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Type-free Approaches to Supercharacter Theories of Unipotent Groups

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by

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Type-free approaches to supercharacter theories of unipotent groups

Thesis directed by Associate Professor Nathaniel Thiem

Supercharacter theories are a relatively new tool in studying the representation theory of unipotent groups over finite fields. In this thesis I present two new approaches to constructing supercharacter theories of finite unipotent groups. The first method utilizes group actions to construct supercharacter theories of the unipotent orthogonal, symplectic and unitary groups. The second technique is via the method of little groups, which describes the irreducible characters of a semidirect product with an abelian normal subgroup in terms of the irreducible characters of the factor groups. Motivated by these constructions, I produce supercharacter theories for a large collection of unipotent matrix groups and construct a Hopf monoid on the supercharacters.

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Chapter 1

Introduction

For q a power of a prime, let $UT_n(\mathbb{F}_q)$ denote the group of unipotent $n \times n$ upper triangular matrices over the q element field $\mathbb{F}_q$. Indexing the irreducible representations of $UT_n(\mathbb{F}_q)$ is known to be a "wild" problem (see [13]). In other words, a classification of these representations would be equivalent to classifying pairs of matrices in $UT_n(\mathbb{F}_q)$ up to simultaneous similarity. In [5], André constructs a set of characters, referred to as "basic characters," such that each irreducible character of $UT_n(\mathbb{F}_q)$ occurs with nonzero multiplicity in exactly one basic character. These characters can be thought of as a coarser approximation of the irreducible characters of $UT_n(\mathbb{F}_q)$. Diaconis–Isaacs generalize the idea of a basic character to a "supercharacter" of an arbitrary finite group in [11]. They also construct supercharacter theories for all finite algebra groups G (as defined in Section 2.3). In the case that $G = UT_n(\mathbb{F}_q)$, the constructions of André and of Diaconis–Isaacs produce the same supercharacter theory. The two constructions use different techniques: André constructs basic characters by inducing linear characters from certain subgroups of $UT_n(\mathbb{F}_q)$, whereas Diaconis–Isaacs utilize the two-sided action of $UT_n(\mathbb{F}_q)$ on the associative algebra of strictly upper triangular matrices.

Since their introduction by Diaconis–Isaacs, supercharacter theories have been connected to a number of areas of mathematics. In [14], Hendrickson shows that the supercharacter theories of a finite group G are in bijection with the central Schur rings over G. Brumbaugh et al. construct certain exponential sums of interest in number theory (e.g., Gauss, Ramanujan, and Kloosterman sums) as supercharacters of abelian groups in [9]. Aguiar et al. use the supercharacters of $UT_n(\mathbb{F}_q)$

to construct a Hopf algebra in [1] and show that this structure is isomorphic to the Hopf algebra of symmetric functions in non-commuting variables. There have also been generalizations of the results for $UT_n(\mathbb{F}_q)$ to other types: in [6, 7], André–Neto modify André's earlier construction to the unitriangular groups in types B, C and D. Marberg describes the type B and D supercharacters in terms of type A supercharacters in [20], and in [8], Benedetti constructs an analogous Hopf algebra to that in [1] on the superclass functions of type D.

In Chapter 3, we generalize the supercharacter theories in types B, C, and D in a manner analogous to the type A construction of Diaconis–Isaacs. The construction in [6, 7] uses the idea of a "basic subset of roots" to induce linear characters from certain subgroups of the full unitriangular group. Our construction instead utilizes actions of $UT_n(\mathbb{F}_q)$ on the Lie algebras of the unitriangular groups in types B, C and D to define superclasses and supercharacters. One advantage of our method is that it works in situations where the idea of a basic subset of roots does not make sense, such as the case of the unipotent radical of a parabolic subgroup. The main results of Chapter 3 are as follows.

(1) Given a pattern subgroup G (see Section 2.3) of the unipotent upper triangular matrices and a subgroup U of G defined by an anti-involution of G, we construct a supercharacter theory of U.

(2) We apply this supercharacter theory to the groups

$$\begin{aligned} UO_n(\mathbb{F}_q) &= \{g \in UT_n(\mathbb{F}_q) \mid g^{-1} = Jg^tJ\} \text{ and} \\ USp_{2n}(\mathbb{F}_q) &= \left\{ g \in UT_{2n}(\mathbb{F}_q) \;\middle|\; g^{-1} = -\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} g^t \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \right\}, \end{aligned}$$

where J is the matrix with ones on the antidiagonal and zeroes elsewhere. We recover the supercharacter theories of André–Neto from [6, 7], and obtain new supercharacter theories for a large collection of subgroups.

(3) We construct a previously unknown supercharacter theory of

$$UU_n(\mathbb{F}_{q^2}) = \{g \in UT_n(\mathbb{F}_{q^2}) \mid g^{-1} = J\overline{g}^tJ\},$$

where $\overline{g}_{ij} = (g_{ij})^q$. We describe the supercharacters and superclasses in terms of combinatorial objects and calculate the supercharacter table.

We show that the supercharacter values on superclasses of $UU_n(\mathbb{F}_{q^2})$ demonstrate "Ennola duality," as they are obtained from the supercharacter values of $UT_n(\mathbb{F}_q)$ by formally replacing 'q' with '$-q$'.

In Chapter 4, we describe a method for constructing supercharacter theories of certain groups from those of subgroups. This technique is a modification of the "method of little groups," which describes the irreducible characters of $G = N \rtimes H$ in the case that N is abelian. The method of little groups was developed by Frobenius in the case of finite groups and extended to topological groups by Mackey in [18]. Marberg has presented an analogue of the method of little groups which describes supercharacters of algebra groups in [19]. In Chapter 4 we take a different viewpoint and use the method of little groups as a tool for constructing supercharacter theories.

(1) We describe the method of little groups and present our technique for constructing supercharacter theories of finite groups of the form $G = N \rtimes H$ with N abelian.

(2) We apply our method to reproduce the supercharacter theories of pattern groups and of subgroups of pattern groups defined by involutions.

(3) We construct new, coarser supercharacter theories of $UT_n(\mathbb{F}_q)$ which have nice indexing sets for the supercharacters and superclasses.

In reproducing the supercharacter theory of $UT_n(\mathbb{F}_q)$, we obtain a description of the supercharacters in terms of supercharacters of nice subgroups of $UT_k(\mathbb{F}_q) \times UT_m(\mathbb{F}_q)$ for any k, m with $k + m = n$. In a sense, this gives an iterative construction of the supercharacter theory of $UT_n(\mathbb{F}_q)$ in terms of supercharacter theories of smaller matrix groups. In other types, our method describes the supercharacters of $UO_{2n}(\mathbb{F}_q)$, $UOSp_{2n}(\mathbb{F}_q)$, and $UU_{2n}(\mathbb{F}_q)$ in terms of supercharacters of subgroups of $UT_n(\mathbb{F}_q)$. This gives a construction of these supercharacter theories directly from the type A supercharacter theory.

For a fixed finite group G, the supercharacter theories of G form a lattice, a property which Hendrickson explores in detail in [14]. The lattice theoretic join of the new supercharacter theories of $UT_n(\mathbb{F}_q)$ constructed in Chapter 4 has superclasses and supercharacters indexed by nonnesting $\mathbb{F}_q$-set partitions. In Chapter 5, we present the following results.

(1) We describe this supercharacter theory and calculate its supercharacter table.

(2) We generalize the nonnesting supercharacter theory to pattern subgroups of $UT_n(\mathbb{F}_q)$.

(3) We construct a Hopf monoid from the nonnesting supercharacter theories of pattern subgroups, for which we give a combinatorial description of the product and coproduct.

The Hopf monoid constructed from pattern subgroups is a generalization of the one presented in [2] for unitriangular matrices, and in fact contains that Hopf monoid as a submonoid. Before presenting these results, we develop necessary background material in Chapter 2.

Chapter 2

Preliminaries

In Chapter 2 we develop background material on character theory of finite groups, supercharacter theories, unipotent groups and algebra groups, and labeled set partitions.

2.1 Character theory of finite groups

Let G be a finite group. If V is a finite dimensional complex vector space and $GL(V)$ denotes the group of invertible linear transformations of V, we call a homomorphism $\rho : G \to GL(V)$ a **representation** of G. We refer to V as a G**-module**, and if there is no proper nonzero subspace W of V such that $\rho(g)W = W$ for all $g \in G$, we say that V is an **irreducible** G-module. The representation ρ is also referred to as irreducible.

Given a representation ρ of G and a choice of basis of V, we define the **character** of ρ to be the function

$$\begin{aligned} \chi_\rho : G &\to \mathbb{C} \\ g &\mapsto \operatorname{tr}(\rho(g)). \end{aligned}$$

Note that the trace of a linear transformation is independent of the choice of basis of V, and as such the character of a representation does not depend on the choice of basis. If χ is a character of G corresponding to an irreducible representation, we will refer to χ as an irreducible character.

Let

$$CF(G) = \{\varphi : G \to \mathbb{C} \mid \varphi(hgh^{-1}) = \varphi(g) \text{ for all } g, h \in G\}$$

be the space of class functions of G, with inner product defined by

$$\langle \varphi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\theta(g)}.$$

Proposition 2.1 ([10, Proposition 9.21])**.** *With respect to this inner product, the irreducible characters of G (denoted* $\mathrm{Irr}(G)$*) form an orthonormal basis of $CF(G)$.* □

The space $CF(G)$ can be made into a $\mathbb{C}$-algebra with two different products. The pointwise product of two class functions is defined by

$$(\varphi \cdot \theta)(g) = (\varphi\theta)(g) = \varphi(g)\theta(g),$$

and the convolution product is defined by

$$(\varphi * \theta)(g) = \frac{1}{|G|} \sum_{h \in G} \varphi(h)\theta(h^{-1}g).$$

Unless otherwise noted, the product of two class functions will always mean the pointwise product.

For $g \in G$, let C_g denote the conjugacy class of g, and define

$$\kappa_{C_g}(h) = \begin{cases} 1 & \text{if } C_h = C_g, \\ 0 & \text{otherwise,} \end{cases}$$

to be the indicator function of C_g. Note that $\{\kappa_{C_g} \mid g \in G\}$ is a basis of $CF(G)$ consisting of primitive idempotents with respect to the pointwise product. In particular, $CF(G)$ is a semisimple, commutative algebra with respect to the pointwise product.

There is another basis of $CF(G)$ given by $\{\chi(1)\chi \mid \chi \in \mathrm{Irr}(G)\}$; by [15, Theorem 2.13], this basis consists of primitive idempotents with respect to the convolution product. This means that $CF(G)$ is also a semisimple, commutative algebra with respect to the convolution product.

Let $\mathbb{F}$ be a field, and let

$$\mathbb{F}G = \left\{ \sum_{g \in G} a_g g \;\middle|\; a_g \in \mathbb{F} \right\}$$

be the group algebra of G, where sums are defined pointwise and multiplication is extended by linearity from the group multiplication. If $\mathbb{F} = \mathbb{C}$, the group algebra $\mathbb{C}G$ is semisimple by Maschke's Theorem, hence $Z(\mathbb{C}G)$ has a basis of primitive central idempotents.

Proposition 2.2 ([10, Proposition 9.21])**.** *The primitive central idempotents of* $\mathbb{C}G$ *are the elements*

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g$$

where $\chi \in \mathrm{Irr}(G)$*.* □

By orthogonality, if $\chi_1, \chi_2 \in \mathrm{Irr}(G)$, then

$$\chi_1(e_{\chi_2}) = \begin{cases} \chi_1(1) & \text{if } \chi_1 = \chi_2, \\ 0 & \text{otherwise.} \end{cases}$$

If H is a subgroup of a finite group G, there are maps

$$\begin{aligned} &\mathbf{Ind} : CF(H) \to CF(G) \quad \text{and} \\ &\mathbf{Res} : CF(G) \to CF(H) \end{aligned}$$

defined by

$$\begin{aligned} \mathrm{Ind}_H^G(\alpha)(g) &= \frac{1}{|H|} \sum_{\substack{k \in G \\ kgk^{-1} \in H}} \alpha(kgk^{-1}) \quad \text{and} \\ \mathrm{Res}_H^G(\beta)(h) &= \beta(h), \end{aligned}$$

where $g \in G$, $h \in H$, $\alpha \in CF(H)$, and $\beta \in CF(G)$. These maps are called the **induction map** and **restriction map**, respectively.

Given finite groups G and H and a surjective homomorphism $\pi : G \to H$, define a map

$$\mathbf{Inf} : CF(H) \to CF(G)$$

by

$$\mathrm{Inf}_H^G(\alpha) = \alpha \circ \pi,$$

where $\alpha \in CF(H)$. This map is called the **inflation map**, and depends on the choice of π (although in most situations it is clear what π should be). In particular, if N is a normal subgroup of G, we can inflate class functions of G/N to class functions of G via the quotient map.

2.2 Supercharacter theories

The idea of a supercharacter theory of an arbitrary finite group was introduced by Diaconis–Isaacs in [11].

Definition 2.1. Let G be a finite group, and suppose that $\mathcal{K}$ is a partition of G into unions of conjugacy classes and $\mathcal{X}$ is a set of characters of G. We say that the pair $(\mathcal{K}, \mathcal{X})$ is a **supercharacter theory** of G if

(SCT1) $|\mathcal{X}| = |\mathcal{K}|$,

(SCT2) the characters $\chi \in \mathcal{X}$ are constant on the members of $\mathcal{K}$, and

(SCT3) each irreducible character of G is a constituent of exactly one character in $\mathcal{X}$.

The characters $\chi \in \mathcal{X}$ are referred to as **supercharacters** and the sets $K \in \mathcal{K}$ are called **superclasses**.

The following lemma is a consequence of [11, Lemma 2.1].

Lemma 2.3. *Let $(\mathcal{K}, \mathcal{L})$ be a supercharacter theory of G. Then*

(1) $\{1\}$ *is in* $\mathcal{K}$*,*

(2) some multiple of the trivial character 1_G *of* G *is in* $\mathcal{X}$*, and*

(3) each character in $\mathcal{X}$ *is of the form*

$$\chi = a_\chi \sum_{\varphi \in S_\chi} \varphi(1)\varphi$$

for some constant a_χ *and some subset* $S_\chi \subseteq Irr(G)$*.*

Proof. Suppose that $g \in G$ is in the same superclass as 1; then g is in the kernel of χ for each $\chi \in \mathcal{X}$. As every irreducible character of G is a constituent of some supercharacter, g is in the kernel of φ for all $\varphi \in \mathrm{Irr}(G)$, hence $g = 1$, proving (1).

Note that $\mathcal{X}$ is a basis for the space of complex-valued functions that are constant on the blocks of $\mathcal{K}$. Certainly 1_G is in this space, and (2) follows. Finally, as $\{1\}$ is a superclass, the **regular character**

$$\sum_{\varphi \in \mathrm{Irr}(G)} \varphi(1)\varphi$$

is constant on the superclasses, hence we have (3). □

In Chapter 4, we will want to construct the supercharacters of a supercharacter theory without explicitly mentioning the corresponding partition of G. To this end, we present a different characterization of a supercharacter theory only in terms of class functions. Lemmas 2.4, 2.5, and 2.6 are well-known but do not seem to appear anywhere in the literature.

Recall that $CF(G)$ is the space of class functions of G. Suppose $A \subseteq CF(G)$ is a unital subalgebra under both the pointwise and convolution products. Then A is semisimple and commutative with respect to both products, hence has two natural bases of primitive idempotents. The idempotents with respect to the pointwise product will each be of the form

$$\sum \kappa_{C_g},$$

where the sum is over some set of conjugacy classes of G; this defines a partition $\mathcal{K}_A$ of G. Similarly, the idempotents with respect to the convolution product will each be of the form

$$\sum \chi(1)\chi,$$

where the sum is over some set of irreducible characters of G; this defines a set $\mathcal{X}_A$ of orthogonal characters of G.

Lemma 2.4. *Let $A \subseteq CF(G)$ be a unital subalgebra under both the pointwise and convolution products; then $(\mathcal{K}_A, \mathcal{X}_A)$ defines a supercharacter theory of G.*

Proof. As $\mathcal{K}_A$ and $\mathcal{X}_A$ both have size $\dim_{\mathbb{C}}(A)$, (SCT1) is immediate. By considering the idempotents with respect to the pointwise product, A is the subalgebra of $CF(G)$ consisting of functions that are constant on the blocks of $\mathcal{K}_A$, hence we have (SCT2). Finally, note that the regular character of G is the identity with respect to the convolution product, and (SCT3) holds. □

There is also a converse of Lemma 2.4.

Lemma 2.5. *Let $(\mathcal{K}, \mathcal{X})$ be a supercharacter theory of G; then the span of $\mathcal{X}$ is a unital subalgebra of $CF(G)$ under both the pointwise and convolution products.*

Proof. The span of $\mathcal{X}$ is the subspace of $CF(G)$ consisting of functions that are constant on the blocks of $\mathcal{K}$, which is closed under the pointwise product. Furthermore, by Lemma 2.3 the characters in $\mathcal{X}$ are multiples of idempotents of $CF(G)$ under the convolution product. By (SCT3) these idempotents are primitive relative to each other, hence the span of $\mathcal{X}$ is closed under the convolution product. Finally, note that the trivial character and the regular character are both constant on the blocks of $\mathcal{K}$, hence the span of $\mathcal{X}$ is a unital subalgebra of $CF(G)$ under both the pointwise and convolution products. □

Lemmas 2.4 and 2.5 show that there is a bijection

$$\left\{ \begin{array}{l} \text{Subalgebras of } CF(G) \text{ under both the} \\ \text{pointwise and convolution products} \end{array} \right\} \longleftrightarrow \{\text{Supercharacter theories of } G\}.$$

In particular, we have the following alternative characterization of a supercharacter theory.

Lemma 2.6. *Suppose that $\mathcal{X}$ is a set of characters of* $\mathrm{Irr}(G)$ *such that*

(1) the trivial character is in $\mathcal{X}$, and

(2) each irreducible character of G is a constituent of exactly one character of $\mathcal{X}$.

Then there exists a partition $\mathcal{K}$ of G such that the pair $(\mathcal{X}, \mathcal{K})$ forms a supercharacter theory of G if and only if $\mathcal{X}$ is a linear basis of a subalgebra of $CF(G)$ under the pointwise product.

Proof. First suppose that such a partition $\mathcal{K}$ exists; by Lemma 2.5, $\mathcal{X}$ is a linear basis of a subalgebra of $CF(G)$ under the pointwise product.

Conversely, suppose that $\mathcal{X}$ is a linear basis of a subalgebra A of $CF(G)$ under the pointwise product. As $CF(G)$ is semisimple and commutative, so is A; it follows that A has a basis of primitive idempotents. These idempotents must be indicator functions for disjoint subsets of G. As the trivial character is in A, these subsets partition G. Let $\mathcal{K}$ denote this partition of G; conditions (SCT1) and (SCT2) follow from the construction of $\mathcal{K}$. By assumption, each irreducible character of G is a constituent of exactly one character in $\mathcal{X}$, and we have (SCT3). □

This allows for the construction of a supercharacter theory without reference to the superclasses. We can similarly describe a supercharacter theory without reference to the supercharacters; for a subset $K \subseteq G$, define the element $\widehat{K} \in \mathbb{C}G$ by

$$\widehat{K} = \sum_{g \in K} g.$$

Lemma 2.7 ([14, Proposition 2.4]). *Suppose that $\mathcal{K}$ is a partition of G such that*

(1) $\{1\} \in \mathcal{K}$, and

(2) each $K \in \mathcal{K}$ is a union of conjugacy classes of G.

Then $\mathcal{K}$ is the set of superclasses of some supercharacter theory of G if and only if the linear span of $\{\widehat{K} \mid K \in \mathcal{K}\}$ is a subalgebra of $\mathbb{C}G$.

Proof. First suppose that $(\mathcal{K}, \mathcal{X})$ is a supercharacter theory of G. Then the characters $\chi \in \mathcal{X}$ are a basis for the space of complex-valued functions that are constant on the parts of $\mathcal{K}$. As each $\chi \in \mathcal{X}$ is of the form

$$\chi = a_\chi \sum_{\varphi \in S_\chi} \varphi(1)\varphi$$

for some subset $S_\chi \subseteq \mathrm{Irr}(G)$ and some constant a_χ, to each $\chi \in \mathcal{X}$ we can associate a central idempotent

$$f_\chi = \sum_{\varphi \in S_\chi} e_\varphi.$$

Observe that for $\chi_1, \chi_2 \in \mathcal{X}$,

$$\chi_1(f_{\chi_2}) = \begin{cases} \chi_1(1) & \text{if } \chi_1 = \chi_2, \\ 0 & \text{otherwise.} \end{cases}$$

This means that the supercharacters are constant on the f_χ, hence the f_χ are contained in the span of $\{\widehat{K} \mid K \in \mathcal{K}\}$. Note that the f_χ form a basis of the span of $\{\widehat{K} \mid K \in \mathcal{K}\}$ and are closed under multiplication; it follows that the linear span of $\{\widehat{K} \mid K \in \mathcal{K}\}$ is a subalgebra of $\mathbb{C}G$.

Conversely, suppose that the linear span of $\{\widehat{K} \mid K \in \mathcal{K}\}$ is a subalgebra A of $\mathbb{C}G$. This subalgebra is contained in $Z(\mathbb{C}G)$ as each $k \in \mathcal{K}$ is a union of conjugacy classes of G. As $\mathbb{C}G$ is semisimple, so is A, hence A has a basis of primitive central idempotents. These idempotents must be of the form

$$f_\chi = \sum_{\varphi \in S_\chi} e_\varphi,$$

where $S_\chi \subseteq \mathrm{Irr}(G)$ and $\chi = \sum_{\varphi \in S_\chi} \varphi(1)\varphi$. Let

$$\mathcal{X} = \{\chi \mid f_\chi \text{ is a primitive idempotent of } A\}.$$

Then $|\mathcal{K}| = |\mathcal{X}|$, the characters of $\mathcal{X}$ are constant on the parts of $\mathcal{K}$, and each irreducible character of G is a constituent of exactly one character of $\mathcal{X}$ (since $1 \in A$). It follows that we have produced a supercharacter theory of G for which $\mathcal{K}$ is the set of superclasses. □

If $A \subseteq Z(\mathbb{C}G)$ is an algebra with a basis $\{\widehat{K} \mid K \in \mathcal{K}\}$ for some partition $\mathcal{K}$ of G with $\{1\} \in \mathcal{K}$, we call A a **central Schur ring** of G. Lemma 2.7 shows that supercharacter theories of G are in bijection with central Schur rings of G. The connections between central Schur rings and supercharacter theories are explored in detail in [14].

Remark. Lemmas 2.4, 2.5, and 2.6 can be viewed as corollaries of Lemma 2.7 by applying the isomorphism

$$\begin{gathered} Z(\mathbb{C}G) \to CF(G) \\ e_\chi \mapsto \chi. \end{gathered}$$

If G is a finite group with a supercharacter theory, we say that a subgroup H of G is **supernormal** if it is a union of superclasses of G. The following proposition is immediate from Lemma 2.7.

Proposition 2.8. *Let H be a supernormal subgroup of G with respect to some supercharacter theory $(\mathcal{K}, \mathcal{X})$ of G. Then*

$$\{K \in \mathcal{K} \mid K \subseteq H\}$$

is the set of superclasses for a supercharacter theory of H. □

We call this the **supernormal supercharacter theory** of H with respect to G.

2.2.1 The direct product of supercharacter theories

Perhaps the most natural method of building a supercharacter theory of a group from supercharacter theories of subgroups is the direct product of supercharacter theories. Let $G = M \times N$, and suppose that M and N are each equipped with supercharacter theories, denoted $(\mathcal{K}, \mathcal{X})$ and $(\mathcal{L}, \mathcal{Y})$. Define

$$\mathcal{M} = \{A \times B \mid A \in \mathcal{K} \text{ and } B \in \mathcal{L}\} \text{ and}$$
$$\mathcal{Z} = \{\chi \times \psi \mid \chi \in \mathcal{X} \text{ and } \psi \in \mathcal{Y}\}.$$

Proposition 2.9 ([14, Proposition 8.1])**.** *The pair $(\mathcal{M}, \mathcal{Z})$ defines a supercharacter theory of G.*

Proof. Conditions (SCT1) and (SCT2) follow directly from the definitions of $\mathcal{M}$ and $\mathcal{Z}$. Furthermore,

$$\mathrm{Irr}(G) = \{\chi \times \psi \mid \chi \in \mathrm{Irr}(M) \text{ and } \psi \in \mathrm{Irr}(N)\},$$

thus every character of $\mathrm{Irr}(G)$ is a constituent of exactly one character in $\mathcal{Z}$. □

The supercharacter theory $(\mathcal{M}, \mathcal{Z})$ is referred to as the **direct product** of $(\mathcal{K}, \mathcal{X})$ and $(\mathcal{L}, \mathcal{Y})$.

