An Analysis of Student Reasoning Regarding the Sequencing of Mathematical Processes in a Pre-Algebra Course

Lauren Farquhar

University of Colorado at Boulder, lfarquhar@stcroixlutheran.org

Follow this and additional works at: https://scholar.colorado.edu/math_gradetds

Part of the Mathematics Commons, and the Science and Mathematics Education Commons

Recommended Citation
Farquhar, Lauren, 'An Analysis of Student Reasoning Regarding the Sequencing of Mathematical Processes in a Pre-Algebra Course' (2018). Mathematics Graduate Theses & Dissertations. 54.
https://scholar.colorado.edu/math_gradetds/54

This Thesis is brought to you for free and open access by Mathematics at CU Scholar. It has been accepted for inclusion in Mathematics Graduate Theses & Dissertations by an authorized administrator of CU Scholar. For more information, please contact cuscholaradmin@colorado.edu.
AN ANALYSIS OF STUDENT REASONING REGARDING THE SEQUENCING OF MATHEMATICAL PROCESSES IN A PRE-ALGEBRA COURSE

by

LAUREN FARQUHAR

B.A., Bethany Lutheran College, 2016

A thesis submitted to the
Faculty of the Graduate School of the
University of Colorado in partial fulfillment
of the requirement for the degree of
Masters
Department of Mathematics
2018
This thesis entitled:
An Analysis of Student Reasoning Regarding the Sequencing of Mathematical Processes in a Pre-Algebra Course
written by Lauren Farquhar
has been approved for the Department of Mathematics

_________________________________________
Dr. Eric Stade

_________________________________________
Dr. David Grant

_________________________________________
Dr. Kate Stange

Date____________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

IRB protocol # 17-0608
Abstract

Farquhar, Lauren Elizabeth (Masters, Mathematics)

An Analysis of Student Reasoning Regarding the Sequencing of Mathematical Processes in a Pre-Algebra Course

Thesis directed by Professor Eric Stade.

We define sequencing of mathematical processes (SMP) to encompass all scenarios where changing the order in which operations or processes are applied to a math problem affects the result. Student errors in secondary and post-secondary mathematics courses frequently stem from SMP misconceptions. In this paper we explore SMP issues in a middle school Pre-Algebra class. Data is drawn from student assessments, including homework assignments, tests, and semester examinations throughout a year-long course. We find in a largely qualitative analysis that students are most successful when they incorporate a step-by-step process understanding with a big-picture object understanding of mathematical ideas. By working to attain this duality, students can be efficient both storing and recalling information. We make use of learning theories of Dubinsky, Gray and Tall, and Sfard to inform our analysis and conclusions.
Acknowledgements

First I would like to thank Dr. Eric Stade. He went above and beyond in supporting my work from across the country and understanding the difficulties of balancing thesis work with teaching and caring for students in my classroom and dormitory. He pushed me to become both a better mathematician and better at communicating mathematics to my students.

Next I would like to thank Dr. Kate Stange for her mentorship. When I moved away from Boulder after my first year in the program, she showed me how I could still complete my Masters degree from a distance and believed in me as I executed the plan we developed.

I would like to thank Laken Top for her camaraderie and insights into the world of Math Education research.

I am also indebted to the faculty, staff, students, and families of St. Croix Lutheran Academy. I am particularly grateful to Ryan Rathje for his support of this study and recommendations of extra resources to examine. Additionally, I would like to thank the Pre-Algebra students and families who not only participated in this study, but gave me words of encouragement and asked how the project was going throughout the year. I could not have asked for a better educational community to work in.

Dr. Ashley Covell, Professor Laura Buch, and Professor Kyle Jore of Bethany Lutheran College likewise deserve thanks. They gave me a solid foundation of college-level mathematics and encouraged me to continue studying math.

Finally, I would like to thank my family. I would especially like to thank my grandpa Harold Theiste for inspiring a love of mathematics in all of his children and grandchildren, and my parents for encouraging all of my mathematical endeavors.
CONTENTS

1. INTRODUCTION....................................................................................................................... 1

2. LEARNING THEORIES............................................................................................................. 3
   2.1 Dubinsky’s APOS Theory ................................................................................................. 3
   2.2 The Process-Concept Duality of Gray and Tall .............................................................. 5
   2.3 Sfard’s Interiorization, Condensation, and Reification .................................................. 8
   2.4 Synthesis of Theoretical Perspectives .............................................................................. 10

3. REVIEW OF THE LITERATURE ON ERRORS IN THE SEQUENCING OF
   MATHEMATICAL PROCESSES ................................................................................................. 13
   3.1 Matz’s Review of Algebraic Errors .................................................................................. 14
   3.2 Misunderstanding Subtraction and Additive Inverses ................................................... 15
   3.3 Order of Operations Misconceptions .............................................................................. 18
   3.4 Other Errors ..................................................................................................................... 20
   3.5 Understanding Misconceptions ...................................................................................... 21

4. METHODS.................................................................................................................................... 22
   4.1 Participants .......................................................................................................................... 22
   4.2 Data Collection ................................................................................................................... 22
   4.3 Data Analysis ...................................................................................................................... 23

5. SMP ERRORS IN PRE-ALGEBRA ............................................................................................. 24
   5.1 Subtraction Errors .............................................................................................................. 26
   5.2 Solving Equations with Inverse Operations: The Order of Undo-Operations ............... 32

6. ANALYSIS OF ERRORS ............................................................................................................ 39
   6.1 Proceptually Sequencing Processes Involving Subtraction and Additive Inverses ......... 39
6.2 Operationally and Structurally Sequencing the “Order of Undo-Operations” .................................................................................................................. 43

6.3 A Duality Approach to Solving Mathematical Problems ................................. 47

7. CONCLUSION .................................................................................................................. 48

7.1 Understanding Student Thinking Regarding SMP Issues ................................. 48

7.2 Implications for the Mathematics Education Community ................................. 49

8. SUGGESTIONS FOR FUTURE RESEARCH ................................................................. 51

BIBLIOGRAPHY .................................................................................................................. 53

APPENDIX A ....................................................................................................................... 55

APPENDIX B ....................................................................................................................... 69
List of Tables

Table 1: Reasoning level descriptors................................................................. 11
Table 2: Semester 1 Final Examination Error Frequencies.................................. 24
Table 3: Chapter 7 Test Question Scores........................................................... 33
List of Figures

Figure 1: Subtracting mixed numbers on the 5-3 Reteaching assignment..............................27

Figure 2: Incorrectly subtracting a mixed number with a larger fractional part on the 5-3 
Reteaching assignment........................................................................................................28

Figure 3: Incorrectly subtracting a larger mixed number from a smaller mixed number on 
the 5-3 Reteaching assignment................................................................................................28

Figure 4: Incorrectly subtracting decimals on the Semester 1 final examination......................29

Figure 5: Incorrect simplification of an order of operations problem on the Semester 1 final 
examination................................................................................................................................30

Figure 6: Incorrect simplification on the Semester 1 final examination.......................................31

Figure 7: Incorrect Simplification on the Semester 1 final examination........................................31

Figure 8: The Order of Undo-Operations, correctly listed by student S12 on the Chapter 7 
test...........................................................................................................................................32

Figure 9: Work from group A students S6 and S11, respectively, on items 2 and 5 of the 
Chapter 7 test............................................................................................................................34

Figure 10: Student S2 exemplifies a Group B response on item 2 of the Chapter 7 test..............36

Figure 11: Work from Group C students S13 and S5, respectively, on items 3 and 6 of the 
Chapter 7 test............................................................................................................................37
Chapter 1

Introduction

Math is hard.

This short phrase is stated more often than any mathematics teacher would like, yet it can be very true. Between learning rules, applying them, and then using them to learn more new rules, a good deal can go into solving just one problem, leaving students ample opportunity to make mistakes. However, with each mistake that is made, there is an opportunity for growth. In order for growth to occur, one must first understand what went wrong.

Our observation is that in many cases, errors in mathematics can be traced back to applying correct rules or operations in an incorrect order. Students might vertically translate a function before applying a vertical compression when asked to compress before translating. They might incorrectly commute numbers before subtracting with the innocent hope of simplifying a problem. They might even multiply before dividing instead of working from left to right because they too rigidly follow the PEMDAS pneumonic. I found it intriguing that students’ prior familiarity with mathematical rules and operations did not necessarily predict their ability to use them on assessments. Many students in my class were familiar with the PEMDAS or “Please Excuse My Dear Aunt Sally” rules before the first day of class, but still got order of operations problems incorrect. This was especially true when numbers outside the set of natural numbers were used in problems. While PEMDAS is changeless, some students who could apply it to natural numbers could not apply it to decimals, fractions, negative numbers, or mixed numbers without assistance. All these
errors and more fall into the category of issues in the *sequencing of mathematical processes* (SMP). We define SMP to include all scenarios where changing the order in which operations or processes are applied changes the result.

While the prevalence of SMP issues is astounding across grade levels, this study focuses primarily on SMP issues in middle school Pre-Algebra. Specifically, we examine errors with subtraction and additive inverses as well as errors solving for a variable in the context of SMP issues. We seek to investigate the following questions: How do students understand the sequencing of mathematical processes? How can teachers effectively guide students toward higher reasoning levels and a deeper understanding of sequencing of mathematical processes?

In Chapter 2 we give an overview of key mathematical learning theories which may be applied to the analysis of how students think about and understand SMP issues. We follow in Chapter 3 with a review of the literature on errors in mathematics, highlighting SMP connections.

Chapter 4 outlines the methods used in our research study. We include a description of the participants and data sources. Results from our participants are presented in Chapter 5. We aim in Chapter 5 to elucidate patterns in errors, and we emphasize connections of errors to SMP issues. Results are analyzed in light of the most applicable mathematical learning theories in Chapter 6.

Concluding remarks regarding students’ understandings of SMP issues as well as teacher efficacy are in Chapter 7, and Chapter 8 outlines possibilities for future research.
Chapter 2
Learning Theories

Learning theories provide a lens through which we can examine student misconceptions and identify reasoning levels in order to guide students toward deeper understanding. Most applicable to our work are the theories developed in the early 1990’s by Dubinsky, Gray and Tall, and Sfard.

2.1 Dubinsky’s APOS Theory

Dubinsky and McDonald (2001) assert that learning theories have a variety of uses in research. In particular, learning theories can serve as a tool for analyzing and explaining data. They are refined with use, and they allow a researcher to understand learning on a deeper level and communicate results in a common language. Learning theories can additionally serve to unify a group of researchers, helping multiple people to approach a question from the same perspective.

