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OF ARBITRARY WIDTH

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ABSTRACT

An analytic theory for the dispersion of the fundamental mode on wide open microstrip is presented. Only a single basis function is needed to accurately represent each of the charge and current distributions on the strip, thus allowing more efficient determination of the propagation constant as compared to moment-method solutions requiring a larger number of basis functions. The results obtained blend smoothly into results of high-frequency (Wiener-Hopf) theories, and still retain the appealing physical interpretation in terms of capacitance and inductance of the narrow strip theory previously obtained by the authors.

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I. INTRODUCTION

In previous work, the authors [1] have presented an analytic theory of dispersion for narrow open microstrip in terms of a dispersive series inductance and capacitance, generalizing the classical expression for the propagation constant from transmission line theory which involves the static values of these parameters. Because an accurate form for the current and charge distributions (which are the same for this case) was available, it was possible to avoid more cumbersome moment function expansions, and to obtain a relatively simple dispersion relation possessing the clear physical interpretation referred to above. In reviewing numerical results available in the literature for wider microstrip, whose strip width is comparable to substrate thickness, the authors found significant discrepancies between workers who used different methods to attack the problem [2]. The best methods seem to be those which can represent the current and charge distributions (especially the edge singularities) accurately with a minimum number of basis functions.

The goal of the present study is to formulate an analytic theory of dispersion similar to [1] which will be valid for wider strips, yet still retain both analytical and computational straightforwardness as well as clear physical insight into the problem. Crucial to this is the recognition that the charge and current distributions now differ significantly from those in the narrow strip limit, and also differ to some extent from each other. Thus an important part of the discussion entails finding accurate and reasonably simple functional descriptions of these distributions. The results will be examined to see to what degree the difference

of these distributions from the narrow strip case and from each other affects the accuracy of the computed dispersion curves.

Of published numerical work, [3] - [5] offer results that we might classify as applying to "wide" microstrip, and these will be used as the basis for comparison. Also, although we shall consider strips wide compared to the substrate, the strips are not allowed to become electrically large, because in this range of parameters, the physical mechanisms are basically different. These are best treated by the methods of [6] - [9] (which use adaptations of the more appropriate Wiener-Hopf technique).

2. APPROXIMATE CHARGE AND CURRENT DISTRIBUTIONS

FOR $\epsilon_r \gg 1$, $\mu_r = 1$ (STATIC CASE)

We shall proceed from the formulation given in [1]. For an assumed propagation factor of $\exp(i\omega t - ik_0 x)$, where α is a yet unknown normalized propagation constant and x is the distance along the strip axis (see Fig. 1):

$$\int_{-l}^l G_e(y-y')\rho_1(y')dy' = \cosh\sqrt{\alpha^2-1} k_0 y ; \quad |y| \leq l \quad (1)$$

Here $\rho_1(y)$ is a normalized charge distribution on the strip ($-l \leq y \leq l$),

$$G_e(y) = 2 \int_0^\infty \frac{(u_n \tanh u_n T) \cos k_0 \lambda y}{\epsilon_r u_o + u_n \tanh u_n T} \frac{d\lambda}{u_o} \quad (2)$$

$T = k_0 t$ is the substrate thickness normalized to the free-space wave-number, and

$$u_n = (\lambda^2 + \alpha^2 - \mu_r \epsilon_r)^{\frac{1}{2}} ; \quad u_o = (\lambda^2 + \alpha^2 - 1)^{\frac{1}{2}} ; \quad \text{Re}(u_o) \geq 0 \quad (3)$$

where ϵ_r and μ_r are respectively the relative permittivity and permeability of the substrate. Once the solution of (1) is known as a function of α , the longitudinal current density $J_x(y)$ is then found from

$$\int_{-l}^l G_m(y-y')J_x(y')dy' = \frac{i\alpha}{k_0} [\cosh\sqrt{\alpha^2-1} k_0 y + \int_{-l}^l M(y-y')\rho_1(y')dy'] \quad (4)$$

$|y| \leq l$

where

$$G_m(y) = 2\mu_r \int_0^\infty \frac{\cos k_0 \lambda y d\lambda}{\mu_r u_o + u_n \coth u_n T} \quad (5)$$

$$M(y) = 2(\mu_r \epsilon_r - 1) \int_0^{\infty} \frac{\cos k_o \lambda y \quad d\lambda}{(\epsilon_r u_o + u_n \tanh u_n T) (\mu_r u_o + u_n \coth u_n T) u_o} \quad (6)$$

The solutions $\rho_1(y)$ and $J_x(y)$ thus obtained, both functions of α , are then inserted into

$$\int_{-\ell}^{\ell} [ik_o \alpha J_x(y) + \rho_1(y)] dy = 0 \quad (7)$$

(which follows from the requirement that the transverse current density vanish at the edges of the strip) which is then a characteristic equation for determining α .

Let us first examine the static limit of these equations. We find that $\cosh \sqrt{\alpha^2 - 1} k_o y \rightarrow 1$,

$$G_e \rightarrow G_e^{(o)}(y) = 2 \int_0^{\infty} \frac{\tanh \lambda t}{\epsilon_r + \tanh \lambda t} \cos \lambda y \frac{d\lambda}{\lambda} \quad (8)$$

$$G_m \rightarrow G_m^{(o)}(y) = 2\mu_r \int_0^{\infty} \frac{\cos \lambda y}{\mu_r + \coth \lambda t} \frac{d\lambda}{\lambda} \quad (9)$$

and $M \rightarrow 0$, thus decoupling (1) and (4) for the quantities $\rho_1(y)$ and $\rho_2(y) \equiv -ik_o J_x(y)/\alpha$. Now if $\epsilon_r \gg 1$, we can write

$$G_e^{(o)}(y) = 2 \int_0^{\infty} \frac{\tanh \lambda t}{(\epsilon_r + 1)} \cos \lambda y \, d\lambda + 2 \int_0^{\infty} \left[\frac{1}{\epsilon_r + \tanh \lambda t} - \frac{1}{\epsilon_r + 1} \right] \frac{\tanh \lambda t}{\lambda} \cos \lambda y \, d\lambda \quad (10)$$

