Rogue Waves in Optics and Water

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Rogue Waves in Optics and Water

by

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A thesis submitted to the

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Department of Applied Mathematics

2013
This thesis entitled:
Rogue Waves in Optics and Water
written by Tommaso Buvoli
has been approved for the Department of Applied Mathematics

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Mark Ablowitz

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Dr. Chris Curtis

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Prof. Bengt Fornberg

Date ________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
In recent years, large amplitude rogue waves have been studied in water and optical fibers. These large waves occur more frequently than suggested by conventional linear models and nonlinear phenomena are considered by many to be responsible for these waves.

The nonlinear Schrödinger equation (NLS) models a slowly modulated, monochromatic, deep water wave train. Moreover, perturbed plane wave solutions of the nonlinear Schrödinger equation experience growth due to modulational instability. Through repeated numerical simulations, wave height statistics are determined for both NLS and an equation which incorporates the full linear water wave dispersion relation. The latter equation prevents unbounded spectral broadening present in 2D NLS. All equations studied lead to non-Gaussian wave statistics that are well-described by Rayleigh distributions, and support rogue waves with amplitudes up to five times the initial amplitude. The differences between the one and two dimensional results are not substantial, with two dimensional equations leading to wave height distributions with smaller variance and higher mean. This suggests that studying, the one dimensional nonlinear Schrödinger equation plus suitable perturbations may be sufficient for a basic understanding of rogue waves, without having to turn to higher dimensional equations.

Finally, NLS-type equations that model pulse propagation in zero dispersion nonlinear fibers are also studied. In addition to modulational instability, it appears that certain parameter regimes are governed by a nonlinear instability. Both processes cause significant growth that can lead to large amplitudes events.
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Contents

Chapter

1 Water Waves

1.1 Water Waves and The Nonlinear Schrodinger Equation ......................... 3
  1.1.1 Linear Wave Theory ................................................. 5
  1.1.2 Deep Water Nondimensionalization of the Euler Equations ................. 8
  1.1.3 Deriving the Stokes Frequency Shift ................................ 8
  1.1.4 Nonlinear Schrödinger from the Deep Water Dispersion Relation ....... 12
  1.1.5 Keeping the Full Linear Dispersion Relation .......................... 14

1.2 Stability of Deep Water Wave Trains ............................................ 16
  1.2.1 Determining Stability of NLS Type Equations .......................... 16
  1.2.2 Stability of NLS ..................................................... 18
  1.2.3 Stability of FDS ..................................................... 20

1.3 Numerical Experiments ............................................................... 22
  1.3.1 Computationally Verifying Linear Stability Analysis ..................... 22
  1.3.2 Wave Height Statistics .............................................. 32

1.4 Conclusions ..................................................................................... 44

2 Optics ............................................................................................... 45

2.1 Optics and the Nonlinear Schrödinger Equation .................................. 45

2.2 Introduction to Optical Rogue Waves .............................................. 47
Tables

Table

1.1 Growth Rates for 1D NLS .................................................. 25
1.2 Numerical Parameters for 1D Water Wave Experiments ................. 33
1.3 Numerical Parameters for 2D Water Wave Experiments .................. 34
2.1 ZDS Unstable Wave Numbers for Various Domains Sizes ................. 52
2.2 Numerical Parameters for 1D ZDS Experiments ........................... 53
3.1 LI4 Fourier Space Splitting ............................................... 64
## Figures

### Figure

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>An Infinite Wave Tank with Impenetrable Bottom</td>
<td>4</td>
</tr>
<tr>
<td>1.2</td>
<td>Stability Space for 1D and 2D NLS</td>
<td>19</td>
</tr>
<tr>
<td>1.3</td>
<td>Stability Space for 1D and 2D FDS</td>
<td>21</td>
</tr>
<tr>
<td>1.4</td>
<td>Perturbed Plane Wave Solutions of 1D NLS (Physical Space)</td>
<td>23</td>
</tr>
<tr>
<td>1.5</td>
<td>Perturbed Plane Wave Solutions of 1D NLS (Fourier Space)</td>
<td>24</td>
</tr>
<tr>
<td>1.6</td>
<td>Stability Space for 2D NLS and 2D FDS on Periodic Domain of 8\pi</td>
<td>27</td>
</tr>
<tr>
<td>1.7</td>
<td>Short-Time Perturbed Plane Wave Solutions of 2D NLS</td>
<td>28</td>
</tr>
<tr>
<td>1.8</td>
<td>Short-Time Perturbed Plane Wave Solutions of 2D FDS</td>
<td>29</td>
</tr>
<tr>
<td>1.9</td>
<td>Long-Time Perturbed Plane Wave Solutions of 2D NLS</td>
<td>30</td>
</tr>
<tr>
<td>1.10</td>
<td>Long-Time Perturbed Plane Wave Solutions of 2D FDS</td>
<td>31</td>
</tr>
<tr>
<td>1.11</td>
<td>Long-Time Wave Statistics for NLS and FDS on a Domain of 8\pi</td>
<td>37</td>
</tr>
<tr>
<td>1.12</td>
<td>Long-Time Wave Statistics for NLS and FDS on a Domain of 16\pi</td>
<td>38</td>
</tr>
<tr>
<td>1.13</td>
<td>Long-Time Wave Statistics for NLS and FDS on a Domain of 24\pi</td>
<td>39</td>
</tr>
<tr>
<td>1.14</td>
<td>Evolution of Wave Height PDFs NLS and FDS on a Domain of 24\pi</td>
<td>40</td>
</tr>
<tr>
<td>1.15</td>
<td>Evolution of Mean and Standard Deviation of Wave Height PDFs for NLS and FDS</td>
<td>41</td>
</tr>
<tr>
<td>1.16</td>
<td>Long-Time Perturbed Gaussian Solutions of 1D and 2D NLS</td>
<td>42</td>
</tr>
<tr>
<td>1.17</td>
<td>Long-Time Perturbed Plane Wave Solutions of 1D NLS on 50\pi Domain</td>
<td>43</td>
</tr>
<tr>
<td>1.18</td>
<td>Long-Time Perturbed Gaussian Solutions of 1D NLS on 50\pi Domain</td>
<td>43</td>
</tr>
<tr>
<td>2.1</td>
<td>Perturbed Plane Wave Solutions of 1D ZDS (Physical Space)</td>
<td>54</td>
</tr>
</tbody>
</table>
2.2 Perturbed Plane Wave Solutions of 1D ZDS (Fourier Space) . . . . . . . . . . . . . . 55
2.3 Long-Time Wave Height Statistics for ZDS for Domains $8\pi$, $16\pi$, $24\pi$ . . . . . . . . 56
2.4 Evolution of Wave Height PDFs for ZDS on Domains $8\pi$, $16\pi$, $24\pi$ . . . . . . . . 57
2.5 Evolution of Mean and Standard Deviation of Wave Height PDF for ZDS . . . . . . . 58
Introduction

Every year, oceanic rogue waves cause catastrophic damage to the shipping industry. Many denied the existence of these 30m walls of water, but in 1995, the first evidence supporting rogue waves was recorded at the Draupner oil rig in the North Sea. Since then, satellites and wave buoys have collected indisputable evidence that rogue waves occur frequently throughout the ocean [15, 4].

There are many different theories which explain oceanic rogue waves. These include focusing [25], modulational instability [19, 12, 20, 14], and wind excitation [17]. In this thesis, we study rogue waves from the point of view of the nonlinear Schrödinger equation. We begin by deriving the Stokes water wave solution from the Euler equations. From this we are able to derive the 1D and 2D nonlinear Schrödinger equations which govern a slowly modulated deep water wave train. We also derive an equation which incorporates the full deep water linear dispersion relation. This equation controls the spectral broadening present in 2D NLS and satisfies the slow modulation assumption for longer time. Moreover, incorporating the linear dispersion relation allows us to bound the region of instability without having to turn to higher order equations such as the Dysthe equation [23]. We call this new equation the FDS equation, which is short for full dispersion Schrödinger.

The 1D and 2D nonlinear Schrödinger equations have been widely used to study rogue waves in the ocean [19, 21, 26]. To our knowledge, there have been no studies comparing these two equations. In this thesis, we numerically determine wave height statistics for 1D NLS, 2D NLS, and 2D FDS. We find that two dimensional effects lead to small changes in variance and wave amplitude. We therefore conclude that 1D NLS statistics provide a suitable basis for studying the class of rogue waves described by the 2D NLS and 2D FDS equations. We also find that the wave height statistics of all equations are well described by Rayleigh distributions; a result which is consistent with the statistics of oceanic waves [10, 16].

We also turn our attention to the more recently discovered optical rogue waves [22]. These rogue events exhibit non-Gaussian, L-shaped distributions, and are governed by 1D higher order NLS type equations. These findings suggest a connection between the mechanisms that generate
rogue waves in optics and water. We use the physical equations in [22] as the starting point, and study the stability of plane wave solution for an equation governing zero-dispersion fibers. Though the equation is modulationally stable, we find indications that a nonlinear instability is responsible for causing rogue growth. We also numerically determine that for large domains the wave height distribution for this equation is also Rayleigh.
Chapter 1

Water Waves

In the 1990s, oceanographers began gathering evidence supporting 30 m rogue waves in deep water. Today, it is widely believed that these rogue events are triggered by instabilities in the full water wave equations. The nonlinear Schrödinger equation (NLS) is the simplest nonlinear equation governing deep water wave trains. It can be written in both one and two dimensions. The primary goal of this thesis is to quantify the differences between 1D and 2D NLS. We hope to determine whether 1D NLS provides a suitable, basic description of rogue growth in water waves.

In this chapter, we derive the 1D and 2D NLS starting from the Navier-Stokes equations, and analyze the linear stability of plane wave solutions. We also consider an equation which incorporates the full linear dispersion relation. Via numerical integration we determine the probability distributions governing wave height for each equation. We compare these distributions to assess the similarities between the one and two dimensional equations.

1.1 Water Waves and The Nonlinear Schrodinger Equation

In this section we provide a brief derivation of the nonlinear Schrödinger equation starting from the Navier-Stokes equations. The derivation presented here is based on Chapters 4, 5 and 6 of [1].

Consider modeling a fluid in an infinite water tank shown in Figure 1.1. Fluid motion is governed by the Navier-Stokes equations, which are derived from conservation laws. Conservation
Figure 1.1: An infinite wave tank with a flat, impenetrable bottom with depth $h$. We want to model free surface denoted by $z = \eta(x,t)$. We define wave amplitude to the the displacement from the undisturbed fluid level at $z = 0$

of mass requires that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

where $\rho$ is the fluid mass density, and $\mathbf{v}$ is the fluid velocity. Conservation of momentum requires that

$$\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \mathbf{F} - \nabla P + \nu \Delta \mathbf{v}$$

where $\mathbf{F}$ is external forcing, $P$ is pressure, and $\nu$ is the kinematic viscosity. For simplicity, we consider a fluid which is

- Incompressible: constant density, i.e. $\rho = \rho_0$.
- Ideal: no frictional forces, i.e. $\nu = 0$.
- Initially irrotational: infinitesimal fluid particles have no local spin, i.e. $\nabla \times \mathbf{v} = 0$.

Under these assumptions, the Navier-Stokes equations reduce to the Euler equations:

$$\nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{\rho_0} (\mathbf{F} - \nabla P)$$
Inside the fluid region, the momentum equation is trivially satisfied when fluid is initially irrotational. This allows us to rewrite \( \mathbf{v} \) as a gradient of a velocity potential such that \( \mathbf{v} = \nabla \phi \). Rewriting conservation of mass in terms \( \phi \) yields Laplace’s equation. At the bottom of the tank, we assume that fluid cannot flow through hence \( \phi_z = 0 \). On the free surface, we impose two additional conditions: conservation of momentum leads to Bernoulli’s equation, and the kinematic condition can be derived by assuming that a fluid packet on the surface stays on the free surface. In summary, the free-surface water wave equations omitting surface tension and assuming a flat bottom are:

\[
\begin{align*}
\Delta \phi &= 0 & -h < z < \eta(x,y,t) & \text{(Laplace’s Equation)} \\
\frac{\partial \phi}{\partial z} &= 0 & z = -h & \text{(No Flow Through Bottom)} \\
\frac{\partial \phi}{\partial t} + \frac{1}{2} \| \nabla \phi \|^2 + g\eta &= 0 & z = \eta(x,y,t) & \text{(Bernoulli’s Equation)} \\
\frac{\partial \phi}{\partial z} &= \frac{\partial \eta}{\partial t} + \nabla \phi \cdot \nabla \eta & z = \eta(x,y,t) & \text{(Kinematic Condition)}
\end{align*}
\]

See Appendix A.1 for a more complete derivation for each of these equations. This system of equations is a free boundary problem, is particularly difficult to solve because the unknown \( \eta(x,y,t) \) is the location of the free surface.

### 1.1.1 Linear Wave Theory

We briefly look at linear wave theory, and derive the linear dispersion relation for water waves. We begin by assuming that \( \| \nabla \phi \| \ll 1 \) and \( |\eta(x,y,t)| \ll 1 \). Physically, this corresponds to small wave amplitudes and slow fluid speed. Since \( |\eta| \ll 1 \), we approximate the free boundary condition by expanding \( \phi(x,y,z,t) \) in a Taylor series around \( z = 0 \) such that

\[
\begin{align*}
\phi(x,y,z,t) &= \phi(x,y,0,t) + \eta \phi_z(x,y,0,t) + O(|\eta|^2) \\
&= \phi_0(x,y,t) + \eta \phi_{0,z}(x,y,t) + O(|\eta|^2)
\end{align*}
\]
Substituting the leading order Taylor expansion for $\phi$ into Eq. (A.12) and Eq. (A.13) and ignoring small terms, we obtain:

\[
\Delta \phi = 0 \quad -h < z < 0 \quad \text{(Laplace’s Equation)} \quad (1.7)
\]

\[
\frac{\partial \phi}{\partial z} = 0 \quad z = -h \quad \text{(No Flow Through Bottom)} \quad (1.8)
\]

\[
\frac{\partial \phi}{\partial t} = -g \eta \quad z = 0 \quad \text{(Bernoulli’s Equation)} \quad (1.9)
\]

\[
\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad z = 0 \quad \text{(Kinematic Condition)} \quad (1.10)
\]

We assume that $\phi$ and $\eta$ have a mode of the form:

\[
\phi(x, y, z, t) = A(k, l, z, t)e^{ikx + ily}
\]

\[
\eta(x, y, t) = N(k, l, t)e^{ikx + ily}
\]

Substituting $\phi(x, y, z, t)$ into Eq. (1.7), we obtain

\[
A_{zz} - \kappa A = 0
\]

where $\kappa = (k^2 + l^2)$. The solution is

\[
A(k, l, z, t) = \alpha(k, l, t) \cosh(\kappa z) + \beta(k, l, t) \sinh(\kappa z)
\]

We shift the solution by $h$, i.e. $z \to z + h$, and set $\beta(k, l, t) = 0$ in order to satisfy Eq. (1.8). Therefore,

\[
\phi(x, y, z, t) = \hat{\alpha}(k, l, t) \cosh(\kappa(z + h))e^{ikx + ily}
\]