Dubinsky's APOS theory of learning mathematics (Dubinsky and McDonald, 2001) is an extension of the work of Piaget (Beth and Piaget, 1974; Dubinsky, 2002). Piaget wrote based on observations of children, but suggested that his theory could be applied to older students as well, and Dubinsky did just that. APOS theory groups learners into four levels of reasoning based on whether they approach problems by performing actions, carrying out processes, constructing objects, or developing schema.

As students are exposed to new mathematical ideas, they begin by performing steps on concrete mathematical objects without necessarily understanding what they are doing.
They require external stimuli, sometimes memorized and sometimes explicitly prompted, in order to perform the steps in the correct order. Students in this first stage of understanding perform *actions*. In example, a student in the action stage of learning about mean, median, and mode requires specific formulas to calculate these measures of central tendency. For early calculations of mean, I often provide a fraction set-up with a box for each number that should be added in the numerator and a single box in the denominator for the number of numbers. Students are capable of doing the calculations to find the mean, but they need reminders of which calculations to do, and which order to do them in.

When actions are internalized, they become *processes*. The student learning about mean, median, and mode no longer needs my calculation templates to calculate. Instead, a student knows that the mean is always the sum of the numbers divided by the number of numbers, the median is the middle number, and the mode is the most frequent number. Processes are carried out similarly to actions, however, learners carrying out processes no longer need external stimuli. Hence processes can be mentally performed and combined with other processes.

In the next stage of learning, students understand concepts as a whole. They construct *objects* from processes when they are able to perform transformations on the process. If a student is in the object stage of understanding mean, median, and mode, he or she will be successful when asked how the averages change when each number in the set increases by five. The student will not necessarily need to recalculate mean, median, and mode, but will know that adding five to each number in the data set and then calculating average outputs the same answer as calculating average and then adding five at the end.
The student understands how transformations affect the sequence of processes used to calculate the answer.

The highest form of understanding is a *schema*. A schema consists of generalizations about actions, processes, and objects linked by commonalities. This level of understanding is achieved when students are able to link together, manipulate, and extend actions, processes, and objects. If a student understands mean, median, and mode on a schematic level, he or she can estimate each measure of average when looking at the data set, so that calculations can be checked for accuracy, and the sequence of mathematical processes used in calculating an answer can be revised if the answer is not close to the estimate. Additionally, a student who understands these measures of average schematically can evaluate which measure is best in various situations.

It is noteworthy that a single concept may span multiple APOS categories. Students might develop a schematic understanding of addition when they are young by working with counting manipulatives or associating real-life examples of addition. Later, the same student will incorporate addition as a process or object into a sequence of mathematical processes comprising a multiplication schema. Hence student work must be analyzed in context to sort understanding into APOS categories, and even then categorizations may not be black-and-white.

### 2.2 The Process-Concept Duality of Gray and Tall

The notion that one idea can span multiple levels of reasoning is strongly supported by Gray and Tall (1994). With each new presentation of an idea, students should deepen their understanding of the meaning of symbols involved.
Gray and Tall categorize students’ level of understanding with the terms *procedural* and *conceptual*. If a student understands an idea procedurally, he or she can carry out routine manipulation of symbols, and a teacher can easily determine if the student has a correct or incorrect understanding. A conceptual understanding is more difficult to assess, in that it requires a student to recognize relationships between processes and use these relationships to solve problems. As students recognize relationships, they can compress a process of many steps into one big-idea picture in order to easily store and quickly recall a process at a later time. While compression requires a high level of insight, it allows students to build on prior knowledge and hence learn more difficult ideas. Gray and Tall describe the transition from procedural to conceptual understanding as the *encapsulation* of processes as mental objects. *Entification* and *reification* (Harel and Kaput, 2002; Sfard 1991) are also used by other authors to describe the compression of many smaller ideas into a big-picture understanding. When a student can efficiently combine procedural understanding and conceptual understanding, Gray and Tall say the student understands an idea as a *procept* (a combination of the words process and concept) and employs *proceptual thinking*.

An *elementary procept*, according to Gray and Tall, is a single process that can be conceptualized as a mental object in combination with a symbol representing the process or the object. As students encapsulate processes as concepts, meanings of operation symbols shift. Gray and Tall assert that ambiguity of symbols, while potentially annoying to a learner, allows the flexible thinking required for mathematicians to compress ideas. In example, the symbol “−” can be used to represent the process of subtracting a number from another or the concept of a negative number. Additionally, “+” can represent addition or a
positive number, function notation represents the process of calculating an output at a specific value of \( x \) or the concept of a function for a general value of \( x \), and so forth.

With each new example of an elementary procept, students construct meaning for the symbols used. As they develop processes to compute symbols, they begin to relate the single processes and concepts associated with specific elementary procepts and collect the connected elementary procepts into an overarching procept (with the term “elementary” dropped) consisting of multiple processes and concepts. Connecting related ideas allows students to manipulate symbols with higher reasoning levels, and eventually abstract the meaning of the symbols to compress the procept into an object that can be stored and easily recalled. Gray and Tall point out that while students employing procedural problem-solving techniques rely on step-by-step processes, students thinking proceptually are able to manipulate symbols and flexibly decompose and recompose problems to effectively solve new problems.

It is interesting to note that while two students may both end up at the same correct answer to a math problem, students using higher-level and lower-level reasoning might take different steps to determine the solution. Gray and Tall divided students into “above average,” “average,” or “below average” groups based on classroom performance in arithmetic. They note that students they classify as below-average approach knowledge as a hierarchical ladder. These students must fully learn or even conceptualize each idea before incorporating it into a new process. In simple arithmetic problems, low-level students resort to counting if they do not know addition or multiplication facts, which demonstrates procedural thinking. Additionally, students who reason at a lower level often try to hide the evidence of low-level reasoning. They might finger-count under the table or
tap their fingers so that peers do not see them counting. On the other hand, above-average students approach knowledge as a collapsed hierarchy. These learners constantly take in processes and compress them as concepts or objects, developing relationships that are all on the same level and are easily manipulated. Rather than counting, above-average students derive useful facts from known facts to solve problems. Students who reason at a high level are more able to make generalizations, and hence demonstrate conceptual thinking.

While many mathematical ideas are both processes and concepts, students who think proceptually can manipulate symbols and apply higher-level reasoning to solve new problems according to Gray and Tall.

2.3 Sfard’s Interiorization, Condensation, and Reification

Sfard (1991) offers yet another duality to elucidate how students develop an understanding of mathematics. Students can conceive ideas operationally as they understand the processes that are used to solve a problem, or they can conceive ideas structurally when they recognize mathematical notions as concrete objects and are able to manipulate them to more efficiently solve problems. The ontological gap dividing these understandings is significant.

Arguably, most if not all math problems could be completed using an operational approach. Operational reasoning relies on rote processes, algorithms, and actions to solve a problem in a pre-determined sequence of steps. Even the most difficult problems could be solved by building up from very basic processes and performing processes on the processes. However, using solely operational reasoning is highly inefficient. It is cognitively
difficult for a student to recall a long sequence of steps, and it is easy for the student to feel inadequate when a better understanding of what is happening is not attained. Students who reason only operationally will miss big ideas and have issues with SMP tasks. It is much more efficient, not to mention satisfying, for a math student to incorporate structural reasoning. Students who understand a concept structurally see a notion as a static, unified structure. Insights are gained and connections between ideas are made so that students can store information in a more organized way and recall it quickly. In this way, outcomes of processes can be considered without spending a thought on the process itself.

In order for a student to develop a structural understanding from an operational understanding, he or she transitions through three learning stages: Interiorization, condensation, and reification. While they can only be observed indirectly, Sfard recognizes these stages by the behaviors, attitudes, and skills demonstrated by students in each stage. Students in the interiorization stage are focused on becoming familiar with operations on already familiar numbers or other objects. They experiment with processes until they are comfortable carrying them out. When students are comfortable with processes and begin to recognize new outcomes of the processes, they are in the condensation stage. They do not need to scrutinize or detail the process as much as they did in the previous stage. Condensation lasts longer than the other stages. In fact, students who are never pushed to fully reach a structural understanding may stay permanently in this stage. However, this is dangerous as learners in this stage still do not fully understand concepts and may be discouraged by the amount of memorization and recall required to solve a problem.

After lingering in the condensation stage, students finally recognize new results as mathematical objects, hence entering the reification stage. Reification can happen in a
moment as students realize connections that allows for a jump to a new understanding and efficient reorganization of information. Diagrams, graphs, pictures, and other visuals can assist students in entering the reification stage.

Students repeat the process of Interiorization, condensation, and reification over and over. Once they have a structural understanding of one concept, the concept becomes part of a new process, which will in turn be realized as a mathematical object. Sfard describes the learning process as a hierarchy, in which a learner cannot move up to understand the next concept before the first process has been realized as an object. Additionally, a single concept realized as an object by a student at a higher level of reasoning might still be understood only operationally by a student at a lower reasoning level. In order for a mathematician to effectively solve problems, he or she must combine operational and structural understandings. He or she can take a symbol to represent either a process or an abstract idea, and must manipulate symbols as such to be efficient. While the difference in what exists is crucial between operational and structural understandings, operational and structural reasoning are innately tied together, and students should work through the process of interiorization, condensation, and reification to understand both.

2.4 Synthesis of Theoretical Perspectives

The theories of Dubinsky, Gray and Tall, and Sfard all present a learning progression relying on interplay between distinct levels of student understanding. As demonstrated in Table 1 below, each theory has descriptors for lower and higher reasoning levels. While the descriptors are different in each theory, they describe similar ideas. The words action, process, procedural, and operational all encapsulate an understanding of a mathematical
notion requiring prompts and explicit step-by-step instructions to arrive at a correct answer. Students in the lower-level reasoning stages often use rote memorization to recall facts on assessments, seek algorithms for solving problems, and are primarily focused on how to answer a question rather than why the solution works. Object, schema, conceptual, and structural are all words intended to describe a deeper understanding of mathematical notions. Students reasoning their way through problems at a higher level generalize patterns and compress understanding into a few overarching ideas. These students constantly ask “why,” and seek to discover what makes their processes successful.

<table>
<thead>
<tr>
<th>Lower-level reasoning</th>
<th>Higher-level reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action, Process</td>
<td>Dubinsky</td>
</tr>
<tr>
<td>Procedural</td>
<td>Tall and Gray</td>
</tr>
<tr>
<td>Operational</td>
<td>Sfard</td>
</tr>
</tbody>
</table>

Table 1: Reasoning level descriptors.

While action and schema, procedural and conceptual, and operational and structural all appear to be opposites, the authors presenting these theories make it quite clear that for successful mathematicians, these opposites exist in tandem. Students progress from a lower level of reasoning to a higher level, then use newly objectified concepts as the base for repeating the cycle with a new concept. Math educators need to know where students make simple lower-level reasoning mistakes, but perhaps more importantly, they must understand higher-level reasoning mistakes. Students constantly build on knowledge they construct as they incorporate new concepts into existing ideas.