The first integral is known exactly [10, #4.116.2] as

$$- \frac{2}{\epsilon_r + 1} \ln \tanh \frac{\pi |y|}{4t}$$

whereas the second can be estimated to be smaller than about

$$\frac{1.6}{\epsilon_r(\epsilon_r+1)}$$

for any value of y . Thus, for $\epsilon_r^2 \gg 1$ as can typically be found in microstrip, we will provisionally neglect the second integral in (10), but return later to the question of a more accurate evaluation. Equation (1) is now

$$-\frac{2}{\epsilon_r+1} \int_{-\ell}^{\ell} \ln \tanh \frac{\pi|y-y'|}{4t} \rho_1(y') dy' = 1 ; \quad |y| \leq \ell \quad (11)$$

which can be solved explicitly for $\rho_1(y')$ by using the fact that (11) is precisely the integral equation for the charge on a symmetric stripline [11], i.e., with a second ground plane at a height t above the strip (Fig. 2a). Its solution (which can be obtained either by conformal mapping techniques or by the Schwinger transformation and a singular integral equation procedure [12]) is

$$\rho_1(y) = \frac{\rho_0}{\sqrt{\cosh^2\left(\frac{\pi\ell}{2t}\right) - \cosh^2\left(\frac{\pi y}{2t}\right)}} \quad (12)$$

where the constant ρ_0 is obtained by substituting (12) into (11) and setting $y=0$; the result is (see Appendix):

$$\rho_0 = \frac{\epsilon_r+1}{4t} \frac{\cosh\left(\frac{\pi\ell}{2t}\right)}{K\left(\operatorname{sech} \frac{\pi\ell}{2t}\right)} \quad (13)$$

where $K(k)$ is the complete elliptic integral of the first kind [13]. The capacitance of the symmetric strip is then obtained from the total charge per unit length on the strip, which is, from the Appendix

$$q = 2 \int_0^{\ell} \rho_1(y) dy = \frac{\epsilon_r + 1}{\pi} \frac{K(\tanh \frac{\pi \ell}{2t})}{K(\operatorname{sech} \frac{\pi \ell}{2t})} = \frac{\epsilon_r + 1}{\pi} \frac{K(k)}{K(k')} \quad (14)$$

if we let the modulus $k = \tanh \frac{\pi \ell}{2t}$ and $k' = (1 - k^2)^{\frac{1}{2}}$ is the complementary modulus.

The rough equivalence between the open strip and symmetric strip for large ϵ_r was apparently first remarked by Dukes [14], who argued that since most of the electric flux is concentrated in the substrate, the upper ground plane has only a small effect. Wheeler [15] has used this limiting case as a partial basis for his approximate conformal mapping solution of the static capacitance of the open microstrip.

In the case of a nonmagnetic substrate ($\mu_r = 1$), we have

$$G_m^{(0)}(y) = \ell \ln \frac{\sqrt{y^2 + 4t^2}}{|y|} \quad (15)$$

As in [1], we can identify (15) as the kernel of the integral equation for the charge density $f(y)$ on one of two parallel strips in free space, whose width is 2ℓ each and which are separated vertically by a distance $2t$ and maintained at the potentials $+V$ and $-V$ (Fig. 2b):

$$\frac{1}{2\pi} \int_{-\ell}^{\ell} f(y') \ell \ln \frac{\sqrt{(y-y')^2 + 4t^2}}{|y-y'|} dy' = V; \quad |y| \leq \ell \quad (16)$$

The capacitance C_p of this system can be found by the method of conformal transformation (see the references to [16]) to be given by

$$C_p = \frac{1}{2V} \int_{-\ell}^{\ell} f(y) dy = K(k_1)/K(k'_1) \quad (17)$$

where the modulus k_1 is the solution of

$$K(k_1)E(\phi, k_1) - E(k_1)F(\phi, k_1) = \frac{\pi\ell}{2t}; \quad \sin \phi = \left[\frac{K(k_1) - E(k_1)}{k_1^2 K(k_1)} \right]^{\frac{1}{2}} \quad (18)$$

and $F(\phi, k)$ and $E(\phi, k)$ are the incomplete elliptic integrals of the first and second kind, while $E(k)$ is the complete elliptic integral of the second kind [13].

It has been found [16] that a simple, closed-form approximation to the function $f(y)$ (that is to say, the current distribution $J_x(y)$ in the present problem) is given by

$$J_x(y) = \frac{J_0}{\sqrt{\cosh^2\left(\frac{\pi\ell}{4t}\right) - \cosh^2\left(\frac{\pi y}{4t}\right)}} \quad (19)$$

where J_0 is some constant. By comparison with (12) it can be seen that the charge distribution is "flatter," i.e., the charge decays more rapidly away from the singularities at the edges of the strip that does the current. In the next section, these distributions will be used to obtain an approximate dispersion relation, valid for large ϵ_r .

3. DERIVATION OF THE DISPERSION RELATION

The static charge and current distributions from the previous section can be expected to describe the distributions for non-zero frequencies with reasonable accuracy as well, provided the strip does not become electrically large. Even though (12) and (19) are not precisely correct even for the static case, one could use these distributions in an expression which possessed some kind of variational property to solve for the propagation constant α . In the static case, when the integral equations for charge and current decouple, such expressions can be found for capacitance and inductance separately [11]; [17]-[19]. More general (and more complicated) stationary functionals for determining α have also been obtained for non-zero frequency when the equations are coupled [2],[20]-[21];[18]. A much simpler form is obtained if trial functions of the same form are used for both charge and current-- that is, if transverse currents are assumed to be negligible [21], [2]. Later on, we shall examine the merits of a variational expression, and the possibility of neglecting transverse currents, but at first, we shall start from a slightly different point.

The functions (12) and (19) will be viewed as the first in two sets of basis functions into which the charge and current respectively will be expanded, the coefficients (of which ρ_0 and J_0 are the first) to be found by the method of moments. Our approximate solution is then obtained by truncating each expansion to one term only. Specifically, we substitute (12) and (19) into the integral equations (1) and (4), explicitly separating out the static parts $G_e^{(0)}$ and $G_m^{(0)}$ from the kernels