Next, we substitute $\phi$ and our original ansatz for $\eta(k, l, t)$ into Eq. (1.9) and Eq. (1.10) and evaluate at the free surface $z = 0$ to obtain the system

\[
\frac{\partial}{\partial t} \begin{bmatrix} \hat{\alpha} \\ N \end{bmatrix} = \begin{bmatrix} 0 & -\frac{g}{\cosh(\kappa h)} \\ \kappa \sinh(\kappa h) & 0 \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ N \end{bmatrix}
\]

Assuming a solution of the form

\[
\begin{bmatrix} \hat{\alpha} \\ N \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} e^{i\omega t}
\]
where \(a, b\) are constants, and forcing the determinant of the resulting system to be zero, we find that

\[ \omega^2 = g\kappa \tanh(\kappa h) \]

The solution for the Fourier mode will be of the form:

\[ \eta(x, y, t) = N_R(k, l)e^{ikx + ily - i\omega_R(\kappa)t} + N_L(k, l)e^{ikx + ily - i\omega_L(\kappa)t} \]

where \(N_R, N_L\) are determined from the initial conditions \(\eta(x, y, 0)\) and \(\eta_t(x, y, 0)\), and

\[ \omega_R(\kappa) = \sqrt{g\kappa \tanh(\kappa h)} \quad \omega_L(\kappa) = -\sqrt{g\kappa \tanh(\kappa h)} \]

represent the dispersion relation for right and left traveling waves, respectively. For a general initial condition the solution to the free surface problem can be written via Fourier integrals:

\[ \eta(x, y, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} N_R(k, l) \exp(ik + ly - i\omega_Rt) + N_L(k, l) \exp(ik + ly - i\omega_Lt) \, dk \, dl \]

### 1.1.1.1 Linear Dispersion Relation for Water Waves

The dispersion relation relates temporal oscillations to spatial oscillations, and determines the velocity of a plane wave with wavenumber \(\kappa\). If \(\omega(\kappa)\) depends nontrivially on \(\kappa\), then plane waves with different frequencies will travel at different speeds, and the underlying equation is said to be dispersive. As shown above, the linear dispersion relation for water waves is

\[ \omega(\kappa)^2 = g\kappa \tanh(h\kappa) \]

where \(g\) denotes gravitational acceleration, \(h\) denotes the water depth, and \(\kappa\) denotes the wavenumber. For deep water, \(\kappa h \gg 1\) and

\[ \omega(\kappa)^2 \approx g\kappa \]

This is often called the deep water dispersion relation.
1.1.2 Deep Water Nondimensionalization of the Euler Equations

The dimensional Euler water wave equations for infinitely deep water can be written as

\[ \Delta \phi = 0 \quad -\infty < z < \eta(x,t) \] (1.11)
\[ \frac{\partial \phi}{\partial z} = 0 \quad z \to \infty \] (1.12)
\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} ||\nabla \phi||^2 + g\eta = 0 \quad z = \eta(x,t) \] (1.13)
\[ \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + (\nabla \phi \cdot \nabla \eta) \quad z = \eta(x,t) \] (1.14)

where \( \phi \) models fluid velocity potential, \( \eta \) models wave height, and \( g \) is the acceleration due to gravity. We are interested in modeling a small amplitude, monochromatic wave train with wavenumber \( k_0 \) and amplitude \( A \). We begin by introducing the dimensionless parameter \( \epsilon = k_0 A \ll 1 \) and nondimensionalize via the transformations:

\[ x = \frac{x'}{k_0} \quad y = \frac{y'}{k_0} \quad z = \frac{z'}{k_0} \quad \eta = A\eta' \quad \phi = \frac{(k_0A)w_0}{k_0^2} \phi' \quad t = \frac{t'}{w_0} \] (1.15)

After simplifying, we obtain the following set of nondimensional, weakly nonlinear equations:

\[ \Delta \phi' = 0 \quad -\infty < z' < \epsilon\eta'(x,t) \] (1.16)
\[ \frac{\partial \phi'}{\partial z'} = 0 \quad z' \to \infty \] (1.17)
\[ \frac{\partial \phi'}{\partial \tau'} + \frac{\epsilon}{2} ||\nabla' \phi'||^2 + \eta' = 0 \quad z' = \epsilon\eta'(x,t) \] (1.18)
\[ \frac{\partial \phi'}{\partial z'} = \frac{\partial \eta'}{\partial \tau'} + \epsilon(\nabla' \phi' \cdot \nabla' \eta') \quad z' = \epsilon\eta'(x,t) \] (1.19)

For convenience, we drop the primes from here on.

1.1.3 Deriving the Stokes Frequency Shift

The dispersion relation governing water waves is nonlinear. In the case where \( k_0A = \epsilon \ll 1 \), we can obtain the \( O(\epsilon^3) \) dispersion relation for a monochromatic wave train using perturbation analysis on Eqns. 1.16, 1.18, and 1.19. We apply multiple scales in time by defining \( T = \epsilon t \) and denote \( \phi \) and \( \eta \) as \( \phi(x,z,t,T) \) and \( \eta(x,t,T) \). From the chain rule, \( \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T} \). We can rewrite
Eq. (1.18) and Eq. (1.19) as

\[ \phi_t + \epsilon \phi_T + \frac{\epsilon}{2} (\phi_x^2 + \phi_z^2) + \eta = 0 \quad (1.20) \]

\[ \eta_t + \epsilon \eta_T + \epsilon \eta_x \phi_x = \phi_z \quad (1.21) \]

Since \(|\epsilon \eta| \ll 1\), we approximate the free boundary condition by expanding \(\phi(x, z, t, T)\) in a Taylor series around \(z = 0\) such that

\[ \phi(x, \epsilon \eta, t, T) = \phi(x, 0, t, T) + (\epsilon \eta) \phi_z(x, 0, t, T) + \frac{1}{2} (\epsilon \eta)^2 \phi_{zz}(x, 0, t, T) + O(\epsilon^3) \]

Substituting the \(O(\epsilon^3)\) Taylor expansion for \(\phi\) into Eq. (1.20) and Eq. (1.21) we obtain

\[ \phi_t + \epsilon \left( \eta \phi_{tz} + \phi_T + \frac{\phi_x^2}{2} + \frac{\phi_z^2}{2} \right) + \epsilon^2 \left( \eta \phi_{Tz} + \frac{\eta}{2} \phi_{txx} + \eta \phi_x \phi_{xz} + \eta \phi_z \phi_{zz} \right) + \eta = O(\epsilon^3) \quad (1.22) \]

\[ \eta_t + \epsilon (\eta_T + \eta_x \phi_x) + \epsilon^2 (\eta \phi_{x} \phi_{xz}) = \phi_z + \epsilon \eta \phi_{zz} + \frac{(\epsilon \eta)^2}{2} \phi_{zzz} \quad (1.23) \]

Next, we expand \(\phi\) and \(\eta\) as

\[ \phi = \phi^{(0)} + \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \cdots \]

\[ \eta = \eta^{(0)} + \epsilon \eta^{(1)} + \epsilon^2 \eta^{(2)} + \cdots \]

Substituting the asymptotic expansions for \(\phi\) and \(\eta\) into Eq. (1.22) into Eq. (1.23) and collecting terms up to \(O(\epsilon^2)\) we obtain:

**O(1):**

\[ \phi_t^{(0)} + \eta^{(0)} = 0 \quad (1.24) \]

\[ \eta_t^{0} - \phi_z^{(0)} = 0 \quad (1.25) \]

**O(\epsilon):**

\[ \phi_t^{(1)} + \eta^{(1)} = - (\eta^{(0)} \phi_{tz}^{(0)} + \phi_T^{(0)}) - \frac{1}{2} \left( (\phi_x^{(0)})^2 + (\phi_z^{(0)})^2 \right) \quad (1.26) \]

\[ \eta^{(1)} - \phi_z^{(1)} = - \eta_x^{(0)} \phi_x^{(0)} - \eta_T^{(0)} + \eta^{(0)} \phi_{zz}^{(0)} \quad (1.27) \]
O(\epsilon^2):

\[ \phi^{(2)}_t + \eta^{(2)} = -\phi^{(1)}_{tz} - \eta^{(0)} \phi^{(1)}_{zz} - \frac{1}{2} (\eta^{(0)})^2 \phi^{(0)}_{zz} - \eta^{(0)} \phi^{(0)}_{zz} \]

\[ - \phi^{(0)} \phi^{(1)}_{zz} - \eta^{(0)} \phi^{(0)}_{zz} - \phi^{(0)}_{T} - \eta^{(0)} \phi^{(0)}_{Tz} \]

\[ \eta^{(2)}_t - \phi^{(2)}_z = \eta^{(0)} \phi^{(1)}_{zz} + \eta^{(1)} \phi^{(0)}_{zz} - \left( \eta^{(0)} \phi^{(1)}_{zz} + \eta^{(1)} \phi^{(0)}_{zz} + \eta^{(0)} \eta^{(0)} \phi^{(0)}_{zz} \right) \]

\[ + \frac{1}{2} (\eta^{(0)})^2 \phi^{(0)}_{zz} - \eta^{(1)} \]

1.1.3.1 Satisfying Laplace’s Equation

We first consider Eq. (1.16) coupled with the condition \( \frac{\partial \phi}{\partial z} = 0 \) as \( z \to \infty \). It is immediately evident that at each order

\[ \phi^{(j)}_{xx} + \phi^{(j)}_{zz} = 0 \]

In order to satisfy Laplace’s equation and the boundary condition at \( z = -\infty \), we assume the following ansatz for \( \phi^{(j)} \):

\[ \phi^{(j)} = \sum_{m=0}^{\infty} A^{(j)}_{m}(T)e^{im\theta + m|k|z} + \text{c.c.} \]

Moreover, because we are interested in modeling a monochromatic wave train, we assume only one harmonic term at leading order. Hence, we take

\[ \phi^{(0)} = A_1(T)e^{i\theta + |k|z} + \text{c.c.} \]

\[ \eta^{(0)} = N_1(T)e^{i\theta} + \text{c.c.} \]

where \( \theta = kx - \omega t \). Notice, that these formula for \( \phi^{(0)} \) and \( \eta^{(0)} \) are only valid if \( \eta(x, t = 0) \) is a monochromatic wave train, where perturbations from additional plane waves are at most O(\epsilon) in magnitude.
1.1.3.2 Leading Order, $O(1)$

Substituting the asymptotic approximations for $\phi^{(0)}$ and $\eta^{(0)}$ into Eq. (1.24) and Eq. (1.25), and setting $z = 0$, we obtain the system

\[
\begin{bmatrix}
-i\omega & 1 \\
-|k| & -i\omega
\end{bmatrix}
\begin{bmatrix}
A_1 \\
N_1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\] (1.30)

To ensure a unique, nontrivial solution, we set the determinant to zero. This leads to the nondimensional dispersion relation

$$\omega^2(k) = |k|$$

The leading order solution for $\eta$ is therefore

$$\eta^{(0)} \sim N_1(T) e^{ikx - i\sqrt{|k|}t} + c.c.$$ 

1.1.3.3 First Order and Second Order

After removing secular terms and solving $O(\epsilon)$ and $O(\epsilon^2)$ equations, we find that

$$N_{1,\tau} = -2ik^2 \omega |N_1|^2 N_1$$

See appendix A.2 for the complete derivation. The solution to this equation is

$$N_1(\tau) = N_1(0) e^{-2ik^2 \omega |N_1(0)|^2 \tau}$$

Therefore, the free-surface solution is

$$\eta(x,t) = \exp \left[ i k x - i \omega \left( 1 + 2 \epsilon^2 k^2 |N_1(0)|^2 \right) t \right] + c.c.$$ 

and the nonlinear dispersion relation for a monochromatic wave train is

$$\omega_{nl} = \omega \left( 1 + 2 \epsilon^2 k^2 |N_1(0)|^2 \right) + O(\epsilon^3)$$

In two dimensions, the nondimensional dispersion relation is

$$\omega(k, l) = \left( k^2 + l^2 \right)^{\frac{1}{4}} \left( 1 + 2 \epsilon^2 k^2 |N_1(0)|^2 \right) + O(\epsilon^3)$$ (1.31)

This equation is commonly called the stokes water wave solution.
1.1.4 Nonlinear Schrödinger from the Deep Water Dispersion Relation

To derive the 2D nonlinear Schrödinger equation governing deep water wave trains, we begin by considering a nondimensional free surface solution of form

$$
\eta'(x', y', t') = N(X, Y, T)e^{i(x' + \tilde{l}_0 t') - \epsilon t'} + \text{c.c.}
$$

where $N(X, Y, T)$ is a slowly varying amplitude such that $X = \epsilon x'$, $T = \epsilon t'$, $\epsilon = k_0 A \ll 1$ and $\tilde{l}_0 = l_0/k_0$. Note that primed spatiotemporal variables $(x', y', t', \eta')$ are related to their physical counterparts $(x, y, t, \eta)$ via Eq. (1.15). The spatial Fourier transform of a localized envelope function is

$$
\hat{N}(K, L, T) = \int_{\mathbb{R}^2} N(X, Y, T)e^{i(\Omega T - KX - LY)}dXdY \tag{1.32}
$$

and

$$
N(X, Y, T) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{N}(K, L, T)e^{-i(\Omega T - KX - LY)}dXdY \tag{1.33}
$$

From Eq. (1.32) and Eq. (1.33) we obtain the relations:

$$
\mathcal{F}^{-1}[(iK)^n \hat{N}] = \frac{\partial^n N}{\partial X^n}, \quad \mathcal{F}^{-1}[(iL)^n \hat{N}] = \frac{\partial^n N}{\partial Y^n}, \quad \mathcal{F}^{-1}[(i\Omega)^n \hat{N}] = \frac{\partial^n N}{\partial T^n}.
$$

Next we consider the nondimensional, nonlinear dispersion relation for a Stokes wave which we derived from the asymptotic reduction of the Euler water wave equations:

$$
\omega(k, l) = (k^2 + l^2)^{\frac{1}{4}}(1 + 2\epsilon^2 k^2 |N|^2) + O(\epsilon^3)
$$

We begin by Taylor expanding $\omega(k, l)$ around the dimensionless wavenumber $(1, \tilde{l}_0)$ (in dimensional coordinates this corresponds to the wavenumber $(k_0, l_0)$) to obtain

$$
\omega(k, l) = \omega_0 + \omega_{0,k}(k - 1) + \omega_{0,l}(l - \tilde{l}_0) + \omega_{0,kk} \frac{(k - 1)^2}{2} + \omega_{0,ll} \frac{(l - \tilde{l}_0)^2}{2} + \omega_{0,kl}(k - 1)(l - \tilde{l}_0) + \ldots
$$

where $\omega_0 = \omega(1, \tilde{l}_0)$. Consistent with the slowly modulated ansatz, we assume a small perturbation such that $k = 1 + \epsilon K$ and $l = l_0/k_0 + \epsilon L$. For convenience we choose $l_0 = 0$ and truncate our approximation at $O(\epsilon^3)$ to obtain

$$
\omega(K, L) = 1 + \frac{K\epsilon}{2} + \epsilon^2 \left( \frac{L^2}{4} - \frac{K^2}{8} + 2|N|^2 \right) + O(\epsilon^3)
$$
Choosing \( l_0 \neq 0 \) leads to a rotated plane wave which can be reduced to the \( l_0 = 0 \) case through a change of coordinates. Finally, \( \omega(K, L) = w_0 + \epsilon \Omega = 1 + \epsilon \Omega \), therefore,
\[
\epsilon \Omega = \frac{K \epsilon}{2} + \epsilon^2 \left( \frac{L^2}{4} - \frac{K^2}{8} + 2|N|^2 \right) + O(\epsilon^3)
\]
(1.34)