All three of the aforementioned learning theories can be used to better understand student reasoning. Lynn Bowie aptly demonstrates this in her study of Calculus misconceptions (1998). In our analysis, we primarily employ the procedural versus
conceptual theory of Gray and Tall and the operational versus structural theory presented by Sfard. Both of these theories have undertones of Dubinsky’s APOS hierarchy, but additionally highlight elements of student learning that help us further understand student thinking regarding the sequencing of mathematical processes. We proceed to review errors students make in the sequencing of mathematical processes and return to the theory-based analysis in Chapter 5.
Chapter 3

Review of the Literature on Errors in the Sequencing of Mathematical Processes

In seeking to discover how students understand the sequencing of mathematical processes (SMP), it is helpful to know what research has already been done in this area, as well as what learning theories are useful in analysis. In this section, we will begin by highlighting what is commonly known about students’ misconceptions and errors in mathematics, especially in algebra.

Extensive research, both informal and formal, has been done by classroom teachers and by researchers to discover when, where, and why student students make mathematical errors. A quick internet search of “math misconceptions” or “math mistakes” outputs no shortage of blogs, images, and classroom websites highlighting a variety of errors students make on a day-to-day basis. While some of these informal sources list many errors, others focus on one persistent error. Some present procedural solutions for teachers to communicate to students, while others suggest reform of overall environments in math classrooms. Most of the errors listed on blogs of classroom teachers are also examined in peer-reviewed articles.

In most of the articles we surveyed, researchers hone in on a single error category. These categories center around specific concepts and learning objectives students are expected to be able to demonstrate on assessments, and often apply to a particular age group or ability. In example, researchers might study difficulties with negative numbers for secondary school students (Fuadiah, Suryadi, and Turmudi, 2016), misunderstandings of differences for college students (Musgrave, Hatfield, and Thompson, 2015), or even
misconceptions with PEMDAS for prospective elementary school teachers (Glidden, 2008). In this study, we take categorization of errors one step further. While we initially categorized errors by the specific learning objective that students misunderstood, we found similarities in how students failed to master different learning objectives and we recognized thinking patterns that contributed to the misconceptions. In this paper, we focus on how errors demonstrating various learning objectives can be viewed as SMP errors and how employing higher reasoning levels can theoretically reduce confusion. Nevertheless, it is enlightening to understand categories of errors examined previously, hence we proceed to review prior work in error identification and analysis.

3.1 Matz’s Review of Algebraic Errors

One of the most extensive articulations of algebraic errors is provided by Matz (1980). In her paper, Matz presents a list of thirty-three observed algebraic errors. These include errors in evaluation, computation, simplification, and solving for a variable. Matz’s goal is to reconstruct students’ derivations of each error. Her perspective is unique in that her list of errors and derivations are not specifically illustrated by examples of student work, but rather evidence from unsourced student work samples was compiled to construct the generalized errors and derivations. The fact that the same mistakes are made often enough for generalizations is enlightening. While many of the errors are related to SMP issues, Matz’s argument that all errors are based on reasoning is even more applicable to our study. When students solve new problems, they look back to prototype examples. Using information from the prototype examples, they either extend a rule so it applies to the new situation or they attempt to make the new situation look like a familiar situation.
Sometimes this extension of prototype examples is successful, and sometimes there are conditions missing from rules or elements missing from problems that allow room for mis-steps. Hence most errors are the result of reasonable attempts to extrapolate known patterns. We keep this idea in mind as we observe students’ misconceptions in prior literature as well as in our own study, which we will discuss in Chapter 5. In order to better understand the reasoning behind incorrect responses, we proceed to examine categories of student errors and misconceptions.

### 3.2 Misunderstanding Subtraction and Additive Inverses

Introducing additive inverses, typically called negative numbers or subtracted numbers at the time of introduction, can turn the simplest problems into the most challenging for young students at the secondary level. Specific challenges are exposed by Fuadiah, Suryadi, and Turmudi (2016). In a study of seventh grade students in Indonesia, they found that students particularly struggle to adapt their formal understanding of positive numbers to an abstract notion of negative numbers. Additionally, students are largely unable to generate an example of a real-life context in which they might encounter a specific negative number. Students confuse magnitude and direction of numbers, they sequence negative numbers incorrectly, and they mistakenly apply properties of addition to subtraction problems. When working with the Commutative Property and Associative Property, students commute numbers before subtracting rather than associating the subtraction sign with the number that was intended to be negative before commuting. Additionally, students apply the Associate Property liberally, subtracting what they see as the easiest numbers first rather than always subtracting left to right. They are unaware
that commutativity and associativity have to be applied in a particular order in subtraction problems. As a result, students often produce answers with the correct magnitude but opposite sign of the correct answer. This simple incorporation of a subtraction or negative sign causes confusion in how students sequence multiple mathematical processes used to simplify expressions.

In studying how students simplify exponential expressions, Cangelosi et al. (2013) elucidate more sequencing misconceptions involving negative numbers. Students in university math courses ranging from College Algebra to Calculus 2 demonstrate difficulties interpreting negative bases and additive inverses of exponential expressions. When dealing with negative exponential expressions of the form \(-a^b\), students often apply the negative sign before the exponent, despite the fact that exponents should be considered before multiplication, division, addition, or subtraction.

One might suggest that parentheses or other grouping symbols might helps students perform operations in the correct order. Yet even when parentheses are involved, students move the parentheses based on how they interpret the expression, some even claiming that parentheses do not matter or disregarding them. Cangelosi et al. (2013) note that some students using technically incorrect notation are able to arrive at correct answers when their incorrect notation represents correct reasoning. However, this incorrect notation for correct reasoning can still lead to incorrect answers for more complex problems.

While parentheses are not always interpreted correctly, it is striking that how expressions are notated can make a difference in how students understand them. In example, students are more likely to correctly simplify a problem if it involves subtracting a
positive exponential expression than if it requires addition of a negative exponential expression.

Negative exponents also present difficulties for students. These difficulties arise as students wrestle with the question, where does a negative exponent come in the sequence of steps needed to simplify an exponential expression? Most students rely on memorized rules for simplifying expressions with negative exponents. As the memorized rule states that \( a^{-1} = 1/a \), some students mistake the positive exponent as the numerator of the simplified expression, as in \( 2^{-3} = 3/2 \) (Cangelosi et al., 2013). They use the process of finding the multiplicative inverse of 2 and then multiply by 3 instead of correctly sequencing the processes with exponentiation of \( 2^3 \) first followed by a calculation of the multiplicative inverse. Other students exponentiated \( 2^3 \) and then negated the entire exponential expression instead of negating the exponent and then exponentiating. While students may know a correct rule, it is incorrectly extended, demonstrating a lack of full understanding of the sequence of mathematical operations required to simplify the expression.

Musgrave, Hatfield, and Thompson (2015) approach subtraction on a broader level. In an effort to analyze structure sense, they surveyed students’ understandings of the word “difference.” They noticed that many Calculus students complained of memorizing complex differentiation rules but having no idea how to apply them to complex expressions. Explicit focus on computational activities hinders student understanding; students should be pushed to explain examples to develop a functional meaning of differences.

Student errors involving negative signs can often be classified as misconceptions about previously learned mathematical processes. It is not uncommon to find confusion between magnitude and rank of negative numbers, misapplication of the Commutative
Property and Associate Property to subtraction, misinterpretation of inverses, and dropped or forgotten negative signs as multi-step processes are executed. Hence we proceed to examine literature regarding how students understand and use well-known sequences of mathematical processes, specifically the order of operations.

3.3 Order of Operations Misconceptions

Joseph (2014) points out that while the classic order of operations is typically taught in middle school, it is necessary throughout all subsequent mathematics courses. Even among college students, difficulties simplifying expressions involving the order of operations are common. Joseph created a ten-question survey (See Appendix B) requiring use of the order of operations, and used it to analyze errors she recognized as a teacher and as a tutor. Each problem was assigned a difficulty level (easy, medium, or hard) based on the number of steps required to solve the problem. As expected, there were significant differences in the scores the college students earned on easy versus medium problems and medium versus hard problems. Additionally, students scored lower when they were required to simplify variable expressions than when they had to simplify numerical expressions. A surprise came in the discrepancy between how students thought they scored and how they actually scored when they had to simplify $6 ÷ 2(2 + 1)$, but for the most part, Joseph proved her point: college students still have difficulty simplifying and solving expressions with the order of operations that they should have learned in middle school.

Glidden (2008) also surveys college students’ ability to apply the order of operations, but with a different purpose. Prospective elementary, early childhood, and
special education teachers participated in Glidden’s study. Each participant was required to simplify four 2-operation expressions and rank their mathematical ability, confidence, and enjoyment. Surprisingly, less than half of the prospective teachers got at least two of the four questions correct. Many of the participants relied too heavily on the PEMDAS pneumonic, performing multiplication before division and addition before subtraction rather than operating from left to right. The prospective teachers ranked their ability as above average 33 percent of the time, 32 percent of them said they enjoy doing mathematics, and yet 45 percent said they felt confident when doing mathematics. Like many younger students, prospective elementary, early childhood, and special education teachers have difficulty applying the order of operations, and must be taught how to use it and how to effectively teach it.

Errors in the order of operations are often related to students’ over-dependence on memorized orders. Each concept teachers disseminate to students can be categorized as arbitrary or necessary (Zazkis and Rouleau, 2017). Arbitrary items are definition-based and can only be learned through reading or explicit instruction. Memorization ensures that students can recall these items later. Processes are categorized as necessary if they can be discovered or worked out naturally. Necessary items should be understood rather than memorized. Zazkis and Rouleau highlight a discrepancy as these apply to the order of operations. The order of operations is a convention that is frequently taught as though it is arbitrary. Students memorize “PEMDAS”, “Please Excuse My Dear Aunt Sally”, or in Canada “BEDMAS” (brackets, exponents, division, multiplication, addition, subtraction) to remember which operations to perform first, second, and so on. However, multiplication is performed before addition because it is repeated addition, and exponents come before both
because they represent repeated multiplication. Parenthesis, brackets, or grouping symbols come first because, by definition, they denote which operations should come first. Hence the order of operations should be taught as a necessary sequence of processes that can be worked out and understood rather than an arbitrary one that only holds because of how the sequence is defined (Zazkis and Rouleau, 2018).