2.2.2 The star product of supercharacter theories

Suppose that G has a normal subgroup N that is equipped with a supercharacter theory $(\mathcal{K}, \mathcal{X})$. We say that this supercharacter theory is G**-invariant** if for each $g \in G$ and $n \in N$, n and gng^{-1} are in the same superclass. Suppose that N is equipped with a G-invariant supercharacter theory $(\mathcal{K}, \mathcal{X})$ and G/N is equipped with a supercharacter theory $(\mathcal{L}, \mathcal{Y})$. Define

$$\mathcal{M} = \mathcal{K} \cup \{LN \mid L \in \mathcal{L} - \{1\}\} \text{ and}$$
$$\mathcal{Z} = \{\mathrm{Ind}_N^G(\chi) \mid \chi \in \mathcal{X} - \{1_N\}\} \cup \{\mathrm{Inf}_{G/N}^G(\psi) \mid \psi \in \mathcal{Y}\}.$$

Proposition 2.10 ([14, Theorem 4.3])**.** *The pair $(\mathcal{M}, \mathcal{Z})$ defines a supercharacter theory of G.*

Proof. Note that

$$|\mathcal{M}| = |\mathcal{K}| + (|\mathcal{L}| - 1) \text{ and}$$
$$|\mathcal{Z}| = (|\mathcal{X}| - 1) + |\mathcal{Y}|,$$

hence $|\mathcal{M}| = |\mathcal{Z}|$. Furthermore, for all $\chi \in \mathcal{X} - \{1_N\}$,

$$\mathrm{Ind}_N^G(\chi)(g) = \begin{cases} |G : N|\chi(g) & \text{if } g \in N, \\ 0 & \text{otherwise,} \end{cases}$$

as $(\mathcal{K}, \mathcal{X})$ is G-invariant, and

$$\mathrm{Inf}_{G/N}^G(\psi)(g) = (\psi \circ \pi)(g)$$

for all $\psi \in \mathcal{Y}$ and $g \in G$. It follows that the characters of $\mathcal{Z}$ are constant on the blocks of $\mathcal{M}$. The above calculations also show that $\langle \chi, \psi \rangle = 0$ for any $\chi, \psi \in \mathcal{Z}$ with $\chi \neq \psi$. This means that each irreducible character of G is a constituent of at most one character in $\mathcal{Z}$, hence the characters in $\mathcal{Z}$ are linearly independent. As $|\mathcal{M}| = |\mathcal{Z}|$, $\mathcal{Z}$ is a basis for the set of complex-valued functions that are constant on the blocks of $\mathcal{M}$. Note that $\{1\} \in \mathcal{M}$, thus the regular character of G is constant on the blocks of $\mathcal{M}$, and each irreducible character of G is a constituent of exactly one character in $\mathcal{Z}$. □

We refer to this supercharacter theory as the **$*$-product** of the supercharacter theories $(\mathcal{K}, \mathcal{X})$ and $(\mathcal{L}, \mathcal{Y})$, and write

$$(\mathcal{M}, \mathcal{Z}) = (\mathcal{K}, \mathcal{X}) * (\mathcal{L}, \mathcal{Y}).$$

2.2.3 The lattice of supercharacter theories of a finite group

Let G be a finite group, and let $SCT(G)$ be the set of supercharacter theories of G. Recall that by Lemma 2.7, there is a bijection

$$\varphi : \{\text{Central Schur rings of } G\} \longrightarrow SCT(G).$$

Lemma 2.11 ([14, Lemma 3.2])**.** *The set of central Schur rings of G forms a lattice under inclusion with $A \wedge B = A \cap B$.*

Proof. Note that $Z(\mathbb{C}G)$ is the maximal central Schur ring; it suffices to show that $A \cap B$ is a central Schur ring for any two central Schur rings A and B. Suppose that A and B correspond to the partitions $\mathcal{K}$ and $\mathcal{L}$ of G, respectively. Let $\mathcal{M}$ be the finest partition of G such that each block of $\mathcal{M}$ is a union of blocks of $\mathcal{K}$ and a union of blocks of $\mathcal{L}$. Let C denote the linear span of $\{\widehat{M} \mid M \in \mathcal{M}\}$; then $C \subseteq A \cap B$. At the same time, if $\gamma \in A \cap B$, we can write

$$\gamma = \sum_{K \in \mathcal{K}} a_K \widehat{K} = \sum_{L \in \mathcal{L}} b_L \widehat{L}.$$

Note that if $K \cap L \neq \emptyset$, then $a_K = b_L$. As $\mathcal{M}$ is the finest partition of G such that each block of $\mathcal{M}$ is a union of blocks of $\mathcal{K}$ and a union of blocks of $\mathcal{L}$, we can write

$$\gamma = \sum_{M \in \mathcal{M}} c_M \widehat{M}.$$

It follows that $C = A \cap B$. □

Using the bijection φ, we define a lattice structure on $SCT(G)$ with $\varphi(A) \vee \varphi(B) = \varphi(A \cap B)$ (note that this construction is order-reversing). From Lemma 2.11, we have the following.

Lemma 2.12 ([14, Proposition 3.3])**.** *Let $(\mathcal{K}_1, \mathcal{X}_1)$ and $(\mathcal{K}_2, \mathcal{X}_2)$ be supercharacter theories of a finite group G. Then the superclasses of $(\mathcal{K}_1, \mathcal{X}_1) \vee (\mathcal{K}_2, \mathcal{X}_2)$ correspond to the finest partition of G whose blocks are unions of blocks of $\mathcal{K}_1$ and unions of blocks of $\mathcal{K}_2$.* □

2.3 Unipotent groups and algebra groups

Let $\mathbb{F}_q$ denote the field with q elements, $GL_n(\mathbb{F}_q)$ the group of invertible $n \times n$ matrices over $\mathbb{F}_q$, and $UT_n(\mathbb{F}_q)$ the subgroup of upper triangular matrices with ones on the diagonal. An element $u \in GL_n(\mathbb{F}_q)$ is called **unipotent** if the characteristic polynomial of u is $(t-1)^n$, and a subgroup $U \subseteq GL_n(\mathbb{F}_q)$ is called a **unipotent group** if every $u \in U$ is unipotent. It is evident that $UT_n(\mathbb{F}_q)$ is a unipotent group, and the following standard result allows us think of any unipotent group as a subgroup of $UT_n(\mathbb{F}_q)$.

Lemma 2.13. *Suppose $U \subseteq GL_n(\mathbb{F}_q)$ is a unipotent group; then there exists $g \in GL_n(\mathbb{F}_q)$ such that gUg^{-1} is contained in $UT_n(\mathbb{F}_q)$.*

Proof. Note that

$$|GL_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1}) = q^{\binom{n}{2}}(q^n - 1)(q^{n-1} - 1) \dots (q - 1).$$

If p is the prime dividing q, then p does not divide any of $(q^n - 1), (q^{n-1} - 1), \dots, (q - 1)$, hence any subgroup of $GL_n(\mathbb{F}_q)$ of order $q^{\binom{n}{2}}$ is a Sylow p-subgroup of $GL_n(\mathbb{F}_q)$. In particular, $UT_n(\mathbb{F}_q)$ is a Sylow p-subgroup of $GL_n(\mathbb{F}_q)$.

By considering the Jordan normal form, every unipotent element of $GL_n(\mathbb{F}_q)$ is conjugate to an element of $UT_n(\mathbb{F}_q)$, hence has p-power order. It follows that every unipotent subgroup of $GL_n(\mathbb{F}_q)$ is a p-group and is conjugate to some subgroup of $UT_n(\mathbb{F}_q)$. □

Let $\mathbb{F}_q$ be a field and let $\mathfrak{g}$ be a nilpotent associative algebra over $\mathbb{F}_q$. The **algebra group** G associated to $\mathfrak{g}$ is the set of formal sums

$$G = \{1 + x \mid x \in \mathfrak{g}\}$$

with multiplication defined by $(1+x)(1+y) = 1+(x+y+xy)$ (see [16]). As $\mathfrak{g}$ is nilpotent, elements in G have inverses given by

$$(1 + x)^{-1} = 1 + \sum_{i=1}^{\infty}(-x)^i.$$

We will often write $G = 1 + \mathfrak{g}$ to indicate that G is the algebra group associated to $\mathfrak{g}$. For example, if we define $\mathfrak{ut}_n(\mathbb{F}_q)$ to be the algebra of $n \times n$ upper triangular matrices over $\mathbb{F}_q$ with zeroes on the diagonal, then $UT_n(\mathbb{F}_q)$ is the algebra group associated to $\mathfrak{ut}_n(\mathbb{F}_q)$.

Let $G = 1 + \mathfrak{g}$ be an algebra group and $H = 1 + \mathfrak{h}$ be a subgroup.

(1) If $\mathfrak{h}$ is a left ideal of $\mathfrak{g}$, we call H a **left ideal subgroup** of G.
(2) If $\mathfrak{h}$ is a right ideal of $\mathfrak{g}$, we call H a **right ideal subgroup** of G.
(3) If $\mathfrak{h}$ is a two-sided ideal of $\mathfrak{g}$, we call H a **two-sided ideal subgroup** of G.

These ideal subgroups have nice properties which we will later exploit.

Example 2.1. Let $G = UT_7(\mathbb{F}_q)$, and define

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 & * & * & 0 & * \\ 0 & 1 & 0 & 0 & * & 0 & * \\ 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in UT_7(\mathbb{F}_q) \right\},$$

$$K = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in UT_7(\mathbb{F}_q) \right\}, \text{ and}$$

$$M = \left\{ \begin{pmatrix} 1 & 0 & 0 & * & * & * & * \\ 0 & 1 & 0 & 0 & * & * & * \\ 0 & 0 & 1 & 0 & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in UT_7(\mathbb{F}_q) \right\}.$$

Then H is a left ideal subgroup of G, K is a right ideal subgroup of G, and M is a two-sided ideal subgroup of G.

There is a large class of algebra subgroups of $UT_n(\mathbb{F}_q)$ which will be of interest to us. Let $\mathcal{P}$ be a poset on $[n] = \{i \in \mathbb{N} \mid 1 \leq i \leq n\}$ that is a sub-order of the usual linear order. In other words, $\mathcal{P}$ has the properties that

(1) if $i \preceq_{\mathcal{P}} j$ then $i \leq j$,

(2) if $i \preceq_{\mathcal{P}} j$ and $j \preceq_{\mathcal{P}} k$ then $i \preceq_{\mathcal{P}} k$, and

(3) $i \preceq_{\mathcal{P}} j$ and $j \preceq_{\mathcal{P}} i$ if and only if $i = j$.

Corresponding to the poset $\mathcal{P}$ are a **pattern subgroup**

$$U_{\mathcal{P}} = \{g \in UT_n(\mathbb{F}_q) \mid g_{ij} = 0 \text{ unless } i \preceq_{\mathcal{P}} j\}$$

and a **pattern subalgebra**

$$\mathfrak{u}_{\mathcal{P}} = \{x \in \mathfrak{ut}_n(\mathbb{F}_q) \mid x_{ij} = 0 \text{ unless } i \preceq_{\mathcal{P}} j\}.$$

Note that $U_{\mathcal{P}}$ is the algebra group corresponding to $\mathfrak{u}_{\mathcal{P}}$. The groups H, K and M in Example 2.1 are all pattern groups.

There is a strong connection between algebra groups and unipotent groups; the following result is well-known, but does not seem to appear anywhere in the literature.

Proposition 2.14. *Let $G = 1 + \mathfrak{g}$ be an algebra group over the field $\mathbb{F}_q$; then G is isomorphic to a unipotent group contained in $GL_{n+1}(\mathbb{F}_q)$, where $n = \dim_{\mathbb{F}_q}(\mathfrak{g})$.*

Proof. Define an algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g} \oplus \mathbb{F}_q)$$

by

$$\rho(x)(y) = xy \text{ if } y \in \mathfrak{g} \text{ and}$$
$$\rho(x)(\alpha) = \alpha x \text{ if } \alpha \in \mathbb{F}_q,$$

and extending linearly. Note that $\rho(x)(1) = x$ for all $x \in \mathfrak{g}$, hence this is an embedding of $\mathfrak{g}$ into $\mathfrak{gl}(\mathfrak{g} \oplus \mathbb{F}_q)$. It follows that the map

$$1 + x \mapsto I + \rho(x)$$

is an isomorphism, and G is isomorphic to a unipotent subgroup of $GL_{n+1}(\mathbb{F}_q)$. □

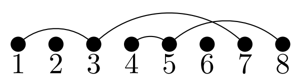
2.4 Labeled set partitions

Let I be a finite set; a **set partition** of I is a collection $\{S_i\}$ of nonempty subsets of I such that I is the disjoint union of the S_i. As every finite set is in bijection with $[n]$ for some n, we will usually think of set partitions as partitions of $[n]$.

For our purposes, it will be useful to represent set partitions as **arc diagrams**. Given a set partition $\nu = \{S_i\}$ of $[n]$, we consider the set of edges

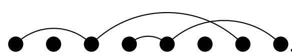
$$\{k \frown l \mid \text{there exists } i \text{ with } k, l \in S_i, \text{ and if } k < j < l, j \notin S_i\},$$

from which we construct an arc diagram. For example, let $\{\{1,3,7\},\{2\},\{4,5,8\}\}$ be a set partition of $[8]$. Corresponding to this set partition is the arc diagram



We will usually omit the labels from the nodes of arc diagrams. From now on we will conflate the notions of set partitions, arc diagrams, and edge sets of arc diagrams. For instance, we will write

$$\nu = \{\{1,3,7\},\{2\},\{4,5,8\}\},$$
$$\nu = \{1 \frown 3, 3 \frown 7, 4 \frown 5, 5 \frown 8\}, \text{ and}$$

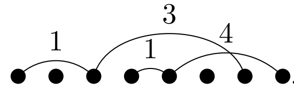
$\nu =$  ,

all referring to the same set partition. Note that the set partition can be recovered from the arc diagram by considering the connected components. An arc diagram ν will correspond to a set partition if and only if for all $i \frown j \in \nu$ and $i < k < j$, $i \frown k, k \frown j \notin \nu$.

One advantage to representing set partitions as arc diagrams is that we can introduce labels to the arcs. If $\mathbb{F}_q$ is the finite field with q elements, an $\mathbb{F}_q$-set partition of $[n]$ is a labeled arc diagram of the form

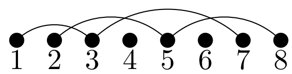
$$\nu = \{i \overset{a}{\frown} j \mid a \in \mathbb{F}_q^\times \text{ and if } i < k < j, i \overset{b}{\frown} k, k \overset{b}{\frown} j \notin \nu\}.$$

For example,

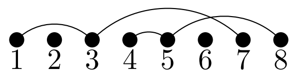


is an $\mathbb{F}_5$-set partition of $\{1,2,3,4,5,6,7,8\}$.

There are several subsets of $\mathbb{F}_q$ set partitions which will be of particular interest to us. If η is a set partition, we will say that η is **nonnesting** if there are no $i < j < k < l$ such that $i \frown l, j \frown k \in \eta$. For example, if

$\eta =$ 

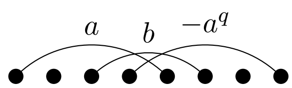
and

$\nu =$ 

,

then η is a nonnesting set partition but ν is not, as $3 \frown 7, 4 \frown 5$ is a nesting. If the underlying set partition of an $\mathbb{F}_q$-set partition is nonnesting, we will call that labeled set partition nonnesting.

If η is an $\mathbb{F}_{q^2}$-set partition with the property that $i \overset{a}{\frown} j \in \eta$ if and only if $\bar{j} \overset{-a^q}{\frown} \bar{i} \in \eta$, where $\bar{i} = n - i + 1$, we call η a **twisted $\mathbb{F}_q$-set partition**. Note that if η is a twisted $\mathbb{F}_q$-set partition and $i \overset{a}{\frown} \bar{i} \in \eta$, then $a + a^q = 0$. For example, if $a, b \in \mathbb{F}_{q^2}$ and $b + b^q = 0$,

$\eta =$ 

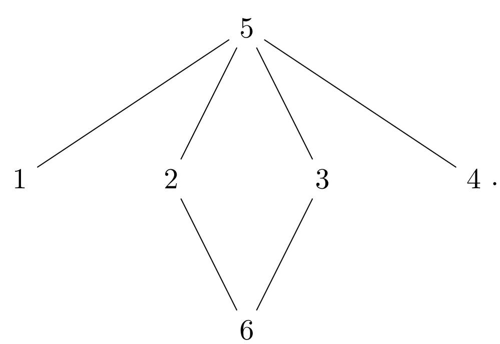
is a twisted $\mathbb{F}_q$-set partition.

Suppose that $\mathcal{P}$ is a poset on I; a **$\mathcal{P}$-set partition** of I is an arc diagram ν such that

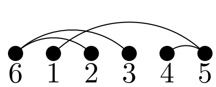
(1) if $i \frown j \in \nu$ then $i \prec_{\mathcal{P}} j$; and

(2) for all $i \frown j \in \nu$ and $i \prec_{\mathcal{P}} k \prec_{\mathcal{P}} j$, we have $i \frown k, k \frown j \notin \nu$.

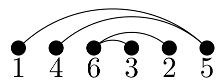
Example 2.2. Let $I = \{1, 2, 3, 4, 5, 6\}$ and let $\mathcal{P}$ be the poset on I with Hasse diagram



We have that

$\nu =$ 

is a valid $\mathcal{P}$-set partition of I. Note that in representing the arc diagram as above we are implicitly viewing $\mathcal{P}$ as a subset of the linear order $6 \prec 1 \prec 2 \prec 3 \prec 4 \prec 5$; we could just as well consider $\mathcal{P}$ as a subset of the linear order $1 \prec 4 \prec 6 \prec 3 \prec 2 \prec 5$ and represent the arc diagram as

$\nu =$  .

Although the representations look different, these arc diagrams have the same edge set.

While our definition of a $\mathcal{P}$-set partition does not require us to view the poset as a subset of some linear order on I, any poset on a finite set is isomorphic to a poset on $[n]$ (for some n) that is a subset of the usual linear order on $[n]$. As such, we will usually assume this and omit the labels on the nodes of our arc diagrams. We will also consider **$(\mathbb{F}_q, \mathcal{P})$-set partitions**, which are $\mathcal{P}$-set partitions with arcs labeled by elements of $\mathbb{F}_q^\times$.

Let ν be a $\mathcal{P}$-set partition; we say that ν is **nonnesting** if there are no $i \frown j, k \frown l \in \nu$ with $i \prec_\mathcal{P} k \prec_\mathcal{P} l \prec_\mathcal{P} j$. Note that if $\mathcal{P}$ is a linear order the nonnesting $\mathcal{P}$-set partitions are the same as the nonnesting set partitions.

Remark. The notion of a $\mathcal{P}$-set partition of I is not the same as that of a **poset partition** (see, for instance, [3, Section 13.1.4]).

Chapter 3

Supercharacter theories of unipotent groups defined via group actions

In [11], Diaconis–Isaacs construct a supercharacter theory of an arbitrary algebra group $G = 1 + \mathfrak{g}$ by utilizing actions of G on $\mathfrak{g}$. Their construction generalizes that of André from [5], which only applies in the case that $G = UT_n(\mathbb{F}_q)$. In Chapter 3, we review the construction of Diaconis–Isaacs and generalize to unipotent orthogonal, symplectic, and unitary groups. We show that our method produces the same supercharacter theories of the unipotent orthogonal and symplectic groups as that of André–Neto from [6, 7]. In the case of the unipotent unitary group, we describe the supercharacters and superclasses in terms of twisted $\mathbb{F}_q$-set partitions.

3.1 Interactions between groups and vector spaces

Before developing the main results of Chapter 3, we establish some necessary background material.

3.1.1 Linear actions of groups on vector spaces

Let G be a finite group acting linearly on a finite dimensional vector space V over a finite field. There is a corresponding linear action on the dual space V^*; for $\lambda \in V^*$, $g \in G$, and $v \in V$, define

$$(g \cdot \lambda)(v) = \lambda(g^{-1} \cdot v).$$

Lemmas 3.1 and 3.2 give properties of the orbits of these actions.

Lemma 3.1 ([11, Lemma 4.1]). *The actions of G on V and V^* have the same number of orbits.*

Proof. For a fixed $g \in G$, define

$$K = \{v \in V \mid g \cdot v = v\}$$

and

$$W = \{v - g \cdot v \mid v \in V\}.$$

Note that K and W are both subspaces of W, and K is the kernel of the surjective linear map

$$\begin{aligned} \varphi : V &\to W \\ v &\mapsto v - g \cdot v. \end{aligned}$$

It follows that $|K| = |V|/|W|$.

Now observe that

$$\begin{aligned} \{\lambda \in V^* \mid g \cdot \lambda = \lambda\} &= \{\lambda \in V^* \mid g^{-1} \cdot \lambda = \lambda\} \\ &= \{\lambda \in V^* \mid \lambda(g \cdot v - v) = 0 \text{ for all } v \in V\} \\ &= \{\lambda \in V^* \mid W \subseteq \ker(\lambda)\}. \end{aligned}$$

It follows that

$$|K| = |V|/|W| = |\{\lambda \in V^* \mid W \subseteq \ker(\lambda)\}| = |\{\lambda \in V^* \mid g \cdot \lambda = \lambda\}|,$$

and g fixes the same number of elements of V and V^* under the respective actions of G. By Burnside's Lemma, the number of orbits of the two actions are equal. □

For $\lambda \in V^*$, define a subspace

$$V_\lambda = \{v \in V \mid g\lambda(v) = \lambda(v) \text{ for all } g \in G\}.$$

Lemma 3.2. *Suppose that $\lambda \in V^*$ is a functional such that $G \cdot \lambda - \lambda$ is a subspace of V^*; then*

$$G \cdot \lambda = \{\mu \in V^* \mid \mu|_{V_\lambda} = \lambda|_{V_\lambda}\}.$$

Proof. There is a bijection

$$\{\text{Subspaces of } V\} \longleftrightarrow \{\text{Subspaces of } V^*\}$$

defined by

$$W \mapsto \{\mu \in V^* \mid \mu(w) = 0 \text{ for all } w \in W\} \text{ and}$$
$$Z \mapsto \{v \in V \mid \mu(v) = 0 \text{ for all } \mu \in Z\},$$

where $W \subseteq V$ and $Z \subseteq V^*$. Note that

$$V_\lambda = \{v \in V \mid \mu(v) = 0 \text{ for all } \mu \in G \cdot \lambda - \lambda\},$$

hence we have

$$G \cdot \lambda - \lambda = \{\mu \in V^* \mid \mu(v) = 0 \text{ for all } v \in V_\lambda\};$$

adding λ to both sides, we have

$$G \cdot \lambda = \{\mu \in V^* \mid \mu|_{V_\lambda} = \lambda|_{V_\lambda}\}. \qquad \square$$

3.1.2 Complex-valued functions of certain p-groups

Let G be a finite group, and let V be a vector space over the finite field $\mathbb{F}_q$ such that there exists a bijection $f : G \to V$. Let $\theta : \mathbb{F}_q^+ \to \mathbb{C}^\times$ be a nontrivial homomorphism. We can use the vector space structure of V to study the space of functions from G to $\mathbb{C}$. Recall that this space has an inner product as defined in Section 2.1; the following lemma is a consequence of [11, Lemma 5.1].

Lemma 3.3. *Let G, V and θ be as above.*

(1) The set of functions $\theta \circ \lambda$, where $\lambda \in V^*$, form an orthonormal basis for the space of functions from V to $\mathbb{C}$.

(2) The set of functions $\theta \circ \lambda \circ f$, where $\lambda \in V^*$, form an orthonormal basis for the space of functions from G to $\mathbb{C}$.

Proof. For $\lambda, \mu \in V^*$,

$$\begin{aligned}\langle \theta \circ \lambda, \theta \circ \mu \rangle &= \frac{1}{|V|} \sum_{v \in V} \theta \circ \lambda(v) \overline{\theta \circ \mu(v)} \\ &= \frac{1}{|V|} \sum_{v \in V} \theta((\lambda - \mu)(v)) \\ &= \delta_{\mu\lambda},\end{aligned}$$

as θ is nontrivial. This implies that the functions $\theta \circ \lambda$ are orthonormal, hence linearly independent, and (1) follows. Similarly,

$$\begin{aligned}\langle \theta \circ \lambda \circ f, \theta \circ \mu \circ f \rangle &= \frac{1}{|G|} \sum_{g \in G} \theta \circ \lambda \circ f(g) \overline{\theta \circ \mu \circ f(g)} \\ &= \frac{1}{|V|} \sum_{v \in V} \theta \circ \lambda(v) \overline{\theta \circ \mu(v)} \\ &= \langle \theta \circ \lambda, \theta \circ \mu \rangle \\ &= \delta_{\mu\lambda},\end{aligned}$$

as θ is nontrivial. This implies that the functions $\theta \circ \lambda \circ f$ are orthonormal, hence linearly independent, and (2) follows. □

The next lemma will be useful in describing certain induced characters.

Lemma 3.4. *Let W be a subspace of a finite dimensional vector space V over $\mathbb{F}_q$ with subspace W, and let $\lambda \in W^*$. Then*

$$\frac{|W|}{|V|} \sum_{\substack{\mu \in V^* \\ \mu|_W = \lambda}} \theta \circ \mu(v) = \begin{cases} (\theta \circ \lambda)(v) & \text{if } v \in W, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let W' be a subspace of V such that $V = W \oplus W'$. Let $v \in V$, and write $v = w + w'$, where $w \in W$ and $w' \in W'$. Then

$$\begin{aligned}\frac{|W|}{|V|} \sum_{\substack{\mu \in V^* \\ \mu|_W = \lambda}} (\theta \circ \mu)(v) &= \frac{|W|}{|V|} \sum_{\substack{\mu \in V^* \\ \mu|_W = \lambda}} (\theta \circ \mu)(w + w') \\ &= \frac{|W|(\theta \circ \lambda)(w)}{|V|} \sum_{\substack{\mu \in V^* \\ \mu|_W = \lambda}} (\theta \circ \mu)(w').\end{aligned}$$

Observe that the set of functionals $\mu|_{W'}$ such that $\mu|_W = \lambda$ is exactly $(W')^*$. Furthermore, for $w' \in W'$,

$$\sum_{\eta \in (W')^*} (\theta \circ \eta)(w') = \begin{cases} |W'| & \text{if } w' = 0, \\ 0 & \text{otherwise,} \end{cases}$$

as θ is nontrivial. □

Corollary 3.5. *If* $f(1) = 0$*, then*

$$\sum_{\lambda \in V^*} \theta \circ \lambda \circ f$$

is the regular character of G*.* □

Each of the unipotent groups we are studying has elements naturally in bijection with the elements of a vector space. We can consider an algebra group $G = 1 + \mathfrak{g}$ along with the bijection

$$\begin{aligned} f : G &\to \mathfrak{g} \\ 1 + x &\mapsto x. \end{aligned}$$

We can also take our group U to be as defined in Section 3.3.1, along with the corresponding Lie algebra $\mathfrak{u}$ and Springer's morphism $f : U \to \mathfrak{u}$ (see Section 3.3.2). In these two cases, we can use the adjoint action of the group on its Lie algebra to understand certain induced representations.

Lemma 3.6. *Suppose that* G *is a finite group,* V *is a vector space over* $\mathbb{F}_q$*, and* $f : G \to V$ *is a bijection. Suppose further that there is an action*

$$\begin{aligned} G \times V &\to V \\ (g, v) &\mapsto g \cdot v \end{aligned}$$

such that $f(hgh^{-1}) = h \cdot f(g)$ *for all* $g, h \in G$*. If* H *is a subgroup of* G *such that* $f(H) = W$ *is a subspace of* V*, and* $\lambda \in W^*$ *is a functional such that* $\theta \circ \lambda \circ f$ *is a class function of* H*, then*

$$\mathrm{Ind}_H^G(\theta \circ \lambda \circ f) = \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{\mu \in V^* \\ \mu|_W = \lambda}} \theta \circ (g \cdot \mu) \circ f.$$

Proof. Define $\gamma : G \to \mathbb{C}$ by

$$\gamma(g) = \begin{cases} (\theta \circ \lambda \circ f)(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 3.4, we have

$$\gamma = \frac{|H|}{|G|} \sum_{\substack{\mu \in V^* \\ \mu|_W = \lambda}} \theta \circ \mu \circ f.$$

For $g \in G$, we have

$$\begin{aligned}\mathrm{Ind}_H^G(\theta\circ\lambda\circ f)(g) &= \frac{1}{|H|}\sum_{\substack{h\in G\\ hgh^{-1}\in H}}(\theta\circ\lambda\circ f)(hgh^{-1})\\ &= \frac{1}{|H|}\sum_{h\in G}\gamma(hgh^{-1})\\ &= \frac{1}{|G|}\sum_{h\in G}\sum_{\substack{\mu\in V^*\\ \mu|_W=\lambda}}(\theta\circ\mu\circ f)(hgh^{-1})\\ &= \frac{1}{|G|}\sum_{h\in G}\sum_{\substack{\mu\in V^*\\ \mu|_W=\lambda}}(\theta\circ(h^{-1}\cdot\mu)\circ f)(g)\\ &= \frac{1}{|G|}\sum_{h\in G}\sum_{\substack{\mu\in V^*\\ \mu|_W=\lambda}}(\theta\circ(h\cdot\mu)\circ f)(g),\end{aligned}$$

using the fact that $f(hgh^{-1}) = h\cdot f(g)$. □

3.2 Supercharacter theories of algebra groups

Let $G = 1+\mathfrak{g}$ be an algebra group over the field $\mathbb{F}_q$, where q is a power of a prime. Diaconis–Isaacs construct a supercharacter theory of G in [11], which we describe here. Define

$$\begin{aligned} f : G &\to \mathfrak{g}\\ g &\mapsto g-1.\end{aligned}$$

Note that G acts on $\mathfrak{g}$ by left and right multiplication; there are corresponding actions of G on the dual $\mathfrak{g}^*$ given by

$$(g\lambda)(x) = \lambda(g^{-1}x) \text{ and } (\lambda g)(x) = \lambda(xg^{-1}),$$

where $g\in G$, $\lambda\in\mathfrak{g}^*$, and $x\in\mathfrak{g}$.