Multiplying Eq. (1.34) by \( \hat{N}(K, L, \Omega) \), applying the inverse Fourier transform, and dividing out a common factor of \( \epsilon \) yields
\[
i N_T + i \frac{N_X}{2} - \epsilon N_{XX} \frac{N}{8} + \epsilon N_{YY} \frac{N}{4} - 2\epsilon |N|^2 N = O(\epsilon)
\]

To obtain the canonical form for NLS, we drop the \( O(\epsilon) \) term, and introduce the variable transformations \( \xi = X - T/2, \tau = \epsilon T \) to obtain
\[
i \tilde{N}_\tau - \frac{N_{\xi\xi}}{8} + \frac{N_{YY}}{4} - 2|N|^2 \tilde{N} = 0
\]

See appendix A.3 for details. Finally, we apply the following scalings for convenience,
\[
\tilde{N} = 2N, \quad \tilde{\tau} = \frac{\tau}{2}, \quad \tilde{\xi} = 2\xi, \quad \tilde{Y} = \sqrt{2}Y
\]

which leads to the PDE
\[
i \tilde{N}_{\tilde{\tau}} - \tilde{N}_{\tilde{\xi}\tilde{\xi}} + \tilde{N}_{\tilde{Y}\tilde{Y}} - |\tilde{N}|^2 \tilde{N} = 0
\]
(1.35)

where the relationships between dimensional and slow, dimensionless variables are
\[
|\tilde{N}| = \frac{|\eta|}{A}, \quad \tilde{\tau} = \frac{\epsilon^2 \sqrt{gk_0 t}}{2}, \quad \tilde{\xi} = 2\epsilon k_0 x - \frac{2\epsilon t}{\sqrt{gk_0}}, \quad \tilde{Y} = \sqrt{2\epsilon k_0 y}
\]
(1.36)

and the dimensional free surface solution is given by
\[
\eta(x, t) = A \tilde{N}(\tilde{\xi}, \tilde{Y}, \tilde{\tau})e^{i(k_0 x - \omega_0 t)} + c.c. = \text{Re} \left[ A \tilde{N}(\tilde{\xi}, \tilde{Y}, \tilde{\tau})e^{i(k_0 x - \omega_0 t)} \right]
\]
(1.37)

where \( \omega_0 = \sqrt{gk_0} \). From here on, we drop the tildes, and refer to Eq. (1.35) as the two dimensional nonlinear Schrödinger equation governing water waves. The one dimensional nonlinear Schrödinger equation governing water waves can derived by repeating this calculation, after having removed dependence in the spatial \( y \) variable. It is given by
\[
i \tilde{N}_{\tilde{\tau}} - \tilde{N}_{\tilde{\xi}\tilde{\xi}} - |\tilde{N}|^2 \tilde{N} = 0 + O(\epsilon)
\]
(1.38)
1.1.4.1 NLS Timescales and Wave Height

We make two additional comments regarding NLS timescales, and physical wave height.

- Eq. (1.35) provides an $O(\epsilon)$ approximation to the Euler water wave equations, and is therefore strictly valid for
  \[ \tilde{\tau} = \frac{1}{\epsilon} \quad \text{or equivalently} \quad t = \frac{2}{\sqrt{gk_0}\epsilon^3} \]
  When evolving past these timescales, higher order effects should be considered.

- Physical wave height can be estimated via the quantity $A|\tilde{N}|$. We know
  \[ \eta(x, y, t) = \frac{A}{2} (\tilde{N}e^{i\theta} + \tilde{N}^*e^{-i\theta}) \]
  where $\theta = kx + ly + \omega t$. Rewriting $\tilde{N}$ in polar form s.t. $\tilde{N} = |\tilde{N}|e^{i\theta_N}$ we obtain
  \[ \eta = \frac{A|\tilde{N}|}{2} (e^{i(\theta + \theta_N)} + e^{-i(\theta + \theta_N)}) = |\tilde{N}|(\cos(\theta + \theta_N)) \leq |\tilde{N}| \]
  Therefore, as long as $\tilde{N}$ slowly modulates the carrier wave, the largest amplitude wave will be $|N|$.

1.1.5 Keeping the Full Linear Dispersion Relation

We also consider an equation which incorporates the full linear dispersion relation for water waves. We call this equation the full dispersion Schrödinger equation (FDS). To derive the 2D nondimensional FDS, we proceed exactly as with the nonlinear Schrödinger equation by considering a free surface solution of form
  \[ \eta'(x', y', t') = N(X, Y, T)e^{i(x'+k_0y'-t')} + c.c. \]
where $X = \epsilon x'$, $Y = \epsilon y'$, $\epsilon = k_0A$, $k_0 = l_0/k_0$ and the nondimensional dispersion relation for deep water waves is given by
  \[ \omega(k, l) = (k^2 + l^2)^{\frac{1}{4}}(1 + 2\epsilon^2 k^2|N|^2) + O(\epsilon^3) \]
To derive FDS, we keep the full linear dispersion relation and expand the nonlinear component such that

$$ \omega(k, l) = (k^2 + l^2)^{1/2} + 2\epsilon^2 k^2 |N|^2 + O(\epsilon^3) $$

Once again we assume a slow modulation around the nondimensional wavenumber (1, 0) such that

$$ k = 1 + \epsilon K $$
$$ l = 0 + \epsilon L $$

which leads to

$$ \omega(K, L) = ((1 + \epsilon K)^2 + (\epsilon L)^2)^{1/2} + 2\epsilon^2 |N|^2 + O(\epsilon^3) $$

Finally, using \( w(K, L) = 1 + \epsilon \Omega \), we obtain,

$$ \epsilon \Omega = -1 + ((1 + \epsilon K)^2 + (\epsilon L)^2)^{1/2} + 2\epsilon^2 |N|^2 + O(\epsilon^3) $$

Multiplying both sides of the last equation by \( \hat{N}(K, L, T) \) and applying the inverse Fourier transform we find

$$ i\epsilon N_T F^{-1}[(\omega_l(K, L) - 1)F[N]] - 2\epsilon^2 |N|^2 N = 0 $$
$$ \omega_l(K, L) = ((1 + \epsilon K)^2 + (\epsilon L)^2)^{1/2} $$

where \( F \) and \( F^{-1} \) denote the forwards and inverse Fourier transform. Alternatively, this can be written as

$$ \epsilon \Omega \hat{N} - (-1 + \omega_l(K, L))\hat{N} - 2\epsilon^2 |N|^2 \hat{N} = 0 $$
$$ \Omega = i\partial T, \ K = -i\partial X, \ L = i\partial Y $$

To properly compare the FDS equation with Eq. (1.35) we must apply the transformations \( \xi = X - T/2 \) and \( \tau = \epsilon T \), and the scalings

$$ \tilde{N} = 2N $$
$$ \tilde{T} = \frac{T}{2} $$
$$ \tilde{\xi} = 2\xi $$
$$ \tilde{Y} = \sqrt{2}Y $$

We obtain:

$$ i\tilde{N}_{\tilde{T}} F^{-1}[\Omega_l(\tilde{K}, \tilde{L})F[\tilde{N}]] - |\tilde{N}|^2 \tilde{N} = 0 + O(\epsilon) $$

(1.39)

$$ \Omega_l(\tilde{K}, \tilde{L}) = \frac{1}{\epsilon^2} \left( 2 + 2\epsilon \tilde{K} - 2((1 + 2\epsilon \tilde{K})^2 + 2(\epsilon \tilde{L})^2)^{1/2} \right) $$
See appendix A.4 for details. Eq. (1.39) is still an $O(\epsilon)$ approximation to the full water wave equations since it lacks any higher order nonlinear terms. The relationship between dimensional variables and slow dimensionless variables are determined by Eq. (1.36), and the physical free surface can be determined from Eq. (1.37). From here on, we drop the tildes, and refer to Eq. (1.39) as the FDS equation.

1.2 Stability of Deep Water Wave Trains

In 1967, Benjamin and Feir discovered that a periodic wave train of frequency $\omega$ is not stable to small perturbations. This phenomenon is driven by an instability that spreads energy from an initially narrow bandwidth to a broader bandwidth and causes temporary exponential growth in solutions. Recently, it has been theorized that this instability, commonly called modulational instability, is a possible mechanism for rogue waves. We can mathematically understand this physical phenomenon by analyzing the stability of the plane wave solution $N(\xi, Y, \tau) = e^{-i\tau}$ for Eqns. (1.35), (1.38), and (1.39). This special solution, leads to the free surface that represents a calm sea state with perfectly oscillatory waves governed by the $O(\epsilon^2)$ nonlinear dispersion relation:

$$\eta(x,t) = \text{Re} \left[ AN(\xi, Y, \tau)e^{i(k_0x-\omega_0\tau)} \right] = \text{Re} \left[ Ae^{i(k_0x-\omega_0(1+\frac{\epsilon^2}{2}t))} \right] = \text{Re} \left[ Ae^{i(k_0x-\omega_0(1+\frac{(k_0A)^2}{2}t))} \right]$$

We now present a complete derivation of modulational instability for Eqs. (1.35), (1.38), and (1.39)

1.2.1 Determining Stability of NLS Type Equations

Let $u : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{C}$ and consider a partial differential equation of the form:

$$iu_t + L[u] + \gamma |u|^2 u = 0$$

(1.40)

where $L[u]$ is a time-independent linear operator. We can analyze the linear stability of the plane wave solution $u(x,t) = ae^{i\gamma a^2t}$ by considering a perturbed solution

$$u(x,t) = ae^{i\gamma a^2t}(1 + \epsilon(x,t))$$

(1.41)
where $\epsilon(x, 0) \ll 1$. Substituting (1.41) into Eq. (1.40) and discarding all nonlinear terms, we arrive at the nonlinear partial differential equation
\[
 i\epsilon(x, t) + L[\epsilon(x, t)] + \gamma a^2(\epsilon(x, t) + \epsilon(x, t)) + O(\epsilon^2) = 0
\]
which describes the temporal evolution of the perturbation $\epsilon(x, t)$. We let $\epsilon(x, t) = \epsilon_R(x, t) + i\epsilon_I(x, t)$ and split the PDE into its real and imaginary components such that:
\[
 \frac{d}{dt} \begin{bmatrix} \epsilon_R \\ \epsilon_I \end{bmatrix} = \begin{bmatrix} -L_I & -L_R \\ L_R + 2a^2\gamma & -L_I \end{bmatrix} \begin{bmatrix} \epsilon_R \\ \epsilon_I \end{bmatrix}
\]
where $L_R$ and $L_I$ are the real and imaginary parts of the operator $L$, respectively. This system can be solved using Fourier transforms. If $\mathbb{I}^N = [-M, M]^N$, we express $\epsilon_R(x, t)$ and $\epsilon_I(x, t)$ via their Fourier series
\[
 \epsilon_R(x, t) = \sum_{n=\mathbb{Z}^N} A(n, t) e^{i\sigma_n \cdot x} \quad \epsilon_I(x, t) = \sum_{n=\mathbb{Z}^N} B(n, t) e^{i\sigma_n \cdot x}
\]
where $\sigma_n = \frac{2\pi n}{M}$ and $n \in \mathbb{Z}$. If $\mathbb{I}^N = (-\infty, \infty)^N$ then we write $\epsilon_I(x, t)$ and $\epsilon_R(x, t)$ via their fourier transforms
\[
 \epsilon_R(x, t) = \int_{\mathbb{R}^N} A(k, t) e^{ik \cdot x} dk \quad \epsilon_I(x, t) = \int_{\mathbb{R}^N} B(k, t) e^{ik \cdot x} dk
\]
In either case our system becomes:
\[
 \frac{d}{dt} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -\hat{L}_I & -\hat{L}_R \\ \hat{L}_R + 2a^2\gamma & -\hat{L}_I \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}
\]
where $\hat{L}_I$ and $\hat{L}_R$ represent the 2 dimensional Fourier transforms of the operators $L_I$ and $L_R$. We now solve the corresponding eigenvalue problem to determine the stability of our linearized equation. We obtain the characteristic polynomial
\[
 \lambda^2 + 2\hat{L}_I \lambda + \hat{L}_R(\hat{L}_R + 2a^2\gamma) + \hat{L}_I^2 = 0
\]
Therefore,
\[
 \lambda = \hat{L}_I \pm \sqrt{-\hat{L}_R(\hat{L}_R + 2a^2\gamma)}
\]
and instability will occur in modes with wavenumber such that \( \hat{L}_R(\hat{L}_R + 2a^2\gamma) < 0 \). Moreover, maximum growth rates occurs either at an interior point where \( \nabla \hat{L}_R(2\hat{L}_R + 2a^2\gamma) = 0 \) or possibly at infinity. Notice that the imaginary operator \( \hat{L}_I \) does not affect the instability region or instability rates.

### 1.2.2 Stability of NLS

We first study the stability of plane wave solutions \( u(x,t) = e^{it} \) for the nondimensionalized 2D NLS (Eq. (1.35)) which we rewrite here for convenience:

\[
iN_\tau - N_{\xi\xi} + N_{YY} - |N|^2N = 0
\]

Eq. (1.35) is equivalent to Eq. (1.40) when:

\[
\gamma = -1 \quad L = \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} \quad a = 1
\]

Therefore,

\[
\hat{L}_R = (il)^2 - (ik)^2 \quad \hat{L}_I = 0
\]

On the infinite line, instability occurs for modes with wave numbers \( k \) and \( l \) such that

\[
(k^2 - l^2)(k^2 - l^2 - 2) < 0
\]

On a periodic domain \( k \to \frac{2\pi k}{L_x} \) and \( l \to \frac{2\pi l}{L_y} \). The maximum growth rate occurs when \( (k^2 - l^2) = 1 \). This corresponds to a growth rate of 1 or \( |2A^2k_0^2w_0|/\epsilon^2 \) in dimensional coordinates. Note that the stability region for the one dimensional nonlinear Schrödinger equation, can be obtained by simply setting \( l = 0 \). Figure 1.2 contains a plot of the instability regions for 1D and 2D NLS. The nondimensional wavenumber \( (k,l) \) is related to its physical counterparts via

\[
k_{\text{physical}} = k_0 + 2\epsilon k_0k
\]

\[
l_{\text{physical}} = l_0 + \sqrt{2}\epsilon k_0l
\]

Therefore, the plots in Figure 1.2 are magnifications of the region surrounding the physical wavenumber \( (k_0,l_0) \).
Figure 1.2: Stability space for the nondimensionalized 1D and 2D nonlinear Schrödinger equation governing water waves. In the right figure, light gray regions denote areas containing unstable modes, and the dark line follows the maximum growth rate curve.
1.2.3 Stability of FDS

We analyze the stability of plane wave solutions \( u(x,t) = e^{it} \) for the nondimensionalized 2D FDS equation (Eq. (1.39)) which we rewrite here for convenience:

\[
iN + F^{-1}[\Omega_t(K, L)F[N]] - |N|^2 N = 0 + O(\epsilon)
\]

\[
\Omega_t(K, L) = \frac{2 + 2\epsilon K - 2((1 + 2\epsilon K)^2 + 2(\epsilon L)^2)}{\epsilon^2}^{1/2}
\]

Eq. (1.39) is equivalent to Eq. (1.40) when:

\[
\gamma = -1 \quad L = F^{-1}[\Omega_t(k, l)F[u]] \quad a = 1
\]

As highlighted in appendix A.5,

\[
\hat{L}_R = \frac{\Omega_t(k, l, \epsilon) + \Omega_t(k, l, -\epsilon)}{2} \quad \hat{L}_I = \frac{\Omega_t(k, l, \epsilon) - \Omega_t(k, l, -\epsilon)}{2}
\]

On the infinite line, instability occurs for modes with wave numbers \( k \) and \( l \) such that

\[
\hat{L}_R(k, l)(\hat{L}_R(k, l) - 2) < 0
\]

On a periodic domain \( k \to \frac{2\pi k}{L_x} \) and \( l \to \frac{2\pi l}{L_y} \). The maximum growth rate occurs when \( \hat{L}_R(k, l) = 1 \). As was the case for NLS, this corresponds to a growth rate of 1 in nondimensional coordinates or \( |2A^2k_0^2w_0|/\epsilon^2 \) in dimensional coordinates. Figure 1.3 contains a plot of the instability regions for several values of \( \epsilon \). In short, the FDS equation preserves the growth rates and behavior of NLS near \( k = 0, l = 0 \) while successfully controlling the unbounded instability region far from the origin. As \( \epsilon \to 0 \), the FDS equation is equivalent to NLS.
Figure 1.3: Stability space for the nondimensional 1D and 2D FDS equations for various $\epsilon$ values. In the top right plot, the gray region denotes areas containing unstable modes, and dark lines represent the maximum growth rate curve. The bottom plot overlays the stability regions that incrementally widen as $\epsilon$ decreases. Note that $\epsilon \to 0$ limits to NLS.
1.3 Numerical Experiments

1.3.1 Computationally Verifying Linear Stability Analysis

Linear instability does not imply nonlinear instability. To test the validity of the linear analysis, we numerically model Eqs. (1.35), (1.38), and (1.39) on a square periodic domain of width $8\pi$, using a perturbed plane wave as the initial condition. We plot the solution in both physical and Fourier space to determine if linearly unstable modes grow exponentially for short time. We use 1024 grid points in each spatial dimension and a time step of .005. All numerical calculations are performed using a pseudospectral scheme in space and ETDRK4 in time.