### 3.4 Other Errors

A plethora of other mathematical errors could be disclosed here, but most would not be worthwhile to read about for the purposes of this study. One last set of misconceptions that we highlight relates to concatenation. We refer to any juxtaposition of two numbers or the juxtaposition of a negative sign and a number as concatenation. As with subtraction, the incorporation of concatenation into a known sequence of processes can confuse students as they determine the effect of concatenation on the sequencing. Lee and Messner (2000) give examples of mixed numbers such as $2\frac{1}{4}$ in which concatenation implies addition, as well as coefficients and variables such as $3x$ in which concatenation implies multiplication. It is no wonder with these multiple meanings of a symbol-less convention that concatenation confuses students who have not developed the flexible thinking required to compress and understand concepts on a deeper level.

The difficulty doubles when a subtraction sign is placed in front of a mixed number. A subtraction sign represents an additional process that students must incorporate into their sequence of processes. Students may attempt to first distribute the subtraction across the whole and fractional parts of the mixed number, or they may only treat the whole part as negative and add the fractional part. Students may also concatenate in the wrong order
when exponents are involved. Lee and Messner point out that when faced with an expression such as $3R^2$, students are prone to interpret it as a sequence requiring exponentiation then addition as in $3 + R^2$, or as a sequence requiring multiplication before exponentiation as in $(3R)^2$. Needless to say, an ability to comfortably manipulate numbers and symbols is key to working with and correctly sequencing concatenation processes.

3.5 Understanding Misconceptions

One can easily find studies of algebraic misconceptions that arise on a day-to-day basis in secondary math classrooms. Similarly, in calculus research, Rasmussen, Marrongelle, and Borba (2014) assert that identifying misconceptions is one of four major research trends, the other three include investigating student learning processes, carrying out classroom studies, and researching teacher beliefs, knowledge, and practices. These trends likewise arise in algebra research. As identification of algebraic errors and misconceptions is well-examined, we align our study with the second trend of Rasmussen, Marrongelle, and Borba: we aim to understand why SMP errors occur and how to guide students to higher reasoning levels by classifying misconceptions and analyzing them through learning theory lenses.
Chapter 4

Methods

4.1 Participants

Pre-Algebra students in grades six through eight at St. Croix Lutheran Academy participated in this study. Students were recruited on a volunteer basis. Thirteen of twenty-one students who completed the course elected to participate. Note that one of these thirteen students transferred into the class partway through the first semester, so for the Lesson 1-2 Pre-Check we only include data from twelve of the students. The students participating represented a broad range of abilities, and as a group, had an average grade in Pre-Algebra that was close to the class average.

4.2 Data Collection

Data collected for this study came from standard Pre-Algebra formative and summative assessments including homework assignments and tests. Students participating were not required to do any work beyond that required for class purposes. Additionally, each of the assessments distributed to students was primarily used for teaching purposes, and only secondarily used as data for this study.

Five assessments were analyzed in this study. Assessments examined include an initial order of operations inventory corresponding to one of the first lessons of the school year, a homework assignment on which students practiced adding and subtracting fractions, a second-semester test on which students demonstrated their ability to solve for a variable, and semester exams from both the first and second semesters. Each assessment was selected for use in the study either because students made more errors than usual on
the assessment or because the errors were closely related to how students understand the sequencing of mathematical processes.

4.3 Data Analysis

Data is analyzed qualitatively in this study. As patterns in student work emerged, we refined our guiding questions and incorporated more assessment data into the study to further elucidate student thinking and use of processes. Our aim in the analysis is to determine how students understand the sequencing of mathematical processes and so that they can eventually be challenged to more effectively employ higher reasoning levels.

Since all participants in this study came from a single Pre-Algebra class, one should be careful not to directly apply results without first considering the effects of differences between the populations. However, results in this study can help elucidate thought processes of a more general population and can provide a starting point for analysis of different populations of students.
Chapter 5

SMP Errors in Pre-Algebra

In Chapter 5, we present and categorize our data. In order to determine how often students make errors in the sequencing of mathematical processes, we began with an analysis of the Semester 1 final examinations taken by Pre-Algebra students participating in the study. Among the thirteen students, 81 total errors were made on the 22 short answer questions. Error frequencies are shown in Table 2 below, where S1, S2, etc. are the students participating in the study.

<table>
<thead>
<tr>
<th>Error Type</th>
<th>Total</th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
<th>S7</th>
<th>S8</th>
<th>S9</th>
<th>S10</th>
<th>S11</th>
<th>S12</th>
<th>S13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negatives and subtraction</td>
<td>22</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sequencing operations</td>
<td>26</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Order of undo-operations</td>
<td>20</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PEMDAS</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inequality errors</td>
<td>20</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exponents</td>
<td>6</td>
<td>1</td>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Did not simplify</td>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Symbol or definitional errors</td>
<td>9</td>
<td>1</td>
<td>1</td>
<td></td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other errors</td>
<td>11</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total errors</td>
<td>81</td>
<td>2</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>2</td>
<td>11</td>
<td>13</td>
<td>5</td>
<td>3</td>
<td>10</td>
<td>8</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

*Table 2: Semester 1 Final Examination Error Frequencies*

Errors ranged from forgetting to write a multiplication symbol (which had no effect on the final answer) to skipping an entire problem. While the impact of the errors varied, patterns clearly emerged and categories were naturally formed. Note that some errors fell into more than one category. Errors involving a dropped negative sign, or a mis-associated or misused subtraction sign accounted for 22 of the total errors. Operations were incorrectly chosen or incorrectly sequenced 26 times when students were simplifying...
expressions or solving for a variable. Inequalities were incorrect 20 times. It was interesting to observe that of the 20 inequality errors, most errors only involved the inequality symbol, and the algebraic solving process was not impacted by the symbol misunderstanding. In example, student S8 simplified $3k \geq -30$ to “$k = -10$ or more” instead of simply $k \geq -10$. While both answers are equivalent, student S8 did not demonstrate proper use of the inequality symbol. In another example, student S13 incorrectly simplified $5 < x - 23$ to $x \geq 29$ without considering non-integer numbers included in the correct solution $x > 28$.

The distribution of errors on the Semester 1 Final Examination prompted further exploration of the most significant errors. Did the same errors occur on homework assignments? Were errors related to the preconceived notions students had before taking the Pre-Algebra class? Did the errors persist into the second semester? How did similar errors show up in the context of new problems?

While conclusions could also be drawn from the inequality, symbol, and definitional errors, for the purposes of this paper we focus our attention on how students understand the items they most often had trouble with: negatives and subtraction as well as sequencing operations. It turns out that both of these categories of common errors stem from SMP issues. Incorporation of subtraction and negative signs adds an additional process that students must sequence, and sequencing operations in solving for a variable relies on memorized and conceptually understood sequencing patterns.
5.1 Subtraction Errors

From an examination of the Lesson 1-2 Pre-Check (See Appendix A), the first homework assignment of the fall semester, it is evident that students are aware that an order of operations exists. On the assignment, students were asked to simplify expressions and determine if given answers were correct or incorrect, explaining their reasoning for all problems. Students were told ahead of time that they were not graded on accuracy, but on how well they explained their thinking. Of the 12 participating students who completed the Lesson 1-2 Pre-Check homework assignment, four specifically mentioned “PEMDAS” or “Please Excuse My Dear Aunt Sally” in an explanation. Other students referred to an order of operations without using the PEMDAS pneumonic. Most students got at least two of the three basic order of operations simplification problems correct on the Lesson 1-2 Pre-Check assignment even before studying the order of operations in class. However, as the order of operations and related sequences of mathematical processes were studied in class, students were introduced to more complex simplification problems and were required to incorporate new details into existing processes. Subtraction and negative numbers provided students with some of the most consistent simplification challenges.

As students practiced subtracting mixed numbers, it became quite clear which ones understood it well and which students needed clarification on the underlying processes or sequencing of the processes. Some students took the subtraction signs on their 5-3 Reteaching homework (See Appendix A) to represent a difference between two numbers rather than taking the second number away from the first. Additionally, some students first divided problems into two smaller problems and found the difference between whole parts and the difference between the fractional parts of the mixed numbers, and second
proceeded to add the two differences together to find the final answer. This two-step difference strategy worked well for problems in which the first number, as well as the first whole and fractional parts, were all bigger than the second number, whole part, and fractional part. Question 5 (See Figure 1) fits this model. Since 3 is larger than 2, the fraction $\frac{4}{9}$ is larger than $\frac{1}{18}$, and the entire first mixed number is larger than the second, finding differences between parts and adding them together is almost foolproof. Students had almost no trouble finding least common denominators to subtract the fractional parts, so all students, save one who made a careless error, got this problem correct.

\[ 5. \quad 3\frac{4}{9} - 2\frac{1}{18} \]

\[ \begin{array}{c}
\frac{4}{9} \\
\frac{1}{18}
\end{array} \]

\[ \begin{array}{c}
3\frac{4}{9} \\
2\frac{1}{18}
\end{array} \]

\[ \text{Figure 1: Subtracting mixed numbers on the 5-3 Reteaching assignment.} \]

Slightly more challenging homework problems came later on the worksheet. Problem 12 (See Figure 2) had a larger first number and the first whole part was larger than the second, but the second fractional part was larger. To solve this problem using the two-step process of finding differences between the parts of the mixed number and adding them together, students were required to add a new process to the sequence. They had to first take away one from 11 to view the problem as $10\frac{8}{5} - 9\frac{17}{20}$. Many students performed this extra step and obtained an answer of $1\frac{3}{4}$. However, some students continued with the unaltered two-step process of finding two positive differences, one between whole parts and one between fractional parts, and adding them together. Those who continued with the two-step differences process arrived at the incorrect answer of $2\frac{1}{4}$ or $2\frac{5}{20}$ before simplification.
Most challenging was the last homework problem on the 5-3 Reteaching assignment. As students attempted to subtract $3\frac{2}{3}$ from the smaller mixed number $3\frac{2}{9}$, those who knew they needed to borrow from the whole number to subtract the fractional parts obtained a positive fractional difference and concatenated it with a negative difference in the whole-number parts (See Figure 3). These students, in strictly following their process for subtracting mixed numbers, were unable to sequentially group two pieces of a number together before applying a negative sign to the number in its entirety. While one could argue that all individual steps in the subtraction process were performed correctly, the sequence of borrowing first then using the two-step difference process caused almost all students to arrive at an incorrect answer. Had they been able to understand the functionality of adding the whole and fractional parts together at the end of the two-step difference process, they might have realized that they still needed to find a difference between the whole and fractional parts of what was listed by most as the final answer.

Confusion was evident as students studied the meaning of the subtraction sign and applied the subtraction sign in new problem situations. Not yet exposed to additive
inverses, students took the subtraction sign to mean “take away,” “how far apart,” or “difference” between two numbers. Students knew only the binary meaning of subtraction, hence could only subtract when two inputs were given, and needed to use a sequence of steps in order to subtract two mixed numbers that each had two parts.