Theorem 3.7 ([11]). *Let $G = 1+\mathfrak{g}$ be an algebra group, and let $\theta : \mathbb{F}_q^+ \to \mathbb{C}^\times$ be a nontrivial homomorphism. For $g\in G$ and $\lambda\in\mathfrak{g}^*$, define*

$$K_g = \{h\in G \mid f(h)\in Gf(g)G\} \quad \textit{and} \quad \chi_\lambda = \frac{|G\lambda|}{|G\lambda G|}\sum_{\mu\in G\lambda G}\theta\circ\mu\circ f.$$

We have the following.

(1) *The functions χ_λ are characters of G.*

(2) *The partition of G given by $\mathcal{K} = \{K_g \mid g \in G\}$, along with $\mathcal{X} = \{\chi_\lambda \mid \lambda \in \mathfrak{g}^*\}$, form a supercharacter theory of G. This supercharacter theory is independent of the choice of θ.*

Remark. Note that the set $G\lambda G$ is the orbit of λ under the action of $G \times G$ on $\mathfrak{g}^*$ defined by $((g,h) \cdot \lambda)(x) = \lambda(g^{-1}xh)$. In particular, $G\lambda$ is the orbit of λ under the action of the normal subgroup $G \times \{1\}$. It follows that $|G\mu| = |G\lambda|$ for all $\mu \in G\lambda G$, and the definition of χ_λ is independent of the choice of representative of $G\lambda G$. The supercharacter theory is independent of θ in that the sets $\mathcal{K}$ and $\{\chi_\lambda \mid \lambda \in \mathfrak{u}^*\}$ do not depend on θ. If a different θ is chosen, the χ_λ will be permuted.

We will present a proof of this result that is somewhat different from that in [11] as motivation for our proof in other types.

Define

$$\mathfrak{j}_\lambda = \{x \in \mathfrak{g} \mid \lambda(yx) = 0 \text{ for all } y \in \mathfrak{g}\},$$

and let $J_\lambda = 1 + \mathfrak{j}_\lambda$.

Lemma 3.8 ([11, Lemma 4.2 (d)]). *With notation as above, we have $G\lambda = \{\mu \in \mathfrak{g}^* \mid \mu|_{\mathfrak{j}_\lambda} = \lambda|_{\mathfrak{j}_\lambda}\}$.*

Proof. By Lemma 3.2, it suffices to show that $G\lambda - \lambda$ is a subspace of $\mathfrak{g}^*$. Let $x, y, z \in \mathfrak{g}$ and $a \in \mathbb{F}_q$; then we have

$$\begin{aligned}(a((1+x)^{-1}\lambda - \lambda) + (1+y)^{-1}\lambda - \lambda)(z) &= \lambda((ax+y)z) \\ &= ((1+ax+y)^{-1}\lambda - \lambda)(z),\end{aligned}$$

and $G\lambda - \lambda$ is a subspace of $\mathfrak{g}^*$. □

Diaconis–Isaacs prove the following result as part of [11, Theorem 5.4].

Lemma 3.9. *The function $\mathrm{Res}^G_{J_\lambda}(\theta \circ \lambda \circ f)$ is a linear character of J_λ.*

Proof. Let $x, y \in \mathfrak{j}_\lambda$; then

$$\begin{aligned}(\theta \circ \lambda \circ f)((1+x)(1+y)) &= \theta(\lambda(x+y+xy)) \\ &= \theta(\lambda(x+y)) \\ &= (\theta \circ \lambda \circ f)(1+x)(\theta \circ \lambda \circ f)(1+y).\end{aligned}$$

□

We can now prove that the functions χ_λ are characters of G.

Proposition 3.10 ([11]). *With χ_λ as defined above, we have*

$$\chi_\lambda = \mathrm{Ind}_{J_\lambda}^G(\mathrm{Res}_{J_\lambda}^G(\theta \circ \lambda \circ f)).$$

Proof. By Lemma 3.6 and Lemma 3.8,

$$\begin{aligned}
\mathrm{Ind}_{J_\lambda}^G(\mathrm{Res}_{J_\lambda}^G(\theta \circ \lambda \circ f)) &= \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{\mu\in\mathfrak{g}^* \\ \mu|_{\mathfrak{j}_\lambda}=\lambda|_{\mathfrak{j}_\lambda}}} \theta \circ g\mu g^{-1} \circ f \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{\mu\in G\lambda} \theta \circ g\mu g^{-1} \circ f \\
&= \frac{|G\lambda|}{|G|^2}\sum_{g\in G}\sum_{h\in G} \theta \circ g(h\lambda)g^{-1} \circ f \\
&= \frac{|G\lambda|}{|G|^2}\sum_{h,g\in G} \theta \circ g\lambda h \circ f \\
&= \frac{|G\lambda|}{|G\lambda G|}\sum_{\mu\in G\lambda G} \theta \circ \mu \circ f.
\end{aligned}$$

□

We now are prepared to prove Theorem 3.7.

Proof of Theorem 3.7. Claim (1) is immediate from Proposition 3.10.

For (2), we need to show that conditions (SCT1)–(SCT3) for a supercharacter theory (see Definition 2.1) are satisfied. For (SCT1), note that $|\mathcal{K}|$ is the number of orbits of the action of $G \times G$ on $\mathfrak{g}$ defined by $(g, h) \cdot x = gxh^{-1}$. At the same time, $|\{\chi_\lambda \mid \lambda \in \mathfrak{g}^*\}|$ is the number of orbits of the corresponding action of $G \times G$ on $\mathfrak{g}^*$. By Lemma 3.1, the number of orbits of the two actions are equal.

To demonstrate that (SCT2) holds, choose $g \in G$ and $\lambda \in \mathfrak{g}^*$; we have that

$$\begin{aligned}\chi_\lambda(g) &= \frac{|G\lambda|}{|G\lambda G|}\sum_{\mu\in G\lambda G}(\theta\circ\mu\circ f)(g)\\ &= \frac{|G\lambda|}{|G|^2}\sum_{h,k\in G}(\theta\circ h\lambda k\circ f)(g)\\ &= \frac{|G\lambda|}{|G|^2}\sum_{h,k\in G}(\theta\circ\lambda)(h^{-1}f(g)k^{-1})\\ &= \frac{|G\lambda|}{|K_g|}\sum_{h\in K_g}(\theta\circ\lambda)(f(h)).\end{aligned}$$

It follows that $\chi_\lambda(g)$ only depends on the superclass of g.

Condition (SCT3) follows from Lemma 3.3 and Corollary 3.5. □

In the case that $G = UT_n(\mathbb{F}_q)$, there is a nice description of the supercharacters and superclasses in terms of $\mathbb{F}_q$-set partitions. Propositions 3.11 and 3.13, as well as Corollary 3.12 and Theorem 3.14, were all initially developed by André (for instance, in [4]), but not phrased in terms of $\mathbb{F}_q$-set partitions. These results can all be found in [22].

Given an $\mathbb{F}_q$-set partition η, define $x_\eta \in \mathfrak{ut}_n(\mathbb{F}_q)$ and $\lambda_\eta \in \mathfrak{ut}_n(\mathbb{F}_q)^*$ by

$$x_\eta = \sum_{i\overset{a}{\frown}j\in\eta} ae_{ij} \quad\text{and}\quad \lambda_\eta(x) = \sum_{i\overset{a}{\frown}j\in\eta} ax_{ij},$$

where $x \in \mathfrak{g}$.

Proposition 3.11 ([22, Section 2.3])**.** *We have the following.*

(1) The set

$$\{x_\eta \mid \eta \text{ is an } \mathbb{F}_q\text{-set partition}\}$$

is a set of orbit representatives for the action of $UT_n(\mathbb{F}_q) \times UT_n(\mathbb{F}_q)$ on $\mathfrak{ut}_n(\mathbb{F}_q)$.

(2) The set

$$\{\lambda_\eta \mid \eta \text{ is an } \mathbb{F}_q\text{-set partition}\}$$

is a set of orbit representatives for the action of $UT_n(\mathbb{F}_q) \times UT_n(\mathbb{F}_q)$ on $\mathfrak{ut}_n(\mathbb{F}_q)^*$. □

For an $\mathbb{F}_q$-set partition η, define $g_\eta \in UT_n(\mathbb{F}_q)$ by $g_\eta = 1 + x_\eta$ and let $\chi_\eta = \chi_{\lambda_\eta}$.

Corollary 3.12 ([22, Section 2.3]). *The sets*

$$\mathcal{X} = \{\chi_\eta \mid \eta \text{ is an } \mathbb{F}_q\text{-set partition}\} \quad \text{and} \quad \mathcal{K} = \{K_{g_\nu} \mid \nu \text{ is an } \mathbb{F}_q\text{-set partition}\}$$

are the supercharacters and superclasses of the supercharacter theory of $UT_n(\mathbb{F}_q)$ defined in Theorem 3.7. □

Next we calculate the dimensions of the supercharacters.

Proposition 3.13 ([22, Equation 2.2]). *Let η be an $\mathbb{F}_q$-set partition; then*

$$\chi_\eta(1) = q^{\sum_{i \overset{a}{\frown} j \in \eta} j-i-1}.$$ □

Finally we calculate the values of the supercharacters on the superclasses.

Theorem 3.14 ([22, Equation 2.1]). *Let η and ν be $\mathbb{F}_q$-set partitions; then we have*

$$\chi_\eta(g_\nu) = \begin{cases} \frac{\chi_\eta(1)}{q^{\mathrm{nst}_\nu^\eta}} \prod\limits_{\substack{i \overset{a}{\frown} j \in \eta \\ i \overset{b}{\frown} j \in \nu}} \theta(ab) & \text{if for } i \overset{a}{\frown} j \in \eta \text{ and } i < k < j, \\ & i \overset{b}{\frown} k, k \overset{b}{\frown} j \notin \nu, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{nst}_\nu^\eta = |\{i < j < k < l \mid j \overset{a}{\frown} k \in \nu, i \overset{b}{\frown} l \in \eta\}|$. □

If $G = 1 + \mathfrak{g}$ is an algebra group and $H = 1 + \mathfrak{h}$ is a left or right ideal subgroup of G, there is a coarser supercharacter theory of H described below.

Proposition 3.15. *Let $\theta : \mathbb{F}_q^+ \to \mathbb{C}^\times$ be a nontrivial homomorphism.*

(1) If H is a left ideal subgroup of G, there is a supercharacter theory of H with superclasses defined by

$$K_g = \{h \in H \mid f(h) \in Gf(g)H\}$$

and supercharacters given by

$$\chi_\lambda = \frac{|G\lambda|}{|G\lambda H|} \sum_{\mu \in G\lambda H} \theta \circ \mu \circ f.$$

(2) *If H is a right ideal subgroup of G, there is a supercharacter theory of H with superclasses defined by*

$$K_g = \{h \in H \mid f(h) \in Hf(g)G\}$$

and supercharacters given by

$$\chi_\lambda = \frac{|G\lambda|}{|H\lambda G|} \sum_{\mu \in H\lambda G} \theta \circ \mu \circ f.$$

Proof. We prove (1); the proof is similar to that of Theorem 3.7.

Let H be a left ideal subgroup of G. For a functional $\lambda \in \mathfrak{h}^*$, define

$$\mathfrak{k}_\lambda = \{x \in \mathfrak{h} \mid \lambda(yx) = 0 \text{ for all } y \in \mathfrak{g}\},$$

and let $K_\lambda = 1 + \mathfrak{k}_\lambda$. We make the following observations.

(1) We have that $G\lambda - \lambda$ is a subspace of $\mathfrak{h}^*$, hence by Lemma 3.2, $G\lambda = \{\mu \in \mathfrak{h}^* \mid \mu|_{\mathfrak{k}_\lambda} = \lambda|_{\mathfrak{k}_\lambda}\}$.

(2) The function $\mathrm{Res}^H_{K_\lambda}(\theta \circ \lambda \circ f)$ is a linear character of K_λ (this is an immediate corollary of Lemma 3.9).

(3) We have that $\chi_\lambda = \mathrm{Ind}^H_{K_\lambda}\mathrm{Res}^H_{K_\lambda}(\theta \circ \lambda \circ f)$ (see Proposition 3.10).

Conditions (SCT1)–(SCT3) follow in an identical manner as in the proof of Theorem 3.7. □

We will refer to this supercharacter theory as the **left (right) ideal supercharacter theory** of H.

Suppose that $G = 1 + \mathfrak{g}$ and that $H = 1 + \mathfrak{h}$ is a two-sided ideal subgroup of G. As two-sided ideals of $\mathfrak{g}$ are unions of orbits of the action of $G \times G$ on $\mathfrak{g}$, H is a supernormal subgroup of G with respect to the algebra group supercharacter theory of G. Note that this is not a characterization of the supernormal subgroups of G unless q is a prime; see [19, Proposition 3.2].

Proposition 3.16. *If H is a two-sided ideal subgroup of G, the supernormal supercharacter theory of H has superclasses*

$$K_g = \{h \in H \mid f(h) \in Gf(g)G\}$$

and supercharacters

$$\chi_\lambda = \sum_{\mu \in G\lambda G} \theta \circ \mu \circ f.$$

This supercharacter theory is also the lattice-theoretic join of the the right and left ideal supercharacter theories of H.

Proof. The fact that this is the supernormal supercharacter theory of H follows directly from Lemma 2.8. Note that the orbits of the action of $G \times G$ on $\mathfrak{h}$ give the smallest partition of $\mathfrak{h}$ into unions of orbits of the actions of $G \times H$ and of $H \times G$ on $\mathfrak{h}$; by Lemma 2.12, this supercharacter theory is the lattice-theoretic join of the the right and left ideal supercharacter theories of H. □

3.3 Supercharacter theories of unipotent groups defined by anti-involutions

In Section 3.3 we construct supercharacter theories of a large collection of unipotent groups. In particular, we apply these constructions to the unipotent orthogonal, symplectic, and unitary groups. In the case of the unipotent unitary groups we calculate the values of the supercharacters on the superclasses.

3.3.1 Subgroups of algebra groups defined by anti-involutions

For q a power of a prime, let $\mathfrak{g}$ be a nilpotent associative algebra of finite dimension over $\mathbb{F}_q$ and define $G = 1 + \mathfrak{g}$. We equip $\mathfrak{g}$ with a Lie algebra structure given by $[x, y] = xy - yx$.

Let

$$\begin{aligned} \dagger : \mathfrak{g} &\to \mathfrak{g} \\ x &\mapsto x^\dagger \end{aligned}$$

be an involutive associative algebra antiautomorphism, and for $x \in \mathfrak{g}$ define $(1+x)^\dagger = 1 + x^\dagger$. Note that this makes $\dagger$ an involutive antiautomorphism of G.

Define

$$U = \{u \in G \mid u^\dagger = u^{-1}\}$$

and

$$\mathfrak{u} = \{x \in \mathfrak{g} \mid x^\dagger = -x\}.$$

Note that $\mathfrak{u}$ is not an associative algebra, although it is closed under the Lie bracket.

For $g \in G$ and $x \in \mathfrak{g}$, define $g \cdot x = gxg^\dagger$. This defines a linear action of G on $\mathfrak{g}$; the action restricts to an action of G on $\mathfrak{u}$, and for $x \in \mathfrak{g}$ and $u \in U$, we have $u \cdot x = uxu^{-1}$.

We can also define a left action of $\mathfrak{g}$ on itself by $x * y = xy + yx^\dagger$. This action restricts to an action of $\mathfrak{g}$ on $\mathfrak{u}$, and for $x \in \mathfrak{u}$ and $y \in \mathfrak{g}$, we have $x * y = [x, y]$.

The motivating examples of groups defined in this manner are the unipotent orthogonal, symplectic, and unitary groups in odd characteristic. For instance, if $G = UT_n(\mathbb{F}_q)$ and $\mathfrak{g} = \mathfrak{ut}_n(\mathbb{F}_q)$, with q odd, we can define an antiautomorphism

$$\begin{aligned} \dagger : \mathfrak{g} &\to \mathfrak{g} \\ x &\mapsto Jx^tJ, \end{aligned}$$

where J is the matrix with ones on the antidiagonal and zeroes elsewhere. Then

$$UO_n(\mathbb{F}_q) = \{u \in UT_n(\mathbb{F}_q) \mid u^\dagger = u^{-1}\}$$

and

$$\mathfrak{uo}_n(\mathbb{F}_q) = \{x \in \mathfrak{ut}_n(\mathbb{F}_q) \mid x^\dagger = -x\}.$$

The unipotent symplectic and unitary groups can be similarly described in terms of antiautomorphisms of the unipotent upper triangular matrices.

3.3.2 Springer morphisms

In order to utilize the Lie algebra structure of $\mathfrak{u}$ to study U, we would like a bijection between U and $\mathfrak{u}$ that preserves useful properties. In the case of an algebra group G, we can use the map $g \mapsto g - 1$ to relate G to $\mathfrak{g}$. In general, however, it is not the case that $U = 1 + \mathfrak{u}$, so we need a variation on this map. André–Neto define a bijection from U to $\mathfrak{u}$ in [6], but we require a map that is invariant under the adjoint action of U.

Definition 3.1. Let $G = 1 + \mathfrak{g}$ be an algebra group, and let $\dagger$ be an antiautomorphism of $\mathfrak{g}$. A **Springer morphism** $f : G \to \mathfrak{g}$ is a bijection such that

(1) $f(U) = \mathfrak{u}$ and $f^{-1}(\mathfrak{u}) = U$;
(2) there exist $a_i \in \mathbb{F}_q$ such that $f(1 + x) = x + \sum_{i=2}^{\infty} a_i x^i$.

The dependence of these conditions on $\dagger$ is implicit in that U and $\mathfrak{u}$ are defined in terms of $\dagger$. Note that condition (2) gives that $f(H) = \mathfrak{h}$ for any algebra subgroup $H = \mathfrak{h} + 1$, and also guarantees that f will be invariant under the adjoint action of G. We require that the coefficient of the x term of $f(1+x)$ be 1 for ease of computation; relaxing this condition would not have any effect on the resulting supercharacter theory.

Springer morphisms are introduced by Springer and Steinberg in [21, III, 3.12] and are utilized by Kawanaka in [17]. Our definition of a Springer morphism is slightly modified from the original definition, but the examples given below are Springer morphisms in the original sense. The logarithm map

$$h(1 + x) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i}$$

is perhaps the most natural choice of a Springer morphism, but is not defined in many characteristics. The map

$$f(1 + x) = 2x(x + 2)^{-1}$$

is, however, a Springer morphism in all odd characteristics. We mention that this is a constant multiple of the map $1 + x \mapsto x(x + 2)^{-1}$, which is often referred to as the **Cayley map** (see, for instance, [17]). We make these statements precise in the following lemma.

Lemma 3.17. *Let q be a power of the prime p, and let f and h be the maps defined above. Let $G = 1 + \mathfrak{g}$ be any algebra group, and let $\dagger$ be any anti-involution of $\mathfrak{g}$. If $x^p = 0$ for all $x \in \mathfrak{g}$, then h is a Springer morphism. If p is odd, then f is a Springer morphism.*

Proof. We prove that f is a Springer morphism in all odd characteristics; the proof that h is a Springer morphism if $x^p = 0$ for all $x \in \mathfrak{g}$ is similar.

Note that f^{-1} is defined by

$$\begin{aligned} f^{-1} : \mathfrak{g} &\to G \\ x &\mapsto \left(1 + \frac{x}{2}\right)\left(1 - \frac{x}{2}\right)^{-1}, \end{aligned}$$

as

$$\begin{aligned} f\left(\left(1 + \frac{x}{2}\right)\left(1 - \frac{x}{2}\right)^{-1}\right) &= 2\left(\left(1 + \frac{x}{2}\right)\left(1 - \frac{x}{2}\right)^{-1} - 1\right)\left(\left(1 + \frac{x}{2}\right)\left(1 - \frac{x}{2}\right)^{-1} + 1\right)^{-1} \\ &= 2\left(\left(1 + \frac{x}{2}\right)\left(1 - \frac{x}{2}\right)^{-1} - 1\right)\frac{1}{2}\left(1 - \frac{x}{2}\right) \\ &= x. \end{aligned}$$

For $x \in \mathfrak{u}$, we have

$$\begin{aligned} (f^{-1}(x))^\dagger &= \left(\left(1 + \frac{x}{2}\right)\left(1 - \frac{x}{2}\right)^{-1}\right)^\dagger \\ &= \left(1 - \frac{x^\dagger}{2}\right)^{-1}\left(1 + \frac{x^\dagger}{2}\right) \\ &= \left(1 + \frac{x}{2}\right)^{-1}\left(1 - \frac{x}{2}\right) \\ &= (f^{-1}(x))^{-1}. \end{aligned}$$

It follows that $f^{-1}(\mathfrak{u}) \subseteq U$.

Conversely, if $x \in \mathfrak{g}$ and $f^{-1}(x) \in U$, then

$$\begin{aligned} f^{-1}(x) &= (f^{-1}(x)^{-1})^\dagger \\ &= \left(\left(1 - \frac{x}{2}\right)\left(1 + \frac{x}{2}\right)^{-1}\right)^\dagger \\ &= \left(\left(1 + \frac{-x^\dagger}{2}\right)\left(1 - \frac{-x^\dagger}{2}\right)^{-1}\right) \\ &= f^{-1}(-x^\dagger). \end{aligned}$$

As f^{-1} is a bijection, $x = -x^\dagger$ and $x \in \mathfrak{u}$. It follows that $f(U) \subseteq \mathfrak{u}$, and in fact $f(U) = \mathfrak{u}$ and $f^{-1}(\mathfrak{u}) = U$.

Condition (2) follows from the fact that

$$f(1+x) = \sum_{i=1}^{\infty} x^i \Big(-\frac{1}{2} \Big)^{i-1}. \qquad \square$$

Lemma 3.17 allows us to assume the existence of a Springer morphism if we are working in odd characteristic, which we will do for the remainder of Section 3.3.

3.3.3 Construction of supercharacter theories

Let q be a power of an odd prime, and let $G = 1 + \mathfrak{g}$ be a pattern subgroup of $UT_n(\mathbb{F}_{q^k})$ for some n and k. For $1 \le i \le n$, recall that $\bar{i} = n + 1 - i$. We consider $\mathfrak{g}$ as an $\mathbb{F}_q$-algebra; let $\dagger$ be an anti-involution of $\mathfrak{g}$ such that $(\alpha e_{ij})^\dagger \in \mathbb{F}_{q^k}^\times e_{\bar{j}\bar{i}}$ for all $\alpha \in \mathbb{F}_{q^k}^\times$. In other words, $\dagger$ reflects the entries of elements of $\mathfrak{g}$ across the antidiagonal, up to a constant multiple; the antiautomorphisms which define the orthogonal, symplectic and unitary groups all have this property. Let

$$U = \{u \in G \mid u^\dagger = u^{-1}\}$$

and

$$\mathfrak{u} = \{x \in \mathfrak{g} \mid x^\dagger = -x\}$$

as above. Recall that there are left actions of G and $\mathfrak{g}$ on $\mathfrak{g}$ defined by

$$g \cdot x = gxg^\dagger \text{ and}$$
$$y * x = yx + xy^\dagger$$

for $g \in G$ and $x, y \in \mathfrak{g}$, along with corresponding contragredient actions on $\mathfrak{g}^*$.

In the construction of the supercharacter theories of algebra groups, the normal subgroup $G \times 1$ of $G \times G$ plays an important role. We need an analogous subgroup of G to construct a supercharacter theory of U. Let

$$\mathfrak{h} = \left\{ x \in \mathfrak{g} \;\middle|\; x_{ij} = 0 \text{ if } j \le \frac{n}{2} \right\},$$

and define $H = \mathfrak{h} + 1$. Note that $\mathfrak{h}$ is a two-sided ideal of $\mathfrak{g}$, hence H is a normal subgroup of G.

Theorem 3.18. *Let f be any Springer morphism and let $\theta : \mathbb{F}_q^+ \to \mathbb{C}^\times$ be a nontrivial homomorphism. For $u \in U$ and $\lambda \in \mathfrak{u}^*$, define*

$$K_u = \{v \in U \mid f(v) \in G \cdot f(u)\} \quad \textit{and} \quad \chi_\lambda = \frac{|H \cdot \lambda|}{|G \cdot \lambda|} \sum_{\mu \in G\cdot\lambda} \theta \circ \mu \circ f.$$

We have the following.

(1) *The functions χ_λ are characters of U.*

(2) *The partition of U given by $\mathcal{K} = \{K_u \mid u \in U\}$, along with $\mathcal{X} = \{\chi_\lambda \mid \lambda \in \mathfrak{u}^*\}$, form a supercharacter theory of U. This supercharacter theory is independent of the choice of θ and f.*

Remark. As H is a normal subgroup of G, $|H\cdot\lambda|$ is independent of the choice of orbit representative of $G \cdot \lambda$. The supercharacter theory is independent of θ and f in that the sets $\mathcal{K}$ and $\{\chi_\lambda \mid \lambda \in \mathfrak{u}^*\}$ do not depend on these functions. If a different θ is chosen or condition (2) in the definition of a Springer morphism is relaxed to allow for other x coefficients, the χ_λ will be permuted. The supercharacter theory is also independent of the choice of subfield of $\mathbb{F}_{q^k}$; that is, if $\mathbb{F}$ is any subfield of $\mathbb{F}_{q^k}$ and $\dagger$ is an antiautomorphism of $\mathfrak{g}$ when viewed as an $\mathbb{F}$-algebra, we get the same supercharacter theory as by considering $\mathfrak{g}$ as an $\mathbb{F}_q$-algebra.

