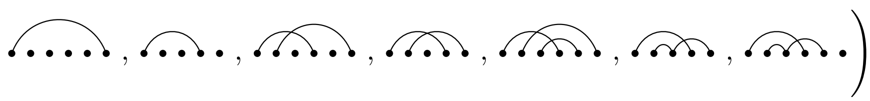
$$\varsigma^r_j = \begin{cases} \varsigma^r_{j+\frac{1}{2}} & \text{if } r \neq r_s \text{ for any } s, \\ \varsigma^r_{j+\frac{1}{2}} - (i_s, l_s; a_s) & \text{if } r = r_s \text{ for some } s. \end{cases}$$

Therefore the inverse of φ is

$$\begin{array}{rcl}\varphi^{-1} : \mathcal{ST}_k(\lambda) & \longrightarrow & \mathcal{P}_k(\lambda) \\ T & \mapsto & P = (\mathrm{sh}(T_1), \mathrm{sh}(T_2), \ldots, \mathrm{sh}(T_k)).\end{array}$$

□

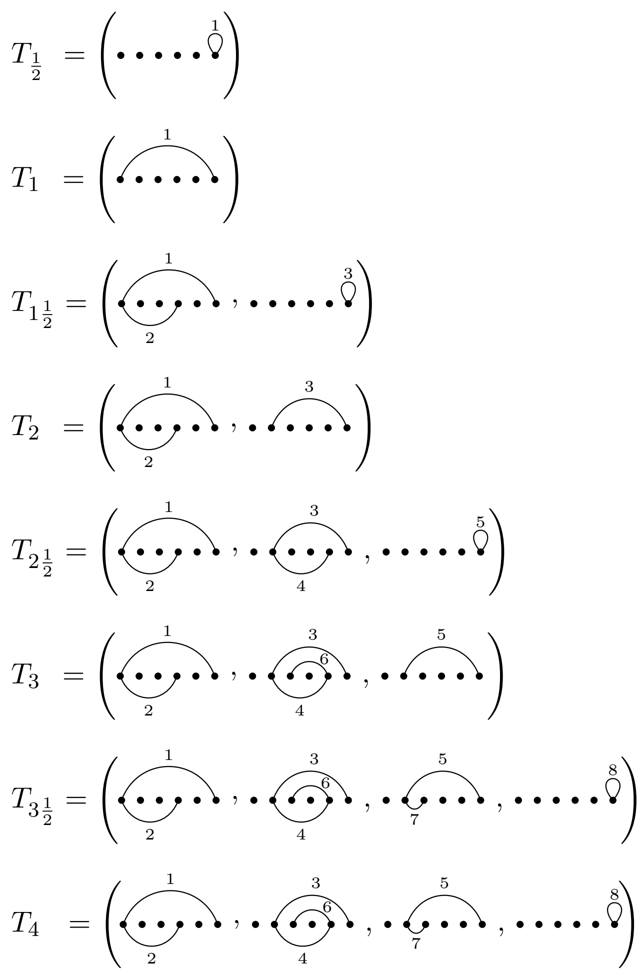
Example 4.1.5. For the path

$P = (\emptyset, \emptyset,$  $)$

the sequence of shell tableaux is

$T_0 = (\,)$

$T_{\frac{1}{2}}, T_1, T_{1\frac{1}{2}}, T_2, T_{2\frac{1}{2}}, T_3, T_{3\frac{1}{2}}, T_4 =$



Note that each shell ς^r keeps track of the arc introduced at the rth row of the Bratteli diagram from inducing $\mathrm{Res}^{U_n}_{U_{n-1}}(V^{\otimes r-1})$.

When $q = 2$, then $q - 1 = 1$ so that many of the edges in the Bratteli diagram have weight 1. In this case, we can account for the weights of paths in the Bratteli diagram by removing the second condition in the definition of a strict labeling. A *semi-strict shell tableau* is a shell tableau where we allow $\tau(j, m) = \tau(i, l) + 1$ for every pair of arcs $(i, l; \tau(i, l))$ and $(j, m; \tau(j, m))$ in a labeled shell with $\dim(i, l) > \dim(j, m)$ and $i = j$ or $k = l$. This is reminiscent of semi-standard Young tableaux where we allow the entries along the rows to be weakly increasing.

Suppose $\mathcal{SST}_k(\lambda)$ is the set of semi-strict shell tableaux of length k and shape λ. Recall the sum of the weights of paths to λ is the multiplicity of χ^λ in V^k. When $q = 2$, this is the number of semi-strict shell tableaux.

Proposition 4.1.6. *Let $q = 2$ and $\lambda \in \hat{Z}_k$. Then*

$$\sum_{P \in \mathcal{P}_k(\lambda)} w(P) = |\mathcal{SST}_k(\lambda)|.$$

Proof. Let $q = 2$ and $\lambda \in \hat{Z}_k$. Let $T = (\varsigma^1, \dots, \varsigma^k) \in \mathcal{ST}_k(\lambda)$ be the shell tableau corresponding to the path $P = (\lambda^0, \lambda^{\frac{1}{2}}, \dots, \lambda^{k-\frac{1}{2}}, \lambda^k)$ via the bijection in the proof of Theorem 4.1.4. Since the weight of P is the product of its edge labels, it suffices to consider the label of a single edge. Recall the label of an edge $(\lambda^r, r) \to (\lambda^{r+\frac{1}{2}}, r + \frac{1}{2})$ in P is

$$m((\lambda^r, r) \to (\lambda^{r+\frac{1}{2}}, r + \tfrac{1}{2})) = \frac{2^{\mathrm{crs}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^r - \lambda^{r+\frac{1}{2}})}}{2^{\mathrm{crs}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^{r+\frac{1}{2}} - \lambda^r)}}.$$

Suppose

$$\bigcup_{s=1}^{t} \{i_s \frown l_s\} \cup \bigcup_{s=1}^{t} \{i_s \smile l_{s+1}\}$$

is the shell created by the symmetric difference of λ^r and $\lambda^{r+\frac{1}{2}}$ where $(i_s, l_{s+1}; a_s) \in \varsigma^{r_s}$. For $1 \le s \le t$, define the set

$$Y_s = \left\{ m \;\middle|\; \begin{array}{c} (j \frown m, i_s \frown l_s) \in \mathrm{CRS}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^r - \lambda^{r+\frac{1}{2}}), \\ (j \smile m, i_s \smile l_{s+1}) \notin \mathrm{CRS}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^{r+\frac{1}{2}} - \lambda^r) \end{array} \right\}.$$

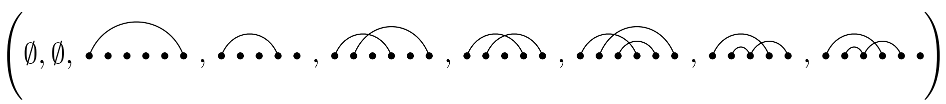
Note that

$$\sum_{s=1}^{t}|Y_s| = \mathrm{crs}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^r - \lambda^{r+\frac{1}{2}}) - \mathrm{crs}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^{r+\frac{1}{2}} - \lambda^r).$$

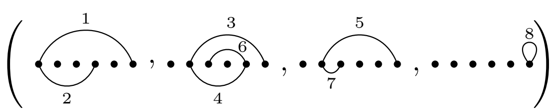
For each subset $X_s \subseteq Y_s$, add the arcs $(i_s, l; b_l) \in \varsigma^{r_s}$ for $l \in X_s$. There is a unique relabeling of the arcs with a distinct label in $\{1, 2, \ldots, \sum_{r=1}^{k}|\varsigma^r| + \sum_{s=1}^{t}|X_s|\}$ so that the order of the labels of the original arcs in T is preserved, and every pair of arcs $(i, l; \tau(i, l))$ and $(j, m; \tau(j, m))$ in a labeled shell with $\dim(i, l) > \dim(j, m)$ satisfies $\tau(i, l) < \tau(j, m)$. Then each $(X_1, \ldots, X_t)$ determines one of the $2^{\sum_{s=1}^{t}|Y_s|}$ semi-strict shell tableaux.

□

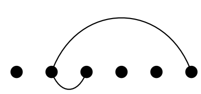
Example 4.1.7. Consider the path P from Example 4.1.5,

$P =$ 

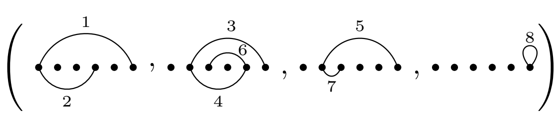
and corresponding tableaux

$T =$ 

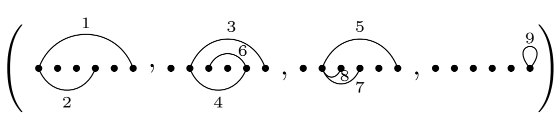
The path P has weight 2 since $m((\lambda^3, 3) \to (\lambda^{3\frac{1}{2}}, 3\tfrac{1}{2})) = 2$. The shell created by the symmetric difference between λ^3 and $\lambda^{3\frac{1}{2}}$ is



and the set $Y_1 = \{4\}$ as $(1 \frown 4, 2 \frown 6) \in \mathrm{CRS}(\lambda^3 \cap \lambda^{3\frac{1}{2}}, \lambda^3 - \lambda^{3\frac{1}{2}})$, but $(1 \smile 4, 2 \smile 3) \notin \mathrm{CRS}(\lambda^3 \cap \lambda^{3\frac{1}{2}}, \lambda^{3\frac{1}{2}} - \lambda^3)$. The two semi-strict tableaux corresponding to $\emptyset$ and Y_1 are

$T =$ 

and

$\tilde{T} =$ 

respectively.

Chapter 5

Schur–Weyl dualities

Schur–Weyl duality is a fundamental framework in combinatorial representation theory [14]. Classically it relates the irreducible representations of the symmetric group S_n to the irreducible representations of the general linear group $GL_n(\mathbb{C})$ via their commuting actions. More recently, the study of new versions of Schur–Weyl duality has led to many remarkable discoveries about algebras of operators on tensor space that are full centralizers of each other [10, 14, 17].

5.1 Classical Schur–Weyl duality

First, we review the classical Schur–Weyl duality of $GL_n(\mathbb{C})$ and S_k. Let V be an n dimensional complex vector space. Consider the tensor space

$$V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ factors}}.$$

The general linear group $GL_n(\mathbb{C})$ acts on $V^{\otimes k}$ diagonally, and the symmetric group S_k acts on $V^{\otimes k}$ by permuting factors. Schur–Weyl duality gives rise to the following two statements.

(a) These actions commute and each action generates the full centralizer of the other. That is, if the representations arising from each action are

$$\mathbb{C}GL_n(\mathbb{C}) \xrightarrow{\rho} \operatorname{End}(V^{\otimes k}) \xleftarrow{\pi} \mathbb{C}S_k$$

then the images of ρ and π are

$$\rho(\mathbb{C}GL_n(\mathbb{C})) = \operatorname{End}_{S_k}(V^{\otimes k}) \quad \text{and} \quad \pi(\mathbb{C}S_k) = \operatorname{End}_{GL_n(\mathbb{C})}(V^{\otimes k}).$$

(b) This double-centralizer relationship in Theorem 2.1.7 produces

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S_k^\lambda \quad \text{as a } (GL_n(\mathbb{C}), S_k)\text{-bimodule},$$

where the G^λ are irreducible $GL_n(\mathbb{C})$-modules and the S_k^λ are irreducible S_k-modules.

The bimodule decomposition of $V^{\otimes k}$ makes studying the irreducible representations of $GL_n(\mathbb{C})$ and the irreducible representations of S_k two sides of the same coin [14].

5.2 The partition algebra

There are many groups besides $GL_n(\mathbb{C})$ and S_k that play analogous Schur–Weyl duality roles. We turn to a version of Schur–Weyl duality obtained by restricting the action of $GL_n(\mathbb{C})$ to one of its subgroups.