G_e and G_m as was done for the narrow strip in [1]. The static terms can be simplified using (11), (14), (16), and (17). We then multiply (1) by (12) and (4) by (19), and integrate from $-\ell$ to ℓ with respect to y . The resulting expressions determine ρ_0 and J_0 as functions of α . These are used with (7) to obtain, after considerable algebra, the characteristic equation for determining α :

$$\alpha^2 = L(\alpha) C(\alpha) \quad (20)$$

where $L(\alpha)$ and $C(\alpha)$ are a dispersive inductance and capacitance per unit length of line, respectively, and each consists of a static part and a dispersive part:

$$L(\alpha) = L_s + L_d(\alpha) \quad (21)$$

$$\frac{1}{C(\alpha)} = \frac{1}{C_s} + \frac{1}{C_d(\alpha)} \quad (22)$$

The static inductance is in a form which was shown in [16] to be an excellent approximation to the exact value for the parallel plates, thus:

$$L_s = \frac{1}{2C_p} \quad (23)$$

while C_s is related to the capacitance of the symmetric stripline mentioned in section 2, plus a moderately small correction term:

$$\frac{1}{C_s} = \frac{1}{2(\epsilon_r + 1)} \frac{K(k')}{K(k)} + \frac{\pi}{4} \left[\frac{1}{k'K(k)} \right]^2 \int_0^\infty \frac{\tanh \lambda e^{-\lambda}}{\lambda \cosh \lambda} \frac{[P_{-\frac{1}{2}+i\lambda/\pi}(\cosh \frac{\pi\ell}{t})]^2}{(\epsilon_r + 1)(\epsilon_r + \tanh \lambda)} d\lambda \quad (24)$$

Here $k = \tanh \frac{\pi\ell}{2t}$ is the same modulus which appears in (14), and $P_\mu(z)$ is the Legendre function of argument z and index μ which is found to be the Fourier transform of the charge distribution(12)--see the Appendix.

Expression (24), using an approximation instead of the Legendre function, was obtained in [11]. The dispersive terms are α -dependent and can be shown to vanish with the frequency:

$$L_d(\alpha) = \frac{\pi}{8} \left[\frac{1}{k_2' K(k_2)} \right]^2 G_m^{(2)}(\alpha) \quad (25)$$

$$\begin{aligned} \frac{1}{C_d(\alpha)} = & \left\{ \frac{k' K(k)}{k_2' K(k_2)} \frac{P_{2\nu-\frac{1}{2}}(\cosh \frac{\pi\ell}{2t})}{P_{\nu-\frac{1}{2}}(\cosh \frac{\pi\ell}{t})} - 1 \right\} \frac{1}{C_s} \\ & + \frac{\pi}{8} \left[\frac{1}{k_2' K(k_2)} \right] \left[\frac{1}{k' K(k)} \right] \left\{ \frac{P_{2\nu-\frac{1}{2}}(\cosh \frac{\pi\ell}{2t})}{P_{\nu-\frac{1}{2}}(\cosh \frac{\pi\ell}{t})} G_e^{(2)}(\alpha) + M_2(\alpha) \right\} \quad (26) \end{aligned}$$

where the modulus $k_2 = \tanh \frac{\pi\ell}{4t}$, while $\nu = T\sqrt{\alpha^2 - 1}/\pi$. The functions $G_e^{(2)}$, $G_m^{(2)}(\alpha)$ and $M_2(\alpha)$ are Sommerfeld integrals similar to those obtained in [1]:

$$G_m^{(2)}(\alpha) = 2 \int_0^\infty \left[\frac{1}{u_o + u_n \coth u_n T} - \frac{1}{\lambda(1 + \coth \lambda T)} \right] \left[P_{-\frac{1}{2} + i2\tau}(\cosh \frac{\pi\ell}{2t}) \right]^2 d\lambda \quad (27)$$

$$G_e^{(2)}(\alpha) = 2 \int_0^\infty \left[\frac{u_n \tanh u_n T}{u_o(\epsilon_r u_o + u_n \tanh u_n T)} - \frac{\tanh \lambda T}{\lambda(\epsilon_r + \tanh \lambda T)} \right] \left[P_{-\frac{1}{2} + i\tau}(\cosh \frac{\pi\ell}{t}) \right]^2 d\lambda \quad (28)$$

$$M_2(\alpha) = 2(\epsilon_r - 1) \int_0^\infty \frac{[P_{-\frac{1}{2} + i\tau}(\cosh \frac{\pi\ell}{t})][P_{-\frac{1}{2} + i2\tau}(\cosh \frac{\pi\ell}{2t})] d\lambda}{u_o(\epsilon_r u_o + u_n \tanh u_n T)(u_o + u_n \coth u_n T)} \quad (29)$$

where $\tau = \lambda T/\pi$. Equations (27)-(29) differ from the corresponding functions in [1] only by the presence of the Fourier transform of the current distribution in $G_m^{(2)}(\alpha)$, that of the charge distribution in $G_e^{(2)}(\alpha)$, and of both in the coupling term $M_2(\alpha)$.

4. AN ALTERNATIVE DISPERSION RELATION

The dispersive part of the capacitance in (26) is rendered somewhat cumbersome by the appearance of the frequency-dependent functions $P_{2\nu-\frac{1}{2}}(\cosh \frac{\pi\ell}{2t})$ and $P_{\nu-\frac{1}{2}}(\cosh \frac{\pi\ell}{t})$ which arise because the forcing (cosh) terms in (1) and (4) are multiplied by different basis functions in the process of obtaining (20) - (29). A somewhat different dispersion relation can be obtained from the variational expression given in [21], which can be shown to be expressible as

$$0 = \int_0^{\infty} \frac{u_n \tanh u_n T}{u_o (\epsilon_r u_o + u_n \tanh u_n T)} [\tilde{\rho}_1(\lambda)]^2 d\lambda + k_o^2 \int_0^{\infty} \frac{\mu_r \{[\tilde{J}_x(\lambda)]^2 + [\tilde{J}_y(\lambda)]^2\}}{\mu_r u_o + u_n \coth u_n T} d\lambda + (\epsilon_r \mu_r - 1) \int_0^{\infty} \frac{[\tilde{\rho}_1(\lambda)]^2}{u_o (\epsilon_r u_o + u_n \tanh u_n T) (\mu_r u_o + u_n \coth u_n T)} d\lambda \quad (30)$$

in the present notation, where $\tilde{\rho}_1(\lambda)$, $\tilde{J}_x(\lambda)$ and $\tilde{J}_y(\lambda)$ are the Fourier transforms of the charge density, longitudinal current density and transverse current density on the strip, respectively. From the continuity equation, the latter can be expressed in terms of the first two:

$$\tilde{J}_y(\lambda) = \frac{\tilde{\rho}_1(\lambda) + ik_o \alpha \tilde{J}_x(\lambda)}{-ik_o \lambda} \quad (31)$$

It should be noted that in order for the second integral in (30) to exist, \tilde{J}_y must not be allowed to have a $1/\lambda$ singularity at $\lambda = 0$, and thus we must have

$$\tilde{\rho}_1(0) + ik_o \alpha \tilde{J}_x(0) = 0$$

This is clearly a restatement of the edge condition (7) on J_y , which is thus implicit in equation (30).