1.3.1.1 1D NLS

Given a periodic domain of width $L_\xi$, we expect perturbed plane wave solutions of Eq. (1.38) to exhibit temporary exponential growth when the perturbation contains modes

$$\exp\left(\frac{2\pi ik}{L_\xi}\right) \text{ where } \left(\frac{2\pi k}{L_\xi}\right)^2 < 2, \ k \in \mathbb{Z}$$

For $L_\xi = 8\pi$, there are 10 unstable modes when $k \in [-5, 5] \setminus \{0\}$. The growth rates for each mode are shown in Table 1.1. To verify the linear analysis, we numerically model the 1D nonlinear Schrödinger equation using the initial conditions:

$$N(\xi, \tau = 0) = 1 + \frac{1}{100} e^{ik\xi}, \quad k = 1, 2, \ldots, 6.$$  \hspace{1cm} (1.42)

We present our numerical results in Figures 1.4 and 1.5. For small time, perturbations containing unstable modes exhibit exponential growth as predicted by the linear stability analysis. Moreover, through additional numerical experiments, we find that growth will not occur on small timescales if the perturbation does not initially contain unstable modes.

1.3.1.2 2D NLS & 2D FDS

We consider the two dimensional NLS (Eq. (1.35)) and FDS (Eq. (1.39)) equations on a doubly periodic domain with width $L_\xi$ and length $L_Y$. We expect perturbed plane wave solutions
Figure 1.4: Numerical solutions of the 1D NLS equation in physical space, using perturbed plane wave initial conditions. Perturbations consist of a single Fourier mode with wavenumber $k$ and amplitude $1/100$. As predicted by linear stability analysis, the first 5 modes experience temporary exponential growth, while the 6th mode does not.
Figure 1.5: Numerical solutions of the 1D NLS equation in Fourier space, using perturbed plane wave initial conditions. Perturbations consist of a single Fourier mode with wavenumber $k$ and amplitude $1/100$. As predicted by linear stability analysis, the first 5 modes experience temporary exponential growth, while the 6th mode does not.
Table 1.1: Mode Growth Rates for 1D NLS, $8\pi$ Periodic Domain

| $k$  | $|\text{Re}(\lambda)|$ |
|------|-----------------------|
| $\pm 1$ | .348                |
| $\pm 2$ | .661                |
| $\pm 3$ | .899                |
| $\pm 4$ | 1                   |
| $\pm 5$ | .827                |

of 2D NLS to exhibit short time exponential growth if the perturbation contains modes

$$
\exp\left(\frac{2\pi ik\xi}{L_\xi} + \frac{2\pi ilY}{L_Y}\right)
$$

where

$$
\left(\left(\frac{2\pi k}{L_\xi}\right)^2 - \left(\frac{2\pi l}{L_Y}\right)^2\right)\left(\left(\frac{2\pi k}{L_\xi}\right)^2 - \left(\frac{2\pi l}{L_Y}\right)^2 - 2\right) < 0, \ k, l \in \mathbb{Z}
$$

Similarly, we expect perturbed plane wave solutions of 2D FDS to grow if the perturbation contains modes where

$$
\hat{L}_R\left(\frac{2\pi ik}{L_\xi}, \frac{2\pi il}{L_Y}\right)\left(\hat{L}_R\left(\frac{2\pi ik}{L_\xi}, \frac{2\pi il}{L_Y}\right) - 2\right) < 0
$$

$$
\hat{L}_R(K, L) = \frac{W_i(K, L, \epsilon) + W_i(K, L, -\epsilon)}{2}
$$

When $L_x = L_y = 8\pi$, NLS and FDS admit 110 and 126 unstable modes, respectively. These modes are plotted in Figure 1.6. Though we do not include any figures, we have numerically confirmed that a plane wave solution exhibits temporary exponential growth if perturbed with any one of these unstable modes. To better understand the role of unstable modes, we consider the initial condition

$$
N(\xi, Y, \tau = 0) = 1 + \sum_{k = -20}^{20} \sum_{l = -20}^{20} A_{k,l} \exp\left(\frac{2\pi k\xi}{L_\xi} + \frac{2\pi lY}{L_Y}\right)
$$

where $A_k$ are random coefficients that have been normalized so that the perturbation has a maximum amplitude of $1/10$. We plot the NLS solution in Figures 1.7 and 1.9, and the FDS solution in Figures 1.8 and 1.10. For short timescales, modulational instability causes exponential growth in
each of unstable modes for NLS and FDS. For longer timescales, the NLS solution undergoes rapid spectral broadening along the X shaped instability region described by the infinite line problem. This excessive broadening not only violates the slowly modulated assumption, but is also difficult to model numerically since many grid points are needed to resolve fast modes. Longer time solutions of FDS do not rapidly broaden. This allows us to evolve to further time without violating the slowly modulated assumption, and makes the equation more suitable for numerical modeling.
Figure 1.6: Stability space for 2D NLS, and 2D FDS on a periodic domain of $8\pi$. For FDS we assume $\epsilon = 1/10$. Each gray dot represents a single unstable mode.
Figure 1.7: Short-time plots of perturbed plane wave solutions for the 2D NLS equation. Top: Solution in Physical Space. Middle: Solution in Fourier Space. Bottom: Magnified Solution in Fourier Space. Notice that the unstable modes grow above the background as predicted by linear stability analysis. Even for small time, the solution spectrally broadens along the X shape seen in the stability analysis for the infinite line problem.
Figure 1.8: Short-time plots for a perturbed plane wave solutions of 2D FDS equation. Top: Solution in Physical Space. Middle: Solution in Fourier Space. Bottom: Magnified Solution in Fourier Space. Unstable modes grow above the background as predicted by linear stability analysis. Notice that even for small times, spectral broadening has been reduced in comparison to two dimensional nonlinear Schrödinger equation.
Figure 1.9: Long-time plots for a perturbed plane wave solution of 2D NLS equation. Top: Solution in Physical Space. Middle: Solution in Fourier Space. Bottom: Magnified Solution in Fourier Space. Notice that the solution continues to broaden outside the range of this figure. On long timescales, we find that time the solution spectrum continues to spread to higher wavenumbers and solutions are no longer slowly modulated.
Figure 1.10: Long-time plots for a perturbed plane wave solution of 2D FDS equation. Top: Solution in Physical Space. Middle: Solution in Fourier Space. Bottom: Magnified Solution in Fourier Space. Once the equation becomes fully nonlinear, the initial instability saturates and the figure eight is no longer visible. Notice that there is less spectral broadening than in the two dimensional nonlinear Schrödinger equation, and the solution at $t = 20$ is much smoother.
1.3.2 Wave Height Statistics

We seek to determine the wave height distributions generated by NLS and FDS. We are particularly interested in understanding the differences between 1D and 2D equations, as well as the underlying wave height distributions. Finally, we also aim understand whether these equations admit rogue growth.

To address these questions we numerically model the evolution of a wave train in a periodic tank. A small random perturbation is initially added to the plane wave in both $x$ and $y$ directions, and numerical integration is repeated over many random trials. We choose a sufficiently large noise amplitude to ensure that modulation instability saturates well within NLS timescales. Maximum wave amplitude is measured at several predetermined times, allowing us to capture the evolving wave height distribution. We describe our experiments and results in detail below.

1.3.2.1 Problem Description

We model a plane wave with amplitude $A$ and wavenumber $k_0$ on a periodic, square domain. We pick the width of the domain to be a multiple $M$ of the wavelength such that $L = \frac{2\pi M}{k_0}$, and only consider plane waves where $\epsilon = \frac{k_0 A}{\epsilon} \ll 1$. This physical system can be modeled using Eq. (1.35), and Eq. (1.39) on a nondimensional rectangular periodic domain where $\xi \in [0, 4\pi M \epsilon]$ and $Y \in [0, 2\sqrt{2}\pi M \epsilon]$. We can also approximate this 2D system using Eq. (1.38) on a 1D periodic domain with with width $L_\xi = 4\pi M \epsilon$.

We consider $\epsilon$ values of .12, .10, and .08, and domain sizes of $8\pi$, $16\pi$, $24\pi$, and $50\pi$. The amplitude of the perturbation is chosen to be $1/10$ the amplitude of the carrier wave, and each equation is integrated to time $\tau = \frac{1}{\epsilon}$. We divide the temporal domain into 100 equispaced points $t_j$, and for each random initial condition, we determine the largest amplitude in the intervals $[t_0, t_j]$ for $j = 1 \ldots 100$.

Modeling this nondimensional problem allows us to capture wave statistics for a wide variety of physical parameters since the nondimensional domain width no longer depends on $k_0$ or $A$. The
domain sizes we have selected correspond to $M$ values between 150 and 20. Furthermore, physical
wave height distribution for any $(A, k_0)$ pair can be recovered by scaling the nondimensional wave
height $|N|$ by $A$.

Finally, we repeat several of our simulations using a perturbed Gaussian as initial conditions,
in order to assess the impact of initial conditions on wave height distributions. We determine
wave height distributions for perturbed “wide” and “narrow” Gaussians with unit amplitude and
variance of $\frac{3L_\xi}{4\pi}$ and $\frac{L_\xi}{8\pi}$ respectively.

1.3.2.2 Numerical Parameters

Numerical experiments are conducted using a pseudospectral scheme in space and the fourth-
order ETDRK4 in time. We integrate the 1D nonlinear Schrödinger equation using the parameters
in Table 1.2, and 2D nonlinear Schrödinger equation and FDS using the parameters in Table 1.3.

Table 1.2: Numerical Parameters for 1D Water Wave Experiments

| Planewave Initial Condition: $N(\xi, Y, \tau = 0) = 1 + \sum_{k=-20}^{20} A_k e^{i\sigma_k x}$ |
| Gaussian Initial Condition: $N(\xi, Y, \tau = 0) = \exp\left(\frac{-x^2}{2s^2}\right) + \sum_{k=-20}^{20} A_k e^{i\sigma_k x}$ |
| Spatial Grid: 1024 points |
| Time Step: $k = .005$, $t \in [0, \frac{1}{.08}]$ |
| Trials: 400000 |

For each trial, the coefficients $A_k$ are randomly chosen, and normalized so that
the maximum magnitude of the perturbation was 1/10. Note that $\sigma_k = \frac{2n}{L_\xi}$. We
consider $s = \frac{3L_\xi}{4\pi}$ and $s = \frac{L_\xi}{8\pi}$. 

Table 1.3: Numerical Parameters for 2D Water Wave Experiments

Planewave Initial Condition: \[ N(\xi, Y, \tau = 0) = 1 + \sum_{k=-20}^{20} \sum_{l=-20}^{20} A_{k,l} e^{i\sigma_k \xi + i\sigma_l Y} \]

Gaussian Initial Condition: \[ N(\xi, Y, \tau = 0) = \exp \left( - \left( \frac{x^2}{2s^2} + \frac{y^2}{2s^2} \right) \right) + \sum_{k=-20}^{20} \sum_{l=-20}^{20} A_{k,l} e^{i\sigma_l \xi + i\sigma_l Y} \]

Spatial Grid: 1024 points \times 1024 points

Time Step: \( k = .01, t \in [0, \frac{1}{30}] \)

Trials: 10000

For each trial, the coefficients \( A_{k,l} \) are randomly chosen, and normalized so that the maximum magnitude of the perturbation was 1/10. Note that \( \sigma_k = \frac{2\pi}{L_x} \) and \( \sigma_l = \frac{2\pi}{L_y} \). We consider \( s = \frac{3L_x}{4\pi} \) and \( s = \frac{L_y}{8\pi} \).
1.3.2.3 Results

Our results are divided into two sections, based on initial conditions:

**Perturbed Planewave Initial Conditions:**

We organize our figures based on domain size and $\epsilon$ values. Figures 1.11, 1.12, and 1.13 contain wave height distributions for domains of width $L_\xi = 8\pi$, 16$\pi$ and 24$\pi$, respectively. Each figure includes plots for 1D NLS, 2D NLS, and 2D FDS integrated out to time $\tau = 1/\epsilon$ for $\epsilon = .12, .10, .08$. Figure 1.17 contains wave height distributions for 1D NLS on a domain of size of 50$\pi$. We do not test 2D equations on this large domain because the numerical simulations become more involved and the effects of domain size are well described by the first three experiments.

All equations admit rogue growth up to four times the initial plane wave amplitude. The resulting wave height distributions are non-Gaussian but are well described by Rayleigh distributions. Larger domains and smaller $\epsilon$ consistently lead to larger mean wave height. This suggests that growth is amplified when the number of unstable modes increases, and when rogue events have more time to form. The PDFs associated with two dimensional equations have somewhat smaller variance and larger mean.

We also consider the time evolution of wave height histograms. In Figure 1.14 we plot the results for $L_\xi = 24\pi$, and $\epsilon = .08$. These plots illustrate the short-time exponential growth due to modulational instability, and the transition to fully nonlinear behavior. We also plot the temporal evolution of the mean and standard deviation of the PDFs in Figure 1.15.

