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THE E-PLANE TAPER JUNCTION IN RECTANGULAR WAVEGUIDE

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Abstract

There is a major discrepancy between results for the junction parameters from earlier results of the author and those in the Waveguide Handbook. The two sets of results are discussed; the handbook result is apparently in error, but the formulas do not seem capable of reconciliation. Limitations and slight improvements on the author's earlier result are given, together with supporting measurements taken at X-band.

Introduction

The straight guide to taper section is a common element in microwave hardware. Its properties were examined by the author in the late forties and results of a rather simple analysis were published in 1949.(1) They were used to design a matched transition at the junction between a waveguide and horn, and measurements confirmed the theoretical predictions quite closely. In 1951, results of a much more sophisticated quasi-static analysis were published,(2) and presumably both sets of results have been used by engineers in the intervening period. It was recently drawn to the author's attention that the two analyses are not in agreement, and attempts at explaining the differences are reported on here.

1. Approximate Field-Matching Formula

Since the earlier quoted publication may not now be readily available and the analysis is quite short, it is repeated here for convenience. Figure 1 shows the guide-to-taper junction, with the vertex of the tapered section located a distance r_o beyond the junction. In the rectangular guide it is assumed that only the dominant mode need be taken into account, so that the relevant fields are

$$\begin{aligned} E_y &= (e^{-jk'z} + Re^{jk'z}) \sin(\pi x/a) \\ H_x &= -\hat{\eta}(e^{-jk'z} - Re^{jk'z}) \sin(\pi x/a)(k'/k) \end{aligned} \qquad (1)$$

where

R = dominant mode reflection coefficient

$k' = 2\pi/\lambda_g$

$\hat{\eta} = (\varepsilon_o/\mu_o)^{\frac{1}{2}}$, the wave admittance in free space.

In the tapered section, again only a dominant, cylindrical, mode is assumed, and it takes the form

$$\begin{aligned} E_\phi &= T\, H_1^{(2)}(k'r)\sin(\pi x/a) \\ H_x &= -\hat{\eta} T\, H_o^{(2)}(k'r)\sin(\pi x/a)(jk'/k) \end{aligned} \qquad (2)$$

where T is a transmission coefficient.

The two field forms must be equated at the junction. The continuity of tangential fields has to be applied at all points of the junction cross-section, and it is clear that this cannot be done exactly with the assumed fields. However, to the extent that r varies little over the junction, an approximate result can be obtained by equating the fields at the junction center $r = r_o$. The resulting equations are

$$1 + R = T\,H_1^{(2)}(k'r_o)\ , \qquad 1 - R = jT\,H_o^{(2)}(k'r_o) \tag{3}$$

Hence the junction admittance Y_{in} (normalized with respect to the waveguide) is

$$Y_{in} = (1-R)/(1+R) = jH_o^{(2)}(k'r_o)/H_1^{(2)}(k'r_o) \tag{4}$$

Now since it has been assumed that r varies little over the junction we must consider $k'r_o$ to be large (just how large is considered later), so that the asymptotic forms of the Hankel functions can be used. Finally, using the relation $r_o = \frac{1}{2}\, b \cot\phi_o$, where ϕ_o is the half angle of the taper, gives the result found in ref. 1.

$$Y_{in} \approx 1 + j/(k'b \cot \phi_o) \tag{5}$$

2. Discussion of the Approximate Formula

The parallel susceptance, from (5), is capacitive; but since k' occurs in the denominator, the slope is opposite of that of a genuine capacitive element. Hence, as discussed in the reference, the taper behaves more like a negative inductance shunted across the guide at the junction, and can be matched on a broad-band basis by a simple inductive diaphragm, whose dimensions are readily calculated. Early experiments indicated that this broad-band cancellation was in fact achieved, and final results on a matched horn were measured with the taper junction matched in this way.[3]