Using the Fourier transforms of (12) and (19) for $\tilde{\rho}_1(\lambda)$ and $\tilde{J}_x(\lambda)$ in (30), and taking (7) into account, we can again obtain a dispersion relation in the form of (20) - (24), except that a slightly different form is obtained for the dispersive part of $C(\alpha)$:

$$\frac{1}{C_d(\alpha)} = \frac{\pi}{8} \left[\frac{1}{k'K(k)} \right]^2 \left\{ G_e^{(2)}(\alpha) + M_3(\alpha) \right\} - \frac{\pi}{8} G_m^{(3)}(\alpha) \quad (32)$$

while the dispersive part of $L(\alpha)$ from (25) remains unchanged. Here $G_e^{(2)}(\alpha)$ is again given by (28), while

$$M_3(\alpha) = 2(\epsilon_r - 1) \int_0^\infty \frac{[P_{-\frac{1}{2}+i\tau}(\cosh \frac{\pi\ell}{t})]^2 d\lambda}{u_o(\epsilon_r u_o + u_n \tanh u_n T)(u_o + u_n \coth u_n T)} \quad (33)$$

and

$$G_m^{(3)}(\alpha) = 2 \int_0^\infty \frac{d\lambda}{u_o + u_n \coth u_n T} \left[\frac{P_{-\frac{1}{2}+i\tau}(\cosh \frac{\pi\ell}{t})}{k'K(k)} - \frac{P_{-\frac{1}{2}+i2\tau}(\cosh \frac{\pi\ell}{2t})}{K_2'K(k_2)} \right] \frac{1}{\lambda} \quad (34)$$

It can be noted that (32) and (26) differ only by terms which would be absent if $\rho_1(y)$ and $J_x(y)$ had the same functional form; that is, if no transverse currents existed. The question that remains to be examined is how great the effect of these currents is on the computed value of α . Results using equation (26) for $C_d(\alpha)$ were compared with those obtained using (32), and also with those using (32) without $G_m^{(3)}(\alpha)$ (i.e., without the direct contribution from the transverse currents). Using $\ell/t = 2.34$, $k_o t = 0.107$, and $\epsilon_r = 9.9$, values obtained for the effective dielectric constant $\epsilon_{r\text{eff}} = \alpha^2$ from these three methods were 8.339, 8.337, and 8.337 respectively, indicating that the difference between the functional forms of $\rho_1(y)$ and $J_x(y)$ seems to have very little effect on results, although, as may be concluded from the comparisons in [2], the difference between these forms and those for the narrow strip are significant.

5. NUMERICAL RESULTS AND DISCUSSION

5.1 Narrow strips

The essential restriction imposed by this theory is that ϵ_r be large compared to one, but not that l/t necessarily be large. It can be shown, in fact, using the limiting form of the Legendre functions for argument equal to unity [22] and the limiting forms for the elliptic integrals as k and $k_2 \rightarrow 0$ that the dispersion equations derived here pass over into those obtained for narrow strips [1] if ϵ_r is large, and thus the present theory is valid for strips of arbitrary width. Results have been computed for narrow strips and compare quite well with those of [1], although of course the latter does not require evaluations of Legendre functions and is altogether more appropriate to the task.

5.2 Wider strips

Of the results available for wider strips, those of [5] seem to have the greatest likelihood of accuracy. As argued in [2], the moment method used in [5] uses a set of basis functions to describe the currents which possess the proper singular behavior at the edges of the strip, and a sufficient number of these is employed to assure an accurate result. Figure 3 gives a comparison between the results for the widest strip from [5] and from the present method (the three methods mentioned in the previous paragraph gave indistinguishable results when displayed graphically--this was true for all results presented here). The agreement is nearly exact; the discrepancy is at least as much as the error involved in reading data from the graph in [5]. Kowalski and Pregla [3] have used a variational approach, but use only the current distribution appropriate to a narrow strip as a trial function. While, as seen in [2], this gives good results

even for strips as wide as the substrate thickness, a comparison of their results for a wider strip with those of our method (Figure 4) shows that the narrow strip current distribution is no longer adequate, although the same general trend for the effective dielectric constant is predicted.

In [4], results for very wide strips are computed by what is also (in essence) a variational technique, but using a constant distribution of the current on the strip. Comparing results with those of our method in Fig. 5, we see that for $\ell/t = 2$, their method seems to predict a reasonable value for α^2 in the static limit, but dispersion effects are considerably underestimated. For a very wide strip with $\ell/t = 5$, no consistent pattern of error seems to be present. A possible explanation of this is that in both methods, Sommerfeld integrals like (27)-(29) must be evaluated with rapidly oscillating integrands (the conical functions oscillate more rapidly with τ as the argument is increased; a similar rapid oscillation occurs in [4] due to trigonometric functions. In support of our result for the static limit, we can offer agreement with the graphically displayed results of Wheeler [15] and many others who have studied this case.

Also displayed in Fig. 5 are the results of Nefedov and Fialkovskii [9], who apply a Wiener-Hopf technique appropriate to very wide strips and rather high frequencies. It can be seen that in both instances good agreement with our result is obtained at frequencies for which $\sqrt{\epsilon_r} k_0 \ell \gtrsim 1$. Below this, there is some difficulty in accurately reading the graph in [9], but this is of little consequence since the theory admittedly breaks down for low frequencies anyway.

6. CONCLUSION

It has been found that accurate results for the dispersion of wider open microstrip can be obtained for dielectrically dense, nonmagnetic substrates using only a single basis function each for the charge and current distributions on the strip. Computing times can be considerably shortened compared to moment-method approaches requiring larger numbers of basis functions to represent these quantities. A smooth transition has been observed between this, low-frequency theory, and the higher frequency (Wiener-Hopf) approaches existing in the literature.