Viewing $S_n \subseteq GL_n$ as the subgroup of permutation matrices, S_n acts on $V^{\otimes k}$ diagonally

$$\begin{array}{ccccc} & & & & \mathbb{C}A_k(n) \\ & & & & \cup\!\!\mid \\ GL_n(\mathbb{C}) & \longrightarrow & V^{\otimes k} & \longleftarrow & \mathbb{C}S_k \\ \cup\!\!\mid & & & & \\ S_n & & & & \end{array}$$

and its corresponding centralizer algebra $\mathrm{End}_{S_n}(V^{\otimes k})$ is an algebra described in terms of set partitions of $\{1, 2, \ldots, 2k\}$ known as the partition algebra $\mathbb{C}A_k(n)$ [14].

The decomposition of $V^{\otimes k}$ as a $(S_n, A_k(n))$-bimodule is

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash \hat{A}_k(n)} S_n^\lambda \otimes A_k^\lambda(n)$$

where the S_n^λ are irreducible S_n-modules and the $A_k^\lambda(n)$ are irreducible $A_k(n)$-modules. As an S_n-module, V is the permutation representation so that

$$V \cong \mathrm{Ind}_{S_{n-1}}^{S_n} \mathrm{Res}_{S_{n-1}}^{S_n} (\mathbb{1}),$$

where $\mathbb{1}$ is the trivial character of S_n [14, (3.16)]. In general, for any S_n-module M,

$$\begin{aligned}\mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n}(M) &\cong \mathbb{C}S_n \otimes_{\mathbb{C}S_{n-1}} \mathrm{Res}_{S_{n-1}}^{S_n}(M)\\ &\cong (\mathbb{C}S_n \otimes_{\mathbb{C}S_{n-1}} \mathbb{1}) \otimes M \qquad (5.1)\\ &\cong V \otimes M\end{aligned}$$

by the definition of induction and the tensor identity (5.1) from [14, (3.18)]. Iterating this identity, it follows that

$$V^{\otimes k} \cong \underbrace{(\mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n})}_{k \text{ times}}{}^{k}(\mathbb{1})$$

where the trivial character is restricted and induced k times.

5.3 A unipotent analogue of Schur–Weyl duality

We examine the analogue of Schur–Weyl duality for the group U_n of unipotent upper-triangular matrices. Let $V = \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1}$ where $\mathbb{1}$ is the trivial supercharacter. By the definition of induction, V is given by

$$V = \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathrm{Res}_{U_{n-1}}^{U_n}(\mathbb{1}) = \mathrm{Ind}_{U_{n-1}}^{U_n}\mathrm{Res}_{U_{n-1}}^{U_n}(\mathbb{1}).$$

More broadly, we have the following generalization of the tensor identity (5.1) from [14, (3.18)].

Lemma 5.3.1. *Let H be a subgroup of a group G. For a G-module M, the map*

$$\begin{aligned}\tau : \mathbb{C}G \otimes_{\mathbb{C}H} \mathrm{Res}_H^G(M) &\longrightarrow (\mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{1}) \otimes M\\ g \otimes m &\mapsto (g \otimes \mathbb{1}) \otimes gm\\ g \otimes g^{-1}m &\leftarrow\!\shortmid (g \otimes \mathbb{1}) \otimes m\end{aligned}$$

is a G-module isomorphism.

Iterating this identity, we obtain

$$V^{\otimes k} \cong \underbrace{(\mathrm{Ind}_{U_{n-1}}^{U_n}\mathrm{Res}_{U_{n-1}}^{U_n})}_{k \text{ times}}{}^{k}(\mathbb{1}),$$

where the trivial supercharacter is restricted and induced k times. This is reminiscent of the situation in the partition algebra from Section 5.2 where the permutation representation of the symmetric group is isomorphic to restricting and then inducing the trivial character.

The group U_n acts diagonally on $V^{\otimes k}$ and the goal of Schur–Weyl duality is to determine the centralizer algebra $\mathrm{End}_{U_n}(V^{\otimes k})$ generated by this action. By double centralizer theory, the decomposition of the centralizer algebra into irreducibles can be obtained by decomposing the tensor product $V^{\otimes k}$ into irreducible U_n-modules. However, we generally cannot decompose $V^{\otimes k}$ into irreducible U_n-modules as the representation theory of U_n is well known to be "wild" [13]. Therefore, we approximate with supercharacters and consider the following centralizer subalgebras

$$\tilde{Z}_{k,n} = \{\varphi \in \mathrm{End}_{U_n}(V^{\otimes k}) \mid \text{for all } (\lambda, k) \in \Lambda(n),\ \varphi(U_n^\lambda) \cong U_n^\lambda \text{ or } \varphi(U_n^\lambda) = 0\}$$

$$\cup\!\mid$$

$$Z_{k,n} = \{\varphi \in \tilde{Z}_{k,n} \mid \text{if } \varphi(U_n^\lambda) = U_n^\lambda, (\lambda, k) \in \Lambda(n),\ \text{then } \varphi|_{U_n^\lambda} = a\mathrm{Id}, a \in \mathbb{C}\}.$$

where $\Lambda(n)$ is the Bratteli diagram.

Note that the first subalgebra $\tilde{Z}_{k,n}$ distinguishes supermodules so that the supermodules satisfy the first statement of Schur's lemma 2.1.1. While this subalgebra may be interesting to study, we use both statements of Schur's lemma in the next section to produce a bimodule decomposition of $V^{\otimes k}$. Hence we focus on Schur–Weyl duality for the second subalgebra $Z_{k,n}$, which distinguishes supermodules and treats them like irreducibles. That is, we seek to determine the centralizer subalgebra $Z_{k,n}$ of the action of U_n on $V^{\otimes k}$.

If all the supercharacters in the decomposition of $V^{\otimes k}$ are irreducible characters then we have the equalities $Z_{k,n} = \tilde{Z}_{k,n} = \mathrm{End}_{U_n}(V^{\otimes k})$ so that the centralizer subalgebras are in fact the full centralizer algebra. By Propositions 2.1.3 and 2.3.2, a supercharacter χ^λ is irreducible if and only if

$$\langle \chi^\lambda, \chi^\lambda \rangle = (q-1)^{|\lambda|} q^{\mathrm{crs}(\lambda,\lambda)} = 1.$$

Thus the supercharacters coincide with irreducible characters when $q = 2$, and $n \leq 3$ or $k < 2$.

However, the containment of subalgebras can be strict. To see this, suppose χ^λ is a supercharacter in the decomposition of $V^{\otimes k}$ that is not irreducible. Then χ^λ is a sum of at least two irreducible characters. Projection onto one of these constituents is in the full centralizer algebra $\mathrm{End}_{U_n}(V^{\otimes k})$, but not the subalgebra $\tilde{Z}_{k,n}$. Meanwhile, if χ^λ has at least two isomorphic constituents then permutation of the isomorphic constituents of χ^λ is in the subalgebra $\tilde{Z}_{k,n}$, but not $Z_{k,n}$.

5.3.1 Dimensions of centralizer subalgebras

We now show decomposing $V^{\otimes k}$ as a U_n-supermodule produces a decomposition of the centralizer subalgebra $Z_{k,n}$ using a supermodule analogue of double centralizer theory.

Theorem 5.3.2 (Double Centralizer Theorem)**.** *Suppose the decomposition of $V^{\otimes k}$ is given by*

$$V^{\otimes k} \cong \bigoplus_{(\lambda,k)\in\Lambda(n)} m_\lambda U_n^\lambda.$$

Then we obtain the decompositions

(a) $Z_{k,n} \cong \bigoplus_{(\lambda,k)\in\Lambda(n)} M_{m_\lambda}(\mathbb{C})$*;*

(b) as a $(U_n, Z_{k,n})$-bimodule

$$V^{\otimes k} \cong \bigoplus_{(\lambda,k)\in\Lambda(n)} U_n^\lambda \otimes Z_{k,n}^\lambda$$

where the Z^λ are irreducible $Z_{k,n}$-modules.

Proof. We follow the proof of the Centralizer Theorem in [14]. Index the components of the decomposition of $V^{\otimes k}$ by dummy variables ϵ_i^λ so that

$$V^{\otimes k} \cong \bigoplus_{(\lambda,k)\in\Lambda(n)} \bigoplus_{i=1}^{m_\lambda} U_n^\lambda \otimes \epsilon_i^\lambda.$$

This implies

$$Z_{k,n} \subseteq \mathrm{Hom}_{U_n}(V^{\otimes k}, V^{\otimes k}) \cong \mathrm{Hom}_{U_n}\Big(\bigoplus_\lambda \bigoplus_j U_n^\lambda \otimes \epsilon_j^\lambda, \bigoplus_\mu \bigoplus_i U_n^\mu \otimes \epsilon_i^\lambda\Big).$$

If $\varphi \in Z_{k,n}$ satisfies $\varphi|_{U_n^\lambda \otimes \epsilon_j^\lambda} \in \mathrm{Hom}_{U_n}(U_n^\lambda \otimes \epsilon_j^\lambda, U_n^\mu \otimes \epsilon_i^\lambda)$, then $\varphi(U_n^\lambda \otimes \epsilon_j^\lambda) \cong U_n^\mu \otimes \epsilon_i^\mu$, so $\lambda = \mu$ or $\varphi(U_n^\lambda \otimes \epsilon_j^\lambda) = 0$. Thus, we have the isomorphism

$$Z_{k,n} \cong \bigoplus_{\lambda,\mu} \bigoplus_{i,j} \delta_{\lambda\mu} \mathrm{Hom}_{U_n}(U_n^\lambda \otimes \epsilon_j^\lambda, U_n^\mu \otimes \epsilon_i^\mu).$$

For each $(\lambda, k) \in \Lambda(n)$ and $1 \leq i, j \leq m_\lambda$, let $\pi_{ij}^\lambda : U_n^\lambda \otimes \epsilon_j \to U_n^\lambda \otimes \epsilon_i$ be the U_n-supermodule isomorphism given by

$$\pi_{ij}^\lambda(u \otimes \epsilon_j^\lambda) = u \otimes \epsilon_i^\lambda, \qquad u \in U_n^\lambda.$$

Suppose $\pi, \varphi \in Z_{k,n}$ are such that $\pi|_{U_n^\lambda \otimes \epsilon_j^\lambda} = \pi_{ij}^\lambda$ and $\varphi|_{U_n^\lambda \otimes \epsilon_j^\lambda} : U_n^\lambda \otimes \epsilon_j^\lambda \to U_n^\lambda \otimes \epsilon_i^\lambda$ is a U_n-supermodule isomorphism. Then $\varphi \circ \pi^{-1} \in Z_{k,n}$ with $\varphi \circ \pi^{-1}(U_n^\lambda \otimes \epsilon_i^\lambda) = U_n^\lambda \otimes \epsilon_i^\lambda$. Thus $\varphi|_{U_n^\lambda \otimes \epsilon_j^\lambda} \circ (\pi_{ij}^\lambda)^{-1}$ is a multiple of the identity and $\varphi|_{U_n^\lambda \otimes \epsilon_j^\lambda} = \varphi|_{U_n^\lambda \otimes \epsilon_j^\lambda} \circ (\pi_{ij}^\lambda)^{-1}\pi_{ij}^\lambda$ is a multiple of π_{ij}^λ. Hence this yields

$$Z_{k,n} \cong \bigoplus_\lambda \bigoplus_{i,j=1}^{m_\lambda} \mathbb{C}\pi_{ij}^\lambda$$

so the π_{ij}^λ are a basis for the centralizer algebra. Therefore each element $\varphi \in Z_{k,n}$ can be written as

$$\varphi = \sum_{(\lambda,k)\in\Lambda(n)} \sum_{i,j=1}^{m_\lambda} z_{ij}^\lambda \pi_{ij}^\lambda \qquad z_{ij}^\lambda \in \mathbb{C},$$

and identified with an element of $\oplus_\lambda M_{m_\lambda}(\mathbb{C})$ by mapping

$$\varphi \mapsto \sum_{(\lambda,k)\in\Lambda(n)} \sum_{i,j=1}^{m_\lambda} z_{ij}^\lambda e_{ij}^\lambda,$$

where e_{ij}^λ is the matrix with a 1 in the (i, j) entry of the λth diagonal block. Since $\pi_{ij}^\lambda \pi_{kl}^\mu = \delta_{\lambda\mu}\delta_{jk}\pi_{il}^\lambda$, then this mapping is a homomorphism so that we have

$$Z_{k,n} \cong \bigoplus_{(\lambda,k)\in\Lambda(n)} M_{m_\lambda}(\mathbb{C}).$$