Before proving Theorem 3.18, we need to establish a number of preliminary results.

For a fixed $\lambda \in \mathfrak{g}^*$, we define several subalgebras of $\mathfrak{g}$. Let

$$\begin{aligned}
\mathfrak{l}_\lambda &= \{x \in \mathfrak{g} \mid \lambda(yx) = 0 \text{ for all } y \in \mathfrak{h}\}, \\
\mathfrak{r}_\lambda &= \{x \in \mathfrak{g} \mid \lambda(xy) = 0 \text{ for all } y \in \mathfrak{h}^\dagger\}, \text{ and} \\
\mathfrak{g}_\lambda &= \mathfrak{l}_\lambda \cap \mathfrak{r}_\lambda.
\end{aligned}$$

We also define the corresponding algebra subgroups

$$\begin{aligned}
L_\lambda &= 1 + \mathfrak{l}_\lambda, \\
R_\lambda &= 1 + \mathfrak{r}_\lambda, \text{ and} \\
G_\lambda &= 1 + \mathfrak{g}_\lambda = L_\lambda \cap R_\lambda.
\end{aligned}$$

Lemma 3.19. *With notation as above, we have $H\lambda = \{\mu \in \mathfrak{g}^* \mid \mu|_{\mathfrak{l}_\lambda} = \lambda|_{\mathfrak{l}_\lambda}\}$.*

Proof. By Lemma 3.2, it suffices to show that $H\lambda - \lambda$ is a subspace of $\mathfrak{g}^*$. Let $x, y \in \mathfrak{h}$, $z \in \mathfrak{g}$, and $a \in \mathbb{F}_q$; then we have

$$\begin{aligned}(a((1+x)^{-1}\lambda - \lambda) + (1+y)^{-1}\lambda - \lambda)(z) &= \lambda((ax+y)z) \\ &= ((1+ax+y)^{-1}\lambda - \lambda)(z),\end{aligned}$$

and $H\lambda - \lambda$ is a subspace of $\mathfrak{g}^*$. □

The following lemma shows that any $\lambda \in \mathfrak{g}^*$ behaves nicely with respect to products when restricted to $\mathfrak{g}_\lambda$.

Lemma 3.20. *For any $\lambda \in \mathfrak{g}^*$, we have that $\lambda(xy) = 0$ for all $x, y \in \mathfrak{g}_\lambda$.*

Proof. For $x, y \in \mathfrak{g}_\lambda$, define elements x' and y' of $\mathfrak{g}$ by

$$(x')_{ij} = \begin{cases} x_{ij} & \text{if } j > \frac{n}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(y')_{ij} = \begin{cases} y_{ij} & \text{if } i \leq \frac{n}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$(x'y)_{ij} = \sum_{k > \frac{n}{2}} x_{ik} y_{kj}$$

and

$$(xy')_{ij} = \sum_{k \leq \frac{n}{2}} x_{ik} y_{kj}.$$

It follows that $xy = x'y + xy'$. Observe that $x' \in \mathfrak{h}$ and $y' \in \mathfrak{h}^\dagger$; as $x, y \in \mathfrak{g}_\lambda$, we have

$$\lambda(xy) = \lambda(x'y) + \lambda(xy') = 0.$$

□

The following corollary of Lemma 3.20 will allow us to conclude that our supercharacter theories are independent of the choice of Springer morphism f.

Corollary 3.21. *Let $\lambda \in \mathfrak{g}^*$; then*

(1) the function $\mathrm{Res}^G_{G_\lambda}(\theta \circ \lambda \circ f)$ is a linear character of G_λ; and

(2) if f' is another Springer morphism, we have $\mathrm{Res}^G_{G_\lambda}(\theta \circ \lambda \circ f) = \mathrm{Res}^G_{G_\lambda}(\theta \circ \lambda \circ f')$.

Proof. Let $x, y \in \mathfrak{g}_\lambda$; then

$$f((1+x)(1+y)) = x + y + p(x,y),$$

where $p(x,y)$ is a polynomial in x and y with all terms of degree at least two. By Lemma 3.20, we have $\lambda(p(x,y)) = 0$. It follows that

$$\begin{aligned}(\theta \circ \lambda \circ f)((1+x)(1+y)) &= \theta(\lambda(x+y)) \\ &= \theta(\lambda(x))\theta(\lambda(y)).\end{aligned}$$

At the same time, we have

$$\begin{aligned}(\theta \circ \lambda \circ f)(1+x)(\theta \circ \lambda \circ f)(1+y) &= \theta\Big(\lambda\Big(x + \sum_{i=2}^{\infty} a_i x^i\Big)\Big)\theta\Big(\lambda\Big(y + \sum_{i=2}^{\infty} a_i y^i\Big)\Big) \\ &= \theta(\lambda(x))\theta(\lambda(y)),\end{aligned}$$

and $\mathrm{Res}^G_{G_\lambda}(\theta \circ \lambda \circ f)$ is a linear character of G_λ. Note that

$$\mathrm{Res}^G_{G_\lambda}(\theta \circ \lambda \circ f)(x) = \theta(\lambda(x)),$$

a formula independent of f, proving (2). □

Lemmas 3.22 and 3.23 establish properties of the subgroup H which will be useful in future calculations.

Lemma 3.22. *For $h \in H$ and $x \in \mathfrak{g}$, $h \cdot x = (h-1) * x + x$.*

Proof. Let $h = 1 + y$; then

$$h \cdot x = yxy^\dagger + yx + xy^\dagger + x$$

and

$$(h-1) * x + x = yx + xy^\dagger + x.$$

It suffices to show that $\mathfrak{h}\mathfrak{g}\mathfrak{h}^\dagger = 0$. Note that $\mathfrak{h}\mathfrak{g}\mathfrak{h}^\dagger$ is generated by elements of the form $e_{ij}e_{kl}e_{rs}$ with $j > \frac{n}{2}$ and $r < \frac{n}{2} + 1$. This means that $j \geq r$, and as $k < l$, $e_{ij}e_{kl}e_{rs} = 0$. □

Lemma 3.23. *We have that $G = HU$.*

Proof. Let $x \in \mathfrak{g}$; define $y \in \mathfrak{u}$ by

$$y_{ij} = \begin{cases} x_{ij} & \text{if } j \leq \frac{n}{2}, \\ -(x^\dagger)_{ij} & \text{if } i \geq \frac{n}{2}+1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $x - y \in \mathfrak{h}$, hence $\mathfrak{g} = \mathfrak{h} + \mathfrak{u}$. It follows that

$$|HU| = \frac{|H||U|}{|H \cap U|} = \frac{|f(H)||f(U)|}{|f(H \cap U)|} = \frac{|\mathfrak{h}||\mathfrak{u}|}{|\mathfrak{h} \cap \mathfrak{u}|} = |\mathfrak{g}| = |G|,$$

and the result follows. □

In order to use the above results to study U, we will need to extend the elements of $\mathfrak{u}^*$ to elements of $\mathfrak{g}^*$. There are of course many possible ways to do this, however there is one natural choice in our case.

Lemma 3.24. *Given any $\lambda \in \mathfrak{u}^*$, there exists a unique $\eta \in \mathfrak{g}^*$ such that $\eta|_{\mathfrak{u}} = \lambda$ and $\eta(x) = -\eta(x^\dagger)$ for all $x \in \mathfrak{g}$.*

Proof. For $x \in \mathfrak{g}$, let $\eta(x) = \frac{1}{2}\lambda(x - x^\dagger)$. This definition makes sense as $x - x^\dagger \in \mathfrak{u}$ for all $x \in \mathfrak{g}$. Note that $\eta \in \mathfrak{g}^*$ and $\eta(x) = \lambda(x)$ for all $x \in \mathfrak{u}$. From the definition of η, we have that $\eta(x) = -\eta(x^\dagger)$ for all $x \in \mathfrak{g}$.

The uniqueness of η follows from the fact that $\mu(x - x^\dagger) = 2\mu(x)$ for any μ satisfying $\mu(x) = -\mu(x^\dagger)$. This means that such μ is determined only by its values on $\mathfrak{u}$, hence η is unique. □

Lemma 3.25. *Let $\lambda \in \mathfrak{u}^*$, and let $\eta \in \mathfrak{g}^*$ be the extension of λ described above. Then the sets $\{\mu|_{\mathfrak{u}} \mid \mu \in H\eta\}$ and $H \cdot \lambda$ are equal.*

Proof. Let $h \in H$ and $x \in \mathfrak{u}$; then

$$\begin{aligned} h^{-1} \cdot \lambda(x) &= \lambda(h \cdot x) \\ &= \eta((h-1) * x + x) \end{aligned}$$

by Lemma 3.22. Note that

$$\begin{aligned}\eta((h-1)*x+x) &= \eta((h-1)x+x(h-1)^\dagger)+\eta(x)\\ &= 2\eta((h-1)x)+\eta(x)\end{aligned}$$

by the fact that $\eta(y)=-\eta(y^\dagger)$ for all $y\in\mathfrak{g}$. Finally, we have

$$\begin{aligned}2\eta((h-1)x)+\eta(x) &= \eta((2(h-1)+1)x)\\ &= (2h-1)^{-1}\eta(x).\end{aligned}$$

The claim follows from the fact that the map $h\mapsto(2h-1)$ is a bijection from H to H. □

We are now ready to prove that for $\lambda\in\mathfrak{u}^*$, the function

$$\chi_\lambda=\frac{|H\cdot\lambda|}{|G\cdot\lambda|}\sum_{\mu\in G\cdot\lambda}\theta\circ\mu\circ f$$

is a character of U. Let η be the element of $\mathfrak{g}^*$ associated to λ as in Lemma 3.24, and define $U_\lambda=U\cap G_\eta$ and $\mathfrak{u}_\lambda=f(U_\lambda)$.

Proposition 3.26. *We have that*

$$\chi_\lambda=\mathrm{Ind}_{U_\lambda}^{U}(\mathrm{Res}_{U_\lambda}^{U}(\theta\circ\lambda\circ f)).$$

Proof. Note that $\mathrm{Res}_{U_\lambda}^{U}(\theta\circ\lambda\circ f)=\mathrm{Res}_{U_\lambda}^{G}(\theta\circ\eta\circ f)$; as $U_\lambda\subseteq G_\eta$, by Corollary 3.21 the function $\mathrm{Res}_{U_\lambda}^{U}(\theta\circ\lambda\circ f)$ is a linear character of U_λ.

From the definitions of $\mathfrak{u}_\lambda$ and $\mathfrak{g}_\eta$, we have $\mathfrak{u}_\lambda=\mathfrak{u}\cap\mathfrak{g}_\eta$; in fact, it is the case that $\mathfrak{u}_\lambda=\mathfrak{u}\cap\mathfrak{l}_\eta$. This is a consequence of the fact that if $x\in\mathfrak{u}$ and $x\in\mathfrak{l}_\eta$, then $x\in\mathfrak{r}_\eta$. It follows that

$$\begin{aligned}\{\mu\in\mathfrak{u}^*\mid\mu(x)=\lambda(x)\text{ for all }x\in\mathfrak{u}_\lambda\} &= \{\kappa|_{\mathfrak{u}}\mid\kappa(x)=\eta(x)\text{ for all }x\in\mathfrak{l}_\eta\}\\ &= \{\kappa|_{\mathfrak{u}}\mid\kappa\in H\eta\}\\ &= H\cdot\lambda\end{aligned}$$

by Lemma 3.19 and Lemma 3.25. By Lemma 3.6, we have

$$\begin{aligned}\mathrm{Ind}_{U_\lambda}^{U}(\mathrm{Res}_{U_\lambda}^{U}(\theta\circ\lambda\circ f)) &= \frac{1}{|U|}\sum_{u\in U}\sum_{\substack{\mu\in\mathfrak{u}^*\\ \mu|_{\mathfrak{u}_\lambda}=\lambda|_{\mathfrak{u}_\lambda}}}\theta\circ u\mu u^{-1}\circ f\\ &= \frac{1}{|U|}\sum_{u\in U}\sum_{\mu\in H\cdot\lambda}\theta\circ u\mu u^{-1}\circ f\\ &= \frac{|H\cdot\lambda|}{|H||U|}\sum_{u\in U}\sum_{h\in H}\theta\circ u(h\cdot\lambda)u^{-1}\circ f.\end{aligned}$$

Recall that $u\cdot x = uxu^{-1}$ for all $u\in U$ and $x\in\mathfrak{g}$, and by Lemma 3.23, we have $HU = G$. It follows that

$$\begin{aligned}\frac{|H\cdot\lambda|}{|H||U|}\sum_{u\in U}\sum_{h\in H}\theta\circ u(h\cdot\lambda)u^{-1}\circ f &= \frac{|H\cdot\lambda|}{|H||U|}\sum_{u\in U}\sum_{h\in H}\theta\circ(uh\cdot\lambda)\circ f\\ &= \frac{|H\cdot\lambda|}{|G|}\sum_{g\in G}\theta\circ(g\cdot\lambda)\circ f\\ &= \frac{|H\cdot\lambda|}{|G\cdot\lambda|}\sum_{\mu\in G\cdot\lambda}\theta\circ\mu\circ f,\end{aligned}$$

which is by definition χ_λ. □

We are now prepared to prove Theorem 3.18.

Proof of Theorem 3.18. Claim (1) is immediate from Proposition 3.26.

For (2), we need to show that conditions (SCT1)–(SCT3) for a supercharacter theory (see Definition 2.1) are satisfied. For (SCT1), note that $|\mathcal{K}|$ is the number of orbits of the action of G on $\mathfrak{u}$. At the same time, $|\{\chi_\lambda \mid \lambda\in\mathfrak{u}^*\}|$ is the number of orbits of the corresponding action of G on $\mathfrak{u}^*$. By Lemma 3.1, the number of orbits of the two actions are equal.

To demonstrate that (SCT2) holds, choose $u\in U$ and $\lambda\in\mathfrak{u}^*$; we have that

$$\begin{aligned}\chi_\lambda(u) &= \frac{|H\cdot\lambda|}{|G\cdot\lambda|}\sum_{\mu\in G\cdot\lambda}(\theta\circ\mu\circ f)(u)\\ &= \frac{|H\cdot\lambda|}{|G|}\sum_{g\in G}(\theta\circ g\cdot\lambda\circ f)(u)\\ &= \frac{|H\cdot\lambda|}{|G|}\sum_{g\in G}(\theta\circ\lambda)(g^{-1}\cdot f(u))\\ &= \frac{|H\cdot\lambda|}{|K_u|}\sum_{v\in K_u}(\theta\circ\lambda)(f(v)).\end{aligned}$$

It follows that $\chi_\lambda(u)$ only depends on the superclass of u.

Condition (SCT3) follows from Lemma 3.3 and Corollary 3.5. □

There is a connection between the supercharacter theory of U given by Theorem 3.18 and the supercharacter theory of the algebra group G given by Theorem 3.7.

Theorem 3.27. *The superclasses of U are exactly the sets of the form $U\cap K_g$, where K_g is some superclass of G.*

Proof. Note that for each $u\in U$ there exists $g\in G$ such that $f(u)=(u-1)g$. It follows that each superclass of U is contained in some superclass of G. We want to show that each superclass of G contains at most one superclass of U.

Note that for $g,h\in G$ and $x\in\mathfrak{g}$, $gxh=h^\dagger(h^{-\dagger}gx)h$. As such, it suffices to show that if $x\in\mathfrak{u}$ and $gx\in\mathfrak{u}$ for some $g\in G$, then $gx=hxh^\dagger$ for some $h\in G$.

Assume that $x,gx\in\mathfrak{u}$; then

$$gx=-(gx)^\dagger=-x^\dagger g^\dagger=xg^\dagger.$$

Let k be an odd integer such that $g^{2k}=g$ (such k must exist as g has odd order). Then

$$gx=g^{2k}x=g^kx(g^k)^\dagger;$$

let $h=g^k$. □

3.3.4 Supercharacter theories of unipotent orthogonal groups

Let J be the $n \times n$ matrix with ones on the antidiagonal and zeroes elsewhere, and let x^t denote the transpose of a matrix x. For q a power of an odd prime, define

$$O_n(\mathbb{F}_q) = \{g \in GL_n(\mathbb{F}_q) \mid g^{-1} = Jg^tJ\}$$

along with the corresponding Lie algebra

$$\mathfrak{o}_n(\mathbb{F}_q) = \{x \in \mathfrak{gl}_n(\mathbb{F}_q) \mid -x = Jx^tJ\}.$$

Let $UT_n(\mathbb{F}_q)$ be the set of unipotent upper triangular matrices of $GL_n(\mathbb{F}_q)$ and $\mathfrak{ut}_n(\mathbb{F}_q)$ be the set of strictly upper triangular matrices of $\mathfrak{gl}_n(\mathbb{F}_q)$. Define

$$UO_n(\mathbb{F}_q) = UT_n(\mathbb{F}_q) \cap O_n(\mathbb{F}_q) \text{ and}$$
$$\mathfrak{uo}_n(\mathbb{F}_q) = \mathfrak{ut}_n(\mathbb{F}_q) \cap \mathfrak{o}_n(\mathbb{F}_q).$$

Define an antiautomorphism $\dagger$ of $\mathfrak{ut}_n(\mathbb{F}_q)$ by $x^\dagger = Jx^tJ$. We have that

$$UO_n(\mathbb{F}_q) = \{g \in UT_{2n}(\mathbb{F}_q) \mid g^{-1} = g^\dagger\} \text{ and}$$
$$\mathfrak{uo}_n(\mathbb{F}_q) = \{x \in \mathfrak{ut}_{2n}(\mathbb{F}_q) \mid -x = x^\dagger\}.$$

Define K_u and χ_λ as in Theorem 3.18 with $U = UO_n(\mathbb{F}_q)$ and $\mathfrak{u} = \mathfrak{uo}_n(\mathbb{F}_q)$. By Theorem 3.18, there is a supercharacter theory of $UO_n(\mathbb{F}_q)$ with superclasses $\{K_u\}$ and supercharacters $\{\chi_\lambda\}$.

In [7], André–Neto construct a supercharacter theory of $UO_n(\mathbb{F}_q)$. They show that their superclasses are the sets of the form $UO_n(\mathbb{F}_q) \cap K_g$, where K_g is a superclass of $UT_n(\mathbb{F}_q)$ under the algebra group supercharacter theory. In particular, we have the following corollary of Theorem 3.27.

Corollary 3.28. *The supercharacter theory of $UO_n(\mathbb{F}_q)$ defined above coincides with that of André–Neto in [7].* □

We can also construct supercharacter theories of certain subgroups of $UO_n(\mathbb{F}_q)$ using this method. We will call a poset $\mathcal{P}$ a **mirror poset** if $i \preceq_{\mathcal{P}} j$ implies that $\bar{j} \preceq_{\mathcal{P}} \bar{i}$ (recall that

$\bar{i} = n - i + 1$). The antiautomorphism $\dagger$ as defined above restricts to an antiautomorphism of $U_\mathcal{P}$ for any mirror poset. Furthermore,

$$UO_n(\mathbb{F}_q) \cap U_\mathcal{P} = \{g \in U_\mathcal{P} \mid g^{-1} = g^\dagger\} \text{ and}$$
$$\mathfrak{uo}_n(\mathbb{F}_q) \cap \mathfrak{u}_\mathcal{P} = \{x \in \mathfrak{u}_\mathcal{P} \mid -x = x^\dagger\}.$$

Define K_u and χ_λ as in Theorem 3.18 with $U = UO_n(\mathbb{F}_q) \cap U_\mathcal{P}$ and $\mathfrak{u} = \mathfrak{uo}_n(\mathbb{F}_q) \cap \mathfrak{u}_\mathcal{P}$. By Theorem 3.18, there is a supercharacter theory of $UO_n(\mathbb{F}_q) \cap U_\mathcal{P}$ with superclasses $\{K_u\}$ and supercharacters $\{\chi_\lambda\}$. By Theorem 3.27, the superclasses are of the form $K_g \cap UO_n(\mathbb{F}_q)$ where K_g is a superclass of $U_\mathcal{P}$ in the algebra group supercharacter theory. In particular, if U is the unipotent radical of a parabolic subgroup of $O_n(\mathbb{F}_q)$ then $U = UO_n(\mathbb{F}_q) \cap U_\mathcal{P}$ for some mirror poset $\mathcal{P}$.

There are two important examples of a subgroup obtained from a mirror poset in type D. First, let $\mathcal{P}$ be the mirror poset on $[2n]$ defined by

$$i \preceq_\mathcal{P} j \text{ if } i \leq j \text{ and } (i, j) \neq (n, n+1).$$

Then $UO_{2n}(\mathbb{F}_q) \cap U_\mathcal{P} = UO_{2n}(\mathbb{F}_q)$, and we get a second supercharacter theory of $UO_{2n}(\mathbb{F}_q)$ that is at least as fine as the one originally defined. This new supercharacter theory is in fact strictly finer than the original; the elements $e_{1,n} - e_{n+1,2n}$ and $(e_{1,n} - e_{n+1,2n}) + (e_{1,n+1} - e_{n,2n})$ of $\mathfrak{u}$ are in the same orbit under the action of $UT_{2n}(\mathbb{F}_q)$ on $\mathfrak{uo}_{2n}(\mathbb{F}_q)$, but in different orbits under the action of $U_\mathcal{P}$ on $\mathfrak{uo}_{2n}(\mathbb{F}_q)$.

We can also consider the poset $\mathcal{P}$ on $[2n]$ defined by

$$i \preceq_\mathcal{P} j \text{ if } i \leq j \leq n \text{ or } n+1 \leq i \leq j.$$

In this case, $UO_{2n}(\mathbb{F}_q) \cap U_\mathcal{P} \cong UT_n(\mathbb{F}_q)$, and the supercharacter theory obtained is the algebra group supercharacter theory.

3.3.5 Supercharacter theories of unipotent symplectic groups

Let J be the $n \times n$ matrix with ones on the antidiagonal and zeroes elsewhere, and define

$$\Omega = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}.$$

For q a power of an odd prime, define

$$Sp_{2n}(\mathbb{F}_q) = \{g \in GL_{2n}(\mathbb{F}_q) \mid g^{-1} = -\Omega g^t \Omega\}$$

along with the corresponding Lie algebra

$$\mathfrak{sp}_{2n}(\mathbb{F}_q) = \{x \in \mathfrak{gl}_{2n}(\mathbb{F}_q) \mid -x = -\Omega x^t \Omega\}.$$

Let $UT_{2n}(\mathbb{F}_q)$ be the set of unipotent upper triangular matrices of $GL_{2n}(\mathbb{F}_q)$, and $\mathfrak{ut}_{2n}(\mathbb{F}_q)$ be the set of strictly upper triangular matrices of $\mathfrak{gl}_{2n}(\mathbb{F}_q)$. Define

$$USp_{2n}(\mathbb{F}_q) = UT_{2n}(\mathbb{F}_q) \cap Sp_{2n}(\mathbb{F}_q) \text{ and}$$
$$\mathfrak{usp}_{2n}(\mathbb{F}_q) = \mathfrak{ut}_{2n}(\mathbb{F}_q) \cap \mathfrak{sp}_{2n}(\mathbb{F}_q).$$

Define an antiautomorphism $\dagger$ of $\mathfrak{ut}_{2n}(\mathbb{F}_q)$ by $x^\dagger = -\Omega x^t \Omega$. We have that

$$USp_{2n}(\mathbb{F}_q) = \{g \in UT_{2n}(\mathbb{F}_q) \mid g^{-1} = g^\dagger\} \text{ and}$$
$$\mathfrak{usp}_{2n}(\mathbb{F}_q) = \{x \in \mathfrak{ut}_{2n}(\mathbb{F}_q) \mid -x = x^\dagger\}.$$

Define K_u and χ_λ as in Theorem 3.18 with $U = USp_{2n}(\mathbb{F}_q)$ and $\mathfrak{u} = \mathfrak{usp}_{2n}(\mathbb{F}_q)$. By Theorem 3.18, there is a supercharacter theory of $USp_{2n}(\mathbb{F}_q)$ with superclasses $\{K_u\}$ and supercharacters $\{\chi_\lambda\}$.

In [7], André–Neto have also constructed supercharacter theories of $USp_{2n}(\mathbb{F}_q)$. As was the case with the unipotent orthogonal groups, their supercharacter theories are the sets of the form $USp_{2n}(\mathbb{F}_q) \cap K_g$, where K_g is a superclass of $UT_{2n}(\mathbb{F}_q)$ under the algebra group supercharacter theory. In particular, we have the following corollary of Theorem 3.27.

Corollary 3.29. *The supercharacter theory of $USp_{2n}(\mathbb{F}_q)$ defined above coincides with that of André–Neto in [7].* □

We can also construct supercharacter theories of certain subgroups of $USp_{2n}(\mathbb{F}_q)$ just as we did for $UO_n(\mathbb{F}_q)$. The antiautomorphism $\dagger$ as defined above restricts to an antiautomorphism of $U_{\mathcal{P}}$ for any mirror poset. Furthermore,

$$USp_{2n}(\mathbb{F}_q) \cap U_{\mathcal{P}} = \{g \in U_{\mathcal{P}} \mid g^{-1} = g^{\dagger}\} \text{ and}$$
$$\mathfrak{usp}_{2n}(\mathbb{F}_q) \cap \mathfrak{u}_{\mathcal{P}} = \{x \in \mathfrak{u}_{\mathcal{P}} \mid -x = x^{\dagger}\}.$$

Define K_u and χ_λ as in Theorem 3.18 with $U = USp_{2n}(\mathbb{F}_q) \cap U_{\mathcal{P}}$ and $\mathfrak{u} = \mathfrak{usp}_{2n}(\mathbb{F}_q) \cap \mathfrak{u}_{\mathcal{P}}$. By Theorem 3.18, there is a supercharacter theory of $USp_{2n}(\mathbb{F}_q) \cap U_{\mathcal{P}}$ with superclasses $\{K_u\}$ and supercharacters $\{\chi_\lambda\}$. By Theorem 3.27, the superclasses are of the form $K_g \cap USp_{2n}(\mathbb{F}_q)$ where K_g is a superclass of $U_{\mathcal{P}}$ in the algebra group supercharacter theory. In particular, if U is the unipotent radical of a parabolic subgroup of $Sp_{2n}(\mathbb{F}_q)$ then $U = USp_{2n}(\mathbb{F}_q) \cap U_{\mathcal{P}}$ for some mirror poset $\mathcal{P}$.