Clearly, a true negative inductance cannot exist, and the above result is partially due to the approximations used. The matter can be rectified very simply by inverting eq. (5) to give, to the same degree of approximation as used in going from eq. (4) to eq. (5),

$$Z_{in} = 1/Y_{in} \approx 1 - j/(k'b \cot \phi_o) \tag{6}$$

The interpretation now is simply that the junction behaves like an ordinary series capacitance. Fairly broad-band matching with a shunt inductance follows from the approximate constant-resistance network equivalence of the arrangement.[3]

For the results to be valid, two conditions must be satisfied. For the asymptotic formulas of the Hankel functions to be used we need

$$k'r_o = \tfrac{1}{2}k'b \cot \phi_o \gg 1. \tag{7}$$

For r to be approximately constant over the cross-section we need, in addition,

$$k'[(r_o^2 + b^2/4)^{\frac{1}{2}} - r_o] \ll 1 \tag{8}$$

which can also be written

$$k'b \tan(\phi_o/2) \ll 1 \tag{9}$$

For most rectangular waveguides, used near the center of their range, we have $a \approx 2b \approx \lambda_g/2$ so that $k'b$ in the above equations is around $\pi/2$. Thus both eq. (7) and eq. (9), in the usual guide range, can be considered as setting limits on ϕ_o. The limits set by eq. (7) are easy to assess from tables of Bessel functions. Thus, we find

$$H_1^{(2)}(x)/jH_o^{(2)}(x) = 1.02 - j0.24 \text{ when } x = 2$$

$$= 1.07 - j0.45 \text{ when } x = 1$$

whereas eq. (6) would give 1 - j0.25 and 1 - j0.5 respectively in the two cases. Clearly, if we can tolerate an error of a few percent, we can take a value $x = \pi/2$, to give

$$k'b \cot \phi_o \geqslant \pi \tag{10}$$

as one limit.

The expression on the left of eq. (9) is twice the phase deviation from uniformity over the aperture. If we denote the permissible phase error by $\Delta\psi$ and use the lower limit in eq. (10), then eq. (9) gives

$$\tan \phi_o \tan(\phi_o/2) \leqslant 2\Delta\psi/\pi \tag{11}$$

Thus, $\phi_o = 20°$ corresponds to a phase error of $\Delta\psi = 5.7°$, which seems a reasonably small error for a total taper angle of 40°. (Most of the tapers used in the experiments reported in ref. 3 were well below this size). If we use $k'b \approx \pi/2$ in eq.(10) we would need $\tan \phi_o \lesssim 1/2$, or $\phi_o \lesssim 27°$. Thus, for the usual values of waveguide dimensions, it is eq. (9) that sets the more severe limit on ϕ_o.

In summary, eq. (6) for the junction impedance is to be preferred to eq. (5) for the admittance; eqs. (7) and (9) set limits for b and ϕ_o such that, for most guides, an upper limit to ϕ_o around 20° is encountered. But for <u>any</u> value of $k'b$ there will always be a value of ϕ_o small enough to ensure that both restrictions are met, and as $\phi_o \to 0$ the results should in any case be quite accurate. Results for comparison with the handbook equations will therefore be taken for fixed b/λ_g

and small taper angle, where the above formulas should be the most accurate.

It should, perhaps, be pointed out that eq. (8) is taken from a phase requirement, and that there will also be a corresponding amplitude variation. This, however, will be quite small when eq. (8) is satisfied, and it is the phase discrepancy which sets the limit to the accuracy of the field matching. Of course, the present method does not yield the increased field concentration adjacent to the convex corner at the junction, such as would be expected to exist from a consideration of the static case.