As a vector space, $Z_{k,n}^\mu = \mathbb{C}\text{-span}\{\epsilon_i^\mu \mid 1 \leq i \leq m_\mu\}$ is isomorphic to the irreducible $\bigoplus_\lambda M_{m_\lambda}(\mathbb{C})$-module of column vectors of length m_μ. It follows that $V^{\otimes k}$ is given by

$$V^{\otimes k} \cong \bigoplus_{(\lambda,k)\in\Lambda(n)} U_n^\lambda \otimes Z_{k,n}^\lambda$$

as a $(U_n, Z_{k,n})$-bimodule with the action

$$(g \otimes \pi_{ij}^\lambda)(u \otimes \epsilon_k^\mu) = \delta_{\lambda\mu}\delta_{jk}(gu \otimes \epsilon_i^\mu) \qquad u \in U_n^\mu, g \in U_n.$$

$\square$

The Bratteli diagram $\Lambda(n)$ encodes the decomposition of $V^{\otimes k}$ as a U_n-supermodule so it yields a decomposition of the centralizer subalgebra $Z_{k,n}$. By construction, the sum of the weights of paths to λ in row k is the multiplicity m_λ of each U_n-supermodule U_n^λ. Since

$$\begin{aligned} V^{\otimes k} &\cong \bigoplus_{(\lambda,k)\in\Lambda(n)} m_\lambda U_n^\lambda && \text{as a } U_n\text{-supermodule} \\ &\cong \bigoplus_{(\lambda,k)\in\Lambda(n)} U_n^\lambda \otimes Z_{k,n}^\lambda && \text{as a } (U_n, Z_{k,n})\text{-bimodule} \end{aligned}$$

by the supermodule version of the Double Centralizer theorem 5.3.2, it follows that the dimension of $Z_{k,n}^\lambda$ is m_λ. Therefore, the dimension of $Z_{k,n}^\lambda$ is

$$\dim Z_{k,n}^\lambda = m_\lambda = \sum_{P\in\mathcal{P}_k(\lambda)} \mathrm{wt}(P)$$

where $\mathcal{P}_k(\lambda)$ is the set of paths in $\Lambda(n)$ to λ in row k. By part (a) of the Double Centralizer Theorem $Z_{k,n} \cong \bigoplus_{(\lambda,k)\in\Lambda(n)} M_{m_\lambda}(\mathbb{C})$, so the dimension of the centralizer subalgebra is

$$\begin{aligned} \dim Z_{k,n} &= \sum_{(\lambda,k)\in\Lambda(n)} (\dim Z_{k,n}^\lambda)^2 \\ &= \sum_{(\lambda,k)\in\Lambda(n)} m_\lambda^2 \\ &= \sum_{(\lambda,k)\in\Lambda(n)} \Big(\sum_{P\in\mathcal{P}_k(\lambda)} \mathrm{wt}(P) \Big)^2 = \sum_{(\lambda,k)\in\Lambda(n)} \sum_{P,Q\in\mathcal{P}_k(\lambda)} \mathrm{wt}(P)\,\mathrm{wt}(Q). \end{aligned}$$

Thus we can calculate the dimension of the centralizer subalgebra from the Bratteli diagram.

Theorem 5.3.3. *The dimension of the centralizer subalgebra $Z_{k,n}$ is a polynomial in q.*

Proof. By definition, the weight of a path is the product of its edge labels, which are products of powers of $(q-1)$ and q. Thus for any path $P \in \mathcal{P}_k(\lambda)$, $\mathrm{wt}(P) = (q-1)^{a_P} q^{b_P}$ for some $a_P, b_P \in \mathbb{Z}_{\geq 0}$. It follows that

$$\dim Z_{k,n} = \sum_{(\lambda,k)\in\Lambda(n)} \sum_{P,Q\in\mathcal{P}_k(\lambda)} (q-1)^{a_P} q^{b_P} (q-1)^{a_Q} q^{b_Q}$$

is a polynomial in q. □

Some dimensions of centralizer subalgebras $Z_{k,n}$ for $q=2$ are shown below.

	$k=1$	$k=1\frac{1}{2}$	$k=2$	$k=2\frac{1}{2}$	$k=3$
$n=2$	2	4	8	16	32
$n=3$	3	10	36	136	528
$n=4$	4	19	105	676	4600
$n=5$	5	31	235	2257	24125

When the centralizer subalgebra $Z_{k,n}$ is the full centralizer algebra $\mathrm{End}_{U_n}(V^{\otimes k})$, we have nice formulas for its dimension.

Proposition 5.3.4. *For $q=2$, $\dim \mathrm{End}_{U_{n-1}}(\mathrm{Res}^{U_n}_{U_{n-1}}(V)) = 3n(n-1)/2+1$, the nth centered triangular number (sequence A005448 in OEIS).*

Proof. Let $q=2$. By the branching rules 3.2.5, the restriction $\mathrm{Res}^{U_n}_{U_{n-1}}(V)$ is

$$\mathrm{Res}^{U_n}_{U_{n-1}}(V) = n\mathbb{1} + \sum_{1\le i<k<n} \chi^{i\frown k}$$

which implies

$$\dim \mathrm{End}_{U_{n-1}}(\mathrm{Res}^{U_n}_{U_{n-1}}(V)) = n^2 + \sum_{1\le i<k<n} 1 = \frac{3n(n-1)}{2} + 1.$$

□

Proposition 5.3.5. *For $q=2$, $\dim \mathrm{End}_{U_3}(V^{\otimes k}) = s(2k-1)$ and $\dim \mathrm{End}_{U_2}(\mathrm{Res}^{U_3}_{U_2}(V^{\otimes k})) = s(2k)$ where $s(n)=2^{n-1}(1+2^n)$ (sequence A007582 in OEIS).*

Proof. Let $q=2$ and $n=3$. By the branching rules 3.1.10, $V^{\otimes k}$ is given by

$$\begin{aligned} V^{\otimes k} \quad\cong\quad & 2^{k-2}(2^{k-1}+1)\mathbb{1} + 2^{k-2}(2^{k-1}+1)\chi^{2\frown 3} + 2^{2k-2}\chi^{1\frown 3} \\ & + 2^{k-2}(2^{k-1}-1)\chi^{1\frown 2} + 2^{k-2}(2^{k-1}-1)\chi^{1\frown 2\frown 3}, \end{aligned}$$

and so the dimension of the centralizer algebra is

$$\dim \mathrm{End}_{U_3}(V^{\otimes k}) = 2(2^{k-2}(2^{k-1}+1))^2 + (2^{2k-2})^2 + 2(2^{k-2}(2^{k-1}-1))^2 = 2^{2k-2}(1+2^{2k-1}).$$

Similarly, the restriction of $V^{\otimes k}$ is

$$\mathrm{Res}_{U_2}^{U_3}(V^{\otimes k}) \cong 2^{k-1}(2^k+1)\mathbb{1} + 2^{k-1}(2^k-1)\chi^{1\frown 2},$$

which implies

$$\dim \mathrm{End}_{U_2}(\mathrm{Res}_{U_2}^{U_3} V^{\otimes k}) = (2^{k-1}(2^k+1))^2 + (2^{k-1}(2^k-1))^2 = 2^{2k-1}(1+2^{2k}).$$

□

5.3.2 The path basis

Since summing the product of the weights of pairs of paths in the Bratteli diagram gives the dimension of the centralizer subalgebra, these paths also help index a basis. In this section we will explicitly construct part of this basis of the centralizer subalgebra.

Recall, from the proof of the supermodule version of the Double Centralizer Theorem 5.3.2 that if

$$V^{\otimes k} \cong \bigoplus_{(\lambda,k)\in\Lambda(n)} \bigoplus_{i=1}^{m_\lambda} U_n^\lambda \otimes \epsilon_i^\lambda$$

a basis for $Z_{k,n}$ is given by the maps

$$\pi_{ij}^\lambda : U_n^\lambda \otimes \epsilon_j^\lambda \to U_n^\lambda \otimes \epsilon_i^\lambda.$$

We seek to determine each copy of $U_n^\lambda \otimes \epsilon_i$ inside $V^{\otimes k}$ in order to provide explicit formulas for these maps.

More precisely, since

$$V^{\otimes k} = (\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1})^{\otimes k} = (\mathbb{C}U_n/\mathbb{C}U_{n-1} \otimes_{\mathbb{C}U_{n-1}} \mathbb{1})^{\otimes k} \cong (\mathbb{C}\mathbb{F}_q^{n-1})^{\otimes k},$$

a standard basis for $V^{\otimes k}$ is

$$\{v_1 \otimes v_2 \otimes \cdots \otimes v_k \mid v_1, v_2, \ldots, v_k \in \mathbb{F}_q^{n-1}\}.$$

We work towards expressing $U_n^\lambda \otimes \epsilon_i$ as

$$U_n^\lambda \otimes \epsilon_i \cong \mathbb{C}\text{-span}\{u_1^{\lambda,i}, u_2^{\lambda,i}, \ldots, u_{\dim U_n^\lambda}^{\lambda,i}\}$$

where

$$u_s^{\lambda,i} = \sum_{(v_1,\ldots,v_k)\in(\mathbb{F}_q^{n-1})^k} d^s_{(v_1,\ldots,v_k)} v_1 \otimes v_2 \otimes \cdots \otimes v_k, \quad \text{and } d^s_{(v_1,\ldots,v_k)} \in \mathbb{C}.$$

By writing each basis element $u_s^{\lambda,i}$ in terms of the standard basis we obtain the maps σ and σ^{-1} as shown below.

$$\begin{array}{ccc}
\begin{matrix} V^{\otimes k} \\ v_{j_1} \otimes \cdots \otimes v_{j_k} \end{matrix} & \xrightarrow{\sigma^{-1}} & \begin{matrix} V^{\otimes k} \\ \displaystyle\sum_{(\lambda,k)\in\Lambda(n)} \sum_{\substack{1\le j\le m_\lambda \\ 1\le s\le \dim U_n^\lambda}} c_s^{\lambda,j} u_s^{\lambda,j} \end{matrix} \\
\Big\downarrow & & \Big\downarrow \pi_{ij}^{\lambda} \\
\begin{matrix} V^{\otimes k} \\ \displaystyle\sum_{s=1}^{\dim U_n^\lambda} \sum_{(v_1,\ldots,v_k)\in(\mathbb{F}_q^{n-1})^k} c_s^{\lambda,j} d^s_{(i_1,\ldots,i_k)} v_{i_1} \otimes \cdots \otimes v_{i_k} \end{matrix} & \xleftarrow{\sigma} & \begin{matrix} V^{\otimes k} \\ \displaystyle\sum_{s=1}^{\dim U_n^\lambda} c_s^{\lambda,j} u_s^{\lambda,i} \end{matrix}
\end{array}$$

This induces projection maps of $V^{\otimes k}$ that form a basis for the centralizer subalgebra and act on the standard basis of $V^{\otimes k}$. While determining these maps appears to be generally unattainable, such as in the classical or partition algebra versions of Schur–Weyl duality, it seems within reach in this case.