REFERENCES

- [1] D.C. Chang and E.F. Kuester, "An analytic theory for narrow open microstrip," Sci. Rept. No. 28, Electromagnetics Laboratory, Dept. of Elec. Eng., Univ. of Colorado, Boulder (1978).
- [2] E.F. Kuester and D.C. Chang, "An appraisal of methods for computation of the dispersion characteristics of open microstrip," submitted to IEEE Trans. Micr. Theory Tech.
- [3] G. Kowalski and R. Pregla, "Dispersion characteristics of single and coupled microstrips" Arch. Elek. Übertragungstech. v. 26, pp. 276-280 (1972).
- [4] A.P. Gorobets, L.N. Deryugin, O.A. Kurdyumov and V.E. Sotin, "Dispersion equations and retardation calculations for microstrip line," Izv. VUZ Radiofizika, v. 18, pp. 1173-1177 (1975) [in Russian; Engl. transl. in Radiophys. Quantum Electron. v. 18, pp. 866-869 (1975)].
- [5] R.H. Jansen, "High-speed computation of single and coupled microstrip parameters including dispersion, high-order modes, loss and finite strip thickness," IEEE Trans. Micr. Theory Tech. v. 26, pp. 75-82 (1978).
- [6] T.T. Fong and S.W. Lee, "Modal analysis of a planar dielectric strip waveguide for millimeter-wave integrated circuits," IEEE Trans. Micr. Theory Tech. v. 22, pp. 776-783 (1974).
- [7] A.T. Fialkovskii, "Theory of higher-order modes in a nonsymmetrical stripline," Radiotekh. Elektron. v. 21, pp. 683-690 (1976) [in Russian; Engl. transl. in Radio Eng. Electron, Phys. v. 21, no. 4, pp. 15-21 (1976)].
- [8] A.T. Fialkovskii, "Classification of modes in an asymmetrical strip-line," Radiotekh. Elektron. v. 22, pp. 613-615 (1977) [in Russian; Engl. transl. in Radio Eng. Electron Phys. v. 22, no. 3, pp. 128-129 (1977)].
- [9] E.I. Nefedov and A.T. Fialkovskii, "Dispersion characteristics of microstrip waveguide," Dokl. Akad. Nauk SSSR. v. 235, pp. 1303-1305 (1977) [in Russian; Engl. transl. in Sov. Phys. Doklady. v. 22, pp. 448-450 (1977)].
- [10] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products. New York: Academic Press, 1965.
- [11] G. Kowalski and R. Pregla, "Calculation of the distributed capacitances of coupled microstrips using a variational integral," Arch. Elek. Übertragungstech. v. 27, pp. 51-52 (1973).
- [12] L. Lewin, Theory of Waveguides. London: Newnes-Butterworths, 1975.

- [13] P.F. Byrd and M.D. Friedman, Handbook of Elliptic Integrals for Engineers and Scientists. Berlin: Springer-Verlag, 1971.
- [14] J.M.C. Dukes, "An investigation into some fundamental properties of strip transmission lines with the aid of an electrolytic tank," Proc. IEE (London), part B, v. 103, pp. 319-333 (1956).
- [15] H. A. Wheeler, "Transmission-line properties of parallel strips separated by a dielectric sheet," IEEE Trans. Micr. Theory Tech. v. 13, pp. 172-185 (1965).
- [16] E.F. Kuester and D.C. Chang, "A simple, approximate expression for the charge distribution on parallel strips," in preparation.
- [17] E. Yamashita and R. Mittra, "Variational method for the analysis of microstrip lines," IEEE Trans. Micr. Theory Tech. v. 16, pp. 251-256 (1968).
- [18] L.A. Vaynshteyn and A.T. Fialkovskii, "Modes in slotted and stripline waveguides: Variational method and simpler results," Radiotekh. Elektron. v. 21, pp. 1137-1148 (1976) [in Russian; Engl. transl. in Radio Eng. Electron. Phys. v. 21, no. 6, pp. 1-11 (1976)].
- [19] R.F. Fikhmanas and P. Sh. Fridberg, "Two-sided variational estimates for the electrical parameters of microstrip," Radiotekh. Elektron. v. 21, pp. 1137-1148 (1977) [in Russian; Engl. transl. in Radio Eng. Electron. Phys. v. 22, no. 11, to appear].
- [20] G. Kowalski and R. Pregla, "Dispersion characteristics of single and coupled microstrips with double-layer substrates," Arch. Elek. Übertragungstech. v. 27, pp. 125-130 (1973).
- [21] R. Pregla and G. Kowalski, "Simple formulas for the determination of the characteristic constants of microstrips," Arch. Elek. Übertragungstech. v. 28, pp. 339-340 (1974).
- [22] M. I. Zhurina and L. N. Karmazina, Tables and Formulae for the Spherical Functions $P_{-\frac{1}{2}+i\tau}^m(z)$. Oxford: Pergamon Press, 1966.
- [23] F.W.J. Olver, Asymptotics and Special Functions. New York: Academic Press, 1974.
- [24] R.C. Thorne, "The asymptotic expansion of Legendre functions of large degree and order," Phil. Trans. Roy. Soc. London ser. A, v. 249, pp. 597-620 (1957).
- [25] W. Bartky, "Numerical calculation of a generalized complete elliptic integral," Rev. Mod. Phys. v. 10, pp. 264-269 (1938).

APPENDIX

EVALUATION OF SOME INTEGRALS

In this Appendix we derive some integrals which arise in section 2 and 3 in connection with the charge or current distribution

$$\frac{1}{\sqrt{\cosh^2\left(\frac{\pi\ell}{2t}\right) - \cosh^2\left(\frac{\pi y}{2t}\right)}}$$

In particular, from eqns. (11) and (12) we need the value of

$$\begin{aligned} I &= \int_{-\ell}^{\ell} \ln \tanh \frac{\pi|y'|}{4t} \frac{dy'}{\sqrt{\cosh^2\left(\frac{\pi\ell}{2t}\right) - \cosh^2\left(\frac{\pi y'}{2t}\right)}} \quad (\text{A.1}) \\ &= 2 \int_0^{\ell} \ln \tanh \frac{\pi y'}{4t} \frac{dy'}{\sqrt{\cosh^2\left(\frac{\pi\ell}{2t}\right) - \cosh^2\left(\frac{\pi y'}{2t}\right)}} \\ &= \frac{2t}{\pi} \int_0^{\phi_0} \ln \left[\frac{\cosh \phi - 1}{\cosh \phi + 1} \right] \frac{d\phi}{\sqrt{\cosh^2 \phi_0 - \cosh^2 \phi}} \end{aligned}$$

where $\phi = \frac{\pi y'}{2t}$ and $\phi_0 = \frac{\pi\ell}{2t}$. But the logarithm in (A.1) can be expressed as an integral, thus (see, e.g., [12, p. 189]):