**Gaussian Initial Conditions:**

Figure 1.16 contains plots of perturbed Gaussian initial conditions for 1D and 2D NLS. We pick a domain size of 24$\pi$ and $\epsilon = .08$. We repeat this experiment for 1D NLS on a domain of 50$\pi$,
and present our results in Figures 1.18. For both the 1D and 2D equations, wide Gaussian initial conditions lead to Rayleigh wave height statistics, while narrow Gaussian initial conditions lead to more Gaussian-like statistics. We conclude that initial conditions whose “mass” is distributed throughout spatial domain lead to Rayleigh statistics. In the extreme case where initial conditions are highly localized in space (i.e. the narrow Gaussian) statistics will become more Gaussian. The latter case is not likely to exist in the ocean, where sea states are relatively uniform.
Figure 1.11: Long-time maximum amplitude PDF of the 1D NLS (top row), 2D NLS (middle row), and 2D FDS (bottom row) for $\xi \in [-4\pi, 4\pi]$. We consider perturbed plane wave initial conditions and $\epsilon$ values of .12 (left column), .10 (middle column), and .08 (right column). The black curve indicates a Rayleigh PDF and the red curve denotes the Gaussian PDF.
Figure 1.12: Long-time maximum amplitude PDF of the 1D NLS (top row), 2D NLS (middle row), and 2D FDS (bottom row) $\xi \in [-8\pi, 8\pi]$. We consider perturbed plane wave initial conditions and $\epsilon$ values of .12 (left column), .10 (middle column), and .08 (right column). The black curve indicates a Rayleigh PDF and the red curve denotes the Gaussian PDF.
Figure 1.13: Long-time maximum amplitude PDF of the 1D NLS (top row), 2D NLS (middle row), and 2D FDS (bottom row) $\xi \in [-12\pi, 12\pi]$. We consider perturbed plane wave initial conditions and $\epsilon$ values of .12 (left column), .10 (middle column), and .08 (right column). The black curve indicates a Rayleigh PDF and the red curve denotes the Gaussian PDF.
Figure 1.14: Time evolution of the PDFs governing maximum amplitude for 1D NLS, 2D NLS, and 2D FDS. We consider perturbed plane wave initial conditions, $\xi \in [-12\pi, 12\pi]$, and $\epsilon = .08$. We present two different perspectives of the evolving PDF for each equation.
Figure 1.15: Time evolution of the mean and standard deviation of the PDFs governing maximum amplitude for 1D NLS, 2D NLS, and 2D FDS. We consider perturbed plane wave initial conditions, $\xi \in [-12\pi, 12\pi]$, and $\epsilon = .08$. 

\[ \text{1D NLS Evolution of Max Wave Height Mean} \]

\[ \text{1D NLS Evolution of Max Wave Height Standard Deviation} \]

\[ \text{2D NLS Evolution of Max Wave Height Mean} \]

\[ \text{2D NLS Evolution of Max Wave Height Standard Deviation} \]

\[ \text{2D FDS Evolution of Max Wave Height Mean} \]

\[ \text{2D FDS Evolution of Max Wave Height Standard Deviation} \]
Wide Gaussian \( \left( \sigma = \frac{3L\xi}{4\pi} \right) \)

Figure 1.16: Long-time maximum amplitude PDF of 1D NLS (top rows) and 2D NLS (bottom rows) for \( \xi \in [-12\pi, 12\pi] \). We consider perturbed Gaussian initial conditions with standard distribution \( \sigma \) and \( \epsilon \) values of .12 (right column), .10 (middle column), and .08 (right column). The black curve indicates a Rayleigh PDF and the red curve denotes the Gaussian PDF.
Figure 1.17: Long-time maximum amplitude PDF of the 1D NLS equation and \( \xi \in [-25\pi, 25\pi] \). We consider perturbed plane wave initial conditions and \( \epsilon \) values of .12 (right), .10 (middle, and .08 (left). The black curve indicates a Rayleigh PDF and the red curve denotes the Gaussian PDF.

Wide Gaussian \( \left( \sigma = \frac{3L\xi}{4\pi} \right) \)

Figure 1.18: Long-time maximum amplitude PDF of the 1D NLS and \( \xi \in [-25\pi, 25\pi] \). We consider perturbed Gaussian initial conditions with standard distribution \( \sigma \) and \( \epsilon \) values of .12, .10, .08. The black curve indicates a Rayleigh PDF and the red curve denotes the Gaussian PDF.

Narrow Gaussian \( \left( \sigma = \frac{L\xi}{8\pi} \right) \)
1.4 Conclusions

In this chapter we studied the 1D and 2D nonlinear Schrödinger equations. We found that solutions of 2D NLS spectrally broaden and violate the slow modulation assumption. We introduced the 2D FDS equation which overcomes this limitation by incorporating the full linear dispersion relation for water waves. This simple modification allows us to numerically integrate further in time without turning to higher order equations.

From our numerical studies, we conclude that modulational instability causes growth in the 1D NLS, 2D NLS and 2D FDS equations. We observed rogue waves with four times the initial amplitude in 1D NLS and rogue waves with six times the initial amplitude in the 2D equations. In general, two dimensional effects lead to larger mean wave height and smaller variance. Larger domains also lead to higher mean, but the overall effects are small. The wave height distributions for 1D NLS closely approximate their two dimensional counterparts, and all equations respond similarly to parameter changes. This suggests that studying the one dimensional nonlinear Schrödinger equation plus suitable perturbations may be sufficient for a basic understanding of rogue waves.

In short, there are three main conclusion from our work on rogue waves in the nonlinear Schrödinger equations.

- The one-dimensional nonlinear Schrödinger equation provides a suitable description of rogue waves dynamics in all two dimensional equations (2D NLS, 2D FDS).

- All NLS equations generate Rayleigh-like wave height distributions that are consistent with the statistics of ocean waves.

- Modulational instability leads to rogue growth. Parameters which increase the number of unstable modes lead to wave height statistics with larger mean.
Rogue waves also form in zero-dispersion optical fibers, and were first observed in [22]. This physical system can be modeled using higher-order NLS equations. Moreover, many of the phenomena used to explain rogue waves in water, including currents, winds, focusing, and landmass effects, are not present in optical fibers. These two facts suggest that certain rogue dynamics may originate from nonlinear behavior that can be characterized by NLS-type equations.

In this chapter we present the nonlinear Schrödinger equation for optical fibers, and several higher order perturbations. We reproduce the experiment in [22], and perform stability analysis on an equation governing zero-dispersion fibers. We also perform repeated numerical integration to determine the probability distributions governing power in an optical fiber.

2.1 Optics and the Nonlinear Schrödinger Equation

The nonlinear Schrödinger equation also governs pulse propagation in an optical fiber. Here we provide a brief derivation based on [2] and refer the reader to [3] for a more detailed description.

When light passes through a material, the phase velocity is

\[ v_{\text{phase}} = \frac{c}{n} \]  \hspace{1cm} (2.1)

where \( c \) is the speed light in a vacuum and \( n \) is the index of refraction of the medium. In a fiber, the index of refraction depends linearly on frequency and nonlinearly on the amplitude of the electric
field. Ignoring the effects of polarization, the index of refraction is

\[ n(\omega, \mathbf{E}) = n_0(\omega) + n_2|\mathbf{E}|^2 \]  

(2.2)

where \( \omega \) and \( \mathbf{E} \) are the frequency and the electric field of the light; \( n_0(\omega) \) is the frequency-dependent index of refraction and \( n_2 \) is the Kerr coefficient. This formula is the optics equivalent of the nonlinear dispersion relation for deep water wave trains.

The phase velocity of light can also be written in terms of the frequency and wavenumber in the following manner:

\[ v = \frac{\omega}{k} \]  

(2.3)

Substituting \( v \) with \( c/n \), and using the definition of the index of refraction, we arrive at the nonlinear dispersion relation for an optical fiber:

\[ k(\omega, \mathbf{E}) = \frac{\omega}{c} (n_0(\omega) + n_2|\mathbf{E}|^2) \]  

(2.4)

In this context, we assume a linearly polarized, slowly modulated electric field

\[ E(z, t) = \epsilon A(Z, T)e^{i(k_0 z + \omega_0 t)} + \text{c.c.} \]

where \( A(Z, T) \) is a slowly varying amplitude such that \( Z = \epsilon z, T = \epsilon t \). We Taylor expand Eq. (2.4) about the frequency of the carrier wave, \( \omega_0 \) such that

\[ k - k_0 = k'(\omega_0)(\omega - \omega_0) + \frac{k''(\omega_0)}{2} (\omega - \omega_0)^2 + \frac{\omega_0 n_2}{c} |\mathbf{E}|^2 \]  

(2.5)

where \( k_0 = k(\omega_0) \) and \( ' \) means differentiation with respect to \( \omega \). We assume a slow modulation, i.e. \( \omega \rightarrow \omega_0 + \epsilon\Omega \), and \( k \rightarrow k_0 + \epsilon K \), and let Eq. (2.5) operate on \( A \). Replacing \( K \) and \( \Omega \) with their Fourier representation of \( i\partial/\partial T \) and \( i\partial/\partial Z \), we arrive at

\[ i\epsilon \left( \frac{\partial A}{\partial Z} + k'_0(\omega_0) \frac{\partial A}{\partial T} \right) - \epsilon^2 \frac{k''(\omega_0)}{2} \frac{\partial^2 A}{\partial T^2} + \epsilon^2 \nu |A|^2 A = 0 \]  

(2.6)

where \( \nu = \frac{\omega_0 n_2}{3a_{\text{eff}}} \) where \( a_{\text{eff}} \) is the effective cross-sectional area of the fiber. To obtain the canonical NLS equation, we transform into the retarded reference frame

\[ \tau = T - k'_0(\omega_0)z \]

\[ \zeta = \epsilon Z \]
and divide by $\epsilon^2$ to arrive at the NLS equation

$$i \frac{\partial A}{\partial \zeta} + \frac{\text{sgn}(-k''_0(\omega_0))}{2} \frac{\partial^2 A}{\partial \tau^2} + \nu|A|^2 A = 0$$

(2.7)

where $|A|^2$ represents power. By making the appropriate nondimensionalizations and scalings, this equation can be reduced to NLS in its standard form:

$$iq_z + q_{tt} \pm 2|q|^2 q = 0$$

(2.8)

A positive nonlinear term leads to anomalous dispersion, while a negative nonlinear term leads to normal dispersion.

As described in [3], a higher-order equation governing pulse propagation is:

$$iu_z + \sum_{n=2}^{N} i^n B_n u_{nt} + \gamma_1 |u|^2 u + i \gamma_2 (|u|^2 |u|)_t + \gamma_3 (|u|^2 |u|)_t = 0$$

(2.9)

Eq. (2.9) includes higher order dispersion and three nonlinearities:

- $\gamma_1 |u|^2 u$ models the Kerr nonlinearity, which causes spectral broadening during propagation. Notice that the Kerr nonlinearity is also included in Eq. (2.8).

- $i \gamma_2 (|u|^2 |u|)_t$ models the self-steepening nonlinearity, which causes higher intensity pulses to propagate faster than those lower in intensity.

- $\gamma_3 (|u|^2 |u|)_t$ models the Raman nonlinearity, which also causes spectral broadening due to a transfer of energy from high frequency to low frequency modes.

When $N = 6$, Eq. (2.9) becomes the equation used in [22].

2.2 Introduction to Optical Rogue Waves

Using a novel detector, Solli et al. [22] propagated a monochromatic laser pulse down a zero-dispersion fiber, measured power output, and observed large optical rogue waves. This discovery provided the first evidence of rogue events outside the realm of hydrodynamics. Since 2007, several
other papers exploring rogue waves in other physical systems have been published \[6, 27\]. Here, we provide a brief introduction to Solli’s work and the underlying high-order 1D NLS equations.

In \[22\], a supercontinuum was generated by launching picosecond seed pulses at 1064 nm down 15 m of nonlinear fiber with matching zero-dispersion wavelength. Output was redpass filtered at 1450 nm and fed to a novel detector capable of temporally stretching individual events in order to effectively measure intensity. Normally, output intensity was small, but occasionally, rare events with intensities 30 to 40 times the mean were measured. Intensity histograms revealed an L-shaped distribution suggesting that the probability of rogue events was non-Gaussian.

Experimental results were reproduced numerically using a high-order 1D nonlinear Schrödinger equation. Rogue waves dynamics were observed when modeling perturbed Gaussian initial conditions. Small perturbations with amplitude 0.1 percent the instantaneous pulse amplitude were found to be sufficient to trigger rogue waves. It was suggested that an effect like modulational instability may be responsible for exponentially amplifying small perturbations in the Gaussian pulse. In the same way that initial noise causes a periodic train of water waves to rapidly deteriorate, it is possible that the optical pulse mixed with several low-amplitude light rays as it entered the fiber, causing instability and rogue behavior.
2.2.1 Nondimensionalization

The dimensional equation used in [22] is:

\[
\frac{\partial A}{\partial z} - i \sum_{m=2}^{6} \frac{i^m B_m}{m!} \frac{\partial^m A}{\partial \tau^m} = i \gamma \left[ |A|^2 A + \frac{i}{w_0} \frac{\partial}{\partial \tau} (|A|^2 A) - T_r A \frac{\partial |A|^2}{\partial \tau} \right]
\]

(2.10)

| \(B_2\) | 0 |
| \(B_3\) | \(7.67 \times 10^{-5}\) ps\(^3\)m\(^{-1}\) |
| \(B_4\) | \(-1.37 \times 10^{-7}\) ps\(^4\)m\(^{-1}\) |
| \(B_5\) | \(3.61 \times 10^{-10}\) ps\(^5\)m\(^{-1}\) |
| \(B_6\) | \(5.06 \times 10^{-13}\) ps\(^6\)m\(^{-1}\) |
| \(\gamma\) | .011 W\(^{-1}\)m\(^{-1}\) |
| \(T_R\) | .005 ps |
| \(\omega_0\) | \(1.7782 \times 10^{15}\) Hz |

This partial differential equation models the slowly varying electric field envelope. \(|A(\tau, z)|^2\) has units of Watts, \(\tau\) is measured in picoseconds (ps) and \(z\) has units of meters (m). The coefficient \(B_m\) describe dispersion and are in units of \(\frac{ps^n}{m}\). \(\gamma\) governs the strength of the nonlinearity, \(T_r\) measures the Raman response, and \(\omega_0\) is the central carrier frequency. All coefficients were taken directly from [22] with the exception of \(\omega_0\), which we calculate below.