5.3.3 Decomposing $V^{\otimes k}$

We start decomposing $V^{\otimes k}$ by considering the modules $U_n^{i\frown n}$ inside of V. From the construction of the supercharacters of U_n in Section 2.3.2, for $1 \le i \le n$, the supermodule $U_n^{i\frown n}$ is

$$U_n^{i\frown n} \cong \mathbb{C}\text{-span}\left\{\overline{ae_{n,i}} \mid a \in B_n\right\} \cong \mathbb{C}\text{-span}\left\{\overline{\sum_{i<j\le n} a_j e_{j,i}} \mid a_{i+1},\ldots,a_{n-1} \in \mathbb{F}_q,\ a_n \in \mathbb{F}_q^\times\right\}.$$

Note that $U_n^{n\frown n}$ corresponds to the trivial module $U_n^{\varnothing}$. For example, the module $U_3^{1\frown 3}$ is given by

$$U_3^{1\frown 3} \cong \mathbb{C}\text{-span}\left\{ \overline{\begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{bmatrix}} \,\middle|\, a \in \mathbb{F}_q, b \in \mathbb{F}_q^\times \right\}.$$

We embed U_{n-1} into U_n by

$$\begin{array}{cccc} \varepsilon : & U_{n-1} & \hookrightarrow & U_n \\ & \begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & 1 \end{bmatrix} & \mapsto & \begin{bmatrix} 1 & * & \cdots & * & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & & \ddots & * & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \end{array}$$

so that

$$U_{n-1} = \{u \in U_n \mid (u-1)_{ij} \neq 0 \text{ implies } i < j < n\}.$$

When restricting to U_{n-1}, the trivial supermodule $U_n^{n\frown n}$ is itself and we denote the restriction as $\mathrm{Res}_{U_{n-1}}^{U_n}(U_n^{n\frown n}) = U_{n-1}^{n\frown n}$. For $i < n$, the supermodule $U_n^{i\frown n}$ decomposes as

$$\mathrm{Res}_{U_{n-1}}^{U_n}(U_n^{i\frown n}) = \bigoplus_{i \leq m < n} (q-1)\, U_{n-1}^{i\frown m}$$

by Corollary 3.1.12, where $U_{n-1}^{i\frown i}$ corresponds to the trivial supermodule. The following proposition explicitly determines these supermodules inside the restricted supermodule.

Proposition 5.3.6. *Fix $a_n \in \mathbb{F}_q^\times$. The map*

$$\begin{array}{cccc} \mathbf{R} : & U_{n-1}^{i\frown m} & \hookrightarrow & \mathrm{Res}_{U_{n-1}}^{U_n}\big(U_n^{i\frown n}\big) \\ & \overline{\sum_{i<j\leq m} a_j e_{j,i}} & \mapsto & \overline{a_n e_{n,i} + \sum_{i<j\leq m} a_j e_{j,i}} \end{array}$$

is an injective U_{n-1}-supermodule homomorphism.

Proof. Fix $a_n \in \mathbb{F}_q^\times$. For $g \in U_{n-1}$, we have

$$\begin{aligned}\mathbf{R}\big(g\overline{\sum_{i<j\leq m} a_j e_{j,i}}\big) &= \mathbf{R}\big(\vartheta(\sum_{i<j\leq m} a_j g_{i,j})\overline{\sum_{i<j\leq k\leq m} a_k g_{j,k} e_{j,i}}\big) \\ &= \vartheta(\sum_{i<j\leq m} a_j g_{i,j})\mathbf{R}\big(\overline{\sum_{i<j\leq k\leq m} a_k g_{j,k} e_{j,i}}\big)\end{aligned}$$

by the left action of U_{n-1} and the linearity of $\mathbf{R}$. Applying $\mathbf{R}$ and using the definition of the left action of U_n, shows

$$\begin{aligned}\mathbf{R}\big(g\overline{\sum_{i<j\leq m} a_j e_{j,i}}\big) &= \vartheta(\sum_{i<j\leq m} a_j g_{i,j})\big(\overline{a_n e_{n,i} + \sum_{i<j\leq k\leq m} a_k g_{j,k} e_{j,i}}\big) \\ &= \varepsilon(g)\big(\overline{a_n e_{n,i} + \sum_{i<j\leq m} a_j e_{j,i}}\big) \\ &= \varepsilon(g)\mathbf{R}\big(\overline{\sum_{i<j\leq m} a_j e_{j,i}}\big),\end{aligned}$$

where ε is the embedding of U_{n-1} into U_n and $\vartheta : \mathbb{F}_q^+ \to \mathbb{C}^\times$ is a nontrivial homomorphism. $\square$

It follows that each $a_n \in \mathbb{F}_q^\times$ specifies a copy of the supermodule $U_{n-1}^{i\frown m} \subseteq \mathrm{Res}_{U_{n-1}}^{U_n}\big(U_n^{i\frown n}\big)$ given by

$$\mathbb{C}\text{-span}\left\{\overline{a_n e_{n,i} + \sum_{i<j\leq m} a_j e_{j,i}} \mid a_{i+1}, \ldots, a_{m-1} \in \mathbb{F}_q,\ a_m, a_n \in \mathbb{F}_q^\times\right\}.$$

For instance, the copy of $U_3^{1\frown 3} \otimes \epsilon_1^{1\frown 3}$ indexed by the dummy variable $\epsilon_1^{1\frown 3}$ corresponding to the element $1 \in \mathbb{F}_q^\times$ is

$$\mathbf{R}(U_3^{1\frown 3}) = \mathbb{C}\text{-span}\left\{\overline{\begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}\,\middle|\, a \in \mathbb{F}_q, b \in \mathbb{F}_q^\times\right\} \subseteq \mathrm{Res}_{U_3}^{U_4}(U_4^{1\frown 4}).$$

By the branching rules 3.2.5, V is given by

$$V = \mathrm{Ind}_{U_{n-1}}^{U_n}\mathrm{Res}_{U_{n-1}}^{U_n}(\mathbb{1}) \cong \bigoplus_{i\leq n} U_n^{i\frown n}.$$

We construct this isomorphism in order to realize the supermodules $U_n^{i\frown n}$ inside V. For vectors

$u = (u_1, \ldots, u_s) \in \mathbb{F}_q^s$ and $v = (v_1, \ldots, v_t) \in \mathbb{F}_q^t$ define the stack function $\mathrm{stk} : \mathbb{F}_q^s \times \mathbb{F}_q^t \to U_{s+t+1}$ as

$$\mathrm{stk}(u,v) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & u_1 \\ & \ddots & \ddots & & \vdots & & \vdots \\ & & 1 & 0 & 0 & \cdots & u_s \\ & & & 1 & 0 & \cdots & v_1 \\ & & & & \ddots & \ddots & \vdots \\ & & & & & 1 & v_t \\ & & & & & & 1 \end{bmatrix}.$$

As an example, for $q = 5$ stacking the vectors $(2,3)$ and (4) yields

$$\mathrm{stk}((2,3),(4)) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proposition 5.3.7. *Let $\vartheta : \mathbb{F}_q^+ \to \mathbb{C}^\times$ be a nontrivial homomorphism. The map*

$$\begin{array}{rccc} \mathbf{I}: & \bigoplus_{i \le n} U_n^{i \frown n} & \longrightarrow & \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1} \\ & \overline{0} & \mapsto & \displaystyle\sum_{u \in \mathbb{F}_q^{n-1}} \mathrm{stk}(u, ()) \otimes \mathbb{1} \\ & \overline{\displaystyle\sum_{i<j\le n} a_j e_{j,i}} & \mapsto & \displaystyle\sum_{u \in \mathbb{F}_q^i} \vartheta(a_n u_i)\,\mathrm{stk}(u, (a_{i+1}, \ldots, a_{n-1})) \otimes \mathbb{1} \end{array}$$

is a U_n-supermodule isomorphism.

Proof. Fix a nontrivial homomorphism $\vartheta : \mathbb{F}_q^+ \to \mathbb{C}^\times$. Let $g \in U_n$. Then the image of $g\overline{0}$ is

$$\mathbf{I}(g\overline{0}) = \mathbf{I}(\overline{0}) = \sum_{u \in \mathbb{F}_q^{n-1}} \mathrm{stk}(u, ()) \otimes \mathbb{1} = g \sum_{u \in \mathbb{F}_q^{n-1}} \mathrm{stk}(u, ()) \otimes \mathbb{1} = g\mathbf{I}(\overline{0}),$$

and for $i < n$, we have

$$\begin{aligned} \mathbf{I}\big(g \overline{\sum_{i<j\le n} a_j e_{j,i}}\big) &= \mathbf{I}\big(\vartheta(\sum_{i<j\le n} a_j g_{i,j}) \overline{\sum_{i<j\le k\le n} a_k g_{j,k} e_{j,i}}\big) \\ &= \vartheta(\sum_{i<j\le n} a_j g_{i,j}) \mathbf{I}\big(\overline{\sum_{i<j\le k\le n} a_k g_{j,k} e_{j,i}}\big) \end{aligned}$$

by the left action of U_n and the linearity of $\mathbf{I}$. Applying $\mathbf{I}$ and again using the definition of the left action of U_n yields

$$\begin{aligned}
&\vartheta(\sum_{i<j\le n} a_j g_{i,j})\mathbf{I}(\overline{\sum_{i<j\le k\le n} a_k g_{j,k} e_{j,i}}) \\
&\quad = \vartheta(\sum_{i<j\le n} a_j g_{i,j}) \sum_{u\in\mathbb{F}_q^i} \vartheta(a_n u_i)\,\mathrm{stk}\Big(u, (\sum_{i+1\le k\le n} a_k g_{i+1,k}, \ldots, \sum_{n-1\le k\le n} a_k g_{n-1,k})\Big) \otimes \mathbb{1} \\
&\quad = g \sum_{u\in\mathbb{F}_q^i} \vartheta(a_n u_i)\,\mathrm{stk}(u, (a_{i+1}, \ldots, a_{n-1})) \otimes \mathbb{1} \\
&\quad = g\mathbf{I}(\overline{\sum_{i<j\le n} a_j e_{j,i}}),
\end{aligned}$$

as desired. □

Example 5.3.8. When $q = 2$, $U_3^{1\frown 3}$ is given by

$$U_3^{1\frown 3} \cong \mathbb{C}\text{-span}\left\{ \overline{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}, \quad \overline{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \right\}$$

and

$$\mathbf{I}(U_3^{1\frown 3}) \cong \mathbb{C}\text{-span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \mathbb{1} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \mathbb{1}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \mathbb{1} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \mathbb{1} \right\}.$$

Now that we have decomposed V as a direct sum of U_n-supermodules, we work towards decomposing $V^{\otimes k}$ as a direct sum of U_n-supermodules. The branching rules specify the coefficients m_λ for which

$$V^{\otimes k} \cong (\mathrm{Ind}_{U_{n-1}}^{U_n} \mathrm{Res}_{U_{n-1}}^{U_n})^k(\mathbb{1}) \cong \bigoplus_{\lambda\vdash[n]} m_\lambda U_n^\lambda$$

so we aim to determine each copy of U_n^λ inside $V^{\otimes k}$.