The issue of students finding differences or distances between parts of numbers and then adding those parts together instead of correctly subtracting numbers in their entirety persisted on the Semester 1 exam (See Appendix A), even without the presence of fractions. Item 28 on the exam asked students to solve an equation, which required subtraction of 4.01 from 3.92. While calculators were permitted on the exam, student S10 elected to do the work by hand, clearly showing her sequence of steps. After determining that 4.01 must be subtracted from both sides of the equation, she first found the differences between the digits in each place value. Second, she concatenated the differences and left the negative sign in place to arrive at an incorrect answer of -1.91 instead of -0.09 (See Figure 4). Had the student been required to add instead of subtract, she likely would have gotten the problem correct as any sum greater than 9 would have more naturally prompted the process of carrying over to the next place value in the sequence of processes. As the problems stood, the student chose an incorrect sequence of steps when subtracting 4.01 from 3.92.

![Incorrectly subtracting decimals on the Semester 1 final examination.](image)

Subtraction caused difficulty yet again for students simplifying expressions on the Semester 1 exam. On item 6 alone, specifically designed to test students' understanding of
the order of operations, three students made the same mistake shown in Figure 5 below. It is interesting to observe that the overall sequence of processes was correct. Students calculated the value inside parentheses first, then multiplied $3 \cdot 5$ and $7 \cdot 3$, then performed addition and subtraction from left to right. However, inputs for the subtraction process were sequenced incorrectly, thus causing the entire problem to be incorrect. Students, likely without realizing it, mentally applied the Commutative Property before subtracting and took 2 away from 15 instead of taking 15 away from 2. They found $2 - 15$ to equal 13 instead of -13. While this negative sign is physically small enough to appear almost insignificant, it prompts students to consider the additive inverse of 13 before adding to 21 instead of simply finding the sum of 13 and 21. Students knew the basic order of operations, but the three who made this mistake demonstrated the importance of also understanding the processes that a sequence is comprised of in order to perform the sequence correctly.

Figure 5: Incorrect simplification of an order of operations problem on the Semester 1 final examination.

Another subtraction error surfaced when students were required to simplify expressions with like terms. Yet again on item 15, positive differences were found between each pair of terms, with no regard for the placement of the subtraction symbols (See Figure 6). Symbols and their placement are the clues for which processes and which sequence to use to simplify an expression, so disregarding them naturally leads to incorrect answers.
While some students seemingly disregarded symbol placement as unimportant in sequencing operations, others misassociated it. It is not uncommon to find students interpreting the binary subtraction symbol as a prompt to “take away” something from the number preceding the subtraction symbol. While this process seems harmless in simple subtraction exercises, it becomes problematic when the numbers on two sides of a subtraction symbol are not like terms. As is demonstrated in Figure 7, students have a tendency to use the symbol just to the right of a number when combining it with the next like term. They often forget that a subtraction sign is short hand for adding the opposite of something, and hence should be associated with the following number, not the preceding.

While students generally know the order of operations and can simplify basic expressions, negative signs often present obstacles to accurate completion of exercises. Negative signs disrupt processes and expose how students understand or misunderstand sequencing mathematical processes.
5.2 Solving Equations with Inverse Operations: The Order of Undo-Operations

As the school year progressed and new concepts and processes were introduced, errors in sequencing mathematical processes persisted in various manifestations. This became quite obvious during a unit on solving equations for a variable. Since the goal of solving an equation is to isolate a variable by getting rid of all numbers and operations acting on a variable, I coined the term “Order of Undo-Operations,” which the Pre-Algebra students used to refer to the four-step solving process (See Figure 8).

1. What are the four steps for solving equations in the “Order of Undo Operations”?  

- Use the distributive prop if needed
- Combine like terms
- Undo +/–
- Undo x/÷

*Figure 8: The Order of Undo-Operations, correctly listed by student S12 on the Chapter 7 test.*

In many cases, errors on homework assignments contradicted what students had written in their notes. Rather than look back at in-class examples and steps to check their reasoning, students found their own way to make sense of mathematical symbols and instructions. In some cases, all a student needed to correct his or her work was the question, “What step comes next?” Students knew the sequence of steps, but often selected more comfortable reasoning techniques that did not rely on the all-new “Order of Undo Operations”. Hence the question arose: could students be more successful on assessments if they named their steps before performing them? While some students named the steps perfectly and still did not perform them in the correct order while solving problems,
comparing what students initially wrote for steps and what they wrote to show their work illuminated how they reasoned their way to what they wrote for final answers.

Distinct student reasoning levels emerged from an analysis of the Pre-Algebra Chapter 7 test on solving multi-step equations and inequalities (See Appendix A). The first item on the test asked students to name the four steps in the Order of Undo-Operations. The next four items required students to use the Order of Undo-Operations to solve for a variable, and the sixth item challenged students to solve a two-variable equation for one of the variables. Of the thirteen participating students, nine were able to perfectly name the four steps in the correct sequence. These nine can be recognized by a score of 2 out of 2 points on item 1 in Table 3 below. While naming the steps perfectly did not guarantee a perfect score on items 2 through 5, all students missing any part of the Order of Undo-Operations in question 1 got at least one of the next four questions wrong. As could be expected, a positive correlation exists between score on item 1 and score on items 2 through 5. The three groups of students are divided by numerical scores, and we will elucidate later in this section and in Section 6.2 how the division is reinforced by the work and explanations students show on assessments. Students are in group A if they scored less than 2 out of 2 on item 1. Those who scored 2 out of 2 on item 1 are in group B if they scored 7 or less out of 8 on items 2 through 5, and they are in group C if they scored 7.5 or 8 out of 8 total points on items 2 through 5.

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
<th>S7</th>
<th>S8</th>
<th>S9</th>
<th>S10</th>
<th>S11</th>
<th>S12</th>
<th>S13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score on item 1 out of 2 points</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1.5</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1.5</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Total score on items 2-5 out of 8 points</td>
<td>8</td>
<td>6</td>
<td>7.5</td>
<td>4</td>
<td>7.5</td>
<td>6</td>
<td>6.5</td>
<td>8</td>
<td>6</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 3: Chapter 7 Test Question Scores
Group A consists of students who incorrectly named or incorrectly sequenced the mathematical processes in the Order of Undo-Operations. While students in group A did not use fully correct steps, they applied parts of the Order of Undo-Operations and showed valid attempts to reason their way through problems. It was students in this group whose tests took the longest to grade as it was often clear that students were working hard to find the correct solution and parts of incorrect solutions were on the right track. However, the basic processes were sequenced incorrectly, hence it was tough for students in this group to arrive at correct answers no matter how well they could apply the processes that they were comfortable with. Figure 9 shows examples of work from students in Group A.

![Example work from students in Group A](image)

*Figure 9: Work from group A students S6 and S11, respectively, on items 2 and 4 of the Chapter 7 test.*

It is interesting to note that while student S6 did not list the distributive property as part of the Order of Undo-Operations in item 1, he used it correctly on item 2. However, his “undo addition and subtraction” step was done incorrectly. Additionally, his notation indicates that he divided by the variable k in the “undo multiplication and division” step, but his answer indicates that he correctly left the variable alone and divided only by the coefficient of the variable. Student S11 started out correctly by using an inverse operation to combine like terms, but then combined unlike terms in the next step before dividing and
then switching the variable \( n \) to the left side of the equal sign. Both students drew a vertical line through the equal signs of the equations to show the balance between the left and right expressions and remind themselves to perform the same operations on both sides.

In Group B, students had a better grasp on the Order of Undo-Operations. Students could sequence the four processes perfectly in item 1. Difficulty came in applying the Order of Undo-Operations. Students in Group B often used processes correctly, but changed the sequence or even performed a process incorrectly when a component made an equation look unlike those that could be solved comfortably. In example, while all students in Group 2 showed an understanding of the Distributive Property in item 5 and correctly sequenced it in the Order of Undo-Operations in item 1, students S2 and S4 in Group B both skipped the distribution step on item 2 (See Figure 10). Had the students used the Distributive Property first as the Order of Undo-Operations calls for, they would have been closer to the correct solution, however, the odd-looking fraction caused both students to neglect the known sequence in favor of experimental reasoning. Note that all students in Group B consistently represent the separation and equality between the two sides of the equation with a line down the center equal signs, and all students in this group correctly use notation and symbols to show their steps even when their steps are incorrect. Students know the processes and are aware that the processes should be performed in a specific sequence, but cannot necessarily apply a known sequence of processes to an unfamiliar situation.
Like those in Group B, students in Group C can perfectly name and sequence the steps in the Order of Undo-Operations. However, students in Group C can also apply and correctly execute the Order of Undo-Operations for the most part. Group C students make minor errors from time to time, but none of them lost more than half of a point on items 2 through 5 of the Chapter 7 test. In Figure 11 below, students S13 and S5 of Group C demonstrate proper application of the Order of Undo Operations in solving the given problems. In studying the work shown by students in Group C, one of the most unique patterns is that this is the only group in which students either neglect to draw the balance line through the equal signs of the equations or neglect to draw the equal signs on the line. During lessons and on homework students seemed to understand problems better when they drew lines through the equal signs, yet on the summative chapter assessment the students with the most problems correct were the ones who neglected to write or draw notational reminders.

Figure 10: Student S2 exemplifies a Group B response on item 2 of the Chapter 7 test.
Another unique feature of Group C is that no student outside the group used any variation of the Order of Undo-Operations to solve problem 6 on the test. Students in Groups A and B offered at most a vertical line through the equal sign and a best-guess one-line answer. Most students in Group C offered at least two lines of work, and some determined the correct solution for this challenging two-variable equation. While student S5 incorrectly subtracted instead of adding 5 on item 6, she demonstrates a mostly correct extension of her Order of Undo-Operations to an unfamiliar situation. Note that while she multiplies by -4 first instead of distributing the -4 to \( y \) and -5 in the numerator first, she preserves properties of operations and selects a correct sequence of processes.