3. The Waveguide Handbook Formula

The circuit parameters were stated to be obtained by a "simple equivalent static method" but no details were published. It leads to a more complicated equivalent circuit than the method of section 1; for convenience it is repeated here in figure 2, with the circuit parameters given by

$$Y(r_2)/Y_1 = \sin(\theta)/\theta \tag{12}$$

$$B_a/Y_1 = (2\pi b'/\lambda_g\theta)\log(\theta \operatorname{cosec}\theta) \tag{13}$$

$$B_c/Y_1 = (2b'/\lambda_g)[\gamma + \Psi(\theta/\pi)] \tag{14}$$

$$B_b/Y_1 = (\lambda_g/\pi b')\,\frac{\sin\theta}{\theta}\,\frac{\sin\theta}{1 - \sin(2\theta)/2\theta} \tag{15}$$

where γ is Euler's constant, 0.5772 ....

The notation of the handbook has been retained; θ is the same as ϕ_o, and b' is the same as $b/2$ of the previous sections. (The handbook notation is here changed to b' to prevent confusion with b in comparing results). The Ψ function is not, as commonly used, the logarithmic derivative of the gamma function, but of the factorial function.

Thus $\Psi(x) = d/dx \log \Gamma(1+x)$, and is the function tabulated in Jahnke and Emde's "Table of Functions."[4] It differs from the usual logarithmic derivative by an extra term $1/x$ coming from $\Gamma(1+x) = x\Gamma(x)$. From the known properties of the gamma function[5] we have

$$\frac{d}{dx} \log \Gamma(1+x) = -\gamma + x \sum_1^\infty \frac{1}{n(n+x)} = -\gamma \quad \text{at} \quad x = 0$$

$$\frac{d^2}{dx^2} \log \Gamma(1+x) = \sum_1^\infty \frac{1}{(x+n)^2} = \pi^2/6 \quad \text{at} \quad x = 0$$

Hence, using a Taylor expansion for small x

$$\Psi(x) = -\gamma + x\pi^2/6 + \cdots \tag{16}$$

Using these results we find, for small θ, that

$$\begin{aligned} Y(r_2)/Y_1 &= 1 + 0(\theta^2) \\ B_a = B_c &= (2\pi b'\theta/6\lambda_g)[1 + 0(\theta^2)] \\ B_b &= (3\lambda_g/2\pi b'\theta)[1 + 0(\theta^2)] \end{aligned} \tag{17}$$

As $\theta \to 0$, B_a and B_c approach zero, B_b (which is a series element) becomes infinite, and the circuit reduces to a matched termination. But for θ small, but not zero, the equivalent circuit gives, for the input admittance,

$$\begin{aligned} Y_{in} &= jB_a + [1/(1+jB_c) + 1/(-jB_b)]^{-1} \\ &\approx 1 + j(B_a + B_c - 1/B_b) \end{aligned} \tag{18}$$

where B_a, B_c and $1/B_b$ are all small. For small θ, this is the result for comparison with eq. (5). Unfortunately, when (17) is used in (18) we get a susceptance $(2\pi b'\theta/\lambda_g)(1/6 + 1/6 - 1/3) = 0$, so there is no surviving term which is first order in θ and a proper comparison is not possible.

4. Discussion of the Handbook Formula

According to the Waveguide Handbook the formulas are valid for $2b'/\lambda_g \ll 1$, with an estimated accuracy of a few percent when $2b'/\lambda_g < 0.1$. There is no stated restriction on θ, and it should certainly be permissible to use the small θ limits.

In correspondence, Dr. Marcurvitz, who was kind enough to go through his early notes on the subject, writes "I suspect the equivalent circuit parameter for B_b may be written down incorrectly, since a result in my notes for a single lumped susceptance approximation is $j(B_a + B_c)$, which apparently is not the result derivable from the equivalent circuit and appears to indicate that the B_b term in the handbook is not correct."