Using Lemma 5.3.1 we construct a map from $V \otimes \mathrm{Inf}_{U_{n-1}}^{U_n} \mathrm{Res}_{U_{n-1}}^{U_n}(V^{\otimes k-1})$ to $V^{\otimes k}$. Let

$$U_{[i]\times n} = \{u \in U_n \mid (u-1)_{j,k} \neq 0 \text{ implies } 1 \le j \le i, k = n\}.$$

The group $U_{[n]\times n}$ is a normal subgroup of U_n with $U_n/U_{[n]\times n} \cong U_{n-1}$ so that

$$\mathrm{Inf}_{U_{n-1}}^{U_n}(U_{n-1}^{\lambda}) = \mathbb{C}z_{U_{[n]\times n}} \otimes_{U_{n-1}} U_{n-1}^{\lambda}, \qquad \text{where} \qquad z_{U_{[n]\times n}} = \frac{1}{q^{n-1}} \sum_{u \in U_{[n]\times n}} u.$$

Proposition 5.3.9. *For a U_{n-1}-module $M \subseteq (\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1})^{\otimes k-1}$, the map*

$$\begin{array}{rcl} \Psi : (\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1}) \otimes \mathrm{Inf}_{U_{n-1}}^{U_n}(M) & \hookrightarrow & (\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1})^{\otimes k} \\ (g \otimes \mathbb{1}) \otimes (z_{U_{[n]\times n}} \otimes m) & \mapsto & (g \otimes \mathbb{1}) \otimes gm \end{array}$$

is an injective homomorphism.

Proof. Applying Lemma 5.3.1, gives

$$\begin{array}{rcl} (\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1}) \otimes (\mathbb{C}z_{U_{[n]\times n}} \otimes_{\mathbb{C}U_{n-1}} M) & \xrightarrow{\tau^{-1}} & \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{C}z_{U_{[n]\times n}} \otimes_{\mathbb{C}U_{n-1}} M \\ (g \otimes \mathbb{1}) \otimes (z_{U_{[n]\times n}} \otimes m) & \mapsto & g \otimes g^{-1} z_{U_{[n]\times n}} \otimes m. \end{array}$$

Since $\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1} \cong \mathbb{C}U_n/\mathbb{C}U_{n-1} \otimes_{\mathbb{C}U_{n-1}} \mathbb{1}$, we can assume $g \in U_{[n]\times n}$ to obtain

$$g \otimes g^{-1} z_{U_{[n]\times n}} \otimes m = g \otimes z_{U_{[n]\times n}} \otimes m.$$

In the tensor product $\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{C}z_{U_{[n]\times n}} \otimes_{\mathbb{C}U_{n-1}} M$, any $h \in U_{n-1}$ passes through $\mathbb{C}z_{U_{[n]\times n}}$ as

$$h \otimes z_{U_{[n]\times n}} \otimes m \quad = \quad 1 \otimes h\, z_{U_{[n]\times n}} \otimes m \quad = \quad 1 \otimes z_{U_{[n]\times n}}\, h \otimes 1 \quad = \quad 1 \otimes z_{U_{[n]\times n}} \otimes hm.$$

This implies there is an isomorphism $\phi : \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{C}z_{U_{[n]\times n}} \otimes_{\mathbb{C}U_{n-1}} M \to \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} M$ given by removing $z_{U_{[n]\times n}}$. Therefore, we have

$$\begin{array}{rcl} \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{C}z_{U_{[n]\times n}} \otimes_{\mathbb{C}U_{n-1}} M & \xrightarrow{\phi} & \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} M \\ g \otimes z_{U_{[n]\times n}} \otimes m & \mapsto & g \otimes m \\ & \xrightarrow{\iota} & \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathrm{Res}_{U_{n-1}}^{U_n}((\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1})^{\otimes k-1}) \\ & \mapsto & g \otimes m \\ & \xrightarrow{\tau} & (\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1}) \otimes (\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1})^{\otimes k-1} \\ & \mapsto & (g \otimes \mathbb{1}) \otimes gm, \end{array}$$

where ι is the inclusion and τ is the map in Lemma 5.3.1. Hence $\Psi = \tau \circ \iota \circ \phi \circ \tau^{-1}$ is an injective homomorphism. □

Using Proposition 5.3.9 we can start to recursively build up the decomposition of $V^{\otimes k}$ as a U_n-supermodule in some cases. The following theorem gives a copy of certain U_{n-1} submodules $M \subseteq \mathrm{Res}_{U_n}^{U_{n-1}}(V^{\otimes k-1})$. As a corollary we obtain a copy of $U_n^{i \frown n} \otimes \mathrm{Inf}_{U_n}^{U_{n-1}}(M)$ in $V^{\otimes k}$.

Theorem 5.3.10. *Let $(i_1 \frown l_1, i_2, \frown l_2, \ldots i_k \frown l_k)$ be a sequence such that $1 \leq i_r \leq l_r \leq n$, and $l_r = n$ if and only if $i_r = n$. Let $(b_1, b_2, \ldots, b_k) \in \mathbb{F}_q^k$ where $b_r = 0$ if and only if $i_r = n$. The supermodule*

$$\bigotimes_{r=1}^{k} U_{n-1}^{i_r \frown l_r} \otimes \epsilon_{b_r}^{i_r \frown l_r} \subseteq \mathrm{Res}_{U_{n-1}}^{U_n}\left((\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1})^{\otimes k}\right)$$

indexed by the dummy variables $\epsilon_{b_1}^{i_1 \frown l_1}, \ldots, \epsilon_{b_k}^{i_k \frown l_k}$ is isomorphic to

$$\mathbb{C}\text{-span}\left\{ \sum_{(u_1,\ldots,u_k) \in \mathbb{F}_q^{i'_1} \times \cdots \times \mathbb{F}_q^{i'_k}} \bigotimes_{r=1}^{k} \left(\vartheta(b_r(u_r)_{i'_r}) \prod_{s=1}^{r} \mathrm{stk}(u_s, a_s) \otimes \mathbb{1} \right) \;\middle|\; \begin{array}{l} a_r \in \mathbb{F}_q^{n-i'_r-1} \text{ with} \\ (a_r)_j \neq 0 \text{ if } j = l_r - i_r \\ (a_r)_j = 0 \text{ if } l_r - i_r < j < n \end{array} \right\}$$

where $(a_r)_j$ denotes the jth entry of the rth vector and $i'_r = \min\{i_r, n-1\}$.

Proof. We induct on k. If $k = 1$, consider the copy of $U_{n-1}^{i_1 \frown l_1} \subseteq \mathrm{Res}_{U_{n-1}}^{U_n}(U_n^{i_1 \frown n})$ indexed by the dummy variable $\epsilon_{b_1}^{i_1 \frown l_1}$ for some $b_1 \in \mathbb{F}_q$. If $i_1 = l_1 = n$ and $b_1 = 0$, we have

$$U_{n-1}^{i_1 \frown l_1} \otimes \epsilon_{b_1}^{i_1 \frown l_1} \cong \mathbb{C}\text{-span}\left\{ \sum_{u_1 \in \mathbb{F}_q^{n-1}} \vartheta(b_1(u_1)_{i_1}) \,\mathrm{stk}(u_1, ()) \otimes \mathbb{1} \right\}$$

by Proposition 5.3.7. For $1 \leq i_1 \leq l_1 < n$ and $b_1 \in \mathbb{F}_q^{\times}$, we have

$$U_{n-1}^{i_1 \frown l_1} \otimes \epsilon_{b_1}^{i_1 \frown l_1} \cong \mathbb{C}\text{-span}\left\{ \sum_{u_1 \in \mathbb{F}_q^{i_1}} \vartheta(b_1(u_1)_{i_1}) \,\mathrm{stk}(u_1, a_1) \otimes \mathbb{1} \;\middle|\; \begin{array}{l} a_1 \in \mathbb{F}_q^{n-i_1-1} \text{ with} \\ (a_1)_j \neq 0 \text{ if } j = l_1 - i_1 \\ (a_1)_j = 0 \text{ if } l_1 - i_1 < j < n \end{array} \right\}$$

by Propositions 5.3.6 and 5.3.7.

Let $(i_1 \frown l_1, i_2, \frown l_2, \ldots i_k \frown l_k)$ be a sequence such that $1 \leq i_r \leq l_r \leq n$, and $l_r = n$ if and only if $i_r = n$. Let $(b_1, b_2, \ldots, b_k) \in \mathbb{F}_q^k$ where $b_r = 0$ if and only if $i_r = n$. By the inductive hypothesis the tensor product

$$\bigotimes_{r=2}^{k} U_{n-1}^{i_r\frown l_r} \otimes \epsilon_{b_r}^{i_r\frown l_r} \subseteq \mathrm{Res}_{U_{n-1}}^{U_n}\big((\mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1})^{\otimes k-1}\big)$$

indexed by the dummy variables $\epsilon_{b_2}^{i_2\frown l_2}, \ldots, \epsilon_{b_k}^{i_k\frown l_k}$ is isomorphic to

$$\mathbb{C}\text{-span}\left\{ \sum_{\mathbf{u}\in\mathbb{F}_q^{i'_2}\times\cdots\times\mathbb{F}_q^{i'_k}} \bigotimes_{r=2}^{k}\left(\vartheta(b_r(u_r)_{i'_r})\prod_{s=2}^{r}\mathrm{stk}(u_s,a_s)\otimes\mathbb{1}\right) \,\middle|\, \begin{array}{l}(a_r)_j\in\mathbb{F}_q^{n-i'_r-1} \text{ with}\\ (a_r)_j\neq 0 \text{ if } j=l_r-i_r\\ (a_r)_j=0 \text{ if } l_r-i_r<j<n\end{array}\right\}$$

where $\mathbf{u}=(u_2,\ldots,u_k)$, and $i'_r=\min\{i_r,n-1\}$.

If $U_n^{i_1\frown n}\subseteq \mathbb{C}U_n\otimes_{\mathbb{C}U_{n-1}}\mathbb{1}$, then for $i<n$ we obtain the isomorphism

$$U_n^{i_1\frown n}\cong \mathbb{C}\text{-span}\left\{\sum_{u_1\in\mathbb{F}_q^{i_1}}\vartheta(b_1(u_1)_{i_1})\,\mathrm{stk}(u_1,a_1)\otimes\mathbb{1}\,\middle|\, a_1\in\mathbb{F}_q^{n-i_1-1}, b_1\in\mathbb{F}_q^{\times}\right\}$$

and for $b_1=0$,

$$U_n^{n\frown n}\cong \mathbb{C}\text{-span}\left\{\sum_{u_1\in\mathbb{F}_q^{n-1}}\vartheta(b_1(u_1)_{i_1})\,\mathrm{stk}(u_1,())\otimes\mathbb{1}\right\}$$

by Proposition 5.3.7 . Thus, a basis element of

$$U_n^{i_1\frown n}\otimes \mathrm{Inf}_{U_{n-1}}^{U_n}\Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r\frown l_r}\otimes\epsilon_{b_r}^{i_r\frown l_r}\Big)$$

is

$$\Big(\sum_{u_1\in\mathbb{F}_q^{i'_1}}\vartheta(b_1(u_1)_{i_1})\,\mathrm{stk}(u_1,a_1)\otimes\mathbb{1}\Big)\otimes\left(z_{U_{[n]\times n}}\otimes\sum_{\mathbf{u}\in\mathbb{F}_q^{i_2}\times\cdots\times\mathbb{F}_q^{i_k}}\bigotimes_{r=2}^{k}\left(\vartheta(b_r(u_r)_{i_r})\prod_{s=1}^{r}\mathrm{stk}(u_s,a_s)\otimes\mathbb{1}\right)\right)$$

where $b_1\in\mathbb{F}_q$ with $b_1=0$ if and only if $i_1=n$, $i'_1=\min\{i_1,n-1\}$, and $a_1\in\mathbb{F}_q^{n-i'_1-1}$. Applying the map Ψ in Proposition 5.3.9 to send this element to $(\mathbb{C}U_n\otimes_{\mathbb{C}U_{n-1}}\mathbb{1})^{\otimes k}$, we obtain

$$\Big(\sum_{u_1\in\mathbb{F}_q^{i'_1}}\vartheta(b_1(u_1)_{i'_1})\,\mathrm{stk}(u_1,a_1)\otimes\mathbb{1}\Big)\otimes\sum_{\mathbf{u}\in\mathbb{F}_q^{i'_2}\times\cdots\times\mathbb{F}_q^{i'_k}}\bigotimes_{r=2}^{k}\left(\vartheta(b_1(u_1)_{i'_1})\,\mathrm{stk}(u_1,a_1)\vartheta(b_r(u_r)_{i'_r})\prod_{s=2}^{r}\mathrm{stk}(u_s,a_s)\otimes\mathbb{1}\right)$$

$$=\sum_{(u_1,\ldots,u_k)\in\mathbb{F}_q^{i'_1}\times\cdots\times\mathbb{F}_q^{i'_k}}\bigotimes_{r=1}^{k}\left(\vartheta(b_r(u_r)_{i'_r})\prod_{s=1}^{r}\mathrm{stk}(u_s,a_s)\otimes\mathbb{1}\right).$$