$$\begin{aligned} I &= -\frac{2t}{\pi} \int_0^{\phi_0} d\phi \int_0^1 dw \left[\frac{1}{\cosh \phi - w} + \frac{1}{\cosh \phi + w} \right] \frac{1}{\sqrt{\cosh^2 \phi_0 - \cosh^2 \phi}} \\ &= -\frac{2t}{\pi} \int_0^1 dw \int_0^{\phi_0} \left[\frac{1}{\cosh \phi - w} + \frac{1}{\cosh \phi + w} \right] \frac{d\phi}{\sqrt{\cosh^2 \phi_0 - \cosh^2 \phi}} \quad (\text{A.2}) \\ &= -\frac{4t}{\pi} \int_0^1 dw \int_0^{x_0} \frac{dx}{[x^2 + (1-w^2)]\sqrt{x_0^2 - x^2}} \end{aligned}$$

where $x = \sinh \phi$, and $x_0 = \sinh \phi_0$. The inner integral can be further transformed by $x = x_0 \sin \theta$, so that

$$\begin{aligned}
 I &= -\frac{4t}{\pi} \int_0^1 dw \int_0^{\pi/2} \frac{d\theta}{[x_0^2 \sin^2 \theta + 1 - w^2]} \\
 &= -2t \int_0^1 \frac{dw}{\sqrt{1-w^2} \sqrt{1+x_0^2} \sqrt{1-w^2}} \\
 &= -2t \operatorname{sech} \left(\frac{\pi \ell}{2t} \right) K \left(\operatorname{sech} \frac{\pi \ell}{2t} \right) \quad (\text{A.3})
 \end{aligned}$$

by [10, #3.642.3] and the definition of the elliptic integral [13].

Another integral arises in finding the total charge on the strip:

$$\begin{aligned}
 J &= \int_{-\ell}^{\ell} \frac{dy}{\sqrt{\cosh^2 \left(\frac{\pi \ell}{2t} \right) - \cosh^2 \left(\frac{\pi y}{2t} \right)}} \\
 &= 2 \int_0^{\ell} \frac{dy}{\sqrt{\cosh^2 \left(\frac{\pi \ell}{2t} \right) - \cosh^2 \left(\frac{\pi y}{2t} \right)}} \\
 &= \frac{4t}{\pi} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cosh^2 \phi_0 - \cosh^2 \phi}} \\
 &= \frac{4t}{\pi} \int_0^{x_0} \frac{dx}{\sqrt{x_0^2 - x^2} \sqrt{1+x^2}} \\
 &= \frac{4t}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+x_0^2} \sin^2 \theta} = \frac{4t}{\pi} \operatorname{sech} \left(\frac{\pi \ell}{2t} \right) K \left(\tanh \frac{\pi \ell}{2t} \right) \quad (\text{A.4})
 \end{aligned}$$

by [13, #282.00].

In order to study the frequency dependence of the wide microstrip, we require the Fourier transform

$$\begin{aligned}
 J' &= \int_{-\ell}^{\ell} \frac{\cos k_0 \lambda y \, dy}{\sqrt{\cosh^2\left(\frac{\pi \ell}{2t}\right) - \cosh\left(\frac{\pi y}{2t}\right)}} \\
 &= 2t P_{-\frac{1}{2}+i\tau}\left(\cosh \frac{\pi \ell}{t}\right) \tag{A.5}
 \end{aligned}$$

where $\tau = \lambda T/\pi$ and $P_{-\frac{1}{2}+i\tau}(z)$ is a particular form of Legendre function known as a conical function. The result (A.5) is a standard integral representation for this function [22, p. 14]. Efficient numerical procedures exist for generating these functions: a uniform asymptotic expansion for large τ [22, p. 23]; [23, p. 466]; [24], and for small τ either a power series whose coefficients are tabulated as functions of the argument z [23] or a method similar to the arithmetic-geometric mean algorithm for evaluating elliptic integrals [25].

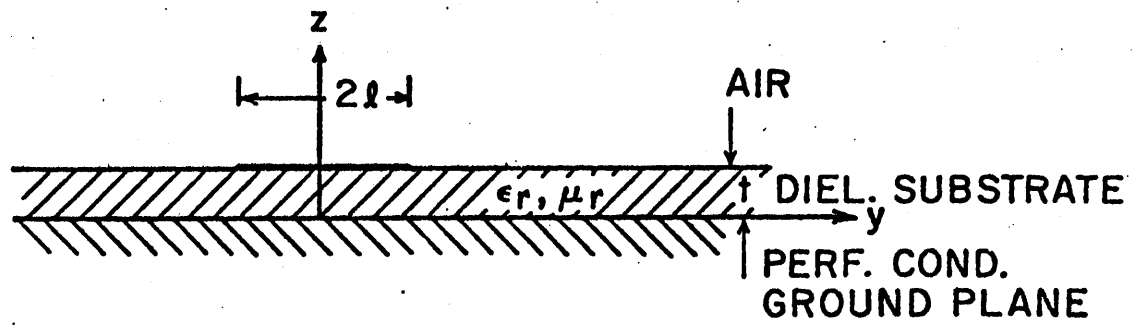
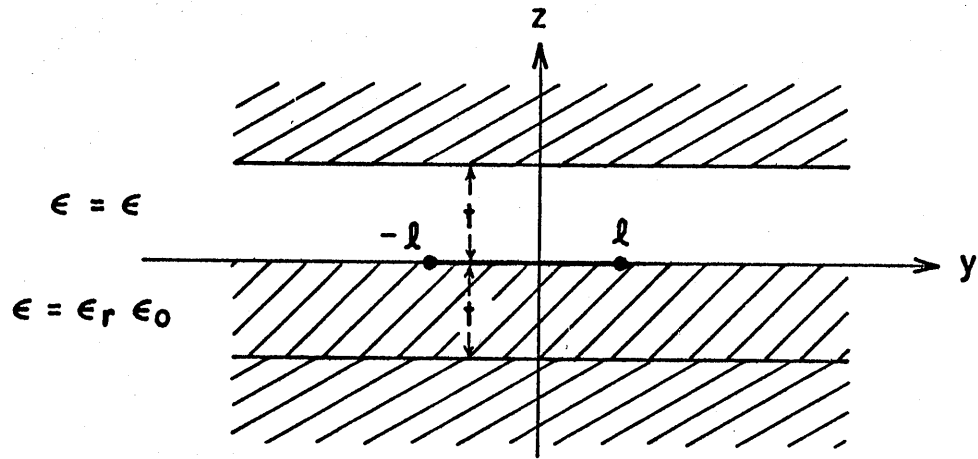
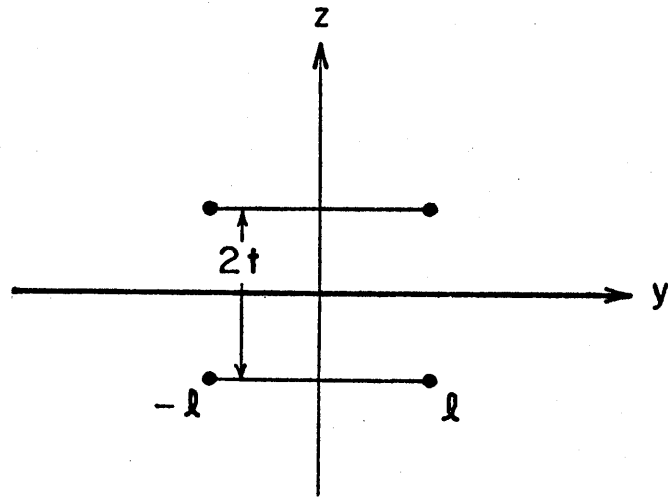