The term $(B_a + B_c)$ alone is $2\pi b'\theta/3\lambda_g$, whereas, in the same notation, equation (5) gives $\theta\lambda_g/4\pi b'$. The former expression is supposedly valid for small θ and small (b'/λ_g), and the latter for small θ and <u>large</u> $(b'/\theta\lambda_g)$. Hence the results can in no way be reconciled in the common range of small b'/λ_g and small enough θ. The coefficients of θ are seen to be the same when $b(= 2b') = 0.39\lambda_g$, which is much larger than the usual guide size. For $b = \lambda_g/4$, the formulas differ by a factor 2.43, the handbook result being the smaller. Both these values of b are greater than the $0.1\lambda_g$ value suggested in the handbook for accuracies of a few percent. For yet smaller values of b/λ_g the discrepancy between the two results is, of course, still greater.

Some theoretical support for the structure of the handbook formula can be obtained in a simple way from a consideration of the static limit ($\lambda_g \to \infty$) as follows. For a given value of electric field the voltage across the guide is proportional to b, whilst the excess electric charge in a region near the junction would also be proportional to b. Thus the excess capacitance C of the junction region is independent of b. The guide impedance Z_g is proportional to $(b\lambda/\lambda_g)$ so the parameter $\omega C Z_g$, which measures the junction susceptance effect, would thus be proportional to (b/λ_g), as found above; but only in the limit for large λ_g. The field-matching formula is contingent on $(b/\phi_o \lambda_g)$ being <u>large</u>, and is not, in this sense, a static formula. In fact it depends essentially on the wave in the tapered section being in a "propagating" condition at the throat of the taper, which is a dynamic, not a static, requirement. This may account for the basic difference in the two results, but does not indicate which should be the more accurate in the common region of moderate b/λ_g and small enough θ or ϕ_o.

5. Measured Results

The measurements reported in reference 3 were taken some 30 years ago, and are no longer available. In order to help settle the validity of either of the two junction formulas, an X-band taper section was constructed with a taper semi-angle ϕ_o of 12°, well within the expected limits of validity for eq. (6). The taper length was 12 inches, as shown in figure 3, and the reflection into the guide was measured from

8.2 to 11.1 GHz. Because of the mis-match at the horn mouth, there is a beat between the horn mouth and the junction, the number of maxima corresponding to the 12" taper length being 8 in the measurement band. The results are shown in figure 4.

The magnitude of the horn reflection is known from the well-known Wiener-Hopf analysis for a parallel plate waveguide, and it should be fairly close for the tapered horn with an aperture in the E-plane of some 6 inches. By combining the figures at the maxima in figure 4, with the values of the minima on either side, values can be obtained for $R_1 \pm R_2$ where R_1 and R_2 are, respectively, the magnitudes of the reflection from the horn mouth and the junction. There is an in-built ambiguity as to which is which, but by comparison with the theoretical figure from the horn, given by

$$R_1 = (2a/\lambda - 2a/\lambda_g)e^{-\frac{1}{2}k'a} \tag{19}$$

the ambiguity can be removed.

The results of this calculation are shown in figure 5, from which it can be seen that

i) The measured and theoretical values for the horn mouth are in close agreement, the measurements being a little on the low side.

ii) The measured values for the junction are close to, but a little lower than, the values indicated by eq. (6). The slopes are comparable.

iii) There is no agreement with the handbook values, either in magnitude or slope. (The handbook figure was taken from $B_a + B_c = 2\pi b'/3\lambda_g$).

The discrepancies between measurements and theory are about 0.01 to 0.02 in reflection coefficient. It was estimated that the experimental error for the equipment used was about 0.01; but the consistency with which the measurements fall below the theoretical values indicates either a small systematic error, or a slight over-estimation by the theory. In any case, for the present purpose of distinguishing between the two available theories, the figures are quite adequate.

6. Conclusions

It would appear that the handbook equations are in error, at least as far as the series element is concerned; but even so, the other susceptance formulas appear to be incompatible with the measurements, since the factor b/λ_g increases with frequency whilst the measured values show a decrease. There seems to be no obvious way of reconciling these results. The measurements are, however, in close agreement with equations (5) and (6). The latter is preferred since the equivalent circuit reduces to a simple series capacitor, with ordinary (positive) capacitance. In the static limit, for a fixed taper angle, neither of these equations will ultimately apply. However, in the usual waveguide range, and for taper angles that are not too great, they are well within their proper range of validity, and can be used with confidence for taper design.