3.3.6 Supercharacter theories of unipotent unitary groups

Let q be a power of an odd prime, and for $x \in \mathfrak{gl}_n(\mathbb{F}_{q^2})$, define $\overline{x}$ by $(\overline{x})_{ij} = (x_{ij})^q$. Let

$$U_n(\mathbb{F}_{q^2}) = \{g \in GL_n(\mathbb{F}_{q^2}) \mid g^{-1} = J\overline{g}^t J\} \text{ and}$$
$$\mathfrak{u}_n(\mathbb{F}_{q^2}) = \{x \in \mathfrak{gl}_n(\mathbb{F}_{q^2}) \mid -x = J\overline{x}^t J\},$$

and let

$$UU_n(\mathbb{F}_{q^2}) = U_n(\mathbb{F}_{q^2}) \cap UT_n(\mathbb{F}_{q^2}) \text{ and}$$
$$\mathfrak{uu}_n(\mathbb{F}_{q^2}) = \mathfrak{u}_n(\mathbb{F}_{q^2}) \cap \mathfrak{ut}_n(\mathbb{F}_{q^2}).$$

The group $U_n(\mathbb{F}_{q^2})$ is the group of unitary $n \times n$ matrices over $\mathbb{F}_{q^2}$. In Section 3.3.6 we construct a supercharacter theory of $UU_n(\mathbb{F}_{q^2})$ using Theorem 3.18 and calculate the values of the supercharacters on the superclasses.

The map $x^{\dagger} = J\overline{x}^t J$ defines an antiautomorphism of $\mathfrak{ut}_n(\mathbb{F}_{q^2})$ if we consider $\mathfrak{ut}_n(\mathbb{F}_{q^2})$ as an $\mathbb{F}_q$-algebra. We have that

$$UU_n(\mathbb{F}_{q^2}) = \{g \in UT_n(\mathbb{F}_{q^2}) \mid g^{-1} = g^\dagger\} \text{ and}$$
$$\mathfrak{uu}_n(\mathbb{F}_{q^2}) = \{x \in \mathfrak{ut}_n(\mathbb{F}_{q^2}) \mid -x = x^\dagger\}.$$

Define K_u and χ_λ as in Theorem 3.18 with $U = UU_n(\mathbb{F}_{q^2})$ and $\mathfrak{u} = \mathfrak{uu}_n(\mathbb{F}_{q^2})$. By Theorem 3.18, there is a supercharacter theory of $UU_n(\mathbb{F}_{q^2})$ with superclasses $\{K_u\}$ and supercharacters $\{\chi_\lambda\}$.

As with the orthogonal and symplectic cases, by Theorem 3.27 the superclasses are of the form $K_g \cap UU_n(\mathbb{F}_{q^2})$, where K_g is a superclass of $UT_n(\mathbb{F}_{q^2})$ under the algebra group supercharacter theory.

We can once again construct supercharacter theories of certain subgroups of $UU_n(\mathbb{F}_{q^2})$. The antiautomorphism $\dagger$ as defined above restricts to an antiautomorphism of $U_{\mathcal{P}}$ for any mirror poset. Furthermore,

$$UU_n(\mathbb{F}_{q^2}) \cap U_{\mathcal{P}} = \{g \in U_{\mathcal{P}} \mid g^{-1} = g^\dagger\} \text{ and}$$
$$\mathfrak{uu}_n(\mathbb{F}_{q^2}) \cap \mathfrak{u}_{\mathcal{P}} = \{x \in \mathfrak{u}_{\mathcal{P}} \mid -x = x^\dagger\}.$$

Define K_u and χ_λ as in Theorem 3.18 with $U = UU_n(\mathbb{F}_{q^2}) \cap U_{\mathcal{P}}$ and $\mathfrak{u} = \mathfrak{uu}_n(\mathbb{F}_{q^2}) \cap \mathfrak{u}_{\mathcal{P}}$. By Theorem 3.18, there is a supercharacter theory of $UU_n(\mathbb{F}_{q^2}) \cap U_{\mathcal{P}}$ with superclasses $\{K_u\}$ and supercharacters $\{\chi_\lambda\}$. By Theorem 3.27, the superclasses are of the form $K_g \cap UU_n(\mathbb{F}_{q^2})$, where K_g is a superclass of $U_{\mathcal{P}}$ in the algebra group supercharacter theory.

We now describe the superclasses and supercharacters of $U = UU_n(\mathbb{F}_{q^2})$ in terms of twisted $\mathbb{F}_q$-set partitions.

Lemma 3.30. *Each superclass of U contains exactly one element u with the property that $f(u)$ has at most one nonzero entry in each row and column.*

Proof. The superclasses of $UT_n(\mathbb{F}_{q^2})$ contain exactly one element u such that $f(u)$ has at most one nonzero entry in each row and column. It follows that the superclasses of U contain at most one element with this property. Let $x \in \mathfrak{u}$; we want to row-reduce x using the action of $UT_n(\mathbb{F}_{q^2})$.

Suppose that (i, j) is such that

(1) $x_{ij} \neq 0$,

(2) there exists $k < i$ with $x_{kj} \neq 0$, and

(3) there is no other pair (l, m) satisfying properties (1) and (2) with $l \geq i$ and $m \leq j$.

If no such (i, j) exists, then x has at most one nonzero entry in each row and column. Assume that such a pair (i, j) exists. If $k \neq \bar{j}$, we consider

$$y = \left(1 - \frac{x_{kj}}{x_{ij}} e_{ki}\right) \cdot x.$$

If $k = \bar{j}$, we consider

$$y = \left(1 - \frac{x_{kj}}{x_{ij} + x_{ij}^q} e_{ki}\right) \cdot x.$$

The element y is in the same superclass as x, but has $y_{kj} = 0$. Repeated application of this process will yield an element with at most one nonzero entry in each row and column. $\square$

To each twisted $\mathbb{F}_q$-set partition η we assign the element $x_\eta \in \mathfrak{u}$ defined by

$$(x_\eta)_{ij} = \begin{cases} a & \text{if } i \overset{a}{\frown} j \in \eta, \\ 0 & \text{otherwise.} \end{cases}$$

and the element $u_\eta \in U$ such that $f(u_\eta) = x_\eta$. Note that x_η is in fact an element of $\mathfrak{u}$ and has at most one entry in each nonzero row and column.

Corollary 3.31. *The elements*

$$\{u_\eta \mid \eta \text{ is a twisted } \mathbb{F}_q\text{-partition}\}$$

are a set of superclass representatives. $\square$

As there are equal numbers of superclasses and supercharacters, the supercharacters can also be indexed by twisted $\mathbb{F}_q$-set partitions. Given a twisted $\mathbb{F}_q$-set partition, define $\lambda_\eta \in \mathfrak{u}^*$ by

$$\lambda_\eta(x) = \sum_{i \overset{a}{\frown} j \in \eta} a x_{ij}.$$

Lemma 3.32. *The set*

$$\{\lambda_\eta \mid \eta \textit{ is a twisted } \mathbb{F}_q\textit{-partition}\}$$

is a set of orbit representatives for the action of $UT_n(\mathbb{F}_{q^2})$ on $\mathfrak{u}^*$.

Proof. It suffices to show that if η and ν are distinct twisted $\mathbb{F}_q$-set partitions then λ_η and λ_ν are in different orbits under the action of $UT_n(\mathbb{F}_{q^2})$ on $\mathfrak{u}^*$.

We can extend λ_η and λ_ν to elements of $\mathfrak{ut}_n(\mathbb{F}_{q^2})^*$ defined by

$$\widetilde{\lambda}_\eta(x) = \sum_{i \overset{a}{\frown} j \in \eta} a x_{ij} \text{ and}$$
$$\widetilde{\lambda}_\nu(x) = \sum_{i \overset{a}{\frown} j \in \nu} a x_{ij}.$$

Suppose that there exists $g \in UT_n(\mathbb{F}_{q^2})$ such that $g \cdot \lambda_\eta = \lambda_\nu$; then $g\widetilde{\lambda}_\eta J \overline{g}^t J = \widetilde{\lambda}_\nu$, and $\widetilde{\lambda}_\eta$ and $\widetilde{\lambda}_\nu$ are in the same orbit of the action of $UT_n(\mathbb{F}_{q^2}) \times UT_n(\mathbb{F}_{q^2})$ on $\mathfrak{ut}_n(\mathbb{F}_{q^2})^*$. This contradicts Lemma 3.11, so in fact λ_η and λ_ν are in different orbits under the action of $UT_n(\mathbb{F}_{q^2})$ on $\mathfrak{u}^*$. □

For a twisted $\mathbb{F}_q$-set partition, we define $\chi_\eta = \chi_{\lambda_\eta}$.

Corollary 3.33. *The sets*

$$\mathcal{K} = \{K_{u_\eta} \mid \eta \textit{ is a twisted } \mathbb{F}_q\textit{-partition}\} \quad \textit{and} \quad \mathcal{X} = \{\chi_\eta \mid \eta \textit{ is a twisted } \mathbb{F}_q\textit{-partition}\}$$

are the superclasses and supercharacters of $UU_n(\mathbb{F}_{q^2})$. □

We now calculate $\chi_\eta(u_\nu)$, where η and ν are twisted $\mathbb{F}_q$-set partitions. We will call a supercharacter **elementary** if it corresponds to a twisted $\mathbb{F}_q$-set partition of the form $\eta = \{i \overset{a}{\frown} j \cup \bar{j} \overset{-a^q}{\frown} \bar{i}\}$ with $i \neq \bar{j}$ or of the form $\eta = \{i \overset{a}{\frown} \bar{i}\}$ with $a^q + a = 0$. In order to simplify calculations, we will show that every supercharacter can be written as a product of distinct elementary supercharacters. This is analogous to the method used in types A, B, C, and D (see [5, 7]).

Recall that, for $\lambda \in \mathfrak{u}^*$, the supercharacter χ_λ is induced from a linear character of the subgroup U_λ (see Section 3.3.3 for specifics). The subgroup U_λ is associated to a subalgebra $\mathfrak{u}_\lambda$ of $\mathfrak{u}$. We can describe this subalgebra in terms of the twisted $\mathbb{F}_q$-set partition associated to χ_λ.

Lemma 3.34. *Let η be a twisted $\mathbb{F}_q$-set partition; then*

$$\mathfrak{u}_{\lambda_\eta} = \left\{ x \in \mathfrak{u} \;\middle|\; x_{ij} = 0 \text{ if } i \frown k \in \eta \text{ with } j < k \text{ and } j \leq \frac{n+1}{2} \right\}.$$

Proof. Recall that $\mathfrak{u}_{\lambda_\eta} = \mathfrak{l}_\mu \cap \mathfrak{u}$, where $\mu \in \mathfrak{g}^*$ is the functional defined by $\mu(x) = \frac{1}{2}\lambda_\eta(x - x^\dagger)$ and

$$\mathfrak{l}_\mu = \{x \in \mathfrak{g} \mid \mu(yx) = 0 \text{ for all } y \in \mathfrak{h}\}$$

as in Section 3.3.3. We have that

$$\mathfrak{l}_\mu = \left\{ x \in \mathfrak{g} \;\middle|\; x_{ij} = 0 \text{ if } i \frown k \in \eta \text{ with } j < k \text{ and } j \leq \frac{n+1}{2} \right\}.$$

□

For a twisted $\mathbb{F}_q$-set partition η, we can write η as a disjoint union of twisted $\mathbb{F}_q$-set partitions of the form $\{i \overset{a}{\frown} j \cup \bar{j} \overset{-a^q}{\frown} \bar{i}\}$ with $i \neq \bar{j}$ or of the form $\{i \overset{a}{\frown} \bar{i}\}$ with $a^q + a = 0$. In other words, there exists m such that

$$\eta = \bigsqcup_{r=1}^{m} \eta_r$$

with each η_r of the described form. For $1 \leq r \leq m$, define $\lambda_r = \lambda_{\eta_r}$.

Lemma 3.35. *With notation as above,*

(1) $\mathfrak{u}_{\lambda_\eta} = \bigcap_{r=1}^{m} \mathfrak{u}_{\lambda_r}$*, and for any* $s \geq 1$, $\mathfrak{u} = \mathfrak{u}_{s+1} + \bigcap_{r \leq s} \mathfrak{u}_{\mathfrak{r}}$*; and*

(2) $U_{\lambda_\eta} = \bigcap_{r=1}^{m} U_{\lambda_r}$*, and for any* $s \geq 1$, $U = U_{s+1}(\bigcap_{r \leq s} U_r)$.

Proof. Part (1) follows directly from Lemma 3.34. Part (2) follows from (1) and the fact that $f(U_\mu) = \mathfrak{u}_\mu$ for any $\mu \in \mathfrak{u}^*$. □

This allows us to write any supercharacter of $UU_n(\mathbb{F}_{q^2})$ as a product of elementary supercharacters.

Lemma 3.36. *With notation as above, we have*

$$\chi_\eta = \prod_{r=1}^{m} \chi_{\eta_r}.$$

Proof. If H_1 and H_2 are subgroups of a finite group G and ψ_1 and ψ_2 are characters of H_1 and H_2, respectively, then

$$\mathrm{Ind}_{H_1}^{G}(\psi_1)\mathrm{Ind}_{H_2}^{G}(\psi_2) = \sum_{x\in X} \mathrm{Ind}_{H_1{}^x\cap H_2}^{G}(\psi_1^x\psi_2),$$

where X is a set of (H_1, H_2) double coset representatives of G. In particular, if $HK = G$, then $\mathrm{Ind}_{H_1}^{G}(\psi_1)\mathrm{Ind}_{H_2}^{G}(\psi_2) = \mathrm{Ind}_{H_1\cap H_2}^{G}(\psi_1\psi_2)$. By induction on k, if $H_1, ..., H_k$ are subgroups of G with $H_{s+1}(\bigcap_{r\le s} H_r) = G$ for all $s \ge 1$, and $\psi_1, ..., \psi_k$ are representations of the H_r, then

$$\prod_{r=1}^{m} \mathrm{Ind}_{H_r}^{G}(\psi_r) = \mathrm{Ind}_{(\bigcap_{r=1}^{m} H_r)}^{G} \prod_{r=1}^{m} \psi_r.$$

The result follows from Lemma 3.35. □

We now calculate the values of the supercharacters on the superclasses. First, we determine the dimensions of the elementary supercharacters.

Proposition 3.37. *Let* $\eta = \{i \overset{a}{\frown} j \cup \bar{j} \overset{-a^q}{\frown} \bar{i}\}$ *(with* $i \neq \bar{j}$*) be a twisted* $\mathbb{F}_q$*-set partition; then*

$$\chi_\eta(1) = |H\cdot\lambda_\eta| = \begin{cases} q^{2(j-i-1)} & \text{if } n \text{ is even,} \\ q^{2(j-i-1)} & \text{if } n \text{ is odd and } j \le \frac{n+1}{2}, \\ q^{2(j-i)} & \text{if } n \text{ is odd and } j > \frac{n+1}{2}. \end{cases}$$

Let $\eta = \{i \overset{a}{\frown} \bar{i}$ *(with* $i \le \frac{n+1}{2}$*) be a twisted* $\mathbb{F}_q$*-set partition; then*

$$\chi_\eta(1) = |H\cdot\lambda_\eta| = \begin{cases} q^{2(n-2i)} & \text{if } n \text{ is even,} \\ q^{2(n+1-2i)} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This follows from the fact that $|H\cdot\lambda_\eta| = |U : U_{\lambda_\eta}|$ and Lemma 3.34. □

We mention that the dimension of an arbitrary supercharacter can be calculated by applying Lemma 3.36. Next we calculate the value of a supercharacter on a superclass.

Theorem 3.38. *Let* η *and* ν *be twisted* $\mathbb{F}_q$*-set partitions. Then*

$$\chi_\eta(u_\nu) = \begin{cases} \dfrac{\chi_\eta(1)}{(-q)^{\mathrm{nst}_\nu^\eta}}\theta\Bigg(\displaystyle\sum_{\substack{i\overset{a}{\frown}j\in\eta \\ i\overset{b}{\frown}j\in\nu}} ab\Bigg) & \text{if for } i\frown j\in\eta \text{ and } i<k<j, \\ & i\frown k, k\frown j\notin\nu, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{nst}_\nu^\eta = |\{i<j<k<l \mid j\frown k\in\nu, i\frown l\in\eta\}|$.

Proof. By Lemma 3.36, the proof reduces to proving that the theorem holds in the case that χ_η is an elementary supercharacter. The technique we use is similar to that employed by Diaconis–Thiem in the proof of [12, Theorem 5.1]. First let $\eta = \{i \overset{a}{\frown} j \cup \bar{j} \overset{-a^q}{\frown} \bar{i}\}$ (with $i \neq \bar{j}$). We have that

$$\begin{aligned}
\chi_\eta(u_\nu) &= \chi^\eta(1)\Bigg(\prod_{\substack{i<k<l<j \\ k\overset{b}{\frown}l\in\nu}} \frac{1}{q^4} \sum_{c_k, d_l \in \mathbb{F}_{q^2}} \theta(abc_kd_l + (abc_kd_l)^q)\Bigg) \\
&\quad \cdot \Bigg(\prod_{\substack{i<k<j \\ k\overset{b}{\frown}j\in\nu}} \frac{1}{q^2} \sum_{c_k \in \mathbb{F}_{q^2}} \theta(abc_k + (abc_k)^q)\Bigg) \\
&\quad \cdot \Bigg(\prod_{\substack{i<l<j \\ i\overset{b}{\frown}l\in\nu}} \frac{1}{q^2} \sum_{d_l \in \mathbb{F}_{q^2}} \theta(abd_l + (abd_l)^q)\Bigg) \cdot \prod_{i\overset{b}{\frown}j\in\nu} \theta(ab + (ab)^q) \\
&= \chi^\eta(1)\Bigg(\prod_{\substack{i<k<l<j \\ k\overset{b}{\frown}l\in\nu}} \frac{1}{q^2}\Bigg) \cdot \Bigg(\prod_{\substack{i<k<j \\ k\overset{b}{\frown}j\in\nu}} 0\Bigg) \cdot \Bigg(\prod_{\substack{i<l<j \\ i\overset{b}{\frown}l\in\nu}} 0\Bigg) \cdot \prod_{i\overset{b}{\frown}j\in\nu} \theta(ab + (ab)^q).
\end{aligned}$$

It follows that

$$\chi_\eta(u_\nu) = \begin{cases} \frac{\chi_\eta(1)}{(q^2)^{\#\{k\frown l\in\nu \mid i<k<l<j\}}} \prod_{i\overset{b}{\frown}j\in\nu} \theta(ab + (ab)^q) & \text{if for } i<k<j,\ i\frown k, k\frown j \notin \nu, \\ 0 & \text{otherwise.} \end{cases}$$

We can rewrite this as

$$\chi_\eta(u_\nu) = \begin{cases} \frac{\chi^\eta(1)}{(-q)^{\mathrm{nst}_\nu^\eta}} \theta\Bigg(\sum_{\substack{r\overset{a}{\frown}s\in\eta \\ r\overset{b}{\frown}s\in\nu}} ab\Bigg) & \begin{array}{l}\text{if for } r\frown s\in\eta \text{ and } r<k<s, \\ r\frown k, k\frown s\notin\nu,\end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\eta = \{i \overset{a}{\frown} \bar{i}\}$ (with $i \leq \frac{n+1}{2}$). Then we have

$$\chi_\eta(u_\nu) = \chi^\eta(1)\Bigg(\prod_{\substack{i<k<\frac{n}{2}\\ k<l<\bar{k}\\ k\overset{b}{\frown}l\in\nu}} \frac{1}{q^2}\sum_{c_k,c_{\bar{l}}\in\mathbb{F}_{q^2}} \theta(abc_kc_{\bar{l}}^q+(abc_kc_{\bar{l}}^q)^q)\Bigg)$$
$$\cdot\Bigg(\prod_{\substack{i<l<\bar{i}\\ i\overset{b}{\frown}l\in\nu}} \frac{1}{q^2}\sum_{c_{\bar{l}}\in\mathbb{F}_{q^2}} \theta(abc_{\bar{l}}+(abc_{\bar{l}})^q)\Bigg)$$
$$\cdot\Bigg(\prod_{\substack{i<k\leq\frac{n}{2}\\ k\overset{b}{\frown}\bar{k}\in\nu}} \frac{1}{q^2}\sum_{c_k\in\mathbb{F}_{q^2}} \theta(ab(c_k)^{q+1})\Bigg)\cdot\prod_{i\overset{b}{\frown}\bar{i}\in\nu}\theta(ab)$$
$$=\chi^\eta(1)\Bigg(\prod_{\substack{i<k<\frac{n}{2}\\ k<l<\bar{k}\\ k\overset{b}{\frown}l\in\nu}}\frac{1}{q^2}\Bigg)\Bigg(\prod_{\substack{i<l<\bar{i}\\ i\overset{b}{\frown}l\in\nu}}0\Bigg)\cdot\Bigg(\prod_{\substack{i<k\leq\frac{n}{2}\\ k\overset{b}{\frown}\bar{k}\in\nu}}\frac{1}{-q}\Bigg)\prod_{i\overset{b}{\frown}\bar{i}\in\nu}\theta(ab).$$

It follows that

$$\chi_\eta(u_\nu) = \begin{cases} \frac{\chi_\eta(1)}{(-q)^{\#\{k\frown l\in\nu \mid i<k<l<\bar{i}\}}}\displaystyle\prod_{i\overset{b}{\frown}j\in\nu}\theta(ab+(ab)^q) & \text{if for } i<k<j,\ i\frown k, k\frown j\notin\nu, \\ 0 & \text{otherwise.}\end{cases}$$

This can be rewritten as

$$\chi_\eta(u_\nu) = \begin{cases} \frac{\chi^\eta(1)}{(-q)^{\mathrm{nst}_\nu^\eta}}\theta\Bigg(\displaystyle\sum_{\substack{r\overset{a}{\frown}s\in\eta\\ r\overset{b}{\frown}s\in\nu}} ab\Bigg) & \begin{array}{l}\text{if for } r\frown s\in\eta \text{ and } r<k<s,\\ r\frown k, k\frown s\notin\nu,\end{array} \\ 0 & \text{otherwise.}\end{cases}$$

□

Note that in the above formula the sum of the terms ab is an element of $\mathbb{F}_q$, even though each individual term might not be. This formula is identical to the type A supercharacter formula (see Theorem 3.14), except with $-q$ replacing q everywhere (note that the supercharacter degrees are all powers of q^2). This idea that information about the unitary group can be obtained from information about the general linear group by replacing q with $-q$ is referred to as **Ennola duality** (see [17]).

Chapter 4

Supercharacter theories constructed via the method of little groups

In Chapter 4, we use the method of little groups to construct supercharacter theories of semidirect products with abelian normal subgroups. In particular, we apply our technique to unipotent groups, reproducing the supercharacter theories constructed in Theorems 3.7 and 3.18. We also construct a collection of coarser supercharacter theories of $UT_n(\mathbb{F}_q)$.

4.1 The method of little groups

Clifford theory studies the relationship between the irreducible characters of a group G and those of a normal subgroup N of G. The method of little groups is one application of Clifford theory; in Section 4.1 we develop background from Clifford theory and describe the method of little groups.

First we will establish some notation. If $g, h \in G$, we will often write ${}^h g = hgh^{-1}$ and $g^h = h^{-1}gh$. Similarly, if $\alpha : G \to \mathbb{C}$, the function ${}^h\alpha$ is defined by $({}^h\alpha)(g) = \alpha(g^h)$.

Let G be a finite group with normal subgroup N. If ψ is a character of N, let $I_G(\psi)$ denote the stabilizer of ψ under the conjugation action of G (this is known as the **inertial subgroup** of ψ). Lemmas 4.1, 4.2, and 4.4, along with Theorem 4.3 and Corollary 4.5, all follow from results in [10, Section 11].

Lemma 4.1. *Let H be a subgroup of $I_G(\psi)$, and let $\psi \in \mathrm{Irr}(N)$ be a character that can be extended to a character $\widetilde{\psi}$ of H. Then:*

(1) $\widetilde{\psi}$ *is an irreducible character of* H*;*

(2) if χ *is an irreducible character of* H/N*, then* $\mathrm{Inf}_{H/N}^{H}(\chi)$ *is an irreducible character of* H*; and*

(3) the set $\{\widetilde{\psi} \cdot \mathrm{Inf}_{H/N}^{H}(\chi) \mid \chi \in \mathrm{Irr}(H/N)\}$ *is a set of distinct irreducible characters of* H *and is independent of the choice of extension* $\widetilde{\psi}$*.*

If $H = I_G(\psi)$, we can say even more.

Lemma 4.2. *Let* $\psi \in \mathrm{Irr}(N)$ *be a character that can be extended to a character* $\widetilde{\psi}$ *of* $I_G(\psi)$*. Then the set*

$$\{\mathrm{Ind}_{I_G(\psi)}^{G}(\widetilde{\psi} \cdot \mathrm{Inf}_{I_G(\psi)/N}^{I_G(\psi)}(\chi)) \mid \chi \in \mathrm{Irr}(I_G(\psi)/N)\}$$

is a set of distinct irreducible characters of G*. This set only depends on the orbit of* ψ *under the conjugation action of* G*, and the sets of irreducible characters of* G *associated to different conjugation orbits of irreducible characters of* N *are distinct.*

If every irreducible character of N can be extended to a character of $I_G(\psi)$, the above construction in fact produces all of the irreducible characters of G.

Theorem 4.3 (**Method of Little Groups**, [10, Proposition 11.8])**.** *Let* G *be a finite group, and let* N *be a normal subgroup of* G *such that each* $\psi \in \mathrm{Irr}(N)$ *can be extended to a character* $\widetilde{\psi}$ *of* $I_G(\psi)$*. Let* $\mathcal{S}$ *be a set of orbit representatives of* $\mathrm{Irr}(N)$ *under the conjugation action of* G*. Then the set of irreducible characters of* G *is given by*

$$\mathrm{Irr}(G) = \{\mathrm{Ind}_{I_G(\psi)}^{G}(\widetilde{\psi} \cdot \mathrm{Inf}_{I_G(\psi)/N}^{I_G(\psi)}(\chi)) \mid \psi \in \mathcal{S}, \chi \in \mathrm{Irr}(I_G(\psi)/N)\}.$$

The most common application of the method of little groups is to semidirect products. If $G = N \rtimes H$, we can identify G/N with H. For a character $\psi \in \mathrm{Irr}(N)$, let $I_H(\psi)$ denote the stabilizer of ψ under the conjugation action of H. Observe that $I_H(\psi) \cong I_G(\psi)/N$. In this situation, linear characters of N can always be extended to characters of $I_G(\psi)$.

Lemma 4.4. *Let* $G = N \rtimes H$*, let* ψ *be a linear character of* N*, and let* $K \subseteq I_H(\psi)$*. For* $n \in N$ *and* $h \in K$*, define* $\widetilde{\psi}(nh) = \psi(n)$*. Then* $\widetilde{\psi}$ *is an extension of* ψ *to* NK*.*

A corollary of Lemma 4.4 is that the method of little groups applies to semidirect products with abelian normal subgroups.