Two months later, sequencing trends first recognized on the Chapter 7 test persisted on the Semester 2 final exam. Items 6 and 7 on the final assessed the same learning objectives that were assessed by items 2 through 5 on the Chapter 7 test. Not surprisingly, all students from Group C correctly answered items 6 and 7 on the final. Students with a full understanding of the Order of Undo-Operations during the Chapter 7 test at the end of March were able to recall their reasoning for the final examination at the end of May. Some students in Group B made a mistake on either problem 6 or problem 7 but correctly completed the other of the two problems. Two students from Group A made
an error both on item 6 and on item 7, and two students fit better with Groups B and C on the final. All students demonstrated similar or higher-level SMP reasoning on the Semester 2 final examination compared to the Chapter 7 test. With a better awareness of where student errors occurred on assessments, we proceed to elucidate students' underlying reasoning by examining the errors through theoretical perspectives outlined in Chapter 2.
Chapter 6

Analysis of Errors

Many of the errors highlighted in Chapter 5 can be better understood by analyzing them through theoretical perspectives outlined in Chapter 2. Mistakes students make when subtraction signs and negative numbers introduce new SMP questions into known sequences of processes can be explained by Gray and Tall’s analysis of procedural and conceptual thinking. SMP misconceptions and errors in Order of Undo-Operations problems are best explained in terms of Sfard’s operational and structural thinking patterns, and the interiorization, condensation, and reification stages students go through as they gain a structural understanding. By analyzing errors through these learning theory perspectives, we can develop a sense of how students currently approach SMP problems and how they can progress toward a better understanding of the sequencing of mathematical processes.

6.1 Proceptually Sequencing Processes Involving Subtraction and Additive Inverses

Gray and Tall heavily emphasize a duality between procedural and conceptual thinking, and they also emphasize a flexible interpretation of symbols that can represent both a process and a concept. Most subtraction errors could eventually be avoided by approaching a problem proceptually. Recall that proceptual thinking requires a person to view a symbol both as a process and as a concept and to choose the most applicable approach at each point in solving the problem.

The easily-recognized symbol "−" can take on two distinct meanings. Students are first taught to interpret it as a binary symbol representing the process of subtracting of one
number from another or a shift to the left along a number line from the first number by a distance of the second number. This is the interpretation students were most familiar with when they completed the 5-3 Reteaching assignment. While this “take away,” “difference,” or “move left” process works in many cases, understanding deepens when students are introduced to the concept of additive inverses or negative numbers. This transition in thinking allows the binary subtraction symbol to be viewed as the unary symbol that produces the opposite of a number. It also allows students to break down and then conceptualize subtraction as a sequence of two processes, first finding the additive inverse of the number following the sign, then adding that to the number preceding the sign.

Since the binary subtraction process works in most cases Pre-Algebra students had been exposed to, primarily subtraction of two integers, they naturally chose to use it on the 5-3 Reteaching assignment. However, concatenation of whole parts and fractional parts of mixed numbers turns simple subtraction into a multi-step process. Likely if you are reading this paper you understand math well enough to approach subtraction conceptually or proceptually. You would automatically combine the opposite of the second mixed number with the first mixed number if you were making an answer key for the 5-3 Reteaching worksheet. Pre-Algebra students, still in the procedural stage of understanding subtraction, were not as able to see the big picture of what they were doing. They solved the problems using the basic sequence of steps that worked for most problems. However, they did not always adapt the sequence to account for differences between some of the problems. Students were aware that concatenation of a whole number and a fraction produced a mixed number equal to the sum of the whole and fractional parts, yet confusion ensued when one mixed number was subtracted from another.
Since students had yet to make sense of subtraction as adding the inverse of a number, they came up with various ways to extend the basic procedure of taking one number away from another. Most often, they used the following sequence of processes: (1) find a common denominator for the fractional parts, (2) find the difference between whole parts, (3) find the difference between fractional parts, and (4) concatenate the differences to get the final answer. For problems similar to item 5 in which the first whole and fractional parts were both larger than the second whole and fractional parts, applying the sequence of processes with a procedural approach led to the correct answer. When the first mixed number and whole part were larger than the second but the second fractional part was larger than the first (i.e. item 12 on the 5-3 Reteaching assignment), some students were able to see that the sequencing of processes needed adaptation. Before the original first process, they had to go through the process of borrowing one unit from the whole number in order to find a positive difference between the fractions. However, when the second mixed number was bigger than the first in item 14, flexibly interpreting the subtraction sign and its implied sequence of processes became too difficult.

Students were unable to see that a subtraction sign in front of a mixed number always made the entire mixed number negative. Many viewed the concatenation as addition of a positive fraction to a positive or negative integer. This can best be explained in terms of SMP. Students either did not distribute the sign to the whole part and the fractional part of the mixed number before separating the parts, i.e. interpreting $3 \frac{2}{9} - 3 \frac{2}{3}$ as $3 + \frac{2}{9} - 3 + \frac{2}{3}$ instead of $\left(3 + \frac{2}{9}\right) - \left(3 + \frac{2}{3}\right)$, or they made the reverse error when concatenating the final answer. Many students who made the reverse error used a reasonable sequence sequence of borrowing, distributing, commuting, and associating to
obtain the idea that $3\frac{2}{9} - 3\frac{2}{3} = (2 - 3) + \left(\frac{11}{9} - \frac{6}{9}\right) = -1 + \frac{5}{9}$, but then were faced with the dilemma: Should $\frac{5}{9}$ be added to 1 before or after negating? Every student who made it to this point chose the wrong option. They all immediately concatenated, hence adding before negating. Many students arrived at an answer of $-1\frac{5}{9}$ for item 14 as in Figure 3 instead of correctly arriving at $-\frac{4}{3}$. Had students been able to conceptually understand the application of subtraction signs as negating mixed numbers and had they proceptually approached the problem, many more would have likely answered item 14 on the 5-3 Reteaching assignment correctly.

A procedural understanding of an almost identical SMP issue was the cause of the error in Figure 4, where student S10 was subtracting decimal numbers in item 28 of the Semester 1 final examination. She subtracted $3.92 - 4.01$ as $(3 - 4) + (0.9 - 0.0) + (0.02 - 0.01) = -1 + 0.9 + 0.001$, but again sequenced addition before the negation, arriving at an incorrect answer of -1.91 instead of -0.09. While a procedural understanding can sometimes prove sufficient, subtraction disrupts the sequence of processes in multi-part problems, hence a conceptual understanding and a proceptual approach is required in order to simplify or solve more complex equations.

Proceptual thinking, including interpreting the symbol "−" flexibly either as a subtraction symbol or as a symbol for the additive inverse, allows students to assess SMP issues as mathematical objects and to apply processes where appropriate to efficiently and correctly solve subtraction problems. Students associate a concept of “difference,” “take-away,” or “move left” with the subtraction symbol early on, and gradually deepen their understanding as examples are incorporated. Students make connections between
examples, and gradually compress processes into a smaller set of guidelines for manipulating sequences of mathematical processes. As seen with item 5, students can theoretically solve more complex problems before fully conceptualizing the sequence of processes involved, but even when a procedural understanding seems sufficient, it can be dangerous for a student to rely on memorized steps that can easily be forgotten. Memorization can also lead to feelings of inadequacy when students do not understand why they are using a particular process or sequence (Gray and Tall, 1994). As new examples are explored, they should always be connected to prior learning to strengthen a conceptual understanding. When new obstacles to solving or simplifying arise, emphasis should remain on the main goal of getting a variable by itself or reducing an expression to the smallest number of terms. By focusing on overarching concepts, proceptual thinking will be strengthened. New obstacles will be easier to approach, and sequences of mathematical processes will be more easily adapted, reducing SMP errors. While most math problems can theoretically be solved using some specific sequence of mathematical processes, that sequence determined by the student is more likely to be correct if the problem is understood conceptually and approached proceptually.

6.2 Operationally and Structurally Sequencing the “Order of Undo-Operations”

Undertones of Anna Sfard’s operational and structural thinking are similar to the undertones of Gray and Tall’s proceptual thinking, yet the differences between these two learning theories make Sfard’s ideal for an analysis of students’ SMP issues with the Order of Undo-Operations. Recall that students who approach a problem operationally rely on step-by-step processes and may need reminders as they sequence processes. Students with
a structural understanding of a problem work with the problem as an object and are able to manipulate rules and symbols to efficiently adapt a known sequence of processes. As students journey from an operational to a structural understanding, they go through three distinct stages of concept formation: interiorization, condensation, and reification (Sfard, 1991). These three stages are clearly demonstrated by the misconceptions and errors of various students on the Chapter 7 test and the Semester 2 final examination.

Recall from Section 5.2 that responses produced by the thirteen participants on the first few items of the Chapter 7 test naturally divided students into three distinct groups. Students in Group A were in Sfard’s interiorization stage. These students demonstrated the broadest range of SMP errors. They incorrectly listed steps in the Order of Undo-Operations since they were still learning this sequence of processes for solving equations. They could usually recall that they needed to undo addition or subtraction and then undo multiplication or division, but forgot the first two newer processes of distributing and combining like terms. Because students were still becoming comfortable with the Distributive Property and had only recently learned what a like term was, it was difficult to treat these individual processes as objects and correctly include them in the Order of Undo-Operations. Because students in Group A were in the interiorization stage and were still getting familiar with naming the sequence of processes in the Order of Undo-Operations, they were unable to move forward and apply the process, hence losing credit by making many SMP errors while solving the equations in test items 2 through 5.

Students in Group B demonstrated a familiarity with the Order of Undo-Operations by sequencing the names of the four processes perfectly on Chapter 7 test item 1. These students had already interiorized the solving process and were in Sfard’s condensation
stage at the time of the assessment. Students could apply the Order of Undo-Operations to some problems they had never seen before. The general sequence of processes was memorized, yet it was not always comfortably applied in every new situation. Students were still working to understand why the memorized sequence worked and was necessary to keep in order. They recognized that addition and subtraction are in the same step since they are inverse operations, but sometimes had difficulty determining inverses when negative numbers and subtraction were both used in the same problem. They demonstrated an understanding of the Distributive Property on test item 5, but for item 2 neglected to distribute first in the sequence of processes, hence arriving at an incorrect answer when a fraction was involved. Students in Group B demonstrated that they had interiorized the Order of Undo-Operations process, but needed to be exposed to more examples before completely eliminating SMP errors when applying the Order of Undo-Operations to new equations.

Interiorization, condensation, and reification stages are recognized by external “behaviors, attitudes, and skills” (Sfard, 1991, pg. 18), yet Groups A, B, and C were divided by numerical test scores. Hence the group divisions do not perfectly align with Sfard’s stages. Some students from Group C were likely still in the condensation stage at the time of the Chapter 7 test, but others clearly demonstrated reification. Students in Group C perfectly recalled the sequence of processes in the Order of Undo-Operations and missed at most half of a point on items 2 through 5 of the Chapter 7 test. Additionally, two students in Group C conquered the challenge of solving a two-variable equation for x in item 6. These two students had not only memorized and become familiar with how to use the Order of Undo-Operations, but demonstrated reification as they were able to extend the sequencing
to the odd situation in item 6 where they had to choose which variable to isolate and obtained an answer with a variable. Interestingly, students in Group C were the only students who either neglected to write equal signs in each step of the solving process or did not draw a line through all equal signs in a problem representing the balance of the two sides. These students could conceptualize the sequencing of processes faster than they could write down steps, and were able to manipulate symbols and notation to solve problems efficiently. It appears that many Group C students understood operations as objects and could use both an object and a process understanding to solve problems. They could store information efficiently, and hence did well even two months after the Chapter 7 test when similar problems appeared on their Semester 2 final examination.