Acknowledgements

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References

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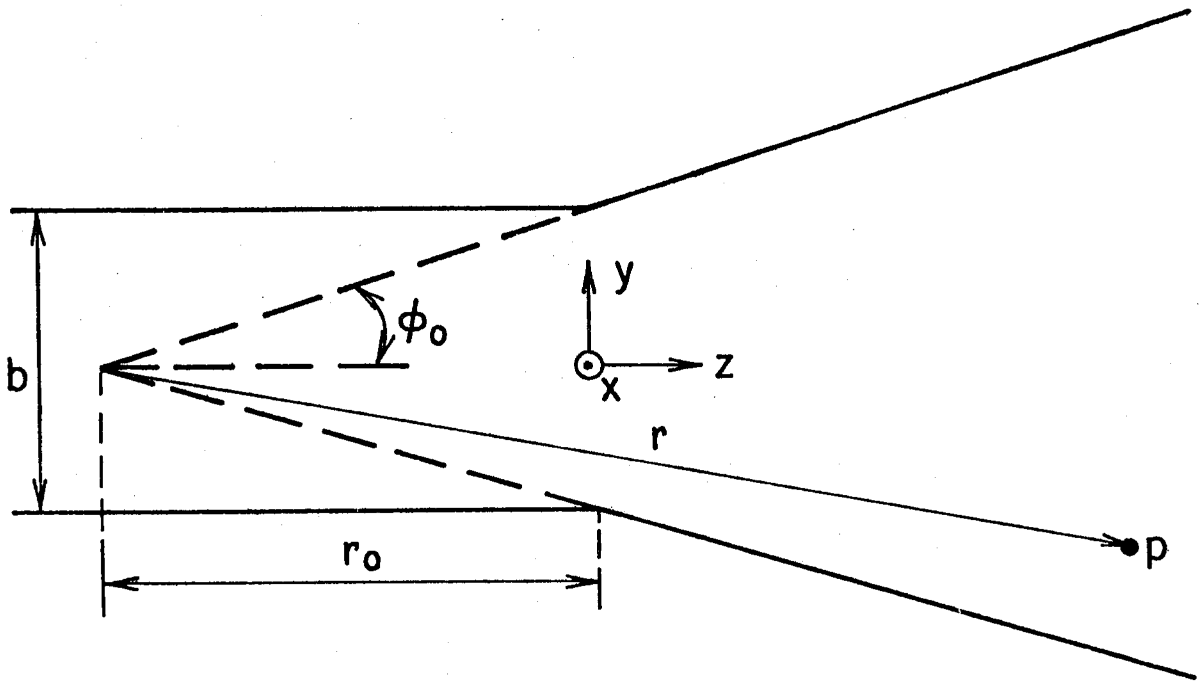