Corollary 4.5. *Let $G = N \rtimes H$, with N abelian. Let $\mathcal{S}$ be a set of orbit representatives of* $\mathrm{Irr}(N)$ *under the conjugation action of G. Then*

$$\mathrm{Irr}(G) = \{\mathrm{Ind}_{I_G(\psi)}^{G}(\widetilde{\psi} \cdot \mathrm{Inf}_{I_G(\psi)/N}^{I_G(\psi)}(\chi)) \mid \psi \in \mathcal{S}, \chi \in \mathrm{Irr}(I_H(\psi))\}.$$

We will now establish some notation for the characters described above. Let $\psi \in \mathrm{Irr}(N)$, $K \subseteq I_H(\psi)$, and $\widetilde{\psi}$ be the extension described in Lemma 4.4. For χ a character of K, define

$$\psi \rtimes \chi = \mathrm{Ind}_{NK}^{G}(\widetilde{\psi} \cdot \mathrm{Inf}_{K}^{NK}(\chi)). \tag{4.1}$$

The dependence on K is implicit in that χ is a character of K.

The remainder of this section will be devoted to establishing a number of properties of the characters $\psi \rtimes \chi$. It is worth mentioning that the definition of $\psi \rtimes \chi$, along with the following results, make sense for any class function χ of K (and not just characters).

Lemma 4.6. *Let ψ be a linear character of N and let χ be a character of K, with $K \subseteq I_H(\psi)$. If $n \in N$ and $h \in H$, then*

$$(\psi \rtimes \chi)(nh) = \frac{1}{|K|} \sum_{\substack{k \in H \\ {}^k h \in K}} \psi({}^k n)\chi({}^k h).$$

Proof. Let $n \in N$ and $h \in H$; then

$$\begin{aligned}
(\psi \rtimes \chi)(nh) &= \mathrm{Ind}_{NK}^{G}(\widetilde{\psi} \cdot \mathrm{Inf}_{K}^{NK}(\chi))(nh) \\
&= \frac{1}{|N||K|} \sum_{\substack{m \in N,\, k \in H \\ {}^{mk}(nh) \in NK}} \psi(m({}^k n)({}^{khk^{-1}}(m^{-1})))\chi({}^k h) \\
&= \frac{1}{|N||K|} \sum_{\substack{m \in N,\, k \in H \\ {}^k h \in K}} \psi(m)\psi({}^k n)\psi({}^{khk^{-1}}(m^{-1}))\chi({}^k h) \\
&= \frac{1}{|K|} \sum_{\substack{k \in H \\ {}^k h \in K}} \psi({}^k n)\chi({}^k h).
\end{aligned}$$

□

This formula leads to the following lemma.

Lemma 4.7. *Let $K_1 \subseteq K_2 \subseteq I_H(\psi)$, and let χ be a character of K_1. Then*

$$\psi \rtimes \mathrm{Ind}_{K_1}^{K_2}(\chi) = \psi \rtimes \chi.$$

Proof. Let $n \in N$ and $h \in H$; then

$$\begin{aligned}(\psi \rtimes \mathrm{Ind}_{K_1}^{K_2}(\chi))(nh) &= \frac{1}{|K_2|} \sum_{\substack{k \in H \\ {}^k h \in K_2}} \psi({}^k n)\mathrm{Ind}_{K_1}^{K_2}(\chi)({}^k h) \\ &= \frac{1}{|K_2|} \sum_{\substack{k \in H \\ {}^k h \in K_2}} \psi({}^k n)\left(\frac{1}{|K_1|} \sum_{\substack{l \in K_2 \\ {}^{lk} h \in K_1}} \chi({}^{lk} h)\right).\end{aligned}$$

If $l \in K_2$ and ${}^{lk}h \in K_1$, then certainly ${}^k h \in K_2$. Furthermore, if $l \in K_2$, then $\psi({}^{lk}h) = \psi({}^k h)$. It follows that

$$\begin{aligned}(\psi \rtimes \mathrm{Ind}_{K_1}^{K_2}(\chi))(nh) &= \frac{1}{|K_2||K_1|} \sum_{\substack{k \in H \\ l \in K_2 \\ {}^{lk} h \in K_1}} \psi({}^k n)\chi({}^{lk} h) \\ &= \frac{1}{|K_2||K_1|} \sum_{\substack{k \in H \\ l \in K_2 \\ {}^{lk} h \in K_1}} \psi({}^{lk} n)\chi({}^{lk} h) \\ &= \frac{1}{|K_1|} \sum_{\substack{k \in H \\ {}^k h \in K_1}} \psi({}^k n)\chi({}^k h) \\ &= (\psi \rtimes \chi)(nh).\end{aligned}$$

□

For our purposes we will need to describe the products of characters in terms of the method of little groups. The following lemma is a general result about the products of induced characters.

Lemma 4.8 ([10, Theorem 10.18])**.** *Let H and K be subgroups of a finite group G, and let χ and ψ be characters of H and K, respectively. If X denotes a set of (H, K) double coset representatives of G, then*

$$\mathrm{Ind}_H^G(\chi)\mathrm{Ind}_K^G(\psi) = \sum_{x \in X} \mathrm{Ind}_{H^x \cap K}^G(\mathrm{Res}_{H^x \cap K}^{H^x}(\chi^x)\mathrm{Res}_{H^x \cap K}^K(\psi)).$$

□

Let $G = N \rtimes H$ and let $\psi_1 \rtimes \chi_1$ and $\psi_2 \rtimes \chi_2$ be characters of G, where χ_1 is a character of K_1 and χ_2 is a character of K_2.

Lemma 4.9. *Let X be a set of (K_1, K_2) double coset representatives of H. Then*

$$(\psi_1 \rtimes \chi_1)(\psi_2 \rtimes \chi_2) = \sum_{x \in X} (\psi_1^x \psi_2) \rtimes (\mathrm{Res}^{K_1^x}_{K_1^x \cap K_2}(\chi_1^x)\mathrm{Res}^{K_2}_{K_1^x \cap K_2}(\chi_2)).$$

Proof. Note that X is also a set of (NK_1, NK_2) double coset representatives of G. By Lemma 4.8,

$$\begin{aligned}
&(\psi_1 \rtimes \chi_1)(\psi_2 \rtimes \chi_2) \\
&\quad = \mathrm{Ind}^G_{NK_1}(\widetilde{\psi_1} \cdot \mathrm{Inf}^{NK_1}_{K_1}(\chi_1))\mathrm{Ind}^G_{NK_2}(\widetilde{\psi_2} \cdot \mathrm{Inf}^{NK_2}_{K_2}(\chi_2)) \\
&\quad = \sum_{x \in X} \mathrm{Ind}^G_{(NK_1)^x \cap NK_2}(\mathrm{Res}^{(NK_1)^x}_{(NK_1)^x \cap NK_2}((\widetilde{\psi_1} \cdot \mathrm{Inf}^{NK_1}_{K_1}(\chi_1))^x)\mathrm{Res}^{NK_2}_{(NK_1)^x \cap NK_2}(\widetilde{\psi_2} \cdot \mathrm{Inf}^{NK_2}_{K_2}(\chi_2))).
\end{aligned}$$

We can rewrite the inner portion of each term in the above sum as

$$\begin{aligned}
\mathrm{Res}^{(NK_1)^x}_{(NK_1)^x \cap NK_2}&((\widetilde{\psi_1} \cdot \mathrm{Inf}^{NK_1}_{K_1}(\chi_1))^x)\mathrm{Res}^{NK_2}_{(NK_1)^x \cap NK_2}(\widetilde{\psi_2} \cdot \mathrm{Inf}^{NK_2}_{K_2}(\chi_2)) \\
&= (\mathrm{Res}^{(NK_1)^x}_{(NK_1)^x \cap NK_2}(\widetilde{\psi_1}^x)\mathrm{Res}^{NK_2}_{(NK_1)^x \cap NK_2}(\widetilde{\psi_2})) \\
&\cdot (\mathrm{Res}^{(NK_1)^x}_{(NK_1)^x \cap NK_2}(\mathrm{Inf}^{NK_1}_{K_1}(\chi_1))^x \mathrm{Res}^{NK_2}_{(NK_1)^x \cap NK_2}\mathrm{Inf}^{NK_2}_{K_2}(\chi_2)).
\end{aligned}$$

The first term can be simplified, as

$$\begin{aligned}
\mathrm{Res}^{(NK_1)^x}_{(NK_1)^x \cap NK_2}(\widetilde{\psi_1}^x)\mathrm{Res}^{NK_2}_{(NK_1)^x \cap NK_2}(\widetilde{\psi_2}) &= \mathrm{Res}^{N(K_1)^x}_{N(K_1^x \cap K_2)}(\widetilde{\psi_1^x})\mathrm{Res}^{NK_2}_{N(K_1^x \cap K_2)}(\widetilde{\psi_2}) \\
&= \widetilde{\psi_1^x \psi_2},
\end{aligned}$$

where $\widetilde{\psi_1^x \psi_2}$ denotes the extension of $\psi_1^x \psi_2$ to $N(K_1^x \cap K_2)$ defined in Lemma 4.4. Furthermore, the second term can be rewritten by noting that

$$\begin{aligned}
\mathrm{Res}^{(NK_1)^x}_{(NK_1)^x \cap NK_2}(\mathrm{Inf}^{NK_1}_{K_1}(\chi_1))^x &= \mathrm{Res}^{N(K_1)^x}_{N(K_1^x \cap K_2)}\mathrm{Inf}^{N(K_1)^x}_{K_1^x}(\chi_1^x) \\
&= \mathrm{Inf}^{N(K_1^x \cap K_2)}_{K_1^x \cap K_2}\mathrm{Res}^{K_1^x}_{K_1^x \cap K_2}(\chi_1^x),
\end{aligned}$$

and similarly

$$\mathrm{Res}^{NK_2}_{(NK_1)^x \cap NK_2}\mathrm{Inf}^{NK_2}_{K_2}(\chi_2) = \mathrm{Inf}^{N(K_1^x \cap K_2)}_{K_1^x \cap K_2}\mathrm{Res}^{K_2}_{K_1^x \cap K_2}(\chi_2).$$

This means that the second term can be simplified to

$$\begin{aligned}\mathrm{Res}^{(NK_1)^x}_{(NK_1)^x\cap NK_2}&(\mathrm{Inf}^{NK_1}_{K_1}(\chi_1))^x\mathrm{Res}^{NK_2}_{(NK_1)^x\cap NK_2}\mathrm{Inf}^{NK_2}_{K_2}(\chi_2)\\ &= \mathrm{Inf}^{N(K_1^x\cap K_2)}_{K_1^x\cap K_2}\mathrm{Res}^{K_1^x}_{K_1^x\cap K_2}(\chi_1^x)\mathrm{Inf}^{N(K_1^x\cap K_2)}_{K_1^x\cap K_2}\mathrm{Res}^{K_2}_{K_1^x\cap K_2}(\chi_2)\\ &= \mathrm{Inf}^{N(K_1^x\cap K_2)}_{K_1^x\cap K_2}(\mathrm{Res}^{K_1^x}_{K_1^x\cap K_2}(\chi_1^x)\mathrm{Res}^{K_2}_{K_1^x\cap K_2}(\chi_2)).\end{aligned}$$

Substituting these simplifications into the initial equation, we get that

$$\begin{aligned}(\psi_1 \rtimes \chi_1)&(\psi_2 \rtimes \chi_2)\\ &= \sum_{x\in X}\mathrm{Ind}^G_{N(K_1^x\cap K_2)}(\widetilde{\psi_1^x\psi_2}\cdot \mathrm{Inf}^{N(K_1^x\cap K_2)}_{K_1^x\cap K_2}(\mathrm{Res}^{K_1^x}_{K_1^x\cap K_2}(\chi_1^x)\mathrm{Res}^{K_2}_{K_1^x\cap K_2}(\chi_2)))\\ &= \sum_{x\in X}(\psi_1^x\psi_2)\rtimes(\mathrm{Res}^{K_1^x}_{K_1^x\cap K_2}(\chi_1^x)\mathrm{Res}^{K_2}_{K_1^x\cap K_2}(\chi_2)).\end{aligned}$$

□

4.2 Supercharacter theories of semidirect products

In Section 4.2 we present the main result of Chapter 4.

Let $G = N \rtimes H$ be a finite group with N abelian, and let $\mathcal{L}$ be a lattice of subgroups of H (under the usual meet and join operations of the subgroup lattice) such that

(L1) H and $\{1\}$ are in $\mathcal{L}$,

(L2) if $K \in \mathcal{L}$ and $h \in H$, then ${}^hK \in \mathcal{L}$,

(L3) each $K \in \mathcal{L}$ is equipped with a supercharacter theory, and

(L4) these supercharacter theories are compatible in the sense that induction and restriction send superclass functions to superclass functions and conjugate subgroups have conjugate supercharacter theories (more specifically, if χ is a supercharacter of K then χ^g is a supercharacter of K^g).

The compatibility property given by (L4) is not true of many known supercharacter theories of families of groups. For example, if G is an algebra group and H is an algebra subgroup, both with the algebra group supercharacter theory (see Section 3.2), it is not always the case that induction sends superclass functions to superclass functions (for a counterexample see [11, Section 6]). However, if H is a left (or right, or two-sided) ideal subgroup of G with the left (or right, or two-sided) ideal supercharacter theory, then induction and restriction will send superclass functions to superclass functions.

Suppose that for each $\psi \in \operatorname{Irr}(N)$ we can choose a subgroup $H_\psi \in \mathcal{L}$ such that our choices of H_ψ satisfy the following properties.

(H1) H_ψ is a normal subgroup of $I_H(\psi)$.

(H2) For all $h \in H$, we have $H_{(\psi^h)} = (H_\psi)^h$.

(H3) If 1_N denotes the trivial character of N, we have $H_{1_N} = H$.

(H4) If $\psi, \varphi \in \operatorname{Irr}(N)$, then we have $H_\psi \cap H_\varphi \subseteq H_{\psi\varphi}$.

These conditions may seem restrictive; however, for any lattice $\mathcal{L}$ satisfying (L1)–(L4), there are maximal and minimal choices of H_ψ.

Lemma 4.10. *Let $G = N \rtimes H$ be a finite group with N abelian. If $\mathcal{L}$ is any lattice of subgroups of H satisfying (L1)–(L4), one can always choose*

(1) $H_{1_N} = H$ and $H_\psi = \{1\}$ for all nontrivial ψ, or

(2) $H_\psi = I_{\mathcal{L}}(\psi)$ (where $I_{\mathcal{L}}(\psi)$ is the maximal element of $\mathcal{L}$ contained in $I_H(\psi)$).

These choices of H_ψ each satisfy (H1)–(H4), and are the minimal and maximal choices of H_ψ, respectively.

Proof. The fact that these are the minimal and maximal choices of H_ψ is clear, and choice (1) trivially satisfies (H1)–(H4).

Let $H_\psi = I_{\mathcal{L}}(\psi)$ for all $\psi \in \operatorname{Irr}(N)$ (note that such $I_{\mathcal{L}}(\psi)$ must exist as $\mathcal{L}$ is a lattice). As $\mathcal{L}$ is closed under conjugation, $I_{\mathcal{L}}(\psi)$ is normal in $I_H(\psi)$ and $I_{\mathcal{L}}(\psi)^h = I_{\mathcal{L}}(\psi^h)$ for all $h \in H$, hence we have (H1) and (H2). Condition (H3) is clear, and if $K_\psi \subseteq I_H(\psi)$ and $K_\varphi \subseteq I_H(\varphi)$, then $K_\psi \cap K_\varphi \subseteq I_H(\psi\varphi)$, hence we have (H4). $\square$

Given a choice of H_ψ, consider the set of characters

$$S_\psi = \{\mathrm{Ind}_{H_\psi}^{I_H(\psi)}(\chi) \mid \chi \text{ is a supercharacter of } H_\psi\}.$$

As H_ψ is a normal subgroup of $I_H(\psi)$ and H-conjugates of supercharacters are also supercharacters, each irreducible character of $I_H(\psi)$ is a constituent of exactly one character in S_ψ (although there may be supercharacters that induce to the same character of $I_H(\psi)$).

The following is the main result of Chapter 4.

Theorem 4.11. *Let S be a set of orbit representatives of Irr(N) under the conjugation action of G. Then the set of characters defined by*

$$\mathrm{SCh}(\mathcal{L}) = \{\psi \rtimes \chi \mid \psi \in S \textit{ and } \chi \in S_\psi\}$$

is a set of supercharacters for a supercharacter theory of G.

Proof. As each irreducible character of $I_H(\psi)$ is a constituent of exactly one character in S_ψ, it follows from the method of little groups that every irreducible character of G is a constituent of exactly one character in $\mathrm{SCh}(\mathcal{L})$. Furthermore, the trivial character is in $\mathrm{SCh}(\mathcal{L})$. It suffices to show that the linear span of the characters in $\mathrm{SCh}(\mathcal{L})$ is closed under the pointwise product, and the result will follow from Lemma 2.6.

Suppose that ψ is an irreducible character of N and $K \in \mathcal{L}$ is a subgroup of H_ψ. If χ is a character of K that is constant on the superclasses, then

$$\psi \rtimes \chi = \psi \rtimes \mathrm{Ind}_K^{H_\psi}(\chi).$$

By the assumptions on $\mathcal{L}$, $\mathrm{Ind}_K^{H_\psi}(\chi)$ is a superclass function of H_ψ, hence $\psi \rtimes \chi$ is a superclass function of G.

Let ψ_1 and ψ_2 be linear characters of N and let $\chi_1 \in S_{\psi_1}$ and $\chi_2 \in S_{\psi_2}$. By Lemma 4.9, if X is a set of (H_{ψ_1}, H_{ψ_2}) double coset representatives of G, then

$$(\psi_1 \rtimes \chi_1)(\psi_2 \rtimes \chi_2) = \sum_{x \in X} (\psi_1^x \psi_2) \rtimes (\mathrm{Res}^{H_{\psi_1}^x}_{H_{\psi_1}^x \cap H_{\psi_2}}(\chi_1^x)\mathrm{Res}^{H_{\psi_2}}_{H_{\psi_1}^x \cap H_{\psi_2}}(\chi_2))$$
$$= \sum_{x \in X} (\psi_1^x \psi_2) \rtimes (\mathrm{Res}^{H_{\psi_1^x}}_{H_{\psi_1^x} \cap H_{\psi_2}}(\chi_1^x)\mathrm{Res}^{H_{\psi_2}}_{H_{\psi_1^x} \cap H_{\psi_2}}(\chi_2)).$$

As χ_1^x is a superclass function of $H_{\psi_1^x}$ and χ_2 is a superclass function of H_{ψ_2},

$$\mathrm{Res}^{H_{\psi_1^x}}_{H_{\psi_1^x} \cap H_{\psi_2}}(\chi_1^x)\mathrm{Res}^{H_{\psi_2}}_{H_{\psi_1^x} \cap H_{\psi_2}}(\chi_2)$$

is a superclass function of $H_{\psi_1^x} \cap H_{\psi_2}$. Furthermore, by the assumptions on the choices of H_ψ we have that $H_{\psi_1^x} \cap H_{\psi_2} \subseteq H_{\psi_1^x \psi_2}$. Hence

$$(\psi_1^x \psi_2) \rtimes (\mathrm{Res}^{H_{\psi_1^x}}_{H_{\psi_1^x} \cap H_{\psi_2}}(\chi_1^x)\mathrm{Res}^{H_{\psi_2}}_{H_{\psi_1^x} \cap H_{\psi_2}}(\chi_2))$$

is a superclass function of G, and the linear span of the elements of $\mathrm{SCh}(\mathcal{L})$ is closed under the pointwise product. □

Remark. In a sense, the choice of $\mathcal{L}$ does not matter; that is, if $\mathcal{L}'$ is a different lattice of subgroups that contains all the H_ψ, where each H_ψ has the same supercharacter theory as it does in $\mathcal{L}$, then the resulting supercharacter theory of G will not change.

4.2.1 Relationship to the star product

Let $G = N \rtimes H$ with N abelian. Let $\mathrm{SCT}(N) = (\mathcal{X}_N, \mathcal{K}_N)$ be the supercharacter theory of N obtained from the action of the inner automorphisms of G, and suppose $\mathrm{SCT}(H) = (\mathcal{X}_H, \mathcal{K}_H)$ is any supercharacter theory of H. The $*$-product of these supercharacter theories has supercharacter set

$$\{\mathrm{Inf}_H^G(\chi) \mid \chi \in \mathcal{X}_H\} \cup \{\mathrm{Ind}_N^G(\chi) \mid \chi \in \mathcal{X}_N - \{1_N\}\},$$

where 1_N is the trivial character of N (see Section 2.2.2). This set can be written as

$$\{1_N \rtimes \chi \mid \chi \in \mathcal{X}_H\} \cup \{\psi \rtimes 1_{\{1\}} \mid \psi \in S\}$$

where S is a set of orbit representatives of N under the conjugation action of G and $1_{\{1\}}$ is the trivial character of the trivial subgroup $\{1\} \subseteq H$.

Suppose that $G = N \rtimes H$ with N abelian and that $\mathcal{L}$ is a sublattice of the subgroup lattice of H satisfying (L1)–(L4). Given choices of H_ψ for each $\psi \in \mathrm{Irr}(N)$, we can consider two supercharacter theories of G. The method of little groups produces $\mathrm{SCh}(\mathcal{L})$; the $*$-product allows us to construct $\mathrm{SCT}(N) * \mathrm{SCT}(H)$, where $\mathrm{SCT}(N)$ is the supercharacter theory of N obtained from the action of the inner automorphisms of G and $\mathrm{SCT}(H)$ is the supercharacter theory of H (as an element of $\mathcal{L}$).

Lemma 4.12. *The supercharacter theory* $\mathrm{SCh}(\mathcal{L})$ *is a (not necessarily strictly) finer supercharacter theory of* G *than* $\mathrm{SCT}(N)*\mathrm{SCT}(H)$*. These supercharacter theories coincide exactly when* $H_\psi = \{1\}$ *for all nontrivial* $\psi \in Irr(N)$*.*

Proof. The characters $\{1_N \rtimes \chi \mid \chi \in \mathcal{X}_H\}$ are all supercharacters of $\mathrm{SCh}(\mathcal{L})$ as $H_{1_N} = H$. Furthermore, if ψ is a nontrivial linear character of N and ρ denotes the regular character of H_ψ, then

$$\psi \rtimes 1_{\{1\}} = \psi \rtimes \mathrm{Ind}_N^{NH_\psi}(1_{\{1\}}) = \psi \rtimes \rho.$$

As ρ is a superclass function of H_ψ, $\psi \rtimes 1_{\{1\}}$ is a superclass function of G. The final claim follows from the definition of the $*$-product. □

4.3 Supercharacter theories of pattern groups

Let G be be a pattern subgroup of $UT_n(\mathbb{F}_q)$ (as in Section 2.3). As G is an algebra group, G has a supercharacter theory given by Theorem 3.7. In Section 4.3, we reproduce this supercharacter theory using the method of little groups. In order to do this, we will consider the sets of left and right ideal subgroups of G. First, we show that these subgroups satisfy (L1)–(L4). Lemmas 4.13, 4.14, and 4.15 are stated in terms of left ideal subgroups but are also true of the set of right ideal subgroups.

Lemma 4.13. *The set of left ideal subgroups of* G *is a sublattice of the subgroup lattice of* G*.*

Proof. The only thing to check is that the join of left ideal subgroups is a left ideal subgroup. Let

$\mathfrak{a}$ and $\mathfrak{b}$ be left ideals of $\mathfrak{g}$; then certainly $\langle (1+\mathfrak{a}), (1+\mathfrak{b})\rangle \subseteq 1+(\mathfrak{a}+\mathfrak{b})$. Let $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$; then

$$1+(a+b) = (1+a)(1+b-ab+a^2b-a^3b+\dots)$$

and $b - ab + a^2b - \dots \in \mathfrak{b}$. Thus the reverse containment holds, and the result follows. □

Lemma 4.14. *Suppose $A \subseteq B$ are left ideal subgroups of G and that α is a superclass function of A and β is a superclass function of B. Then*

(1) $\mathrm{Ind}_A^B(\alpha)$ *is a superclass function of B, and*
(2) $\mathrm{Res}_A^B(\beta)$ *is a superclass function of A.*

Proof. Let $b \in B$, with $b = 1+y$. Suppose $b' \in B$ is in the same superclass as b, and write $b' = 1+gyh$, where $g \in G$ and $h \in B$. If $k \in B$, then we have

$${}^{kh}b' = 1 + {}^{kh}(gyh) = 1 + {}^{k}(hg)^{k}y.$$

It follows that ${}^{k}b \in A$ if and only if ${}^{kh}b' \in A$, in which case ${}^{k}b$ and ${}^{kh}b'$ are in the same superclass of A. Thus we have

$$\begin{aligned}\mathrm{Ind}_A^B(\alpha)(b) &= \frac{1}{|A|}\sum_{\substack{x\in B\\ {}^{x}b\in A}} \alpha({}^{x}b)\\ &= \frac{1}{|A|}\sum_{\substack{x\in B\\ {}^{xh}b'\in A}} \alpha({}^{xh}b')\\ &= \mathrm{Ind}_A^B(\alpha)(b'),\end{aligned}$$

and $\mathrm{Ind}_A^B(\alpha)$ is a superclass function of B. For (2), observe that if two elements of A are in the same superclass in A then they are in the same superclass in B. □

Lemma 4.15. *Suppose that H is a left ideal subgroup of G and χ_λ is a supercharacter of G. Then for $g \in G$, we have $(\chi_\lambda)^g = \chi_{\lambda^g}$. In particular, $(\chi_\lambda)^g$ is a supercharacter of H^g.*

Proof. Note that

$$\chi_\lambda = \frac{|G\lambda|}{|G\lambda H|} \sum_{\mu \in G\lambda H} \theta \circ \mu \circ f$$
$$= \frac{|G\lambda|}{|G||H|} \sum_{\substack{h \in G \\ k \in H}} \theta \circ h\lambda k \circ f,$$

and hence we have

$$\chi_\lambda^g = \frac{|G\lambda|}{|G||H|} \sum_{\substack{h \in G \\ k \in H}} \theta \circ g^{-1}h\lambda kg \circ f$$
$$= \frac{|G\lambda^g|}{|G||H|} \sum_{\substack{h \in G \\ k \in H}} \theta \circ h^g\lambda^g k^g \circ f$$
$$= \chi_{\lambda^g}.$$

□

We can use the above results to reproduce Theorem 3.7 using the method of little groups in the specific case that G is a pattern subgroup of $UT_n(\mathbb{F}_q)$.