While students in Group C built their operational understanding of the Order of Undo-Operations to a structural understanding, they were by no means finished with their learning journey. Once a process becomes a structure, it turns into an object on which a new process is built. Operational and structural understandings exist in tandem and each is constantly used to strengthen the other. Sfard argues that theoretically every problem could be solved operationally. At the Pre-Algebra level, emphasis is primarily on building an operational understanding of mathematical ideas so that in later secondary-level Algebra courses students can develop a strong structural understanding. Yet both operational understanding and structural understanding exist even at this stage. Students need to be comfortable with a structural and operational interplay to adapt sequences of mathematical processes and solve problems efficiently. Students can store and recall more information with a structural understanding, but need to incorporate some operational processes to execute problem-solving methods. As students are introduced to new
examples and ideas, the three-stage learning cycle is repeated, new concepts are understood structurally, and the operational-structural duality strengthens with each repetition. It is for this reason that most students performed just as well or better when presented with Order of Undo-Operations problems on the Semester 2 final examination in comparison with similar problems on the Chapter 7 test.

6.3 A Duality Approach to Solving Mathematical Problems

Gray and Tall as well as Sfard recognize that there is a duality between understanding mathematical processes and understanding mathematical objects. While both theories recognize that the process-oriented procedural or operational understandings could theoretically be sufficient to answer many mathematical questions as in the case of students solving the simplest mixed-number subtraction problems, both theories emphasize the benefit of employing higher reasoning levels. Students in Group C demonstrate this benefit as they maintain a strong performance on Order of Undo-Operations problems on the Semester 2 final examination after performing well on the Chapter 7 test. As students solidify processes and build mathematical objects, they develop deeper understandings and strengthen conceptual or structural understandings. It is for this reason that teachers teach simple examples first and proceed to elucidate general strategies later (DeCaro, 2015). As a more thorough understanding is developed, objects and processes exist together, and Pre-Algebra students are more able to manipulate the sequencing of mathematical processes to correctly answer mathematical questions.
Chapter 7

Conclusion

7.1 Understanding Student Thinking Regarding SMP Issues

The theories of Gray and Tall as well as Sfard serve well to elucidate how students understand SMP issues. In our analysis of Pre-Algebra assessments, it became clear that students can understand sequencing of processes both on an operational or procedural level and on a structural or conceptual level. For the best results on assessments, a student must intertwine both understandings.

Gray and Tall found the duality of procedural and conceptual understandings so important that they coined the term proceptual thinking to emphasize the combination of processes and concepts. Students in Pre-Algebra are most comfortable sticking to a procedural approach in many cases. They find security when they are able to follow a sequence of steps and arrive at a reasonable answer. However, obstacles such as subtraction and additive inverses disrupt sequences of processes that work with addition and strictly positive numbers. Hence students must be able to conceptualize subtraction as addition of an inverse and manipulate sequences of mathematical processes accordingly. They must think proceptually in order to consistently arrive at accurate solutions.

Sfard also emphasizes a duality between operational and structural understandings, and characterizes the stages students go through as they strengthen their understanding of mathematical notions. It is important to note that students constantly progress through the interiorization, condensation, and reification stages with various concepts. At the moment in time that Pre-Algebra students took the Chapter 7 test there were students in each of the three stages, and the same students that were at one time in one stage in their
understanding of the Order of Undo-Operations were at another time in other stages of understanding. While students initially rely on explicit steps to sequence mathematical processes, they eventually develop an understanding of relationships between the processes and an ability to manipulate a sequence of mathematical objects. Sfard’s stages do not rank students, but categorize student understanding so that students can be guided toward higher reasoning levels. As students depend on teachers for guidance in the cyclic learning process, this research naturally impacts the mathematics education community as a whole.

7.2 Implications for the Mathematics Education Community

This research can help teachers guide students toward a better understanding of SMP issues and more effective methods of solving problems. Learning theories can help teachers recognize stages of student understanding. As Vygotsky (1978) pointed out, it is important to know where students are at and to provide them with challenges within their zones of proximal development. New knowledge should ideally be incorporated into prior understanding. While certain mathematical conventions are arbitrary and meanings must be memorized, the conventions discussed in this paper are what Zazkis and Rouleau would consider necessary (Zazkis and Rouleau, 2017). The process of subtracting or adding an inverse as well as the Order of Undo-Operations are sequenced by the properties of the operations and processes involved. While students often memorize the steps for subtracting and solving equations first, students can discover why the processes work. Teachers can help students make connections between examples and compress knowledge to attain a more flexible understanding of mathematical processes and concepts.
It is crucial for educators to keep in mind that learning is itself a process; sharing mere facts is insufficient. Teachers can help students work through Sfard’s stages of understanding to help students strengthen a structural framework that can be interwoven with new operational notions. Teachers should begin by building a solid foundation of processes for students to rely on. They should guide students to experiment with the processes and apply them to various examples as the processes are interiorized. As processes are connected to each other and tied back to previous learning, students condense the processes into simpler rules that can be more easily recalled and manipulated. As teachers continue to ask why, students reach reification and attain a structural understanding of a particular concept. However, the process does not end here. This newly realized object becomes the base of the next process to be discovered, and the duality between an understanding of a process and of an object is continually strengthened. It is the role of teachers to always push students to make connections, explain why processes work, and always expand their understanding of the sequencing of mathematical processes.
Chapter 8

Suggestions for future research

In seeking to understand student thinking regarding SMP issues, many new questions arose. How students interpret notation and mathematical language are intriguing concepts, and searching for more connections between between SMP errors in elementary, secondary, and post-secondary math classes could provide fruitful results.

A more in-depth study of notation became intriguing as I analyzed the Semester 1 final examination. As demonstrated by the results in Table 2, one in nine errors that students made was related to misinterpreting a symbol or definition. By the time students take a final examination they have been exposed to the symbols and definitions countless times through various methods, so one could safely assume that symbol and definitional errors occur more frequently as students are initially learning concepts. I am particularly curious how closely misinterpretation of symbols is related to how well students understand a concept. From time to time I have seen students use incorrect notation as they show their work but arrive at a correct answer, or even show all steps correctly but notate an answer incorrectly. On the other hand, I have had conversations with students who use notation as prompts to carry out a mathematical process, but do not fully understand why the process works. In some cases, placement of symbols seems to affect how students understand their meaning. Negative signs written as superscripts, i.e. $\cdot 3$ instead of $-3$, are perhaps less likely to be interpreted as subtraction. Cangelosi et al. (2013) touch on this notational challenge in a discussion of how students determine whether a symbol is a binary subtraction sign or a unary negative sign.
As with notation, words can be easily misinterpreted. I constantly make decisions about whether to introduce an idea first and teach proper terminology later, or start with vocabulary and then explain what the words mean. Even so, there are likely words that some students interpret incorrectly or associate with different examples. This brings to mind the question: How can using more simple or more complex mathematical language impact students’ ability to reach structural understanding?

To add another layer of intriguing complexity to a study, one might consider international differences in how language is used in mathematics. Could it be possible that Canadians who are taught about BEDMAS have different difficulties with the order of operations than American students who are taught about PEMDAS? A significant amount of the literature reviewed in this study was produced outside the United States. Are there certain international educational systems who use language and teaching styles more conducive to higher reasoning levels than others?

Connections between SMP errors across levels of mathematics also deserve more attention. Teaching various levels of students helps me understand each class better, and the same could be true of studying and comparing various levels of students in a research study. The challenge here comes in designing a study that can be used to elicit connections. With variables such as differing state and district curriculums, levels of teacher efficacy, and student interests and majors, a longitudinal study would likely produce the best results, but it would be difficult to coordinate in terms of student consent and tracking data for many years. If such a study was completed, it could provide great insights into effective interventions and teaching methods.
Bibliography


Appendix A

Pre-Algebra Assignments and Assessments
Lesson 1-2 Pre-Check

This assignment will be graded on how much you explain your answers, not on correctness. Use complete English sentences. I do not expect that you will get them all right. I want to see what you already know on your own, so please do not ask anyone else for help.

1. Jenny put the following answers on her math test. Some of them are wrong. For each wrong answer, write the correct answer on the line after it and explain what Jenny did wrong. If Jenny is correct, explain how you know.
   a. \[4 + 6 + 2 = 5\] Explain:
   b. \[2(3 - 1) = 5\] Explain:
   c. \[9 \times 2 - (6 + 1) = 11\] Explain:

2. Larry says that \[\frac{1}{2 + 3} = \frac{1}{2} + \frac{1}{3}\]
   a. Do you agree with Larry? ______ Yes OR ______ No
   b. Explain why you agree or disagree.
   c. If you do not agree, explain why Larry might think these two are equal.
3. Rewrite the following equation without exponents and explain your answer. (I do not expect that you have seen problems like both of these, but take your best guess!)

\[(2 + 3)^2=\]

\[(x + y)^2=\]

4. Do you think the order of steps for solving a problem should be memorized?

<table>
<thead>
<tr>
<th>Yes</th>
<th>Maybe</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Explain why:

5. What do you do if you do not know the correct order of steps for solving a problem on a quiz or a test?

6. What is an operation in math?
Reteaching 5-3
Adding and Subtracting Fractions

Subtract $3\frac{1}{3} - 1\frac{5}{6}$.

Find a common denominator.

$$
\begin{align*}
3\frac{1}{3} &= \frac{10}{3} \\
-1\frac{5}{6} &= -\frac{11}{6}
\end{align*}
$$

Rename $3\frac{2}{6}$ and subtract.

$$
\begin{align*}
3\frac{2}{6} &= \frac{20}{6} \\
-1\frac{5}{6} &= -\frac{11}{6} \\
\frac{20}{6} - \frac{11}{6} &= 1\frac{4}{6} = 1\frac{2}{3}
\end{align*}
$$

Note: $3\frac{2}{6} = 2 + \frac{2}{6} = 2 + \frac{1}{3} = 2 + 1 = 3$. Simplify.

Find each difference.