Fig. 1. Guide-to-Taper Junction

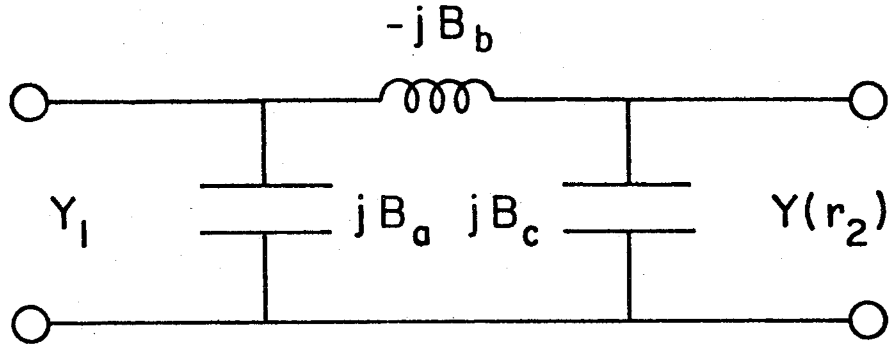

Fig. 2. Quasi-static Equivalent Circuit

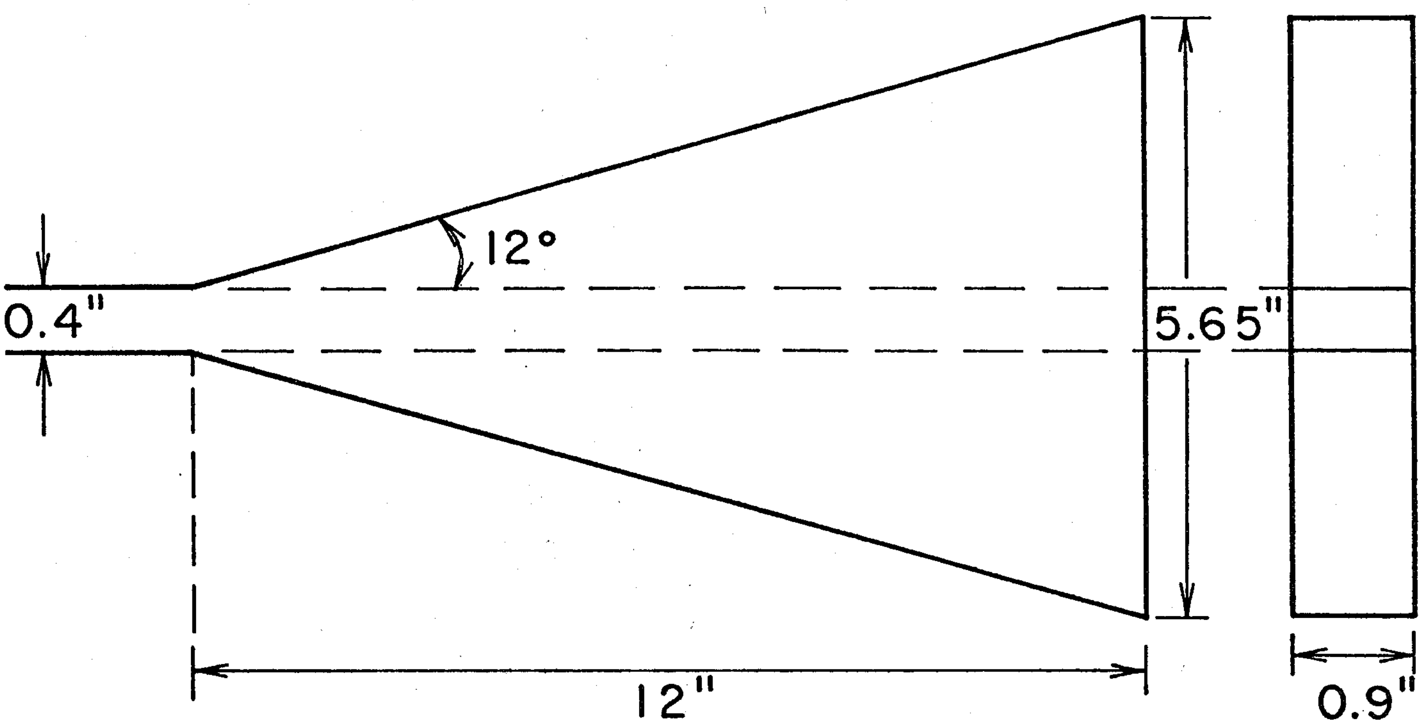

Figure 3. Tapered Section

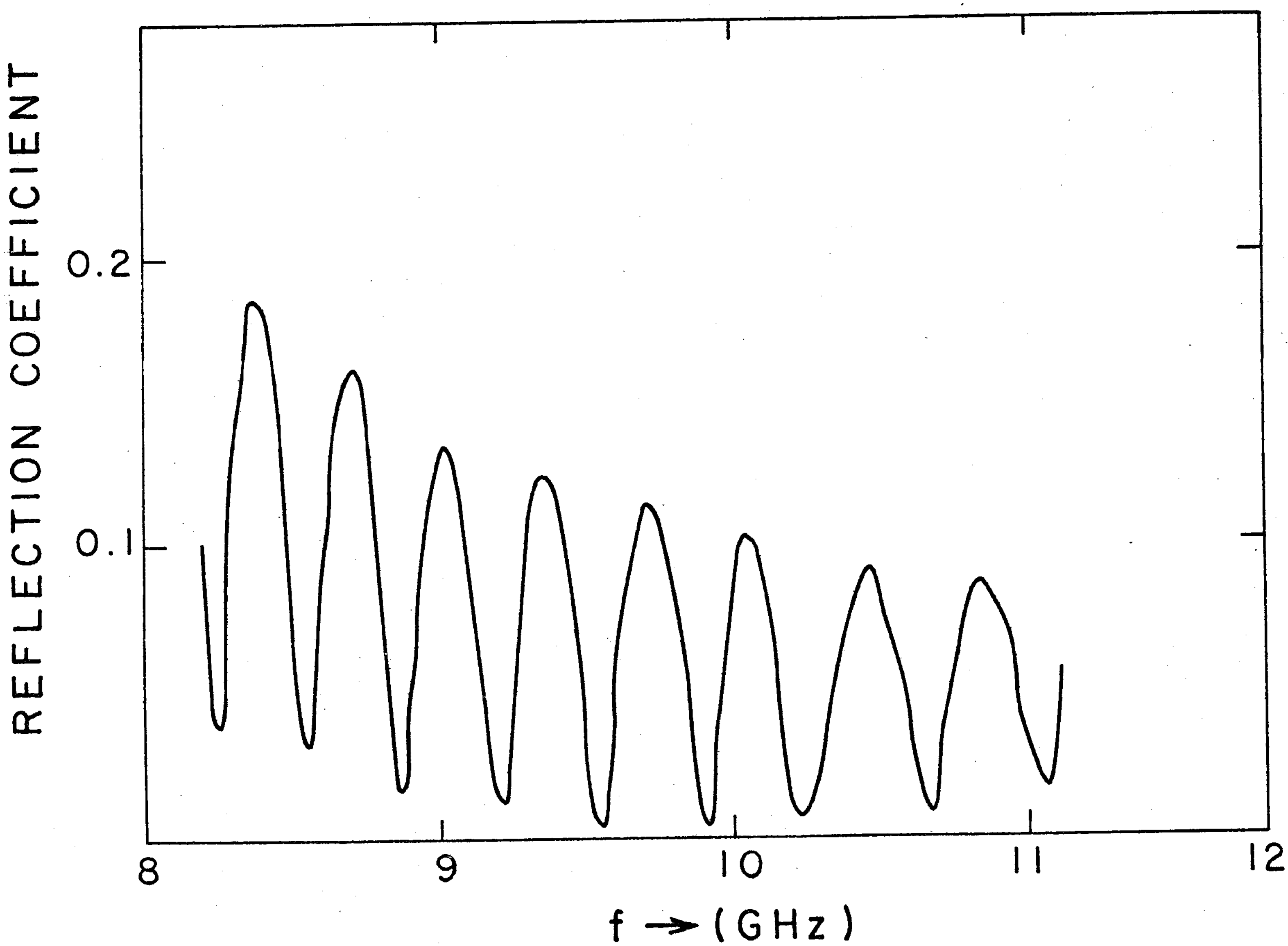

Fig. 4. Measured Reflection