Let $G = 1 + \mathfrak{g}$ be a pattern subgroup of $UT_n(\mathbb{F}_q)$. In order to reconstruct the algebra group supercharacter theory of G using Theorem 4.11, we need to decompose G as a semidirect product $N \rtimes H$ with N abelian. Let $n = k + m$, and define

$$H_k = \{g \in G \mid g_{ij} = 0 \text{ if } j > k \text{ and } j > i\},$$
$$H_m = \{g \in G \mid g_{ij} = 0 \text{ if } i \leq k \text{ and } i < j\}, \text{ and}$$
$$N = \{g \in G \mid g_{ij} = 0 \text{ if } i < j \text{ and } j \leq k \text{ or } i > k\}.$$

Example 4.1. Let $n = 7$, $k = 3$, and $m = 4$, and $G = UT_7(\mathbb{F}_q)$; we have

$$H_k = \left\{ \left(\begin{array}{ccc|cccc} 1 & * & * & 0 & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \in UT_7(\mathbb{F}_q) \right\},$$

$$H_m = \left\{ \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \in UT_7(\mathbb{F}_q) \right\}, \text{ and}$$

$$N = \left\{ \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & * & * & * & * \\ 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \in UT_7(\mathbb{F}_q) \right\}.$$

Observe that

(1) H_k is isomorphic to a pattern subgroup of $UT_k(\mathbb{F}_q)$ and H_m is isomorphic to a pattern subgroup of $UT_m(\mathbb{F}_q)$;

(2) N is abelian; and

(3) if $H = H_k \times H_m$, then $G = N \rtimes H$.

Let $\mathcal{L}$ be the lattice of subgroups of $H_k \times H_m$ of the form $K_k \times K_m$, with K_k a right ideal subgroup of H_k and K_m a left ideal subgroup of H_m. Equip $K_k \times K_m$ with the direct product of the right ideal supercharacter theory of K_k and the left ideal supercharacter theory of K_m.

We will denote by $\mathfrak{h}, \mathfrak{h}_k, \mathfrak{h}_\mathfrak{m}$ and $\mathfrak{n}$ the algebras $f(H), f(H_k)$, $f(H_m)$, and $f(N)$. We have that

(1) the irreducible characters of N are exactly the elements of the form $\psi_\lambda = \theta \circ \lambda \circ f$, where $\lambda \in \mathfrak{n}^*$; and

(2) for $h = (h_k, h_m) \in H$ and $x \in \mathfrak{n}$, ${}^h(1 + x) = 1 + h_k x h_m^{-1}$.

For each $\psi_\lambda \in \mathrm{Irr}(N)$, define $H_{\psi_\lambda} = I_{\mathcal{L}}(\psi_\lambda)$.

Example 4.2. Once again let $n = 7$, $k = 3$, and $m = 4$, and let $G = UT_7(\mathbb{F}_q)$. Define $\lambda \in \mathfrak{n}^*$ by

$$\lambda(x) = x_{15} + x_{37}.$$

In other words, $\lambda(x)$ is determined by the entries denoted by $\circ$ below:

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & * & \circ & * & * \\ 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & * & \circ \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

For this choice of λ, H_{ψ_λ} will be the subgroup

$$H_{\psi_\lambda} = \left\{ \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & \circ & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \circ \\ \hline 0 & 0 & 0 & 1 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \in H_k \times H_m \right\},$$

where once again the entries labeled with $\circ$ are the entries which determine λ. For comparison, note that $I_H(\psi_\lambda)$ is the subgroup

$$I_H(\psi_\lambda) = \left\{ \left(\begin{array}{ccc|cccc} 1 & 0 & a & 0 & \circ & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \circ \\ \hline 0 & 0 & 0 & 1 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \in H_k \times H_m \right\}$$

where a ranges over the elements of $\mathbb{F}_q$. Note that $I_H(\psi_\lambda)$ is an algebra group but not a pattern group.

Theorem 4.16. *For each $\psi_\lambda \in Irr(N)$, define $H_{\psi_\lambda} = I_{\mathcal{L}}(\psi_\lambda)$. These choices of H_{ψ_λ} produce the same supercharacter theory of G as described in Theorem 3.7.*

Proof of Theorem 4.16. Fix an element $\eta \in \mathfrak{g}^*$. Let $\lambda = \eta|_{\mathfrak{n}}$, and define

$$\begin{aligned} K_k &= \{h \in H_k \mid h\lambda = \lambda\} \text{ and} \\ K_m &= \{h \in H_m \mid \lambda h = \lambda\}. \end{aligned}$$

Note that K_k is a right ideal subgroup of H_k, K_m is a left ideal subgroup of H_m, and

$$K = K_k \times K_m = I_{\mathcal{L}}(\psi_\lambda).$$

Denote $f(K), f(K_k)$ and $f(K_m)$ by $\mathfrak{k}, \mathfrak{k}_k$ and $\mathfrak{k}_m$.

Let $\mu = \eta|_{\mathfrak{k}}$; note that μ can be written uniquely as $\mu_k \oplus \mu_m$, where $\mu_k \in \mathfrak{k}_k^*$ and $\mu_m \in \mathfrak{k}_m^*$. Let

$$\begin{aligned} \mathfrak{r}_\mu &= \{x \in \mathfrak{k}_k \mid \mu_k(xy) = 0 \text{ for all } y \in \mathfrak{h}_k\} \text{ and} \\ \mathfrak{l}_\mu &= \{x \in \mathfrak{k}_m \mid \mu_k(yx) = 0 \text{ for all } y \in \mathfrak{h}_m\}, \end{aligned}$$

and denote the corresponding algebra groups by R_μ and L_μ. Then the supercharacters of K under the direct product of the right ideal supercharacter theory of K_k and the left ideal supercharacter theory of K_m are the elements

$$\chi_\mu = \mathrm{Ind}_{R_\mu \times L_\mu}^{K} \mathrm{Res}_{R_\mu \times L_\mu}^{K} (\theta \circ \mu \circ f).$$

Observe that

$$\begin{aligned} \psi_\lambda \rtimes \chi_\mu &= \psi_\lambda \rtimes \mathrm{Res}_{R_\mu \times L_\mu}^{K} (\theta \circ \mu \circ f) \\ &= \mathrm{Ind}_{N(R_\mu \times L_\mu)}^{G} (\theta \circ (\mu|_{\mathfrak{r}_\mu \oplus \mathfrak{l}_\mu} \oplus \lambda) \circ f). \end{aligned}$$

We also have

$$\mathfrak{n} \oplus \mathfrak{r}_\mu \oplus \mathfrak{l}_\mu = \{x \in \mathfrak{g} \mid g\eta h(x) = \eta(x) \text{ for all } g \in NH_m, h \in NH_k\}.$$

Furthermore, for $y \in \mathfrak{n} \oplus \mathfrak{h}_m$ and $z \in \mathfrak{n} \oplus \mathfrak{h}_k$, we have that $y\eta z = 0$, hence $NH_m\eta NH_k - \eta$ is a subspace of $\mathfrak{g}^*$. It follows from Lemma 3.2 that

$$\{\nu \in \mathfrak{g}^* \mid \nu|_{\mathfrak{n}\oplus\mathfrak{r}_\mu\oplus\mathfrak{l}_\mu} = \eta|_{\mathfrak{n}\oplus\mathfrak{r}_\mu\oplus\mathfrak{l}_\mu}\} = NH_m\eta NH_k.$$

We now have

$$\begin{aligned}
\operatorname{Ind}_{N(R_\mu\times L_\mu)}^{G}(\theta\circ(\mu|_{\mathfrak{r}_\mu\oplus\mathfrak{l}_\mu}\oplus\lambda)\circ f) &= \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{\nu\in\mathfrak{g}^*\\ \nu|_{\mathfrak{n}\oplus\mathfrak{r}_\mu\oplus\mathfrak{l}_\mu}=\eta|_{\mathfrak{n}\oplus\mathfrak{r}_\mu\oplus\mathfrak{l}_\mu}}}\theta\circ g\nu g^{-1}\circ f\\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{\nu\in NH_m\eta NH_k}\theta\circ g\nu g^{-1}\circ f\\
&= \frac{|NH_m\eta NH_k|}{|G||H||N|^2}\sum_{g\in G}\sum_{\substack{(h_k,h_m)\in H\\ a,b\in N}}\theta\circ g(ah_k\eta bh_m)g^{-1}\circ f\\
&= \frac{|NH_m\eta NH_k|}{|G|^2}\sum_{g_1,g_2\in G}\theta\circ g_1\eta g_2\circ f\\
&= \frac{|NH_m\eta NH_k|}{|G\eta G|}\sum_{\nu\in G\eta G}\theta\circ\nu\circ f.
\end{aligned}$$

Note that $NH_m\times NH_k$ is a normal subgroup of $G\times G$, hence orbit representative choice does not affect $|NH_m\eta NH_k|$. □

Our construction yields supercharacters which might differ from those defined in Theorem 3.7 by a constant multiple, depending on whether $|NH_m\eta NH_k|$ and $|G\eta|$ are equal. In the case that $G = UT_n(\mathbb{F}_q)$, we in fact have that $|NH_m\eta NH_k| = |G\eta|$, although in general it seems to be unknown whether these orbits are of equal size.

4.4 Supercharacter theories of unipotent groups of other types

As in Section 3.3.3, let q a power of an odd prime and let G be a pattern subgroup of $UT_{2n}(\mathbb{F}_{q^k})$ for some n and k. Let $\mathfrak{g}$ be the corresponding subalgebra of $\mathfrak{ut}_{2n}(\mathbb{F}_{q^k})$, considered as an $\mathbb{F}_q$-algebra. Let $\dagger:\mathfrak{g}\to\mathfrak{g}$ be an algebra anti-involution such that $(\alpha e_{ij})^\dagger\in\mathbb{F}_{q^k}^\times e_{\bar{j}\bar{i}}$ for all $\alpha\in\mathbb{F}_{q^k}^\times$ and $i<j$. For $g=1+x\in G$, define $g^\dagger=1+x^\dagger$; this gives an anti-involution of G. Consider the group

$$U=\{u\in G\mid u^\dagger=u^{-1}\},$$

along with the corresponding Lie algebra

$$\mathfrak{u}=\{x\in\mathfrak{g}\mid x^\dagger=-x\}.$$

In Section 3.3.3, we described a supercharacter theory of U given by Theorem 3.18. In order to reproduce this supercharacter theory via the method of little groups, we need to decompose U as a semidirect product. For each $h \in UT_n(\mathbb{F}_{q^k})$, define $\widetilde{h} \in UT_n(\mathbb{F}_{q^k})$ by

$$\begin{pmatrix} I & 0 \\ 0 & \widetilde{h} \end{pmatrix} = \begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix}^{\dagger}.$$

Note that if

$$\begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix}$$

is an element of G, then

$$\begin{pmatrix} h & 0 \\ 0 & \widetilde{h}^{-1} \end{pmatrix}$$

is in U. Consider the subgroups of U given by

$$H = \left\{ \begin{pmatrix} h & 0 \\ 0 & \widetilde{h}^{-1} \end{pmatrix} \;\middle|\; \begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix} \in G \right\}$$

and

$$N = \left\{ \begin{pmatrix} I & x \\ 0 & I \end{pmatrix} \;\middle|\; \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{u} \right\}.$$

Observe that N is an abelian normal subgroup of U and $U = N \rtimes H$ as an internal semidirect product. Furthermore, H is isomorphic to a pattern subgroup of $UT_n(\mathbb{F}_q)$ under the isomorphism

$$\varphi : \begin{pmatrix} h & 0 \\ 0 & \widetilde{h}^{-1} \end{pmatrix} \mapsto h.$$

Define $\mathfrak{h} = f(H)$ and $\mathfrak{n} = f(N)$. Note that the irreducible characters of N are exactly the elements of the form $\psi_\lambda = \theta \circ \lambda \circ f$, where $\lambda \in \mathfrak{n}^*$. Let

$$\bar{\mathfrak{n}} = \{x \in \mathfrak{g} \mid x_{ij} = 0 \text{ if } i > n \text{ or } j \leq n\}.$$

Given $\lambda \in \mathfrak{n}^*$, define $\bar{\lambda} \in \bar{\mathfrak{n}}^*$ by $\bar{\lambda}(x) = \frac{1}{2}\lambda(x - x^\dagger)$. Note that $\bar{\lambda}|_{\mathfrak{n}} = \lambda$.

Let $\mathcal{L}$ be the lattice of subgroups of H that map to right ideal subgroups under the isomorphism φ. Each $A \in \mathcal{L}$ is equipped with a supercharacter theory with supercharacters

$$\{\chi \circ \varphi \mid \chi \text{ is a right ideal supercharacter of } \varphi(A)\}.$$

It will be useful to describe these supercharacters in terms of the functionals $\mu \in \mathfrak{a}^*$, where $\mathfrak{a} = f(A)$. Let M be the subgroup of G defined by

$$M = \left\{ \begin{pmatrix} I & 0 \\ 0 & h \end{pmatrix} \in G \right\},$$

and let

$$R_\lambda = f^{-1}(\mathfrak{r}_\lambda), \quad \text{where} \quad \mathfrak{r}_\lambda = \{x \in \mathfrak{a} \mid (m \cdot \lambda)(x) = \lambda(x) \text{ for all } m \in M\}.$$

Then the supercharacters of A are the functions

$$\chi_\lambda = \mathrm{Ind}_{R_\lambda}^{A}(\theta \circ \lambda \circ f).$$

(We should mention at this point that the supercharacters of algebra groups can be constructed using any Springer morphism f in place of the usual $g \mapsto g - 1$ map. The choice of Springer morphism has no effect on the resulting supercharacter theory.)

For $\psi_\lambda \in \mathrm{Irr}(N)$, let

$$H_{\psi_\lambda} = \{h \in H \mid h\bar{\lambda} = \bar{\lambda}\}.$$

Note that $\varphi(H_{\psi_\lambda})$ is a right ideal subgroup of $UT_n(\mathbb{F}_q)$, and for $h^{-1} \in H_{\psi_\lambda}$ and $x \in \mathfrak{n}$,

$$\begin{aligned} (h^{-1} \cdot \psi_\lambda)(f^{-1}(x)) &= \theta(\lambda(hxh^{-1})) \\ &= \theta(\bar{\lambda}(hxh^{-1})) \\ &= \theta(\bar{\lambda}(xh^{-1})) \\ &= \theta(\bar{\lambda}(h(-x^\dagger))) \\ &= \theta(\bar{\lambda}(-x^\dagger)) \\ &= \psi_\lambda(f^{-1}(x)). \end{aligned}$$

It follows that $H_{\psi_\lambda} \subseteq I_{\mathcal{L}}(\psi_\lambda)$. Furthermore, if $k \in I_{\mathcal{L}}(\psi_\lambda)$, then $k\bar{\lambda}k^{-1} = \bar{\lambda}$, hence $k^{-1}\bar{\lambda} = \bar{\lambda}k^{-1}$. This means that, for $h \in H_{\psi_\lambda}$,

$$khk^{-1}\bar{\lambda} = kh\bar{\lambda}k^{-1} = \bar{\lambda},$$

and H_{ψ_λ} is in fact normal in $I_{\mathcal{L}}(\psi_\lambda)$, satifying condition (H1). If $\mu = k\lambda k^{-1}$ for some $k \in H$, then

$$\bar{\mu} = k\bar{\lambda}k^{-1},$$

from which it follows that

$$H_{(\psi_\lambda)^k} = H_{\psi_{(\lambda^k)}} = H^k_{\psi_\lambda},$$

and condition (H2) holds. Condition (H3) is clear, and (H4) follows from the fact that $\psi_{\lambda+\mu} = \psi_\lambda \psi_\mu$.

Theorem 4.17. *The above choices of H_ψ produce the same supercharacter theory of U as in Theorem 3.18.*

Proof. Let $\eta \in \mathfrak{u}^*$ be a functional and let $\lambda = \eta|_{\mathfrak{n}}$. Define $\mathfrak{h}_{\psi_\lambda} = f(H_{\psi_\lambda})$ and let $\mu = \eta|_{\mathfrak{h}_{\psi_\lambda}}$. We claim that

$$\chi_\eta = \psi_\lambda \rtimes \chi_\mu.$$

We have that

$$\begin{aligned} \psi_\lambda \rtimes \chi_\mu &= \psi_\lambda \rtimes \mathrm{Ind}_{R_\mu}^{H_{\psi_\lambda}} \mathrm{Res}_{R_\mu}^{H_{\psi_\lambda}} (\theta \circ \mu \circ f) \\ &= \psi_\lambda \rtimes \mathrm{Res}_{R_\mu}^{H_{\psi_\lambda}} (\theta \circ \mu \circ f) \\ &= \mathrm{Ind}_{NR_\mu}^{U} \mathrm{Res}_{NR_\mu}^{NH_{\psi_\lambda}} (\theta \circ (\lambda \oplus \mu) \circ f) \\ &= \mathrm{Ind}_{NR_\mu}^{U} \mathrm{Res}_{NR_\mu}^{U} (\theta \circ \eta \circ f). \end{aligned}$$

Note that

$$\mathfrak{n} \oplus \mathfrak{r}_\mu = \{x \in \mathfrak{u} \mid (k \cdot \eta)(x) = \eta(x) \text{ for all } k \in K\},$$

hence $\mathfrak{n} \oplus \mathfrak{r}_\mu = \mathfrak{u}_\eta$, where $\mathfrak{u}_\eta = f(U_\eta)$ is as defined in Section 3.3.3. This means that $NR_\mu = U_\eta$, and

$$\psi_\lambda \rtimes \chi_\mu = \mathrm{Ind}_{U_\eta}^{U} \mathrm{Res}_{U_\lambda}^{U} (\theta \circ \eta \circ f) = \chi_\eta.$$

□

The motivations for this construction are the unipotent orthogonal, symplectic and unitary groups in even dimension. Each of these groups is defined by an anti-involution with the required properties.

4.4.1 Unipotent orthogonal groups

Let

$$U = UO_{2n}(\mathbb{F}_q) = \{g \in UT_{2n}(\mathbb{F}_q) \mid g^{-1} = Jg^tJ\}$$

be the unipotent orthogonal group, where q is a power of an odd prime. As we saw in Section 3.3.4, U is defined by the anti-involution $x \mapsto Jx^tJ$ of $\mathfrak{ut}_{2n}(\mathbb{F}_q)$. It follows that there is a supercharacter theory of U as described in Theorem 4.17, and furthermore that this supercharacter theory coincides with that produced in Section 3.3.4 and with the earlier construction of André–Neto in [6, 7]. The semidirect product decomposition of U as $U = N \rtimes H$ has

$$H = \left\{ \left(\begin{array}{cc} h & 0 \\ 0 & Jh^{-t}J \end{array} \right) \;\middle|\; h \in UT_n(\mathbb{F}_q) \right\}$$

and

$$N = \left\{ \left(\begin{array}{cc} I & x \\ 0 & I \end{array} \right) \;\middle|\; x \in \mathfrak{o}_n(\mathbb{F}_q) \right\},$$

where $\mathfrak{o}_n(\mathbb{F}_q) = \{x \in \mathfrak{gl}_n(\mathbb{F}_q) \mid Jx^tJ = -x\}$. This means that U is isomorphic to the external semidirect product $UT_n(\mathbb{F}_q) \ltimes \mathfrak{o}_n(\mathbb{F}_q)$, where $\mathfrak{o}_n(\mathbb{F}_q)$ is considered as an additive group and the left action of $UT_n(\mathbb{F}_q)$ on $\mathfrak{o}_n(\mathbb{F}_q)$ is given by $h \cdot x = hxJh^tJ$.

4.4.2 Unipotent symplectic groups

Let

$$U = USp_{2n}(\mathbb{F}_q) = \{g \in UT_{2n}(\mathbb{F}_q) \mid g^{-1} = -\Omega g^t \Omega\}$$

be the unipotent symplectic group, where

$$\Omega = \left(\begin{array}{cc} 0 & J \\ -J & 0 \end{array} \right).$$

As we saw in Section 3.3.5, U is defined by the anti-involution $x \mapsto -\Omega x^t \Omega$ of $\mathfrak{ut}_{2n}(\mathbb{F}_q)$. It follows that there is a supercharacter theory of U as described in Theorem 4.17, and furthermore that this supercharacter theory coincides with that produced in Section 3.3.5 and with the earlier construction of André–Neto in [6, 7]. The semidirect product decomposition of U as $U = N \rtimes H$ has

$$H = \left\{ \begin{pmatrix} h & 0 \\ 0 & Jh^{-t}J \end{pmatrix} \;\middle|\; h \in UT_n(\mathbb{F}_q) \right\}$$

and

$$N = \left\{ \begin{pmatrix} I & x \\ 0 & I \end{pmatrix} \;\middle|\; x \in \mathfrak{o}_n^{\perp}(\mathbb{F}_q) \right\},$$

where $\mathfrak{o}_n^{\perp}(\mathbb{F}_q) = \{x \in \mathfrak{gl}_n(\mathbb{F}_q) \mid Jx^tJ = x\}$. This means that U is isomorphic to the external semidirect product $UT_n(\mathbb{F}_q) \ltimes \mathfrak{o}_n^{\perp}(\mathbb{F}_q)$, where $\mathfrak{o}_n^{\perp}(\mathbb{F}_q)$ is considered as an additive group and the left action of $UT_n(\mathbb{F}_q)$ on $\mathfrak{o}_n^{\perp}(\mathbb{F}_q)$ is given by $h \cdot x = hxJh^tJ$.

4.4.3 Unipotent unitary groups

Let

$$U = UU_{2n}(\mathbb{F}_{q^2}) = \{g \in UT_{2n}(\mathbb{F}_{q^2}) \mid g^{-1} = J\overline{g}^tJ\}$$

be the unipotent unitary group, where $\overline{g}_{ij} = (g_{ij})^q$ and q is a power of an odd prime. As we saw in Section 3.3.6, U is defined by the anti-involution $x \mapsto J\overline{x}^tJ$ of $\mathfrak{ut}_{2n}(\mathbb{F}_{q^2})$ (considered as an $\mathbb{F}_q$-algebra). It follows that there is a supercharacter theory of U as described in Theorem 4.17, and furthermore that this supercharacter theory coincides with that produced in Section 3.3.6. The semidirect product decomposition of U as $U = N \rtimes H$ has

$$H = \left\{ \begin{pmatrix} h & 0 \\ 0 & J\overline{h}^{-t}J \end{pmatrix} \;\middle|\; h \in UT_n(\mathbb{F}_{q^2}) \right\}$$

and

$$N = \left\{ \begin{pmatrix} I & x \\ 0 & I \end{pmatrix} \;\middle|\; x \in \mathfrak{u}_n(\mathbb{F}_{q^2}) \right\},$$

where $\mathfrak{u}_n(\mathbb{F}_{q^2}) = \{x \in \mathfrak{gl}_n(\mathbb{F}_{q^2}) \mid J\overline{x}^tJ = -x\}$. This means that U is isomorphic to the external semidirect product $UT_n(\mathbb{F}_{q^2}) \ltimes \mathfrak{u}_n(\mathbb{F}_{q^2})$, where $\mathfrak{u}_n(\mathbb{F}_{q^2})$ is considered as an additive group and the left action of $UT_n(\mathbb{F}_q)$ on $\mathfrak{u}_n(\mathbb{F}_{q^2})$ is given by $h \cdot x = hxJ\overline{h}^tJ$.

4.5 Coarser supercharacter theories of $UT_n(\mathbb{F}_q)$

One advantage of considering supercharacter theories in terms of the method of little groups is that it is often easy to modify supercharacter theories to obtain coarser or finer supercharacter theories. Suppose $G = N \rtimes H$ with N abelian and that $\mathcal{L}' \subseteq \mathcal{L}$ are two sublattices of the subgroup lattice of H satisfying (L1)–(L4). Furthermore, suppose that

(1) the supercharacter theory assigned to each $K \in \mathcal{L}'$ is (not necessarily strictly) coarser than the supercharacter theory assigned to K as an element of $\mathcal{L}$; and

(2) for each $\psi \in \mathrm{Irr}(N)$, the subgroups $H'_\psi \in \mathcal{L}'$ and $H_\psi \in \mathcal{L}$ have $H'_\psi \subseteq H_\psi$.

Lemma 4.18. *The supercharacter theory of G corresponding to the lattice $\mathcal{L}'$ and the subgroups H'_ψ is (not necessarily strictly) coarser than the the supercharacter theory of G corresponding to the lattice $\mathcal{L}$ and the subgroups H_ψ.*

Proof. Let $\psi \in \mathrm{Irr}(N)$ and let χ be a supercharacter of H'_ψ. Then $\psi \rtimes \chi = \psi \rtimes \mathrm{Ind}_{H'_\psi}^{H_\psi}(\chi)$; as $\mathrm{Ind}_{H'_\psi}^{H_\psi}(\chi)$ is a superclass function of H_ψ, $\psi \rtimes \chi$ is a superclass function of G in the supercharacter theory of G corresponding to the lattice $\mathcal{L}$ and the subgroups H_ψ. It follows that the supercharacter theory of G corresponding to the lattice $\mathcal{L}'$ and the subgroups H'_ψ is at least as coarse as the supercharacter theory of G corresponding to the lattice $\mathcal{L}$ and the subgroups H_ψ. □

Cautionary Example. Let $G = N \rtimes H$ be any semidirect product with N abelian; we can choose

(1) $\mathcal{L} = \{$Normal subgroups of $H\}$ under their usual character theory, and

(2) $\mathcal{L}' = \{$Normal subgroups of $H\}$ under the supercharacter theories formed from conjugation action of H.

As long as $H_\psi = H'_\psi$ for all $\psi \in \mathrm{Irr}(N)$, these two choices of lattice will give the same supercharacter theory of G, even though in general the supercharacter theories of the elements of $\mathcal{L}'$ can be strictly coarser than those in $\mathcal{L}$.

As an example, we construct a collection of supercharacter theories of $UT_n(\mathbb{F}_q)$ which are coarser than the usual supercharacter theory and have a nice indexing set for the superclasses and supercharacters. Recall that if $G = 1 + \mathfrak{g}$ is an algebra group and $K = 1 + \mathfrak{k}$ is a subgroup with $\mathfrak{k}$ a two-sided ideal of $\mathfrak{g}$, by Proposition 3.16 there is a supercharacter theory of K with superclasses

$$K_g = \{h \in K \mid f(h) \in Gf(g)G\}$$

and supercharacters

$$\chi_\lambda = \sum_{\mu \in G\lambda G} \theta \circ \mu \circ f.$$

To construct a supercharacter theory of $UT_n(\mathbb{F}_q)$, let $n = m + k$, and let $H = H_k \times H_m$ and N be as in Section 4.3. Let $\mathcal{L}$ be the lattice of subgroups of H of the form $K_k \times K_m$ with $K_k = 1 + \mathfrak{k}_k$, $K_m = 1 + \mathfrak{k}_m$, and $\mathfrak{k}_k$ and $\mathfrak{k}_m$ two-sided ideals of $\mathfrak{ut}_k(\mathbb{F}_q)$ and $\mathfrak{ut}_m(\mathbb{F}_q)$, respectively. We equip such a subgroup $K_k \times K_m$ with the direct product of the supernormal supercharacter theories of K_k and K_m. For each $\psi \in \mathrm{Irr}(N)$, let $H_\psi = I_{\mathcal{L}}(\psi)$. Observe that

(1) this choice of lattice $\mathcal{L}$ and of H_ψ satisfies (L1)–(L4) and (H1)–(H4); and

(2) by Lemma 4.18 the resulting supercharacter theory is coarser than the usual supercharacter theory.