2.2.1.1 Calculating \(\omega_0\)

We are modeling a pulse with wavelength \(\lambda_0 = 1064\)nm. The central carrier frequency \(\omega_0\), can be calculated via \(\omega_0 = k_0 c\) where \(k_0 = \frac{2\pi}{\lambda_0}\) is the angular wave number of the pulse and \(c\) is the speed of light. Hence:

- The frequency \(w_0 = \frac{2\pi c}{\lambda_0} \approx 1778.2\) ps\(^{-1}\) = \(1.7782 \times 10^{15}\) Hz
- The period \(T = \frac{2\pi}{w_0} \approx 3.53 \times 10^{-15}\)s.
2.2.1.2 Nondimensionalization

We rearrange Eq. (2.10) and nondimensionalize by defining new variables \( A' = \frac{A}{\alpha} \), \( z' = \frac{z}{\beta} \), and \( t' = \frac{t}{\delta} \) to obtain

\[
i \frac{\partial A'}{\partial z'} + \sum_{m=2}^{6} \frac{\beta}{\delta^m} \frac{i^m B_m}{m!} \frac{\partial^m A'}{\partial t'^m} + \gamma \alpha^2 \beta \left[ |A'|^2 A' + \frac{i}{\omega_0} \frac{1}{\delta} \frac{\partial}{\partial t'} (|A'|^2 A') - T_{r} \frac{A'}{\delta} \frac{\partial |A'|^2}{\partial t'} \right] = 0 \tag{2.11}
\]

For convenience we set

\[
\frac{\beta B_3}{6\delta^3} = \text{sgn}(B_3) \quad \gamma \alpha^2 \beta = \text{sgn}(\gamma)
\]

We choose \( \alpha = \sqrt{P_0} \), where \( P_0 \) is the peak power of the initial optical pulse. We then solve for \( \beta \) and \( \delta \) to obtain:

\[
\beta = \left| \frac{1}{P_0 \gamma} \right| \quad \delta = \sqrt[3]{\left| \frac{B_3}{6P_0 \gamma} \right|}
\]

In [22], a 150W Gaussian pulse with width of 3ps was send down 15m of fiber. We set \( P_0 = 150 \) to obtain the equation

\[
i \frac{\partial A}{\partial z'} - iu_{tt'} + \sum_{m=4}^{6} \frac{\beta}{\delta^m} \frac{i^m B_m}{m!} \frac{\partial^m A}{\partial t'^m} + |A|^2 A + \frac{i}{\omega_0} \frac{1}{\delta} \frac{\partial}{\partial t'} (|A|^2 A) - T_{r} \frac{A}{\delta} \frac{\partial |A|^2}{\partial t'} = 0 \tag{2.12}
\]

where:

\[
\frac{\beta B_4}{\delta^4} \approx -2.256 \times 10^{-2} \quad \frac{\beta B_5}{\delta^5} \approx 5.992 \times 10^{-4} \quad \frac{\beta B_6}{\delta^6} \approx 7.069 \times 10^{-6} \quad \frac{1}{\omega_0^6} \approx 0.0284 \quad T_{r} \approx 0.2525
\]

The dimensional experimental parameters \( z = [0, 15m] \), \( t_{\text{width}} = 3\text{ps} \), map to the following nondimensional variables

\[
z' = \frac{15m}{\beta} = \frac{15}{0.6061} \approx 24.75
\]

\[
t'_{\text{width}} = \frac{3\text{ps}}{\delta} = \frac{3}{0.0198} \approx 151.61
\]

2.2.1.3 Remarks

Eq. (2.12) can be interpreted as perturbation of the nonlinear zero-dispersion optics equation

\[
i u_z - i u_{tt} + |u|^2 u = 0 \tag{2.13}
\]
From here on, we refer to Eq. (2.13) as the zero-dispersion Schrödinger equation (ZDS). This equation is commonly used to model pulse propagation in zero-dispersion [3]. Rather than analyze growth in Eq. (2.12), we choose to study the stability of the ZDS equation.

### 2.3 Stability Analysis for the ZDS Equation

We calculate the linear stability of Eq. (2.13), about the plane wave solution \( u(z,t) = ae^{ia^2z} \).

We substitute the perturbed solution \( u_p(z,t) = ae^{ia^2z}(1+\epsilon(z,t)) \) into (2.13) and linearize to obtain:

\[
i\epsilon_z(z,t) + i\epsilon_{ttt}(z,t) + a^2\epsilon^*(z,t) + a^2\epsilon(z,t) + O(\epsilon^2) = 0
\]  
(2.14)

We split (2.14) into real and imaginary parts such that

\[
-d\frac{d\epsilon_I}{dz} - \frac{d^3}{dt^3}\epsilon_I + 2a^2\gamma\epsilon_R = 0
\]  
(2.15)

\[
d\frac{d\epsilon_R}{dz} + \frac{d^3}{dt^3}\epsilon_R = 0
\]  
(2.16)

where \( \epsilon(z,t) = \epsilon_R(z,t) + i\epsilon_I(z,t) \). We assume

\[
\epsilon_R(z,t) = A_{k,1}(z)e^{i\sigma_k t} + c.c.
\]

\[
\epsilon_I(z,t) = A_{k,2}(z)e^{i\sigma_k t} + c.c.
\]

where \( \sigma_k = \frac{2\pi k}{L_L} \). Substituting this ansatz into (2.15) and (2.16) leads to the system:

\[
\frac{d}{dz} \begin{pmatrix} A_{k,2} \\ A_{k,1} \end{pmatrix} = \begin{pmatrix} i\sigma_k^3 & 2a^2 \\ 0 & i\sigma_k^3 \end{pmatrix} \begin{pmatrix} A_{k,2} \\ A_{k,1} \end{pmatrix}
\]  
(2.17)

Solving this linear system, we find

\[
A_{k,2}(z) = \exp(i\sigma_k^3 z) + 2a^2 z \exp(i\sigma_k^3 z)
\]  
(2.18)

\[
A_{k,1}(z) = \exp(i\sigma_k^3 z)
\]

\( A_{k,1}(t), A_{k,2}(t) \) will not grow exponentially; however, \( A_{k,2}(t) \) grows linearly in time for all \( k \). The ZDS equation is therefore modulationally stable.
2.3.1 Numerical Experiments

We numerically model Eq. (2.13) to determine whether linear growth can trigger nonlinear instabilities. We consider the initial conditions:

\[ u(z = 0, t) = 1 + \frac{1}{100} e^{i k t} \quad k = 1, 2, \ldots, 6 \quad (2.19) \]

on a domain of width \( L_t = 8\pi \). Our numerical results are summarized in Figures 2.1 and 2.2. Perturbations containing modes where \( k = 1, 2 \) cause significant growth in the solution, while initial conditions where \( k > 2 \) do not. We determine the number of growing modes using different domain sizes and different initial conditions. Our results are summarized in Table 2.1. In short, we find that larger domains and larger initial amplitudes lead to more unstable modes. Remarkably, this is corresponds exactly with the behavior exhibited by modulational instability in the standard 1D NLS equation.

<table>
<thead>
<tr>
<th>( L_t )</th>
<th>Unstable ( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24\pi</td>
<td>1, 2, 3, 4, 5, 6</td>
</tr>
<tr>
<td>16\pi</td>
<td>1, 2, 3, 4</td>
</tr>
<tr>
<td>8\pi</td>
<td>1, 2</td>
</tr>
<tr>
<td>4\pi</td>
<td>1</td>
</tr>
<tr>
<td>2\pi</td>
<td>\emptyset</td>
</tr>
</tbody>
</table>

Table 2.1: ZDS Unstable Wave Numbers \( k \) For Various Domain Sizes

Unstable wave numbers \( k \) for Eq. (2.13) using a various domain sizes \( L_t \) and Eq. (2.19) as the initial condition.

We also numerically determine the amplitude PDF associated with the ZDS equation. We consider domains of width \( L_t = 8\pi, 16\pi \) and \( 24\pi \), and integrate to nondimensional distance \( z = 20 \). Table 2.2 highlights our numerical parameters. In our numerical experiments, we consider the amplitude \( |u|^2 \) which corresponds to power in optics. We present the results in Figure 2.3. The ZDS equation admits optical rogue waves with powers up to 10 times the initial value. PDFs for the
8π domain lead to Gaussian statistics, while PDFs for the 24π domain are Rayleigh distributed. As domain size increases, there are more unstable modes and the distribution becomes more Rayleigh-like. For each domain size, we plot the evolution of the amplitude PDF for each domain size in Figure 2.4, and the evolution of the amplitude mean and standard deviation in Figure 2.5. Notice that on the spatial scales we study, the amplitude PDFs have not converged to a final state. It would be interesting to repeat this experiment using longer z distance.

Table 2.2: Numerical Parameters for 1D ZDS Experiments

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Condition:</td>
<td>( u(z = 0, t) = 1 + \sum_{k=-20}^{20} A_k e^{i\sigma_k t} )</td>
</tr>
<tr>
<td>Spatial Grid:</td>
<td>1024 points</td>
</tr>
<tr>
<td>Time Step:</td>
<td>( k = .005, t \in [0, 20] )</td>
</tr>
<tr>
<td>Trials:</td>
<td>400000</td>
</tr>
</tbody>
</table>

For each trial, the coefficients \( A_k \) are randomly chosen, and normalized so that the maximum magnitude of the perturbation is 1/10. Note that \( \sigma_k = \frac{2\pi l}{L_t} \) where \( L_t \) represents width of the temporal domain.
Figure 2.1: Numerical solutions of the ZDS equation in physical space, using perturbed plane wave initial conditions on a periodic domain with width $8\pi$. Perturbations consist of a single Fourier mode with wavenumber $k$ and amplitude $1/100$. For $k = 1, 2$ a nonlinear instability causes growth, even though the linearization is stable.
Figure 2.2: Numerical solutions of the ZDS equation in Fourier space, using perturbed plane wave initial conditions on a periodic domain with width $8\pi$. Perturbations consist of a single Fourier mode with wavenumber $k$ and amplitude $1/100$. For $k = 1, 2$ a nonlinear instability causes growth even though the linearization is stable.
Figure 2.3: Long-time maximum power PDF of the 1D ZDS equation for domain sizes of \(8\pi\) (top), \(16\pi\) (middle), and \(24\pi\) (bottom) and perturbed plane wave initial conditions. The black curve indicates a Rayleigh PDF and the red curve denotes the Gaussian PDF.
Figure 2.4: Time evolution of the PDFs governing maximum amplitude for ZDS for domain sizes of $8\pi$ (top), $16\pi$ (middle), $24\pi$ (bottom). In all cases, we consider perturbed plane wave initial conditions, where perturbation has a maximum amplitude of $1/10$. We present two different perspectives of the evolving PDF for each domain size.
Figure 2.5: Time evolution of the mean and standard deviation of the PDFs governing maximum amplitude for the 1D ZDS equation for domain sizes of $8\pi$ (top), $16\pi$ (middle), and $24\pi$ (bottom). We consider perturbed plane wave initial conditions, and $\epsilon = .08$. 


2.3.2 Conclusions

In this chapter we briefly considered rogue waves in equations governing optical pulses. We find that the equation used by Solli et al. is a perturbation of the modulationally stable zero-dispersion Schrödinger equation. Remarkably, this equation admits unstable plane wave solutions that lead to rogue growth. This behavior is identical to that seen in the modulationally unstable NLS equation governing water waves. However, unlike NLS, we believe that growth in the zero-dispersion equation is due to a nonlinear interaction between linearly growing perturbations and the dominant zeroth mode. Although growth in the zero-dispersion equation is governed by a different instability, we find that wave height statistics on large domains are still Rayleigh-like.

In short, there are four main conclusions to draw from our work on optical rogue waves:

- Modulational instability is not the only phenomenon that causes rogue growth in NLS type equations.
- Rogue waves in zero-dispersion optical fibers are likely caused by a nonlinear instability, rather than modulational instability.
- There could a larger class of nonlinear equations that lead to Rayleigh-like wave statistics.
- The number of unstable modes influences wave height statistics.
Chapter 3

Numerical Methods

In this chapter, we describe the numerical schemes used. To complete our numerical studies, we require an efficient numerical scheme capable of integrating stiff, nonlinear equations. We tested several different methods in order to determine which was the most efficient for our problem.

3.1 Equation Structure

We are interested in numerically integrating equations of form

\[ u_t + L[u] + N[u] = 0 \]  \hspace{1cm} (3.1)

where \( L \) is a linear operator and \( N \) is a nonlinear operator. We consider numerical schemes that exploit the linear-nonlinear structure when the operator \( L \) contains stiff terms.

3.2 Approximating Spatial Derivatives

There are several well-known techniques for approximating spatial derivatives. On periodic domains, the discrete Fourier transform (DFT) can be used to accurately approximate derivatives of smooth functions. The formula for the forwards and inverse discrete transform for an even number of grid points points is

\[ a_k = h \sum_{j=-N/2}^{N/2} e^{-ikx_j} f(x_j) \]  \hspace{1cm} (3.2)

\[ f(x_j) = \frac{1}{2\pi} \sum_{k=-N/2}^{N/2} a_k e^{ikx_j} \]  \hspace{1cm} (3.3)
where the prime indicates that the terms \( k, j = \pm \frac{N}{2} \) are multiplied by \( \frac{1}{2} \). Once a set of function values are transformed into Fourier space, differentiation becomes a linear operation:

\[
 f^{(m)}(x_j) = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} (ik)^m a_{k,m} e^{ikx_j}
\]  

(3.4)

Using the fast Fourier transform, a DFT or IDFT can be evaluated in \( O(N\log(N)) \) time. To highlight the simplicity of this approach, we provide Matlab code for evaluating derivatives spectrally on grid with an even number of points. For simplicity, the wavenumber \( k = \frac{N}{2} \) is simply filtered out.

**Sample Matlab Code for Spectral Differentiation**

```matlab
function df=derivative(f,m,Lxd)
    Nx = length(f);
    k = [0:Nx/2 -Nx/2+1:-1] * (2*pi/(Lxd));
    fhat = fft(f); fhat(Nx/2 + 1) = 0;
    df = ifft((1i*k).^m.*fhat);
end
```

### 3.3 Method of Lines in Fourier Space

We choose to solve Eq. (3.1) in Fourier space where differentiation matrices are diagonal. We discretize space using an equispaced grid, and use method of lines (MOL) to integrate in time. Using these techniques, Eq. (3.1) becomes

\[
 \dot{u}_t + \hat{L} \dot{u} + \hat{N}(\dot{u}, t) = 0
\]  

(3.5)

where \( \hat{L} \) and \( \hat{N} \) are \( N \times N \) matrices. In other words, Eq. (3.5) is now a set of coupled differential equations. We can apply this technique to Eq. (1.38) to obtain

\[
 \hat{L}_{i,j} = \begin{cases} 
 i \frac{k^2}{2} & \text{if } i = j \\
 0 & \text{if } i \neq j
\end{cases}
\]

\[
 \hat{N}(\dot{u}, t) = -iF(|u|^2 u) \text{ where } u = F^{-1}\dot{u}
\]
where \( k \in \left[ \frac{L_{\xi}}{2\pi} \left[ \frac{-N}{2} + 1, \frac{N}{2} \right] \right], \) and \( L_{\xi} \) is the width of the periodic domain. Notice that for large \( N, \) the problem becomes stiff.

### 3.4 Numerical Schemes for Integrating in Time

We explore three principal approaches for solving Eq. (3.5): implicit-explicit schemes (IMEX), sliders, and ETD methods. In this section, we provide a brief description of each method.

#### 3.4.1 IMEX

IMEX schemes integrate Eq. (3.5) by treating the linear term implicitly and the nonlinear term explicitly, such that

\[
u(t_{n+1}) - u(t_n) = \text{Implicit Approx. for } \int_{t_n}^{t_{n+1}} \mathcal{L}u + \text{Explicit Approx. for } \int_{t_n}^{t_{n+1}} \mathcal{N}(u)
\]

This technique can be applied to solve any PDE where the linear term is stiffer than the nonlinear term. One of the simplest IMEX schemes is Crank-Nicolson Adams-Bashforth (CNAB). The method can be written as

\[
u^{n+1} = \left(1 - \frac{hL}{2}\right)^{-1} \left[ u^n + \frac{3h}{2} \mathcal{N}(u^n) - \frac{h}{2} \mathcal{N}(u^{n-1}) + \frac{hLu^n}{2} \right]
\]

(3.6)

where \( h \) denotes time step. We can derive CNAB by considering the ODE \( \dot{u} = f(u) \), the second-order Crank-Nicolson approximation

\[
u^{n+1} - u^n = \frac{hf(u^{n+1}) + hf(u^n)}{2}
\]

and the second-order Adams-Bashforth approximation

\[
u^{n+1} - u^n = \frac{3h}{2} f(u^n) - \frac{h}{2} f(u^{n-1})
\]

These two methods can be combined to integrate Eq. (3.5) such that

\[
u^{n+1} - u^n = \left( \frac{3h}{2} \mathcal{N}(u^n) - \frac{h}{2} \mathcal{N}(u^{n-1}) \right) + \left( \frac{hLu^{n+1} + hLu^n}{2} \right)
\]
Solving for $u^{n+1}$ and simplifying leads to Eq. (3.6). The cost of inverting the matrix $L$ should also be considered. However, all the PDEs in this thesis lead to diagonal $\hat{L}$, and inversion is therefore trivial.