1. $2\frac{4}{3} = 2\frac{4}{3} = 3\frac{2}{3}$
   
   $\quad -1\frac{1}{10} = \quad -1\frac{1}{10}$

2. $4\frac{3}{4} = 4\frac{3}{4} = 3\frac{2}{4}$
   
   $\quad -2\frac{11}{12} = \quad -2\frac{11}{12}$

3. $5\frac{1}{2} = 5\frac{1}{2} = 6\frac{1}{2}$
   
   $\quad -2\frac{1}{4} = \quad -2\frac{1}{4}$

4. $7\frac{2}{13} = 7\frac{2}{13} = 6\frac{2}{13}$
   
   $\quad -1\frac{7}{10} = \quad -1\frac{7}{10}$

5. $3\frac{5}{6} - 2\frac{1}{6}$

6. $6\frac{1}{3} - 2\frac{2}{3}$

7. $7\frac{2}{3} - 3\frac{4}{6}$

8. $2\frac{7}{18} - 1\frac{3}{14}$

9. $10\frac{1}{4} - 5\frac{1}{4}$

10. $1\frac{5}{6} - 1\frac{1}{6}$

11. $2\frac{3}{5} - 1\frac{3}{5}$

12. $11\frac{3}{5} - 9\frac{3}{5}$

13. $5\frac{5}{10} - 4\frac{8}{10}$

14. $3\frac{2}{3} - 3\frac{2}{3}$
Middle School Pre-Algebra
 Semester 1 Exam

Answer each question to the best of your God-given ability. Show your work to receive full credit.

**Choose the best answer.**

1. Evaluate $6 - x$ for $x = 4$.
   a. 10
   b. 2
   c. -4
   d. -2

2. Find $5 + (-3)$.
   a. -2
   b. -8
   c. 8
   d. 2

3. Which expression has a different value from the others?
   a. $4 - 7$
   b. $-7 - (-4)$
   c. $|7 - (-4)|$
   d. $-|7 - 4|$

4. Which expression is false?
   a. $|-4| > -4$
   b. $-|-7 + 3| \leq 0$
   c. $-|2 - 4| = |4 - 2|$
   d. $6 = |6|$

5. In which quadrant of the coordinate plane is the point (-2,2)?
   a. I
   b. II
   c. III
   d. IV

**Simplify. Show your work.**

6. $2 - 3 \cdot 5 + 7(-1 + 4)$

**Write your answer. Show your work.**

7. Evaluate $\frac{x + 6}{y}$ for $x = -1$, $y = 5$.

8. $6(-3) = \hspace{1cm}$

9. $12 \div (-4) = \hspace{1cm}$

10. Order 0, 1, -2, -3, 4 from least to greatest.

11. Use the coordinate plane to the right for 11.
   a. Graph the point $A(4,-2)$.
12. Which of the following is an example of the Associative Property of Addition?
   a. $3 + 4 + 5 = 4 + 3 + 5$  
   b. $3(1) = 3$  
   c. $3x + 15 = 3(x + 5)$  
   d. $(3 + 4) + 5 = 3 + (4 + 5)$

13. Which word describes the equation $12x = 4$?
   a. True  
   b. False  
   c. Open

14. Which of the following is a solution to the equation $18 = 2(a + 12)$?
   a. 3  
   b. 4  
   c. -3  
   d. -4

Simplify. Show your work.
15. $6 - 7a + 4 - 3a$  

16. $b + 3(10 - b)$

Solve each equation. Show your work.
17. $20 = 14 + k$  

18. $\frac{n}{-2} = 8$

Solve each inequality.
19. $5 < x - 23$  

20. $3k \geq -30$

Graph the following inequality.
21. $x < 2$

Write an inequality for the following graph.
22.  

Choose the best answer.
23. Round 5.83219 to the nearest tenth.
   a. 6  
   b. 5.8  
   c. 5.83  
   d. 5.84

24. With which unit should I measure the amount of milk in my glass?
   a. milliliters  
   b. meters  
   c. centimeters  
   d. kilograms
25. 14 kg = _____ g
   a. 0.014 b. 0.14 c. 14000 d. 14000000

26. 5km = 5000 _____
   a. mm b. cm c. g d. m

Use rounding to estimate. Show your work.

27. 11.432 · 4.7398 ≈ ____________________________

Solve each equation. Find the exact answer. Show your work.

28. 4.01 + x = 3.92 ______________________________

29. \( \frac{y}{3.2} = 25 \) ______________________________

Find the area using the area formula.
   30. __________________

Find the averages.
   31. 2, 7, 9, 2, 6

   Mean: __________________

   Median: __________________

   Mode: __________________

Choose the best answer.
   32. Which of the following numbers has 3 as a factor?
       a. 83 b. 75 c. 940 d. 62

33. What is the GCF of \( 3x^2 \) and \( 9x \)?
    a. \( 9x^2 \) b. 9x c. 3x d. 3

Write your answer. Show your work.
   34. Write \( -3 \cdot x \cdot y \cdot 5 \cdot y \cdot y \cdot x \) using exponents.

   ______________________ 3
35. Write the prime factorization of 24 using exponents. First, make a factor tree.

   ________________________

36. Write $\frac{10}{b}$ in simplest form.

   ________________________

Insurance:

I. Write an inequality for the situation. Graph the solutions.
The outside temperature was at most -1°C.

   ________________________

II. Simplify. Show your work.

\[-(7 + b) - 3a + 2(a + b)\]

III. Find the area and perimeter of the figure to the right.

   Area: __________
   Perimeter: __________
A. Write your answer.
    1. What are the four steps for solving equations in the “Order of Undo Operations”?
       __________________________
       __________________________
       __________________________

B. Solve each equation.
   2. \( \frac{1}{2}(k - 5) = 3 \)  \hspace{2cm} 3. \( 7n - 3 = 2n + 12 \)

   4. \( 0.2n + 13 = 1.3n - 14.5 \)  \hspace{2cm} 5. \( -17 + 3n = 7(n - 2) \)

   6. Solve \( m = \frac{y - 5}{-4} \) for \( y \).

   7. Find four consecutive integers whose sum is 290. Write an equation, then solve to show your work.

   8. Five less than three times the number \( k \) is greater than 19. Find \( k \). Write an equation, then solve to show your work.
9. The fuel economy of Izzy’s car is 40 mi/gal. This is 6 mi/gal more than Stella’s minivan. What is the fuel economy of Stella’s minivan? **Write an equation**, then solve to show your work.

C. Solve each inequality and graph the solution on a number line.

10. \(4 - 7x \leq 11\)  
11. \(9k + 13 > 4k - 12\)

D. Interest

13. The rate of interest on Owen’s savings account is 7%. What is the simple interest for three years on $352?

14. Grace deposits $600 into a savings account that earns 4% interest compounded yearly.
   a. What is Grace’s balance after three years?

   b. How much interest did Grace’s account earn?

**Insurance.** Which pays more: 5.2% simple interest for 4 years or 5% interest compounded annually for 4 years? Explain.

**Equations**

\[ B = p(1 + r)^n \]
\[ A = \frac{1}{2}bh \]
\[ I = prt \]
\[ C = 2\pi r \]
Middle School Pre-Algebra

Semester 2 Exam

Name: ______________________________

Answer each problem to the best of your ability. Show your work where required. Draw a parallelogram next to your name to earn a bonus point.

Chapter 5

1. Convert the repeating decimal to a fraction.

   \[ 0.1\overline{8} = \frac{\text{________}}{\text{9}} \]

2. Solve the following equations. Show your work.
   a. \[ b + \frac{1}{2} = \frac{3}{4} \quad b = \text{________} \]
   b. \[ -\frac{2}{7}x = 10 \quad x = \text{________} \]

Chapter 6

3. The two figures below are similar. Use a proportion to find side length \( x \).

   \[ \frac{6}{15} = \frac{4}{x} \]

4. Use a proportion to solve: 75% of \( k \) is 12. Find \( k \). \( \text{______} \)

5. A t-shirt originally priced at $25 is sold at a 40% discount. Find the sale price. Show your work.
Chapter 7
Solve the following equations. Show your work.

6. \(0.7 = 1.6x - 2.5\)
7. \(3(5 - x) = 4(x + 2)\)

Solve the inequality and graph it on the number line. Show your work.
8. \(6x + 10 \leq 3x + 1\)

Chapter 8
9. Solve \(y - \frac{2}{3}x = 2\) for \(y\). Then graph.
   \(y\)-intercept: _______ slope: _______

10. Solve the following system using the graph.
    \(y = \frac{1}{2}x - 4\)
    \(y = -2x + 1\)
    solution: (______, ______)

Chapter 9
11. Classify the triangle by sides AND angles by circling words from the word bank.
12. Fill in the blanks to complete the congruence statement.
   \[ \overline{AB} \cong \overline{XW} \]
   \[ \overline{AC} \cong \overline{XY} \]
   \[ \overline{BC} \cong \overline{WY} \]
   \[ \triangle ABC \cong \text{__________} \text{ by } \text{__________} \]

13. What type of transformation is shown on the graph to the right? Circle one: translation, reflection, or rotation?

Chapter 10/11
Find the areas of the figures below. Show your work.

14. 

15. 

16. Solve for \( x \) using the Pythagorean Theorem. Show your work.

Insurance: What do you think was the most important thing you learned this year in Pre-Algebra?
Formulas

\[ A = bh \]
\[ A = \frac{1}{2} bh \]
\[ A = \frac{1}{2} h(b_1 + b_2) \]
\[ A = \pi r^2 \]
\[ a^2 + b^2 = c^2 \]
### Appendix B

Joseph's Order of Operations Survey

<table>
<thead>
<tr>
<th>Problem</th>
<th>Number and type of operations included</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>Three operations: multiplication, addition and parentheses</td>
</tr>
<tr>
<td>Problem 2</td>
<td>Four operations: division, multiplication, parentheses and addition</td>
</tr>
<tr>
<td>Problem 3</td>
<td>Four operations: parentheses, multiplication, addition and division</td>
</tr>
<tr>
<td>Problem 4</td>
<td>Five operations: parentheses, exponents, multiplication, addition and subtraction</td>
</tr>
<tr>
<td>Problem 5</td>
<td>All six operations in PEMDAS: parentheses, exponents, multiplication, division, addition and subtraction</td>
</tr>
<tr>
<td>Problem 6</td>
<td>Five steps are needed to solve for the unknown value</td>
</tr>
<tr>
<td>Problem 7</td>
<td>Six steps are needed to solve for the unknown value</td>
</tr>
<tr>
<td>Problem 8</td>
<td>Six steps are needed to solve for the unknown</td>
</tr>
<tr>
<td>Problem 9</td>
<td>Six steps are needed to solve for the unknown value</td>
</tr>
<tr>
<td>Problem 10</td>
<td>Eight steps are needed to solve for the unknown</td>
</tr>
</tbody>
</table>

*Figure 12: "Instrument Problems and Justifications" (Joseph, 2014, p. 25)*