from the Taper and Horn Mouth

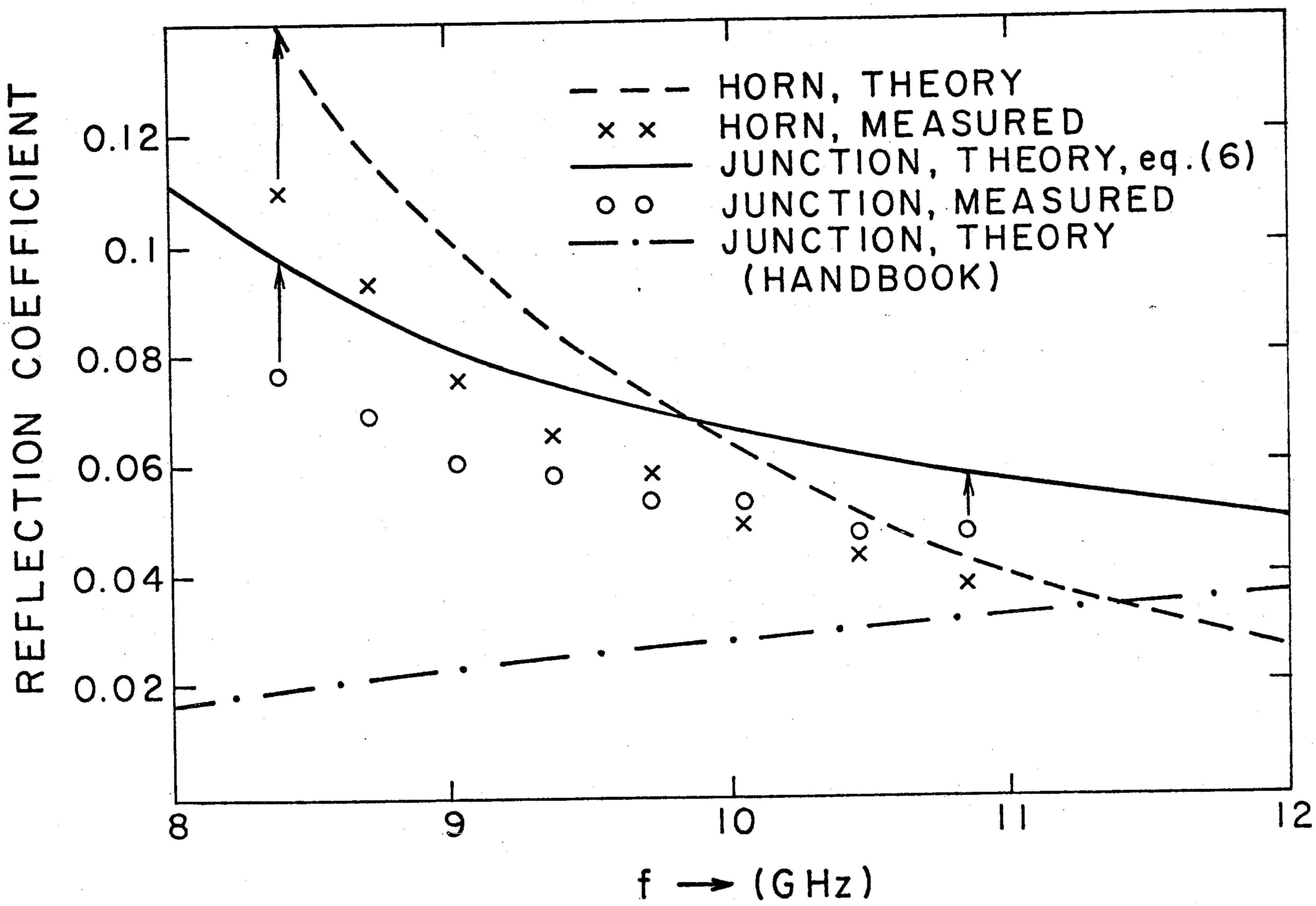

Fig. 5. Comparison of Measured and Theoretical Values