Fig. 1

Fig. 1 Open microstrip



(a)



(b)

Fig. 2 (a) Symmetric stripline
 (b) Parallel-plate line

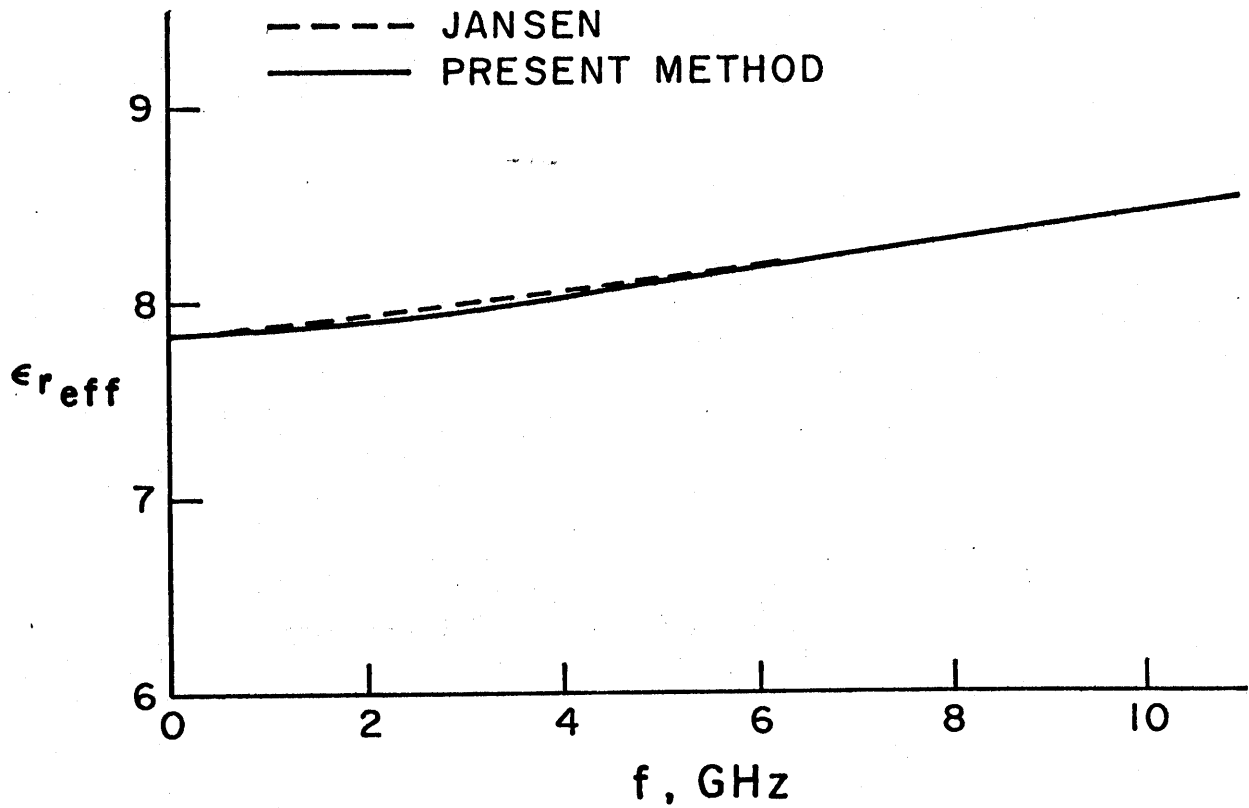


Fig. 3: Effective dielectric constant $\epsilon_{\text{eff}} \equiv \alpha^2$ for open microstrip: $t = 0.64$ mm, $l = 1.5$ mm, $\epsilon_r = 9.9$ as computed by Jansen [5] and by the present method.