Different IMEX schemes can be derived from various integration formulas. We chose to implement CNAB and an improved fourth order equivalent (SBDF4) which uses fourth-order Adams-Bashforth for the nonlinear term and fourth-order backwards differences for the linear term. SBDF4 was first proposed in [24] and found to perform significantly better than CNAB. The formula for SBDF4 is given by:

$$u^{n+1} = (25 - 12hL)^{-1}(48u^n - 36u^{n-1} + 16u^{n-2} - 3u^{n-3} + 48hN(u^n)$$

$$- 72hN(u^{n-1}) + 48hN(u^{n-2}) - 12hN(u^{n-3}))$$

### 3.4.2 Sliders

Sliders use different methods for different ranges of Fourier space. Eq. (3.5), contains ODEs governing both slow and fast wavenumbers. Slow wavenumbers are not stiff and can be integrated using fully explicit, high-order methods. In fact, it is only necessary to use an A-stable or L-stable method for integrating fast wavenumbers. By dividing the Fourier space appropriately, it is possible to simultaneously use multiple numerical schemes.

We implement the LI4 slider method developed in [9]. LI4 splits Fourier in three ways: slow wave numbers are treated with fourth-order Adams-Bashforth (AB4), medium wave numbers are treated with a fourth-order AB4/AM6 IMEX scheme, and fast wave numbers are treated with a second-order IMEX scheme based on AB4 and a specially modified AM2. LI4 can be schematically written as:

**AB4/AB4:**

$$u_{n+1} = u_n + \frac{h}{24} (-9Lu_{n-3} + 37Lu_{n-2} - 59Lu_{n-1} + 55Lu_n)$$

$$+ \frac{h}{24} (-9N(u_{n-3}) + 37N(u_{n-2}) - 59N(u_{n-1}) + 55N(u_n))$$
Table 3.1: LI4 Fourier Space Splitting

| $|K|$ | Low | Medium | High $|K|$ |
|------|-----|--------|--------|
| AB4/AB4 | — | AB4/AM6 | — | AB4/AM2* |

Cutoff | Cutoff
---|---
$|K| = \frac{43}{h}$ | $|K| = \frac{1.36}{h}$

AB4/AM6:

$$u_{n+1} = \left[1 - h \frac{725}{1440} L\right]^{-1} \left[\frac{1}{1440} h (-146L u_{n-3} + 482L u_{n-2} - 798L u_{n-1} + 1427L u_{n}) + \frac{h}{24} (-9N(u_{n-3}) + 37N(u_{n-2}) - 59N(u_{n-1}) + 55N(u_{n}))\right]$$

AB4/AM2*:

$$u_{n+1} = u_n + \frac{h}{2} \left(\frac{3}{2} L(u_{n+1}) + \frac{1}{2} L(u_{n-1})\right)$$

We also consider the Runge-Kutta slider method CRK43 developed in [8]. CRK43 treats slow wavenumbers with traditional fourth-order Runge-Kutta, and fast wavenumbers with an L-stable third-order RK method. To use CRK43, Eq. (3.5) must be partitioned into fast wave numbers ($y$) and slow wavelengths ($z$) such that:

$$y_t = \hat{N}(y(t), z(t)) + \hat{L}y;$$

$$z_t = \hat{G}(y(t), z(t))$$

The solution is then integrated using a composite Runge-Kutta method. We avoid writing out the entire method and refer the reader to p.361 in [8].

3.4.3 ETD Methods

ETD multistep methods can be derived by rewriting Eq. (3.5) in integral form. For simplicity, we consider the ordinary differential equation

$$u_t = ku + N(u, t)$$
Using integrating factors, the solution can be written in integral form as

\[
    u(t_{n+1}) = u(t_n) e^{kh} + e^{kh} \int_{t_n}^{t_{n+1}} e^{-kt} N(u(t), t) dt
\]

where \( h = t_{i+1} - t_i \). Next, let \( P_D(t) \) be the Lagrange interpolating polynomial of degree \( D \) which passes through the points

\[
    \{t_j, N(u(t_j), t_j)\}_{j=n-D}^n
\]

We can approximate Eq. (3.7) by replacing \( N(u,t) \) with \( P_D(t) \) to obtain

\[
    u_{n+1} \approx u_n e^{kh} + e^{kh} \int_{t_n}^{t_{n+1}} e^{-kt} P_D(t) dt
\]

where \( u_n = u(t_n) \). This methodology leads to the ETD multistep schemes which have been extensively studied in [5]. The ETD Euler method can be derived by taking \( D = 0 \) such that

\[
    u_{n+1} = u_n e^{kh} + e^{kh} - \frac{1}{k} N(u_n, t_n)
\]

Multistep ETD methods have been superseded by ETD Runge-Kutta methods [7]. These new schemes provide greater stability, better truncation error, and, like traditional Runge-Kutta methods, they can be constructed from the corresponding Euler method. We highlight the derivation of an ETDRK2 scheme:

- Let \( a_n \) denote the result from the Euler ETD scheme
  
  \[
    a_n = u_n e^{kh} + e^{kdt} - \frac{1}{k} N(u_n, t_n)
  \]

- Let \( p_1(t) \) be the Lagrange interpolating polynomial that passes through the points \((t_n, N(u_n))\) and \((t_{n+1}, a_n)\) such that
  
  \[
    p_1(t) = \left( \frac{N(a_n, t_{n+1}) - N(u_n, t_n)}{t_{n+1} - t_n} \right) t + \left( N(a_n, t_{n+1}) - \frac{N(a_n, t_{n+1}) - N(u_n, t_n)}{t_{n+1} - t_n} \right)
  \]

- We arrive at an ETDRK2 scheme by substituting \( p_1(t) \) into (3.8) and simplifying to obtain:

  \[
    a_n = u_n e^{kh} + \frac{e^{kdt} - 1}{k} N(u_n, t_n)
  \]

  \[
    u_{n+1} = a_n + \frac{N(a_n, t_{n+1})}{hk^2} - \frac{N(u_n, t_n)(e^{kh} - 1 - kh)}{hk^2}
  \]
ETDRK methods up to fifth-order have been developed. The ETDRK4 scheme \[7\] can be written as:

\[
a_n = e^{kh/2}u_n + \frac{(e^{kh/2} - I)}{k} N(u_n, t_n)
\]

\[
b_n = e^{kh/2}u_n + \frac{(e^{kh/2} - I)}{k} N(a_n, t_n + h/2)
\]

\[
c_n = e^{kh/2}a_n + \frac{(e^{kh/2} - I)}{k} (2N(b_n, t_n + h/2) - N(u_n, t_n))
\]

\[
u_{n+1} = e^{kh/2}u_n + h^{-2}k^{-3}\left\{[-4 - kh + e^{kh}(4 - 3kh + (kh)^2)]N(u_n, t_n) + \\
2[2 + kh + e^{kh}(-2 + kh)](N(a_n, t_n + h/2) + N(b_n, t_n + h/2)) + \\
[-4 - 3kh - (kh)^2 + e^{kh}(4 - kh)]N(c_n, t_n + h)\right\}
\]

It has been shown that ETDRK4 is a competitive scheme for approximating PDEs of form Eq. (3.5) \[13, 7, 11\].

### 3.4.4 Evaluating ETDRK4 Coefficients

The ETDRK4 scheme utilizes the functions

\[
f_0(k) = \frac{e^{kh/2} - 1}{hk}
\]

\[
f_1(k) = \frac{e^{kh}(4 - 3hk + (hk)^2) - kh - 4}{(hk)^3}
\]

\[
f_2(k) = \frac{e^{kh}(kh - 2) + hk + 2}{(hk)^3}
\]

\[
f_3(k) = \frac{e^{kh}(4 - hk) - (hk)^2 - 3kh - 4}{(hk)^3}
\]

In each case, there is a removable singularity at \(k = 0\). However, these functions are prone to catastrophic rounding error if evaluated directly for \(k \ll 1\). Kassam and Trefethen proposed to overcome this problem by evaluating each \(f_i\) using Cauchy’s integral formula. For any analytic function \(f(z)\), Cauchy’s integral formula states that

\[
f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz
\]

Using the transformation \(z = Re^{i\theta} + a\) we obtain

\[
f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta} + a) d\theta
\]
We can now approximate this integral by discretizing $\theta \in [0, 2\pi]$ on an equispaced grid of $N$ points which we denote by $\theta_i$. Moreover, because $f(Re^{i\theta}+a)$ is periodic in $\theta$, the trapezoidal rule converges exponentially. The trapezoidal rule approximates the integral via:

$$f(a) \approx \frac{1}{N} [f_1 + f_2 + \ldots + f_{N-1}]$$

where $f_i = f(Re^{i\theta_i} + 1)$

This approximation is computationally fast, and was shown to work effectively for ETDRK4 [13].

We provide sample Matlab code to demonstrate this simplicity of evaluating these coefficients.

```matlab
function [f0, f1, f2, f3] = evalETDRK4Coefficients(ks,h)
Np = 256;
P = exp(2i*pi*(0:(Np-1) + .5)/Np);
for i=1:length(ks)
    k = P + h*ks(i);
    f0(i) = mean((exp(k/2)-1)./k);
    f1(i) = mean((-4 - k + exp(k).* (4-3*k + k.^2))./k.^3);
    f2(i) = mean((2 + k + exp(k).*(-2+k))./k.^3);
    f3(i) = mean((-4 - 3*k - k.^2 + exp(k).*(4-k))./k.^3);
end
end
```

### 3.5 Remarks

We find that the performance of each scheme is consistent with [13, 11]. The two most accurate schemes are CRK43 and ETDRK4. IMEX schemes were the worst performing methods, requiring extremely small step sizes for large grids, and LI4 was not competitive with either ETDRK4 or CRK43.
Bibliography


A.1 Reduction of Navier-Stokes To Euler Water Equations

We begin with the Euler Water wave Equations:

\[ \nabla \cdot v = 0 \]  
(A.1)

\[ \frac{\partial v}{\partial t} + (v \cdot \nabla) v = \frac{1}{\rho_0} (F - \nabla P) \]  
(A.2)

where each term has the following physical interpretation:

- \( \frac{\partial v}{\partial t} \) is acceleration
- \( (v \cdot \nabla) v \) is the convective acceleration.
- \( F \) represents external forcing; if \( F \) is conservative, it can be represented as the gradient of a scalar potential: \( F = \nabla U \)
- \( \nabla P \) is the pressure gradient.

We will now show that inside the fluid region, the equations governing an ideal fluid reduce to Laplace’s equation. We begin by assuming that all external forces are conservative such that \( F = \nabla U \), and rewrite the momentum equation:

\[ \frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla \left( \frac{U + P}{\rho_0} \right) \]  
(A.3)

Next, using the vector identity

\[ (v \cdot \nabla) v = \frac{1}{2} \nabla (v \cdot v) - v \times (\nabla \times v) \]
we rewrite the convective acceleration term, resulting in
\[ \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \frac{U + P}{\rho_0} \right) \]
We define the vorticity to be \( \omega = \nabla \times \mathbf{v} \). In physical terms, the vorticity \( \omega \) represents the local spin of an infinitesimal fluid element. We proceed by taking the curl of the last equation to obtain
\[
\frac{\partial \omega}{\partial t} - \nabla \times (\mathbf{v} \times \omega) = 0 \quad (A.4)
\]
and use the vector identity
\[
\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G} + (\nabla \cdot \mathbf{G}) \mathbf{F} - (\nabla \cdot \mathbf{F}) \mathbf{G}
\]
to obtain
\[
\frac{\partial \omega}{\partial t} = (\omega \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \omega
\]
or equivalently
\[
\frac{D \omega}{Dt} = (\omega \cdot \nabla) \mathbf{v} \quad (A.5)
\]
where
\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)
\]
is the convective derivative, which measures the rate of change following a fluid particle with velocity \( \mathbf{v} \). Eq. (A.5) is known as the vorticity equation. It can be shown that when the fluid is initially irrotational, then \( \omega = 0 \) for all time. Under these conditions, we can express fluid velocity \( \mathbf{v} \) as the gradient of a velocity potential such that \( \mathbf{v} = \nabla \phi \). We could not make this assumption if the fluid was not initially irrotational, since the curl of the gradient vanishes and Eq. (A.4) would be trivially satisfied. However, since we assume zero initial vorticity, Eq. (A.1) becomes
\[
\nabla \cdot \mathbf{v} = \nabla \cdot \nabla \phi = \Delta \phi = 0 \quad (A.7)
\]
Eq. (A.7) is valid for \(-h < z < \eta(x, t, y)\) where \(\eta(x, y, t)\), denotes denotes the free surface height above \( z = 0 \). We now discuss the boundary conditions associated with the Euler water wave
equations. We assume a flat, impenetrable bottom at \( z = -h \) leading to the condition

\[
\frac{\partial \phi}{\partial z} = 0, \quad z = -h
\]  
(A.8)

There are two additional boundary conditions that must be imposed on the free surface \( z = \eta(x, t, y) \).

The first can be derived by rewriting Eq. (A.3) in terms of \( \phi \) to obtain.

\[
\nabla \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} \| \nabla \phi \| ^2 + \frac{U + P}{\rho_0} \right) = 0
\]

Notice that the term \( v \times (\nabla \times v) \) in Eq. (A.3) disappears since \( (\nabla \times \nabla \phi) = 0 \). Therefore, since the gradient acts only on spatial variables,

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} \| \nabla \phi \| ^2 + \frac{U + P}{\rho_0} = f(t)
\]

where \( f(t) \) is an arbitrary function that depends only on \( t \). We make the substitution

\[
\phi = \phi' + \int_{t'}^t f(t') dt'
\]

To obtain the system

\[
\frac{\partial \phi'}{\partial t} + \frac{1}{2} \| \nabla \phi' \| ^2 + \frac{U + P}{\rho_0} = 0
\]

Notice that fluid velocity \( v \) can be obtained from either \( \phi \) or \( \phi' \) since \( \nabla \phi = \nabla \phi' \). Finally, we make the following assumptions

(1) Neglect the effects of surface tension.

(2) Assume that all external forcing is due to the buoyant force, s.t. \( F = -\nabla (\rho_0 g z) \), \( U = \rho_0 g z \).

(3) Assume pressure at the free surface is zero, such that \( P = 0 \).