Therefore,

$$U_n^{i_1\frown n}\otimes \mathrm{Inf}_{U_{n-1}}^{U_n}\Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r\frown l_r}\otimes \epsilon_{b_r}^{i_r\frown l_r}\Big)\subseteq (\mathbb{C}U_n\otimes_{\mathbb{C}U_{n-1}}\mathbb{1})^{\otimes k}$$

is isomorphic to

$$\mathbb{C}\text{-span}\left\{\sum_{(u_1,\dots,u_k)\in\mathbb{F}_q^{i'_1}\times\cdots\times\mathbb{F}_q^{i'_k}}\bigotimes_{r=1}^{k}\left(\vartheta(b_r(u_r)_{i'_r})\prod_{s=1}^{r}\mathrm{stk}(u_s,a_s)\otimes\mathbb{1}\right)\;\middle|\;\begin{array}{l}(a_r)_j\in\mathbb{F}_q \text{ with}\\ (a_r)_j\neq 0 \text{ if } j=l_r-i_r\\ (a_r)_j=0 \text{ if } l_r-i_r<j<n\end{array}\right\}$$

such that $l_1=n$ and $b_1\in\mathbb{F}_q$ with $b_1=0$ if and only if $i_1=n$.

Restricting to U_{n-1}, gives

$$\begin{aligned}
&\mathrm{Res}_{U_{n-1}}^{U_n}\Big(U_n^{i_1\frown n}\otimes \mathrm{Inf}_{U_{n-1}}^{U_n}\Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r\frown l_r}\otimes \epsilon_{b_1}^{i_1\frown l_1}\Big)\Big)\\
&\quad\cong\ \mathrm{Res}_{U_{n-1}}^{U_n}(U_n^{i_1\frown n})\otimes \mathrm{Res}_{U_{n-1}}^{U_n}\mathrm{Inf}_{U_{n-1}}^{U_n}\Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r\frown l_r}\otimes \epsilon_{b_r}^{i_r\frown l_r}\Big)\\
&\quad\cong\ \mathrm{Res}_{U_{n-1}}^{U_n}(U_n^{i_1\frown n})\otimes \Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r\frown l_r}\otimes \epsilon_{b_r}^{i_r\frown l_r}\Big).
\end{aligned}$$

Recall by Propositions 5.3.6 and 5.3.7, inside the restricted module $\mathrm{Res}_{U_{n-1}}^{U_n}(U_n^{i_1\frown n})$ we have

$$U_{n-1}^{i_1\frown l_1}\otimes\epsilon_{b_1}^{i_1\frown l_1}\cong \mathbb{C}\text{-span}\left\{\sum_{u_1\in\mathbb{F}_q^{n-1}}\vartheta(b_1(u_1)_{i_1})\,\mathrm{stk}(u_1,())\otimes\mathbb{1}\right\}$$

for $i_1=l_1=n$ and $b_1=0$, and for $i_1<n$ and some fixed $b_1\in\mathbb{F}_q^{\times}$ the tensor product is

$$U_{n-1}^{i_1\frown l_1}\otimes\epsilon_{b_1}^{i_1\frown l_1}\cong \mathbb{C}\text{-span}\left\{\sum_{u_1\in\mathbb{F}_q^{i_1}}\vartheta(b_1(u_1)_{i_1})\,\mathrm{stk}(u_1,a_1)\otimes\mathbb{1}\;\middle|\;\begin{array}{l}a_1\in\mathbb{F}_q^{n-i_1-1} \text{ with}\\ (a_1)_j\neq 0 \text{ if } j=l_1-i_1\\ (a_1)_j=0 \text{ if } l_1-i_1<j<n\end{array}\right\}.$$

This gives the inclusion

$$\begin{array}{rcl}
\iota:(U_{n-1}^{i_1\frown l_1}\otimes\epsilon_{b_1}^{i_1\frown l_1})\otimes\Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r\frown l_r}\otimes \epsilon_{b_r}^{i_r\frown l_r}\Big) & \longhookrightarrow & \mathrm{Res}_{U_{n-1}}^{U_n}\Big(U_n^{i_1\frown n}\otimes \mathrm{Inf}_{U_{n-1}}^{U_n}\Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r\frown l_r}\otimes \epsilon_{b_r}^{i_r\frown l_r}\Big)\Big)\\
\sum_{u_1\in\mathbb{F}_q^{i'_1}}\big(\vartheta(b_1(u_1)_{i'_1})\,\mathrm{stk}(u_1,a_1)\otimes\mathbb{1}\big)\otimes v & \mapsto & \sum_{u_1\in\mathbb{F}_q^{i'_1}}\big(\vartheta(b_1(u_1)_{i'_1})\,\mathrm{stk}(u_1,a_1)\otimes\mathbb{1}\big)\otimes\big(z_{U_{[n]\times n}}\otimes v\big).
\end{array}$$

Composing the maps ι and Ψ, we obtain

$$\begin{array}{ccc} \mathrm{Res}^{U_n}_{U_{n-1}}\Big(U_n^{i_1\frown n}\otimes \mathrm{Inf}^{U_n}_{U_{n-1}}\Big(\bigotimes_{r=2}^k U_{n-1}^{i_r\frown l_r}\otimes \epsilon_{b_r}^{i_r\frown l_r}\Big)\Big) & \xrightarrow{\Psi} & \mathrm{Res}^{U_n}_{U_{n-1}}\Big((\mathbb{C}U_n\otimes_{\mathbb{C}U_{n-1}}\mathbb{1})^{\otimes k}\Big) \\ \uparrow{\scriptstyle \iota} & \nearrow{\scriptstyle \Psi\circ\iota} & \\ (U_{n-1}^{i_1\frown l_1}\otimes \epsilon_{b_1}^{i_1\frown l_1})\otimes\Big(\bigotimes_{r=2}^k U_{n-1}^{i_r\frown l_r}\otimes \epsilon_{b_r}^{i_r\frown l_r}\Big) & & \end{array}$$

shows that the module

$$\bigotimes_{r=1}^{k} U_{n-1}^{i_r\frown l_r}\otimes \epsilon_{b_r}^{i_r\frown l_r}\subseteq \mathrm{Res}^{U_n}_{U_{n-1}}\Big((\mathbb{C}U_n\otimes_{\mathbb{C}U_{n-1}}\mathbb{1})^{\otimes k}\Big)$$

indexed by the dummy variables $\epsilon_{b_1}^{i_1\frown l_1},\ldots,\epsilon_{b_k}^{i_k\frown l_k}$ is isomorphic to

$$\mathbb{C}\text{-span}\left\{ \sum_{(u_1,\ldots,u_k)\in\mathbb{F}_q^{i'_1}\times\cdots\times\mathbb{F}_q^{i'_k}} \bigotimes_{r=1}^{k}\left(\vartheta(b_r(u_r)_{i'_r})\prod_{s=1}^{r}\mathrm{stk}(u_s,a_s)\otimes\mathbb{1}\right) \;\middle|\; \begin{array}{l} (a_r)_j\in\mathbb{F}_q^{n-i'_r-1}\text{ with}\\ (a_r)_j\neq 0 \text{ if } j=l_r-i_r\\ (a_r)_j=0 \text{ if } l_r-i_r<j<n \end{array}\right\}.$$

□

In proof of Proposition 5.3.10 we showed the following corollary.

Corollary 5.3.11. *Let $1\le i_1\le n$. Let $(i_2,\frown l_2,\ldots i_k\frown l_k)$ be a sequence such that $1\le i_r\le l_r\le n$, and $l_r=n$ if and only if $i_r=n$. Let $(b_2,\ldots,b_k)\in\mathbb{F}_q^k$ where $b_r=0$ if and only if $i_r=n$. The supermodule*

$$U_n^{i_1\frown n}\otimes \mathrm{Inf}^{U_n}_{U_{n-1}}\Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r\frown l_r}\otimes \epsilon_{b_r}^{i_r\frown l_r}\Big)\subseteq (\mathbb{C}U_n\otimes_{\mathbb{C}U_{n-1}}\mathbb{1})^{\otimes k}$$

indexed by the dummy variables $\epsilon_{b_2}^{i_2\frown l_2},\ldots,\epsilon_{b_k}^{i_r\frown l_r}$ is isomorphic to

$$\mathbb{C}\text{-span}\left\{ \sum_{(u_1,\ldots,u_k)\in\mathbb{F}_q^{i'_1}\times\cdots\times\mathbb{F}_q^{i'_k}} \bigotimes_{r=1}^{k}\left(\vartheta(b_r(u_r)_{i'_r})\prod_{s=1}^{r}\mathrm{stk}(u_s,a_s)\otimes\mathbb{1}\right) \;\middle|\; \begin{array}{l} a_r\in\mathbb{F}_q^{n-i'_r-1}\text{ with}\\ (a_r)_j\neq 0 \text{ if } j=l_r-i_r\\ (a_r)_j=0 \text{ if } l_r-i_r<j<n \end{array}\right\}$$

such that $b_1\in\mathbb{F}_q$ with $b_1=0$ if and only if $i_1=n$, and $i'_r=\min\{i_r,n-1\}$.

Example 5.3.12. When $q = 2$, $U_2^{1\frown 2} \otimes \epsilon_1^{1\frown 2}$ is the only copy of $U_2^{1\frown 2} \subseteq \mathrm{Res}_{U_2}^{U_3}(U_3^{1\frown 3})$ so we supress the dummy variables and simply write $U_2^{1\frown 2}$. Then

$$U_3^{1\frown 3} \otimes \mathrm{Inf}_{U_2}^{U_3}(U_2^{1\frown 2}) \cong \mathbb{C}\text{-span}\{v_1, v_2\}$$

where

$$\begin{aligned} v_1 &= \left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\otimes\left(\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)+\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\otimes\left(-\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\\ &+\left(-\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\otimes\left(\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)+\left(-\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\otimes\left(-\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\end{aligned}$$

and

$$\begin{aligned} v_2 &= \left(\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\otimes\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)+\left(\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\otimes\left(-\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\\ &+\left(-\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\otimes\left(\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)+\left(-\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right)\otimes\left(-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\otimes\mathbb{1}\right).\end{aligned}$$

Restricting to U_2, gives

$$U_2^{1\frown 1} \otimes U_2^{1\frown 2} \cong \mathbb{C}\text{-span}\{v_1\} \quad \text{and} \quad U_2^{1\frown 2} \otimes U_2^{1\frown 2} \cong \mathbb{C}\text{-span}\{v_2\}.$$

Corollary 5.3.11 gives a copy of tensor products of supermodules inside $V^{\otimes k}$, which leads to a decomposition of $V^{\otimes k}$. Recall that $V = \mathrm{Ind}_{U_{n-1}}^{U_n}\mathrm{Res}_{U_{n-1}}^{U_n}(\mathbb{1}) \cong \bigoplus_{i\leq n} U_n^{i\frown n}$. Thus

$$\begin{aligned} V^{\otimes 2} \cong (\mathrm{Ind}_{U_{n-1}}^{U_n}\mathrm{Res}_{U_{n-1}}^{U_n})^2(\mathbb{1}) &\cong \mathrm{Ind}_{U_{n-1}}^{U_n}\mathrm{Res}_{U_{n-1}}^{U_n}\Big(\bigoplus_{i\leq n} U_n^{i\frown n}\Big)\\ &\cong \mathrm{Ind}_{U_{n-1}}^{U_n}\Big(\bigoplus_{i\leq l\leq n} m_{i\frown l}U_{n-1}^{i\frown l}\Big)\\ &\cong \bigoplus_{j\leq n}\bigoplus_{i\leq l\leq n} U_n^{j\frown n}\otimes \mathrm{Inf}_{U_{n-1}}^{U_n}(m_{i\frown l}U_{n-1}^{i\frown l}),\end{aligned}$$

where

$$m_{i\frown l} = \begin{cases} 1 & \text{if } i = n, \\ q-1 & \text{if } i < n. \end{cases}$$