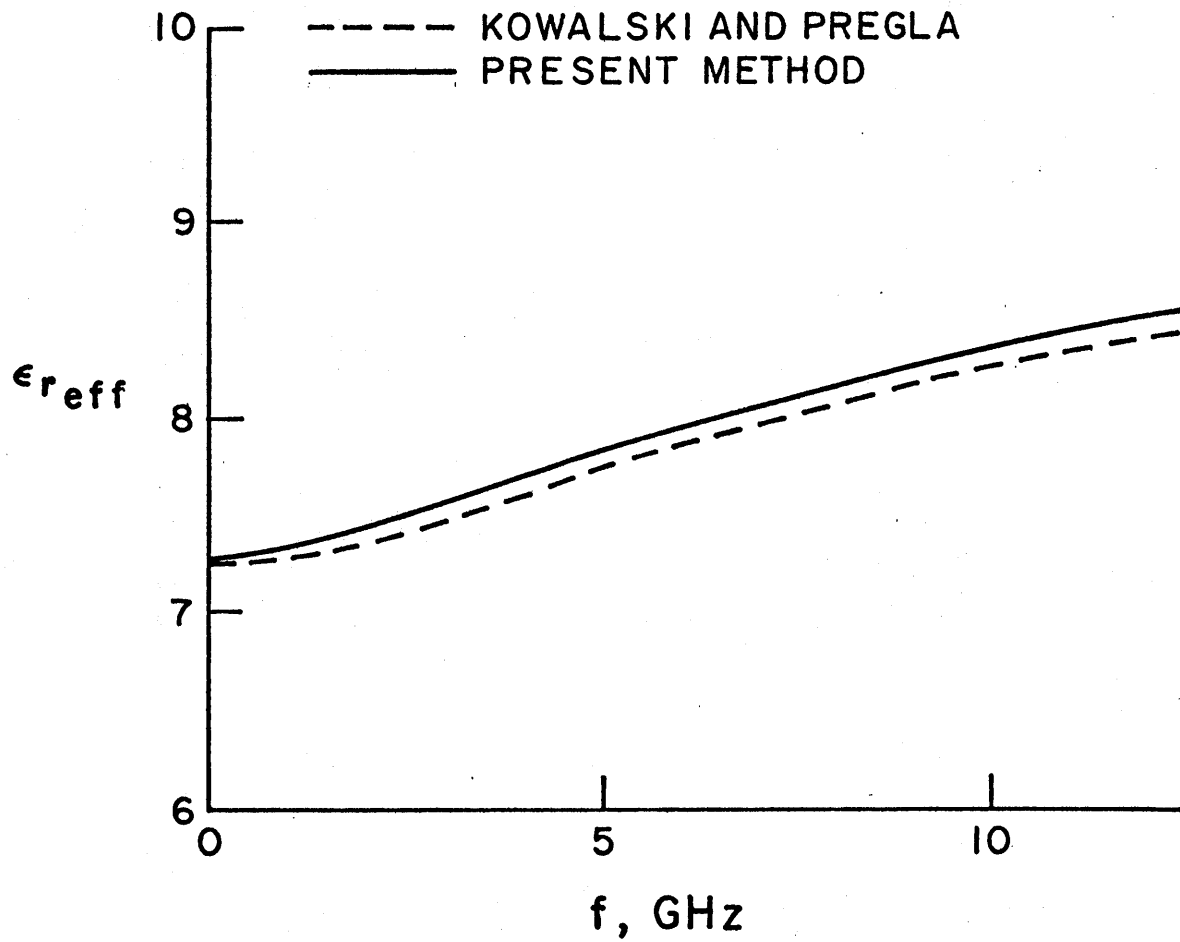


Fig. 4: Effective dielectric constant $\epsilon_{\text{reff}} \equiv \alpha^2$ for open microstrip: $t = 1.27$ mm, $l = 1.905$ mm, $\epsilon_r = 9.7$ as computed by Kowalski and Pregla [3] and by the present method

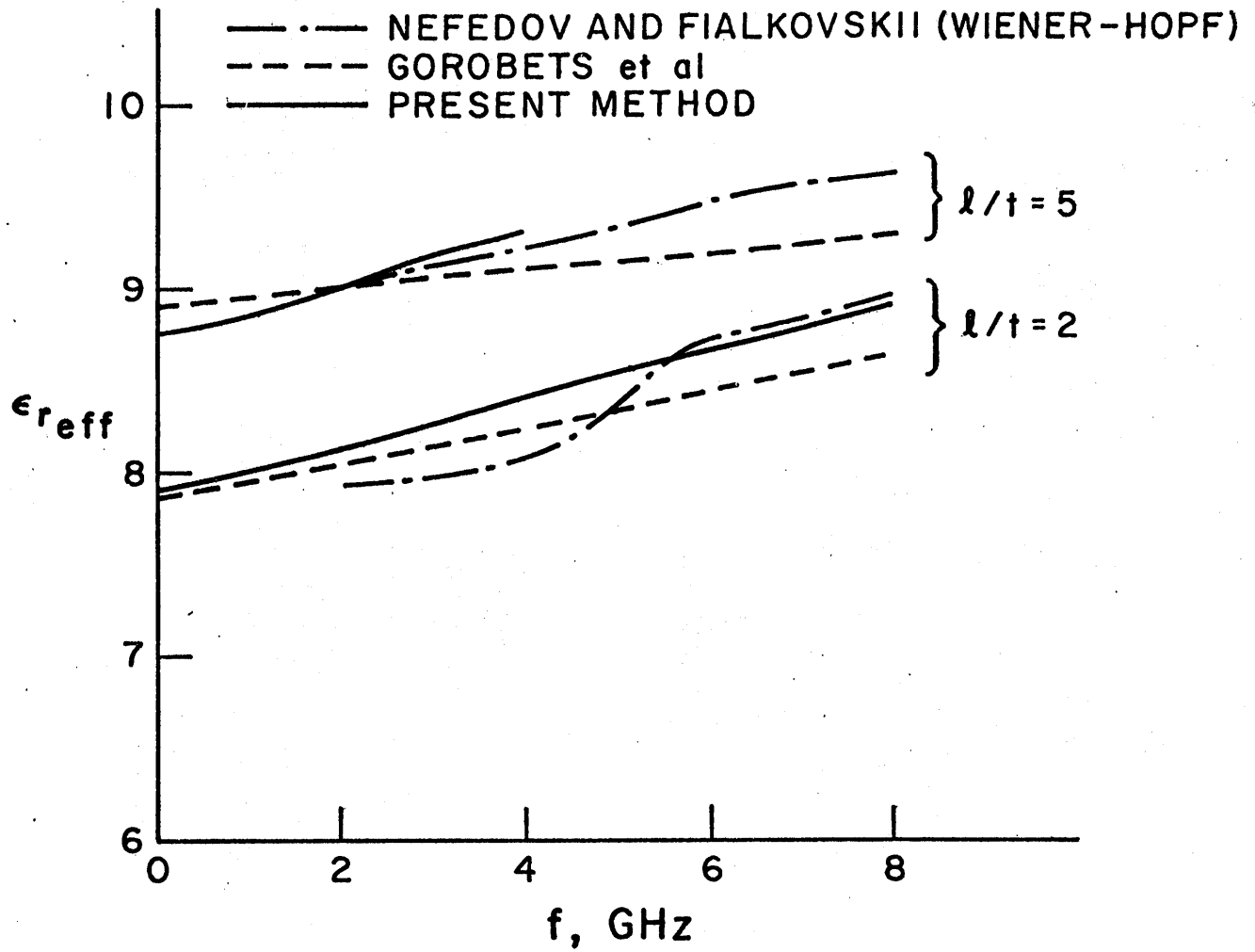


Fig. 5: Effective dielectric constant $\epsilon_{\text{reff}} \equiv \alpha^2$ for open microstrip: $t = 1.27$ mm, $\epsilon_r = 10.2$ for different values of l/t as computed by Gorobets et al. [4], Nefedov and Fialkovskii [9] and by the present method.