These assumptions lead us to the Bernoulli equation:

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} \| \nabla \phi \| ^2 + g\eta = 0, \quad z = \eta(x, y, t)
\]

For the second free surface condition, we assume that a fluid packet that is on the free surface stays on the free surface. It is not sufficient to assume that

\[
\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t}
\]
since fluid may be moving in the $x$ and $y$ directions. We must instead consider the convective derivative. We define the function

$$F(x, y, z, t) = z - \eta(x, y, t)$$

then

$$\frac{DF}{Dt} = \frac{Dz}{Dt} - \frac{D\eta}{Dt} = \frac{\partial \phi}{\partial z} - \left( \frac{\partial \eta}{\partial t} + (\mathbf{v} \cdot \nabla)\eta \right)$$

On the free surface $z = \eta(x, y, t)$, therefore $F(x, y, z, t) = 0$ which leads to the second boundary condition, also called the kinematic condition

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + (\mathbf{v} \cdot \nabla)\eta \quad z = \eta(x, y, t) \quad \text{(A.9)}$$

In summary, the free-surface water wave equations with a flat bottom are

$$\begin{align*}
\Delta \phi &= 0 \quad -h < z < \eta(x, y, t) \quad \text{(Laplace’s Equation)} \quad \text{(A.10)} \\
\frac{\partial \phi}{\partial z} &= 0 \quad z = -h \quad \text{(No Flow Through Bottom)} \quad \text{(A.11)} \\
\frac{\partial \phi}{\partial t} + \frac{1}{2} ||\nabla \phi||^2 + g\eta &= 0 \quad z = \eta(x, y, t) \quad \text{(Bernoulli’s Equation)} \quad \text{(A.12)} \\
\frac{\partial \phi}{\partial z} &= \frac{\partial \eta}{\partial t} + \nabla \phi \cdot \nabla \eta \quad z = \eta(x, y, t) \quad \text{(Kinematic Condition)} \quad \text{(A.13)}
\end{align*}$$
A.2 Solving First and Second Order Equations

First Order, $O(\epsilon)$

We consider the $O(\epsilon)$ equations. From Eq. (1.30) we find that wave height and velocity potential are related to leading order via

$$A_1 = -\frac{i}{\omega}N_1$$  \hspace{1cm} (A.14)

Substituting $\phi^{(0)}$ and $\eta^{(0)}$ into Eq. (1.26) and Eq. (1.27) and applying Eq. (A.14) to nonsecular terms, we obtain the system

$$\dot{\phi}^{(1)} + \eta^{(1)} = |k|N_1^2e^{2i\theta} - A_{1,T}e^{i\theta} + \text{c.c.}$$

$$\eta^{(1)}_t - \phi^{(1)}_z = -\frac{2i}{|k|}e^{2i\theta} - N_{1,T}e^{i\theta} + \text{c.c.}$$

To avoid secularities, we set $A_T = N_{1,T} = 0$, and assume $\phi^{(1)}$ and $\eta^{(1)}$ have a solution of form

$$\phi^{(1)} = A_2e^{2i\theta + 2|k|} + \text{c.c.}$$

$$\eta^{(1)} = N_2e^{2i\theta + 2|k|} + \text{c.c.}$$

these Ansatz lead to the system

$$\begin{bmatrix}
-2i\omega & 1 \\
-2|k| & -2i\omega
\end{bmatrix}
\begin{bmatrix}
A_2 \\
N_2
\end{bmatrix}
= 
\begin{bmatrix}
|k|N_1^2 \\
-\frac{2i}{\omega}k^2N_1^2
\end{bmatrix}
$$

Substituting $\omega = |k|$, and solving this system we find that $A_2 = 0$ and $N_2 = |k|N_1^2$. Because the determinant is nonzero the solution is unique. Therefore we have determined that:

$$\phi^{(1)} = 0$$

$$\eta^{(1)} = |k|N_1^2e^{2i\theta} + \text{c.c.}$$
Second Order, $O(\epsilon^2)$

Next we consider the $O(\epsilon^2)$ equations. To properly apply multiple scales, we must asymptotically expand the coefficients of each secular terms from the previous order

\[ A_{1,T} = \epsilon f_1 + \epsilon^2 f_2^2 + \cdots \]
\[ N_{1,T} = \epsilon g_1 + \epsilon^2 g_2^2 + \cdots \]

and add the resulting $O(\epsilon)$ terms to the right hand side of Eq. (1.28) and Eq. (1.29) to obtain the system

\[
\begin{align*}
\phi_t^{(2)} + \eta^{(2)} &= -\eta^{(1)} \phi_t^{(0)} - \eta^{(0)} \phi_t^{(1)} - \frac{1}{2} (\eta^{(0)})^2 \phi_t^{(0)} - \eta^{(0)} \phi_x^{(0)} \phi_x^{(0)} \\
&\quad - \phi_x^{(0)} \eta^{(1)} - \eta^{(0)} \phi_x^{(0)} \phi_x^{(0)} - \phi_T^{(1)} - \eta^{(0)} \phi_T^{(0)} + (f_1 e^{i\theta} + \text{c.c.}) \\
\eta_t^{(2)} - \phi_z^{(2)} &= \eta^{(0)} \phi_z^{(0)} + \eta^{(1)} \phi_z^{(0)} - (\eta_x^{(0)} \phi_x^{(1)} + \eta_x^{(1)} \phi_x^{(0)} + \eta_x^{(0)} \eta_x^{(0)} \phi_x^{(0)}) \\
&\quad + \frac{1}{2} (\eta^{(0)})^2 \phi_z^{(0)} - \eta_T^{(1)} + (g_1 e^{i\theta} + \text{c.c.})
\end{align*}
\]

We substitute all previous solutions

\[
\begin{align*}
\phi^{(0)} &= A_1(T) e^{i\theta + |k|z} + \text{c.c.} \\
\eta^{(0)} &= N_1(T) e^{i\theta} + \text{c.c.} \\
\phi^{(1)} &= 0 \\
\eta^{(1)} &= |k| N_1^2 e^{2i\theta} + \text{c.c}
\end{align*}
\]

to arrive at a system of the form

\[
\begin{align*}
\phi_t^{(2)} + \eta^{(2)} &= (C_1 e^{i\theta} + C_2 e^{2i\theta} + C_3 e^{3i\theta} + \text{c.c.}) + C_0 \\
\eta_t^{(2)} - \phi_z^{(2)} &= (D_1 e^{i\theta} + D_2 e^{2i\theta} + D_3 e^{3i\theta} + \text{c.c.})
\end{align*}
\]

where the coefficients $C_0, C_1, C_2, C_3, D_1, D_2, D_3$ are combinations of $A_1, N_1, k$ and $\omega$. We consider a solution of form:

\[
\begin{align*}
\phi^{(2)} &= \left(A_1^{(2)} e^{i\theta + |k|z} + A_2^{(2)} e^{2i\theta + |k|z} + A_3^{(2)} e^{3i\theta + |k|z} + \text{c.c.}\right) + A_0 \\
\eta^{(2)} &= \left(N_1^{(2)} e^{i\theta} + N_2^{(2)} e^{2i\theta} + N_3 e^{3i\theta} + \text{c.c.}\right) + N_0
\end{align*}
\]
After removing the secular terms $e^{\pm i\theta}$ and making the substitutions $f_1 = A_{1,\tau}$ and $g_1 = N_{1,\tau}$ where $\tau = \epsilon T$ we obtain the following system:

\[-i\omega A_1^{(2)} + N_1^{(2)} = -\frac{3}{2} k^2 |N_1|^2 N_1 - A_{1,\tau}\]
\[-i\omega N_1^{(2)} - |k| A_1^{(2)} = -\frac{5i}{2} \omega k^2 |N_1|^2 N_1 - N_{1,\tau}\]

Solving for $N_{1,\tau}$, we find

\[N_{1,\tau} = -2ik^2 \omega |N_1|^2 N_1\]
A.3 NLS Transformation

After expanding the $O(\epsilon^2)$ nonlinear deep water dispersion relation and assuming a slowly modulated ansatz, we obtain the equation

\[ iN_T + i \frac{N_X}{2} - \epsilon \frac{N_{XX}}{8} + \epsilon \frac{N_{YY}}{4} - 2\epsilon |N|^2 N = 0 \] (A.15)

Eq. (A.15) is reducible to the canonical NLS by the change of variables

\[ \xi = X - T/2 \]
\[ \tau = \epsilon T \]

From the chain rule we find:

\[ \frac{\partial N}{\partial T} = \frac{\partial N}{\partial \tau} \frac{\partial \tau}{\partial T} + \frac{\partial N}{\partial \xi} \frac{\partial \xi}{\partial T} = \epsilon \frac{\partial N}{\partial \tau} - \frac{1}{2} \frac{\partial N}{\partial \xi} \]
\[ \frac{\partial N}{\partial X} = \frac{\partial N}{\partial \tau} \frac{\partial \tau}{\partial X} + \frac{\partial N}{\partial \xi} \frac{\partial \xi}{\partial X} = \frac{\partial N}{\partial \xi} \]

After substituting appropriately and simplifying, we obtain:

\[ i \left( \epsilon N_T - \frac{N_\xi}{2} \right) + i \frac{N_\xi}{2} - \epsilon \frac{N_{\xi\xi}}{8} + \epsilon \frac{N_{YY}}{4} - 2\epsilon |N|^2 N = 0 \]

By simplifying and dividing through by $\epsilon$ leads to the canonical $O(\epsilon)$ nonlinear Schrödinger equation governing water waves in two dimensions:

\[ iN_T - \frac{N_{\xi\xi}}{8} + \frac{N_{YY}}{4} - 2|N|^2 N = 0 \]
A.4 FDS Transformation

After keeping the full linear deepwater dispersion relation and assuming a slowly modulated ansatz, we obtain the equation

$$\epsilon i N_T - F^{-1}[(w_l(K, L) - 1)F[N]] - 2\epsilon^2|N|^2 N = 0 + O(\epsilon^3)$$

$$w_l(K, L) = \left((1 + \epsilon K)^2 + (\epsilon L)^2\right)^{1/2}$$

where $F$ and $F^{-1}$ denote the forwards and inverse Fourier transform:

$$F[u] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y) e^{i(kx + ly)} \, dx \, dy$$

$$F^{-1}[u] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(k, l) e^{-i(kx + ky)} \, dk \, dl$$

To properly compare this equation to the canonical nonlinear Schrödinger equation, we apply the transformations:

$$\xi = X - T/2$$

$$\tau = \epsilon T$$

and obtain the PDE

$$\epsilon^2 i N_T - \epsilon N_{\xi} - F^{-1}[(w_l(K, L) - 1)F[N]] - 2\epsilon^2|N|^2 N = 0 + O(\epsilon^3)$$

Dividing by $\epsilon^2$, converting spacial derivatives into their Fourier counterparts, and simplifying, we obtain

$$i N_{\tau} + F^{-1} \left[ \frac{1}{\epsilon^2} (1 + \frac{K}{2} - w_l(K, L)) F[N] \right] - 2\epsilon^2|N|^2 N = 0 + O(\epsilon^3)$$

Finally, we apply the following scalings

$$\tilde{N} = 2N \quad \tilde{\tau} = \frac{\tau}{2} \quad \tilde{\xi} = 2\xi \quad \tilde{Y} = \sqrt{2}Y$$

to obtain:

$$i \tilde{N}_{\tilde{\tau}} + F^{-1}[W_l(\tilde{K}, \tilde{L})F[\tilde{N}]] - |\tilde{N}|^2 \tilde{N} = 0 + O(\epsilon)$$

$$W_l(\tilde{K}, \tilde{L}) = \frac{1}{\epsilon^2} \left( 2 + 2\epsilon \tilde{K} - 2((1 + 2\epsilon \tilde{K})^2 + 2(\epsilon \tilde{L})^2)^{1/2} \right)$$
A.5 Real and Imaginary Parts of Full Linear Dispersion Relation

We want to separate the real and imaginary parts of the operator

\[ i\mathcal{N}_\tau + \mathcal{F}^{-1}[W_\ell(K, L)\mathcal{F}[N]] - |N|^2 N = 0 + O(\epsilon) \]

\[ W(K, L) = \frac{1}{\epsilon^2} \left( 2 + 2\epsilon K - 2((1 + 2\epsilon K)^2 + 2(\epsilon L)^2)^{\frac{1}{4}} \right) \]

We expand \( W(k, l) \) in a one dimensional Taylor series in \( \epsilon \) and seek to separate terms containing coefficients \((ik)^a(il)^b\) from terms containing coefficients \(i(ik)^a(il)^b\). These terms correspond to \( \frac{\partial^{a+b}}{\partial x^a \partial y^b} \) and \( i\frac{\partial^{a+b}}{\partial x^a \partial y^b} \) respectively.

\( 2\epsilon^2 \) belongs to the real operator, and \( 2K\epsilon \) belongs to the imaginary operator. We must now determine the real and imaginary parts corresponding to \( w(K, L) = ((1 + 2\epsilon K)^2 + 2(\epsilon L)^2)^{\frac{1}{4}} \). Since \( w(k, l) \) is real-valued, its Taylor expansion will be purely real-valued. Therefore, any terms containing odd powers of \( k \) or \( l \) (but not both) will belong to the imaginary operator, while all other terms will belong to the real operator. We now show that odd and even parts of \( W(k, l, \epsilon) \) contain only odd and even powers respectively. We begin by considering the function

\[ f(x) = (1 + x)^{\frac{1}{4}} \]

For \(|x| < 1\) we can rewrite \( f(x) \) via its Taylor expansion:

\[ f(x) = \sum_{n=0}^{\infty} c_n x^n \quad c_n = \begin{cases} 1 & \text{n=0} \\ \frac{1}{n!} \prod_{i=1}^{n} (\frac{5}{4} - i) & \text{otherwise} \end{cases} \]

Next, let \( x = (\epsilon a + b\epsilon^2) \). From the binomial theorem,

\[ x^n = (\epsilon a + b\epsilon^2)^n = \sum_{i=0}^{n} \binom{n}{i} (\epsilon a)^i (b\epsilon^2)^{n-i} = \sum_{i=0}^{n} \binom{n}{i} \epsilon^{2n-i} a^i b^{n-i} \]

To collect powers of \( \epsilon^\alpha \), we set \( \alpha = 2n - i \) where \( i \in [0, n] \). Therefore, \( \alpha/2 \leq n \leq \alpha \), and we obtain the series

\[ f(\epsilon a + b\epsilon^2) = \sum_{\alpha=0}^{\infty} b_\alpha \epsilon^\alpha \quad b_\alpha = \sum_{i=\left\lfloor \frac{\alpha}{2} \right\rfloor}^{\alpha} c_i \binom{i}{2i-\alpha} a^{2i-\alpha} b^{\alpha-i} \]

When \( \alpha \) is even, the coefficients \( b_\alpha \) contain only even powers of \( a \). Similarly, when \( \alpha \) is odd, the coefficients \( b_\alpha \) contain only odd powers of \( a \). Since \( a \to 4K \) and \( b \to 4K^2 + 2L^2 \), we can see that
the even Taylor coefficients contain even powers of $k$ and $l$, while odd Taylor coefficients contain odd powers of $k$. Therefore:

$$L_R[u] = \mathcal{F}^{-1} \left[ \frac{W(k, l, \epsilon) + W(k, l, -\epsilon)}{2} \mathcal{F}[u] \right]$$

$$L_I[u] = \mathcal{F}^{-1} \left[ \frac{W(k, l, \epsilon) - W(k, l, -\epsilon)}{2} \mathcal{F}[u] \right]$$