More generally, we have

$$V^{\otimes k} \cong (\mathrm{Ind}_{U_{n-1}}^{U_n} \mathrm{Res}_{U_{n-1}}^{U_n})^k(\mathbb{1}) \cong \bigoplus_{i_1 \leq n} \bigoplus_{\substack{1<r\leq k \\ i_r \leq l_r \leq n}} U_n^{i_1 \frown n} \otimes \mathrm{Inf}_{U_{n-1}}^{U_n} \Big(\bigotimes_{r=2}^{k} m_{i_r \frown l_r} U_{n-1}^{i_r \frown l_r} \Big).$$

If $m_{i_r. \frown l_r} = q-1$, we index the $m_{i_r \frown l_r}$ copies of $U_n^{i_r \frown l_r}$ by dummy variables $\epsilon_{b_r}^{i_r \frown l_r}$ for $1 \leq b_r \leq q-1$ otherwise let $b_r = 0$. When each

$$U_n^{i_1 \frown n} \otimes \mathrm{Inf}_{U_{n-1}}^{U_n} \Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r \frown l_r} \otimes \epsilon_{b_r}^{i_r \frown l_r} \Big)$$

is isomorphic to U_n^λ for some $\lambda \vdash [n]$, then we have a decomposition of $V^{\otimes k}$ as a direct sum of U_n-supermodules. This occurs when the supercharacters coincide with irreducible characters.

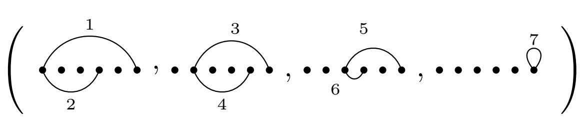
5.3.4 Projections for pairs of paths

In the case when there are no conflicting arcs, we have the isomorphism

$$U_n^{i_1 \frown n} \otimes \mathrm{Inf}_{U_{n-1}}^{U_n} \Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r \frown l_r} \otimes \epsilon_{b_r}^{i_r \frown l_r} \Big) \cong U_n^\lambda$$

and we can construct basis elements of the centralizer subalgebra $Z_{k,n}$. For simplicity, we establish this part of the basis when $q = 2$.

Consider the set $\widetilde{\mathcal{ST}}_k$ of paths in the Bratteli diagram corresponding to shell tableaux $T = (\varsigma^1, \varsigma^2, \ldots, \varsigma^k)$ of length k such that for $1 \leq r < k$, $\varsigma^r = \{(n \frown n; a)\}$ or $|\varsigma^r| = 2$. Then each strictly labeled shell has no inner whorls, so by conditions 4 and 5, there are no conflicts with the outer whorls of each shell. For example, suppose that

$T=$ 

.

If $T = (\varsigma^1, \varsigma^2, \ldots, \varsigma^k) \in \widetilde{\mathcal{ST}}_k$ is given by

$$\varsigma^r = \begin{cases} \{(i_r \frown n; a_r)\} \cup \{(i_r \smile l_r; a_r + 1)\} & \text{for } l_r \neq n, \\ \{(i_r \frown l_r; a_r)\} & \text{for } l_r = n, \end{cases}$$

let $b_r = 1$ if $i_r < n$ and $b_r = 0$ if $i_r = n$ for $r > 1$. Then the module corresponding to T is

$$U_n^{\mathrm{sh}(T)} \cong U_n^{i_1 \frown n} \otimes \mathrm{Inf}_{U_{n-1}}^{U_n} \Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r \frown l_r} \otimes \epsilon_{b_r}^{i_r \frown l_r} \Big)$$

for dummy variables $\epsilon_{b_2}^{i_2 \frown l_2}, \ldots, \epsilon_{b_k}^{i_k \frown l_k}$.

While the dummy variables are superfluous when $q = 2$, we can generalize to arbitrary q by coloring the maximally labeled arcs in each strictly labeled shell ς^r by $b_r \in \mathbb{F}_q$ such that $b_r = 0$ if and only if $\varsigma^r = \{(n \frown n; a)\}$. Then the module corresponding to the colored tableau T is

$$U_n^{\mathrm{sh}(T)} \cong U_n^{i_1 \frown n} \otimes \mathrm{Inf}_{U_{n-1}}^{U_n} \Big(\bigotimes_{r=2}^{k} U_{n-1}^{i_r \frown l_r} \otimes \epsilon_{b_r}^{i_r \frown l_r} \Big)$$

for dummy variables $\epsilon_{b_2}^{i_2 \frown l_2}, \ldots, \epsilon_{b_k}^{i_k \frown l_k}$.

Moreover, a basis for $U_n^{\mathrm{sh}(T)}$ is given by Corollary 5.3.11. Thus for each pair of shell tableau $(T, T') \in \widetilde{\mathcal{ST}}_k \times \widetilde{\mathcal{ST}}_k$ with $\mathrm{sh}(T) = \mathrm{sh}(T')$, we can obtain explicit formulas for the projection map $\pi_{T,T'}$ from $U_n^{\mathrm{sh}(T')}$ to $U_n^{\mathrm{sh}(T)}$. In the next section we illustrate this when $q = 2$ and $n = 2$ to produce a basis for the full centralizer algebra.

5.3.5 The centralizer algebra $\mathrm{End}_{U_2}(V^{\otimes k})$

Let $n = 2$ and $q = 2$. Then

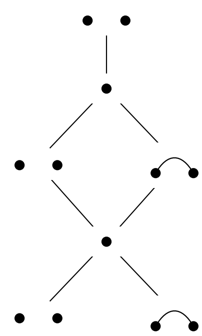
$$V^{\otimes k} \cong \mathbb{C}\text{-span} \left\{ \begin{bmatrix} a_1 \\ 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} a_k \\ 1 \end{bmatrix} \middle| a_1, \ldots, a_k \in \mathbb{F}_2 \right\} \cong \mathbb{C}\mathbb{F}_2^k.$$

We may identify this basis $\mathbb{F}_2^k$ with the k-dimensional hypercube by taking vertices labeled by $(a_1, \ldots, a_k) \in \mathbb{F}_2^k$ and joining two vertices $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ by an edge if and only if $|\{i \mid a_i \neq b_i\}| = 1$. The generator $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ of U_2 acts on $\mathbb{F}_2^k$ by

$$g \cdot (a_1, \ldots, a_k) = (a_1 + 1, \ldots, a_k + 1)$$

to send each vertex to its antipode.

Decomposing $V^{\otimes k}$ as a U_2-module, the first few rows of the Bratteli diagram are given by



and

$$V^{\otimes k} \cong 2^{k-1}\mathbb{1} + 2^{k-1}\chi^{1\frown 2}.$$

Since the centralizer algebra $\mathrm{End}_{U_2}(V^{\otimes k})$ is the subalgebra $Z_{k,2}$, the dimension of the centralizer algebra is

$$(2^{k-1})^2 + (2^{k-1})^2 = 2^{2k-1}.$$

Suppose a path in the Bratteli diagram is given by the shell tableau $T = (\varsigma^1, \varsigma^2, \ldots, \varsigma^k) \in \widetilde{\mathcal{ST}}_k = \mathcal{ST}_k$ where

$$\varsigma^r = \begin{cases} \{(i_r \frown 2; a_r)\} \cup \{(i_r \smile l_r; a_r + 1)\} & \text{for } l_r \neq 2, \\ \{(i_r \frown l_r; a_r)\} & \text{for } l_r = 2. \end{cases}$$

Note that we have

$$\mathrm{sh}(T) = \begin{cases} \bullet \;\; \bullet & \text{if } l_1 = 1, \\ \bullet\!\frown\!\bullet & \text{if } l_1 = 2. \end{cases}$$

Let $b_1 = 1$ if $i_r = 1$, and $b_1 = 0$ if $i_r = 2$. Then we may index $U_2^{i_1 \frown 2}$ by the dummy variable $\epsilon_{b_1}^{i_1 \frown 2}$ so that

$$U_2^{\mathrm{sh}(T)} \cong (U_2^{i_1 \frown 2} \otimes \epsilon_{b_1}^{i_1 \frown 2}) \otimes \mathrm{Inf}_{U_1}^{U_2}\Big(\bigotimes_{r=2}^{k} U_1^{i_r \frown l_r} \otimes \epsilon_{b_r}^{i_r \frown l_r}\Big).$$

When $n = 2$, we have $b_r = 1$ if and only if $i_r = 1$. Thus each tuple $(b_1, \ldots, b_k) \in \mathbb{F}_2^k$ uniquely determines a shell tableau $T \in \mathcal{ST}_k$. Identify T with $(b_1, \ldots, b_k) \in \mathbb{F}_2^k$.

For $\mathbf{b} = (b_1, \ldots, b_k), \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{F}_2^k$, define the "twisted" dot product as

$$\mathbf{a} \circledast \mathbf{b} = (a_1 + a_2, a_2 + a_3, \ldots, a_{k-1} + a_k, a_k) \cdot (b_1, b_2, \ldots, b_k).$$

This dot product counts the number of $a_i = b_i$ and $a_{i+1} = b_i$. Then the basis vector u_T of $U_2^{\mathrm{sh}(T)}$ is written in the standard basis with coefficients given by the twisted dot product.

Theorem 5.3.13. *The map $\sigma : V^{\otimes k} \to V^{\otimes k}$ from the basis of $V^{\otimes k}$ indexed by shell tableaux to the standard basis is defined for the shell tableau $T = (b_1, \ldots, b_k) \in \mathbb{F}_2$ by*

$$\sigma(u^T) = \sum_{(v_1,\ldots,v_k)\in\mathbb{F}_2^k} (-1)^{(b_1,\ldots,b_k)\circledast(v_1,\ldots,v_k)} v_1 \otimes \cdots \otimes v_k.$$

Proof. Corollary 5.3.11 gives

$$U_2^{\mathrm{sh}(T)} \cong \mathbb{C}\text{-span}\{u^T\},$$

where

$$u^T = \sum_{(v_1,\ldots,v_k)\in\mathbb{F}_2^k} \bigotimes_{r=1}^{k} \Big(\vartheta(b_r v_r) \prod_{s=1}^{r} \mathrm{stk}(v_s, ()) \otimes \mathbb{1}\Big)$$

for $\vartheta : \mathbb{F}_2^+ \to \mathbb{C}^\times$ given by $\vartheta(0) = 1$ and $\vartheta(1) = -1$. If $v_s, v_t \in \mathbb{F}_2$, we have

$$\mathrm{stk}(v_s, ())\, \mathrm{stk}(v_t, ()) = \begin{bmatrix} 1 & v_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & v_t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & v_s + v_t \\ 0 & 1 \end{bmatrix} = \mathrm{stk}(v_s + v_t, ()).$$

Therefore, u^T is given in the standard basis as

$$u^T = \sum_{(v_1,\ldots,v_k)\in\mathbb{F}_2^k} \bigotimes_{r=1}^{k} \Big(\vartheta(b_r v_r)\, \mathrm{stk}\,\big(\sum_{s=1}^{r} v_s, ()\big) \otimes \mathbb{1}\Big).$$

Since the sum runs over all $(v_1, \ldots, v_k) \in \mathbb{F}_2^k$, we can reindex to obtain

$$\begin{aligned} u^T &= \sum_{(v_1,\ldots,v_k)\in\mathbb{F}_2^k} \bigotimes_{r=1}^{k} \Big(\vartheta(b_r v_r + b_{r+1} v_r)\, \mathrm{stk}(v_r, ()) \otimes \mathbb{1}\Big) \\ &= \sum_{(v_1,\ldots,v_k)\in\mathbb{F}_2^k} \prod_{r=1}^{k} \vartheta(b_r v_r + b_{r+1} v_r) \bigotimes_{r=1}^{k} \Big(\mathrm{stk}(v_r, ()) \otimes \mathbb{1}\Big) \\ &= \sum_{(v_1,\ldots,v_k)\in\mathbb{F}_2^k} \prod_{r=1}^{k} \vartheta(b_r v_r)\vartheta(b_{r+1} v_r) \bigotimes_{r=1}^{k} \Big(\mathrm{stk}(v_r, ()) \otimes \mathbb{1}\Big), \end{aligned}$$

where $b_{k+1} = 0$. Because $\vartheta(b_r v_r) = (-1)^{b_r v_r}$, it follows that

$$u^T = \sum_{(v_1,\ldots,v_k)\in\mathbb{F}_2^k} (-1)^{(b_1+b_2,\ldots,b_k+b_{k+1})\cdot(v_1,\ldots,v_k)} \bigotimes_{r=1}^{k} \Big(\operatorname{stk}(v_r,()) \otimes \mathbb{1}\Big).$$

Identifying $\operatorname{stk}(v_r,()) \otimes \mathbb{1}$ with v_r, we obtain

$$u^T = \sum_{(v_1,\ldots,v_k)\in\mathbb{F}_2^k} (-1)^{(b_1,\ldots,b_k)\circledast(v_1,\ldots,v_k)} v_1 \otimes \cdots \otimes v_k.$$

□

Inverting this maps yields $\sigma^{-1} : V^{\otimes k} \to V^{\otimes k}$ given for $v_1, \ldots v_k \in \mathbb{F}_2$ by

$$\sigma^{-1}(v_1 \otimes \cdots \otimes v_k) = \frac{1}{2^k} \sum_{(b_1,\ldots,b_k)\in\mathbb{F}_2^k} (-1)^{(b_1,\ldots,b_k)\circledast(v_1,\ldots,v_k)} u^T.$$

The maps σ and σ^{-1} allow us to construct a basis for $\operatorname{End}_{U_2}(V^{\otimes k})$ that acts on the standard basis of $V^{\otimes k}$.

Corollary 5.3.14. *Let σ be the transition map between the basis of $V^{\otimes k}$ indexed by shell tableaux and the standard basis. A basis for the centralizer algebra $\operatorname{End}_{U_2}(V^{\otimes k})$ is*

$$\{(\sigma \circ \pi_{T,T'} \circ \sigma^{-1}) \mid (T,T') \in \mathcal{ST}_k \times \mathcal{ST}_k \text{ with } \operatorname{sh}(T) = \operatorname{sh}(T')\}$$

where for $T = (b_1,\ldots,b_k), T' = (a_1,\ldots,a_k) \in \mathbb{F}_2^k$, and $v = (v_1,\ldots,v_k) \in \mathbb{F}_2^k$, the action on the standard basis is given by

$$(\sigma \circ \pi_{T,T'} \circ \sigma^{-1})(v_1 \otimes \cdots \otimes v_k) = \frac{1}{2^k} \sum_{(u_1,\ldots,u_k)\in\mathbb{F}_2^k} (-1)^{(a_1,\ldots,a_k)\circledast v + (b_1,\ldots,b_k)\circledast(u_1,\ldots u_k)} u_1 \otimes \cdots \otimes u_k.$$

Proof. Suppose $S \in \mathcal{ST}_k$ corresponds to $(u_1,\ldots,u_k) \in \mathbb{F}_2^k$. Composing the maps $\sigma \circ \pi_{T,T'} \circ \sigma^{-1}$ yields

$$\begin{aligned}
(\sigma \circ \pi_{T,T'} \circ \sigma^{-1})(v_1 \otimes \cdots \otimes v_k) &= (\sigma \circ \pi_{T,T'})\Big(\frac{1}{2^k} \sum_{(u_1,\ldots,u_k)\in\mathbb{F}_2^k} (-1)^{(u_1,\ldots,u_k)\circledast v} u^S\Big) \\
&= \sigma\left(\frac{1}{2^k} \cdot (-1)^{(a_1,\ldots,a_k)\circledast v} u^T\right) \\
&= \frac{1}{2^k} \cdot (-1)^{(a_1,\ldots,a_k)\circledast v} \sum_{(u_1,\ldots,u_k)\in\mathbb{F}_2^k} (-1)^{(b_1,\ldots,b_k)\circledast(u_1,\ldots,u_k)} u_1 \otimes \cdots \otimes u_k \\
&= \frac{1}{2^k} \sum_{(u_1,\ldots,u_k)\in\mathbb{F}_2^k} (-1)^{(a_1,\ldots,a_k)\circledast v + (b_1,\ldots,b_k)\circledast(u_1,\ldots u_k)} u_1 \otimes \cdots \otimes u_k
\end{aligned}$$

as desired. Since

$$V^{\otimes k} \cong \bigoplus_{T \in \mathcal{ST}_k} U_2^{\mathrm{sh}(T)},$$

the supermodule analogue of the Double Centralizer Theorem 5.3.2 says a basis of the centralizer subalgebra $Z_{k,2}$ is

$$\{\pi_{T,T'} \mid (T,T') \in \mathcal{ST}_k \times \mathcal{ST}_k \text{ with } \mathrm{sh}(T) = \mathrm{sh}(T')\}.$$

In this case $Z_{k,2} = \mathrm{End}_{U_2}(V^{\otimes k})$ so the $\pi_{T,T'}$ form a basis for the centralizer algebra. Conjugating this basis by σ produces the basis of $\mathrm{End}_{U_2}(V^{\otimes k})$

$$\{(\sigma \circ \pi_{T,T'} \circ \sigma^{-1}) \mid (T,T') \in \mathcal{ST}_k \times \mathcal{ST}_k \text{ with } \mathrm{sh}(T) = \mathrm{sh}(T')\}$$

that acts on the standard basis of $V^{\otimes k}$. □

Since we have found formulas for the projections of $V^{\otimes k}$ when $q = 2$ and $n = 2$, it seems tractable to be able to determine these formulas in general.

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Appendix A

Sage Code

```
sage: R = QQ['q'].fraction_field()

sage: q = R.gen()

sage: NCSym = SymmetricFunctionsNonCommutingVariables(R)

sage: chi = NCSym.chi(q)

sage: def crs(x, y):

       crs = 0

       for i in range(len(x)):

           for j in range(len(y)):

               if x[i][0] < y[j][0] < x[i][1] < y[j][1]:

                   crs = crs+1

       return crs

sage: def nesting(x, y):

       if y[0] < x[0] < x[1] < y[1]:

           return True

sage: def res_coeff(x, y):

       return (q-1)^(Set(x).difference(Set(y)).cardinality())*q^(crs(Set(x).

       intersection(Set(y)),Set(x).difference(Set(y))))/q^(crs(Set(y).

       intersection(Set(x)),Set(y).difference(Set(x))))

sage: def ind_coeff(x, y):
```

```
        return (q-1)^(Set(x).difference(Set(y)).cardinality())*q^(crs(Set(x)
        .difference(Set(y)),
Set(x).intersection(Set(y))))/q^(crs(Set(y).difference(Set(x)),Set(y)
.intersection(Set(x))))
sage: def getKey(item):
        return item[1]
sage: def Res(n,self):
sage: # self = supercharacter given by [[partition], coeff]
        arcsort = sorted(self[0], key=getKey)
        if arcsort == []:
            return [self]
        if arcsort[-1][1] != n:
            return [self]
        elif arcsort[-1][1] == n:
            spiral = arcsort.pop(-1)
            l = []
            l.append(spiral)
            for _ in self[0]:
                if _[0] < spiral[0]:
                    l.append(_)
            P = Poset((self[0], nesting), cover_relations=False)
            C = list(P.chains(exclude=[_ for _ in l]))
            S = [_+[spiral] for _ in C]
            res = []
            re = [max(_) for _ in self[0]]
            for i in range(len(S)):
                arcs = self[0][:]
```

```
            for j in range(len(S[i])):
                arcs.remove(S[i][j])
            for j in range(len(S[i])-1):
                arcs.append((S[i][j+1][0], S[i][j][1]))
            res.append(arcs)
            for k in range(S[i][0][1]-S[i][0][0]-1):
                if S[i][0][1]-k-1 not in re:
                    res.append(arcs+[(S[i][0][0],S[i][0][1]-k-1)])
        return [[res[i], res_coeff(self[0], res[i])*self[1]] for i in
         range(len(res))]
sage: def Ind(n, self):
sage: # self = supercharacter given by [[partition], coeff]
       P = Poset((self[0], nesting), cover_relations=False)
       C = list(P.chains())
       ind = [self[0]]
       le = [min(_) for _ in self[0]]
       for j in range(1,n):
           if j not in le:
               ind.append(self[0]+[(j,n)])
       for j in range(1,len(C)):
           arcs = self[0][:]
           for k in range(len(C[j])):
               arcs.remove(C[j][k])
           for k in range(len(C[j])-1):
               arcs.append((C[j][k][0], C[j][k+1][1]))
           arcs.append((C[j][len(C[j])-1][0],n))
           ind.append(arcs)
```

```
            for k in range(C[j][0][1]-C[j][0][0]-1):
                if C[j][0][0]+k+1 not in le:
                    ind.append(arcs+[(C[j][0][0]+k+1,C[j][0][1])])
        return [[ind[i],ind_coeff(self[0], ind[i])*self[1]] for i in
         range(len(ind))]
sage: def ResSum(n, self):
        l = map(Res, [n]*len(self), self)
        d = [[tuple(sorted(item[0], key=getKey)), item[1]] for sublist in l
         for item in sublist]
        simplify = {}
        for k, v in d:
            simplify[k] = simplify.get(k, 0) + v
        return [[list(item[0]), item[1]] for item in list(simplify.items())]
sage: def IndSum(n, self):
        l = map(Ind, [n]*len(self), self)
        d = [[tuple(sorted(item[0], key=getKey)), item[1]] for sublist in l
         for item in sublist]
        simplify = {}
        for k, v in d:
            simplify[k] = simplify.get(k, 0) + v
        return [[list(item[0]), item[1]] for item in list(simplify.items())]
sage: def IndRes(n,k):
        def funk(y):
            return reduce(lambda x, _: IndSum(n,ResSum(n,x)), xrange(k), y)
        return funk

sage: IndRes(4,3)([[[],1]])
```

```
[[[(2, 3), (3, 4)], 2*q^3 - q^2 - 2*q + 1],
 [[(1, 2)], 2*q^3 - q^2 - 2*q + 1],
 [[(2, 3)], 2*q^3 - q^2 - 2*q + 1],
 [[(1, 2), (2, 3)], q^3 - q^2 - q + 1],
 [[(2, 4)], 2*q^4 - q^2],
 [[(3, 4)], 3*q^3 - 3*q + 1],
 [[(1, 3)], q^4 - q^2],
 [[(1, 4)], q^5 + q^4 - q^3],
 [[(1, 2), (2, 3), (3, 4)], q^3 - q^2 - q + 1],
 [[(1, 3), (3, 4)], q^4 - q^2],
 [[(1, 3), (2, 4)], q^4 - q^2],
 [[], 3*q^3 - 3*q + 1],
 [[(1, 2), (2, 4)], q^4 - q^2],
 [[(2, 3), (1, 4)], q^5 - q^3],
 [[(1, 2), (3, 4)], 2*q^3 - q^2 - 2*q + 1]]
```