We will refer to this supercharacter theory as $SCT(n, k)$.

If $\mathfrak{n} = f(N)$, the irreducible characters of N are exactly the elements $\psi_\lambda = \theta \circ \lambda \circ f$, where $\lambda \in \mathfrak{n}^*$. We have that

$$H_{\psi_\lambda} = \left\{ h \in H \;\middle|\; \begin{array}{l} h_{ij} = 0 \text{ if there exist } i' \text{ and } j' \text{ with} \\ \lambda(e_{i'j'}) \neq 0 \text{ and } i' \leq i < k \text{ or } k < j \leq j' \end{array} \right\}.$$

Example 4.3. Let $n = 7$, $k = 3$, and $m = 4$, and let $G = UT_7(\mathbb{F}_q)$. Define $\lambda \in \mathfrak{n}^*$ by

$$\lambda(x) = x_{15} + x_{36}.$$

In other words, $\lambda(x)$ is determined by the entries denoted by $\circ$ below:

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & * & \circ & * & * \\ 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & \circ & * \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

For this choice of λ, H_{ψ_λ} will be the subgroup

$$H_{\psi_\lambda} = \left\{ \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & \circ & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \circ & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \in H \right\},$$

where once again the entries labeled with $\circ$ are the entries that determine λ. For comparison, note that in the usual supercharacter theory of $UT_7(\mathbb{F}_q)$, the subgroup H_{ψ_λ} consists of the elements

$$H_{\psi_\lambda} = \left\{ \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & \circ & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \circ & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \in H \right\}.$$

Recall from Section 3.2 that the supercharacters and superclasses of $UT_n(\mathbb{F}_q)$ under the algebra group supercharacter theory are indexed by $\mathbb{F}_q$-set partitions. To each $\mathbb{F}_q$-set partition η, we assign representatives $g_\eta \in UT_n(\mathbb{F}_q)$ and $\lambda_\eta \in \mathfrak{ut}_n(\mathbb{F}_q)^*$ and define supercharacters and superclasses corresponding to η. For the purposes of this construction, define

$$K_\eta = K_{g_\eta} \quad \text{and} \quad \chi_\eta = \frac{|G\lambda G|}{|G\lambda|}\chi_{\lambda_\eta}.$$

We mention that this is not the definition of χ_η given in Section 3.2, but for many purposes choosing χ_η as the above constant multiple of χ_{λ_η} can be useful. Our choice has the advantage that

$$\chi_\eta = \sum_{\mu \in G\lambda_\eta G} \theta \circ \mu \circ f$$

and

$$\chi_\eta = \sum \psi(1)\psi,$$

where the sum is over some set of irreducible characters ψ.

We can describe the supercharacters and superclasses of $SCT(n,k)$ in a similar manner to those of the algebra group supercharacter theory. If an $\mathbb{F}_q$-set partition η has the property that if $i \overset{a}{\frown} j$ is an arc of η with $i \le k < j$, then there are no arcs in λ of the form $i' \overset{a'}{\frown} j'$ with $i < i' < j' \le k$ or $k < i' < j' < j$, we will call η a **k-nonnesting $\mathbb{F}_q$-set partition**.

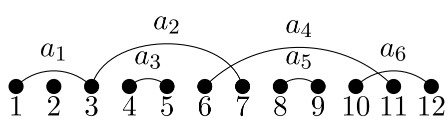
Given an $\mathbb{F}_q$-set partition η, let

$$\overline{\eta}_k = \left\{ i \overset{a}{\frown} j \in \eta \;\middle|\; \begin{array}{c} \text{there are no } i' \overset{a'}{\frown} j' \in \eta \text{ with} \\ i < i' < j' \le k < j \text{ or } i \le k < i' < j' < j \end{array} \right\} \quad \text{and}$$

$$\widetilde{\eta}_k = \left\{ i \overset{a}{\frown} j \in \eta \;\middle|\; \begin{array}{c} \text{there are no } i' \overset{a'}{\frown} j' \in \eta \text{ with} \\ i' < i < j \le k < j' \text{ or } i' \le k < i < j < j' \end{array} \right\}.$$

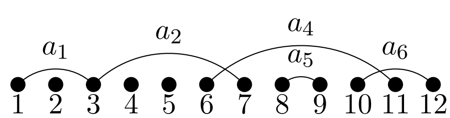
In other words, we obtain $\overline{\eta}_k$ and $\widetilde{\eta}_k$ from η by two different methods of removing arcs that prevent η from being a k-nonnesting $\mathbb{F}_q$-set partition.

Example 4.4. Let $n = 12$, $k = 5$, and $m = 7$, and

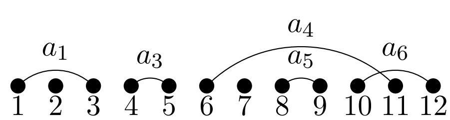
$\eta =$ 

;

then we have

$\overline{\eta}_k =$ 

and

$\widetilde{\eta}_k =$ 

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The constructions $\overline{\eta}_k$ and $\widetilde{\eta}_k$ each define an equivalence relation on the set of $\mathbb{F}_q$-labeled set partitions. In both cases, the equivalence classes are indexed by the k-nonnesting $\mathbb{F}_q$-set partitions. It follows that the two relations have the same number of equivalence classes.

If K_η denotes the superclass of $UT_n(\mathbb{F}_q)$ associated to the $\mathbb{F}_q$-set partition η in the algebra group supercharacter theory, let

$$K_{[\eta]_k} = \bigcup_{\widetilde{\nu}_k = \widetilde{\eta}_k} K_\nu.$$

Similarly, if χ_η is the supercharacter associated to η in the algebra group supercharacter theory as defined above, let

$$\chi_{[\eta]_k} = \sum_{\overline{\nu}_k = \overline{\eta}_k} \chi_\nu.$$

Theorem 4.19. *The characters $\chi_{[\eta]_k}$, along with the subsets $K_{[\eta]_k}$, are the supercharacters and superclasses of $SCT(n,k)$. If η and ν are $\mathbb{F}_q$-set partitions and $g \in K_{[\nu]_k}$, we have that*

$$\chi_{[\eta]_k}(g) = \begin{cases} \frac{\chi_{[\eta]_k}(1)}{\chi_\eta(1)}\chi_\eta(g) & \text{if there are no } i \overset{a}{\frown} j \in \eta \text{ and } i' \overset{a'}{\frown} j' \in \nu \\ & \text{with } i < i' < j' \leq k < j \text{ or } i \leq k < i' < j' < j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $G = UT_n(\mathbb{F}_q)$ and $\mathfrak{g} = \mathfrak{ut}_n(\mathbb{F}_q)$. Choose $\alpha \in \mathfrak{g}^*$, and let $\lambda = \alpha|_{\mathfrak{n}}$. We have that

$$H_{\psi_\lambda} = \left\{ h \in H \;\middle|\; \begin{array}{l} h_{ij} = 0 \text{ if there exist } i' \text{ and } j' \text{ with} \\ \lambda(e_{i'j'}) \neq 0 \text{ and } i' \leq i < k \text{ or } k < j \leq j' \end{array} \right\},$$

where $\psi_\lambda = \theta \circ \lambda \circ f$. Define $\mathfrak{h}_{\psi_\lambda} = f(H_{\psi_\lambda})$, and let $\mu = \alpha|_{\mathfrak{h}_{\psi_\lambda}}$. The supercharacter of $SCT(n,k)$ associated to λ and μ is

$$\begin{aligned} \psi_\lambda \rtimes \sum_{\beta \in H_{\psi_\lambda}\mu H_{\psi_\lambda}} \theta \circ \beta \circ f &= \mathrm{Ind}_{NH_{\psi_\lambda}}^G \left(\sum_{\beta \in S} \theta \circ \beta \circ f \right) \\ &= \sum_{\substack{\beta \in \mathfrak{g}^* \\ \beta|_{\mathfrak{n} \oplus \mathfrak{h}_{\psi_\lambda}} \in S}} \theta \circ \beta \circ f, \end{aligned}$$

where $S = NH_{\psi_\lambda}(\lambda \oplus \mu)NH_{\psi_\lambda}$, by Lemma 3.6. Note that if $\beta \in \mathfrak{g}^*$ and $\theta \circ \beta \circ f$ is a constituent of χ_η, then $\beta \in \{\gamma \in \mathfrak{g}^* \mid \gamma|_{\mathfrak{n} \oplus \mathfrak{h}_{\psi_\lambda}} \in NH_{\psi_\lambda}(\lambda \oplus \mu)NH_{\psi_\lambda}\}$ if and only if $\overline{\eta}_k = \overline{\nu}_k$, where $\theta \circ \alpha \circ f$ is a constituent of χ_ν. It follows that the supercharacters of $SCT(n,k)$ are $\{\chi_{[\eta]_k}\}$.

There are no $i \overset{a}{\frown} j \in \eta$ and $i' \overset{a'}{\frown} j' \in \nu$ with $i < i' < j' \leq k < j$ or $i \leq k < i' < j' < j$ if and only if $g \in NH_{\psi_\lambda}$. Furthermore, $\chi_{[\eta]_k}$ is the character of G obtained by inducing the supernormal supercharacter of NH_{ψ_λ} associated to $\lambda \oplus \mu$ to G. It follows that

$$\chi_{[\eta]_k}(g) = \begin{cases} \frac{\chi_{[\eta]_k}(1)}{\chi_\eta(1)}\chi_\eta(g) & \text{if there are no } i \overset{a}{\frown} j \in \eta \text{ and } i' \overset{a'}{\frown} j' \in \nu \\ & \text{with } i < i' < j' \leq k < j \text{ or } i \leq k < i' < j' < j, \\ 0 & \text{otherwise,} \end{cases}$$

and the superclasses of $SCT(n,k)$ are the sets $K_{[\nu]_k}$. ∎

Observe that

(1) $SCT(n,0) = SCT(n,n)$, and this supercharacter theory is just the usual algebra group supercharacter theory,

(2) $SCT(n,k)$ is strictly coarser than the usual algebra group supercharacter theory for all $1 \leq k \leq n-1$, and

(3) If $1 \leq k < k' \leq n-1$, then $SCT(n,k)$ and $SCT(n,k')$ are incomparable.

4.6 Further directions

Unfortunately, the method of little groups cannot be applied to the groups $UO_{2n+1}(\mathbb{F}_q)$ or $UU_{2n+1}(\mathbb{F}_{q^2})$ as there is no semidirect product decomposition with an abelian normal subgroup. There is some hope that a modification of the method could work, however; $UO_{2n+1}(\mathbb{F}_q)$ has a normal subgroup $N \cong \mathfrak{o}_n(\mathbb{F}_q)$ with $G/N \cong UT_{n+1}(\mathbb{F}_q)$ (a similar statement can be made for $UU_{2n+1}(\mathbb{F}_{q^2})$). In these cases it is not in general true that each $\psi \in \mathrm{Irr}(N)$ can be extended to a character of $I_G(\psi)$. Hopefully the construction can be modified to a group G with abelian normal subgroup N even when N is not the normal complement of any subgroup of G and irreducible characters of N cannot be extended to their inertial subgroups. If so, it seems that the supercharacter theories of the unipotent orthogonal and unitary groups in odd dimension could be recovered from right ideal supercharacter theories.

It would be also of interest to investigate finer supercharacter theories in the orthogonal, symplectic, and unitary types. Note that the choices of H_ψ in Theorem 4.17 are not always maximal, and as such there is a finer supercharacter theory that can be constructed from the same lattice. Furthermore, these supercharacters will each be of the form

$$\chi = c \sum_{\lambda \in S} \theta \circ \lambda \circ f$$

for some constant c and some subset $S \subseteq \mathfrak{u}^*$.

Chapter 5

Supercharacter theories indexed by nonnesting set partitions

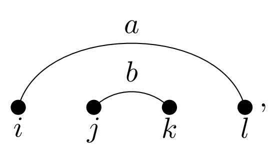
For a fixed n, the lattice-theoretic join of the supercharacter theories $SCT(n,k)$ constructed in Chapter 4 has supercharacters and superclasses indexed by nonnesting set partitions. In Chapter 5 we present this supercharacter theory and generalize it to arbitrary pattern groups. We use these supercharacter theories to construct a Hopf monoid, and give a combinatorial description of the product and coproduct.

5.1 The nonnesting supercharacter theory of $UT_n(\mathbb{F}_q)$

Let η be an $\mathbb{F}_q$-set partition, and define

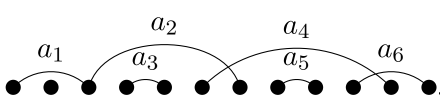
$$\begin{aligned}\overline{\eta} &= \{i \overset{a}{\frown} j \in \eta \mid \text{ there are no } k \overset{b}{\frown} l \in \eta \text{ with } i<k<l<j\} \text{ and}\\ \widetilde{\eta} &= \{i \overset{a}{\frown} j \in \eta \mid \text{ there are no } k \overset{b}{\frown} l \in \eta \text{ with } k<i<j<l\}.\end{aligned}$$

In other words, if η contains two arcs of the form



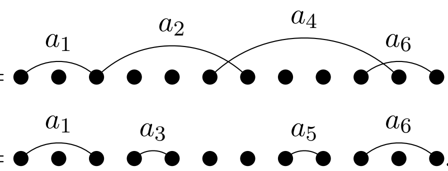
then the arc $j \overset{b}{\frown} k$ is removed in $\overline{\eta}$ and the arc $i \overset{a}{\frown} l$ is removed in $\widetilde{\eta}$.

Example 5.1. If $n = 12$ and

$\eta =$ 

then we have

$\overline{\eta} =$ ... and $\widetilde{\eta} =$ ...



Note that both $\overline{\eta}$ and $\widetilde{\eta}$ are nonnesting $\mathbb{F}_q$-set partitions. These methods of producing a nonnesting $\mathbb{F}_q$-set partition from an arbitrary $\mathbb{F}_q$-set partition each define an equivalence relation on the set of $\mathbb{F}_q$-set partitions, and in both cases the equivalence classes are indexed by the nonnesting $\mathbb{F}_q$-set partitions. For a nonnesting $\mathbb{F}_q$-set partition η, let

$$K_{[\eta]} = \bigcup_{\widetilde{\nu}=\eta} K_\nu,$$

where K_ν denotes the superclass of $UT_n(\mathbb{F}_q)$ associated to the $\mathbb{F}_q$-set partition ν in the algebra group supercharacter theory (see Corollary 3.12). Similarly define

$$\chi_{[\eta]} = \sum_{\overline{\nu}=\eta} \chi_\nu,$$

where

$$\chi_\nu = \sum_{\mu \in G\lambda_\nu G} \theta \circ \mu \circ f$$

as in Section 4.5.

The following lemma provides an alternative description of the characters $\chi_{[\eta]}$.

Lemma 5.1. *For a nonnesting $\mathbb{F}_q$-set partition η, define*

$$U_\eta = \left\{ g \in UT_n(\mathbb{F}_q) \;\middle|\; \begin{array}{l} g_{ij} = 0 \text{ if there exists } k \overset{a}{\frown} l \in \eta \text{ with} \\ (i,j) \neq (k,l) \text{ and } k \leq i < j \leq l \end{array} \right\},$$

and let $\mathfrak{u}_\eta = f(U_\eta)$. Then

$$\chi_{[\eta]} = \mathrm{Ind}_{U_\eta}^{UT_n(\mathbb{F}_q)} \mathrm{Res}_{U_\eta}^{UT_n(\mathbb{F}_q)} (\theta \circ \lambda_\eta \circ f),$$

where $\lambda_\eta(x) = \sum_{i \overset{a}{\frown} j \in \eta} a x_{ij}$ as in Proposition 3.11.

Proof. Note that $\operatorname{Res}^{UT_n(\mathbb{F}_q)}_{U_\eta}(\theta \circ \lambda_\eta \circ f)$ is invariant under the conjugation action of $UT_n(\mathbb{F}_q)$; by Lemma 3.6, we have that

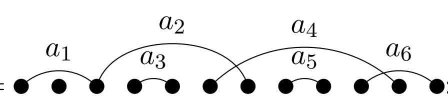
$$\begin{aligned}\operatorname{Ind}^{UT_n(\mathbb{F}_q)}_{U_\eta}\operatorname{Res}^{UT_n(\mathbb{F}_q)}_{U_\eta}(\theta \circ \lambda_\eta \circ f)(g) &= \begin{cases} \frac{|UT_n(\mathbb{F}_q)|}{|U_\eta|}(\theta \circ \lambda_\eta \circ f)(g) & \text{if } g \in U_\eta, \\ 0 & \text{otherwise,} \end{cases} \\ &= \sum_{\substack{\mu \in \mathfrak{ut}_n(\mathbb{F}_q^*) \\ \mu|_{\mathfrak{u}_\eta} = \lambda_\eta|_{\mathfrak{u}_\eta}}} \theta \circ \mu \circ f. \end{aligned}$$

At the same time, we have

$$\chi_{[\eta]} = \sum_{\substack{\mu \in \mathfrak{ut}_n(\mathbb{F}_q^*) \\ \mu|_{\mathfrak{u}_\eta} = \lambda_\eta|_{\mathfrak{u}_\eta}}} \theta \circ \mu \circ f.$$

□

Example 5.2. Once again, let

$\eta =$ ;

then we have

$$U_\eta = \left\{ \begin{pmatrix} 1 & 0 & \mathbf{*} & * & * & * & * & * & * & * & * & * \\ 0 & 1 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{*} & * & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \mathbf{*} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{*} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

where the bold entries correspond to the arcs of $\overline{\eta}$.

Corollary 5.2. *Let η be a nonnesting $\mathbb{F}_q$-set partition. Then $\chi_\eta(1) = q^{|U:U_\eta|}$, where*

$$|U : U_\eta| = \left| \left\{ (i,j) \;\middle|\; \begin{array}{l} i < j \text{ and there exists } k \overset{a}{\frown} l \in \eta \text{ with} \\ (i,j) \neq (k,l) \text{ and } k \leq i < j \leq l \end{array} \right\} \right|.$$

We now calculate the values of the characters χ_η on the elements of the sets K_ν.

Proposition 5.3. *Let η and ν be nonnesting $\mathbb{F}_q$-set partitions and let $g \in K_{[\nu]}$. Then*

$$\chi_{[\eta]}(g) = \begin{cases} \chi_{[\eta]}(1) \prod_{\substack{i \overset{a}{\frown} j \in \eta \\ i \overset{b}{\frown} j \in \nu}} \theta(ab) & \text{if there are no } i \overset{a}{\frown} j \in \eta \text{ and } k \overset{b}{\frown} l \in \nu \\ & \text{with } (i,j) \neq (k,l) \text{ and } i \leq k < l \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. There are no $i \overset{a}{\frown} j \in \eta$ and $k \overset{b}{\frown} l \in \nu$ with $(i,j) \neq (k,l)$ and $i \leq k < l \leq j$ exactly when $K_\nu \subseteq U_\eta$. Assume that $g \in U_\eta$; then if $i \overset{a}{\frown} j \in \eta$ and $g_{ij} \neq 0$, we have that $i \overset{g_{ij}}{\frown} j \in \nu$. The result follows from Lemma 5.1 and Corollary 5.2. $\square$

We can now show that we have constructed a supercharacter theory of $UT_n(\mathbb{F}_q)$.

Theorem 5.4. *The sets*

$$\{\chi_{[\eta]} \mid \eta \text{ is a nonnesting } \mathbb{F}_q\text{-set partition}\} \quad \text{and} \quad \{K_{[\nu]} \mid \nu \text{ is a nonnesting } \mathbb{F}_q\text{-set partition}\}$$

are the supercharacters and superclasses for a supercharacter theory of $UT_n(\mathbb{F}_q)$.

Proof. As both sets are indexed by nonnesting $\mathbb{F}_q$-set partitions, we have (SCT1). Proposition 5.3 demonstrates that (SCT2) holds, and (SCT3) follows from the fact that each algebra group supercharacter is a constituent of exactly one nonnesting supercharacter. $\square$

We call this the **nonnesting supercharacter theory** of $UT_n(\mathbb{F}_q)$; the following corollary is immediate from Lemma 2.12.

Corollary 5.5. *The characters $\chi_{[\eta]}$, along with the superclasses $K_{[\eta]}$, give the lattice theoretic join of the supercharacter theories $SCT(n,k)$, where $0 \leq k \leq n$.* $\square$

5.2 Nonnesting supercharacter theories of pattern groups

Let $\mathcal{P}$ be a poset on $[n]$ such that if $i \prec_{\mathcal{P}} j$ then $i < j$. Recall that the pattern group $U_{\mathcal{P}}$ has a supercharacter theory with superclasses

$$K_g = \{h \in U_{\mathcal{P}} \mid f(h) \in U_{\mathcal{P}} f(g) U_{\mathcal{P}}\}$$

and supercharacters

$$\chi_\lambda = \sum_{\mu \in U_{\mathcal{P}} \lambda U_{\mathcal{P}}} \theta \circ \mu \circ f.$$

Once again, as in Section 4.5, we are taking our supercharacters to be constant multiples of those from Theorem 3.7. Unfortunately, there is no known indexing set for the supercharacters and superclasses of an arbitrary pattern group. We present a coarser supercharacter theory of $U_{\mathcal{P}}$ which is analogous to the nonnesting supercharacter theory on $UT_n(\mathbb{F}_q)$ and has a nice indexing set for the supercharacters and superclasses.

For an element $g \in U_{\mathcal{P}}$, let

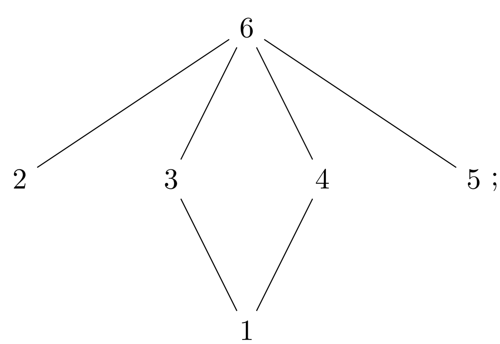
$$\nu_g = \left\{ i \overset{g_{ij}}{\frown} j \;\middle|\; \begin{array}{l} g_{ij} \neq 0 \text{ and if } (k,l) \neq (i,j) \\ \text{and } i \preceq_{\mathcal{P}} k \prec_{\mathcal{P}} l \preceq_{\mathcal{P}} j, \text{ then } g_{kl} = 0 \end{array} \right\}.$$

Similarly, if $\eta \in \mathfrak{u}_{\mathcal{P}}^*$, define

$$\eta_\lambda = \left\{ i \overset{\lambda(e_{ij})}{\frown} j \;\middle|\; \begin{array}{l} \lambda(e_{ij}) \neq 0 \text{ and if } (k,l) \neq (i,j) \\ \text{and } k \preceq_{\mathcal{P}} i \prec_{\mathcal{P}} j \preceq_{\mathcal{P}} l, \text{ then } \lambda(e_{kl}) = 0 \end{array} \right\},$$

where $e_{ij} \in \mathfrak{gl}_n(\mathbb{F}_q)$ is the matrix with a 1 in the (i,j) entry and zeroes elsewhere.

Example 5.3. Let $\mathcal{P}$ be the poset with Hasse diagram



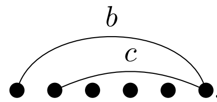
then

$$U_{\mathcal{P}} = \left\{ \begin{pmatrix} 1 & 0 & * & * & 0 & * \\ 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

For $a, b, c, e \in \mathbb{F}_q$, let

$$g = \begin{pmatrix} 1 & 0 & a & 0 & 0 & b \\ 0 & 1 & 0 & 0 & 0 & c \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & e \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

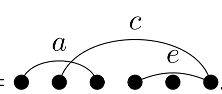
then

$\nu_g =$ 

We can define $\lambda \in \mathfrak{u}_{\mathcal{P}}^*$ by

$$\lambda(x) = ax_{13} + bx_{16} + cx_{26} + ex_{46};$$

then

$\eta_\lambda =$ 

Note that ν_g and η_η are nonnesting $(\mathbb{F}_q, \mathcal{P})$-set partitions for all $g \in U_{\mathcal{P}}$ and $\lambda \in \mathfrak{u}_{\mathcal{P}}^*$. For a nonnesting $(\mathbb{F}_q, \mathcal{P})$-set partition η, define

$$K_\nu = \{g \in U_{\mathcal{P}} \mid \nu_g = \nu\}$$

and

$$\chi_\eta = \sum_{\substack{\lambda \in \mathfrak{u}_{\mathcal{P}}^* \\ \eta_\lambda = \eta}} \theta \circ \lambda \circ f.$$

Lemma 5.6. *We have that*

(1) if $g \in U_{\mathcal{P}}$ and $f(h) \in U_{\mathcal{P}} f(g) U_{\mathcal{P}}$, then $\nu_g = \nu_h$; and

(2) if $\lambda \in \mathfrak{u}_{\mathcal{P}}^*$ and $\mu \in U_{\mathcal{P}} \lambda U_{\mathcal{P}}$, then $\eta_\lambda = \eta_\mu$.

Proof. We prove (1), and mention that the proof of (2) is similar. Let $x \in \mathfrak{u}_{\mathcal{P}}$ and $u, v \in U_{\mathcal{P}}$; then

$$(uxv)_{ij} = \sum_{\substack{k,l \text{ such that} \\ i \preceq_{\mathcal{P}} k \prec_{\mathcal{P}} l \preceq_{\mathcal{P}} j}} u_{ik} x_{kl} v_{lj}.$$

Suppose that $g_{ij} \neq 0$, and that for all $(k,l) \neq (i,j)$ with $i \preceq_{\mathcal{P}} k \prec_{\mathcal{P}} l \preceq_{\mathcal{P}} j$, we have $g_{kl} = 0$. If $f(h) \in U_{\mathcal{P}} f(g) U_{\mathcal{P}}$, then by the above calculation

(1) $h_{ij} = g_{ij}$ and

(2) for all $(k,l) \neq (i,j)$ with $i \preceq_{\mathcal{P}} k \prec_{\mathcal{P}} l \preceq_{\mathcal{P}} j$, we have $h_{kl} = 0$.

It follows that $\nu_g \subseteq \nu_h$, and the same argument shows that $\nu_h \subseteq \nu_g$. □

There is an important corollary of Lemma 5.6.

Corollary 5.7. *The sets K_ν are unions of algebra group superclasses, and the functions χ_η are sums of algebra group supercharacters. In particular, the functions χ_η are characters of $U_{\mathcal{P}}$.*

The following lemma provides an alternative description of the characters χ_η.

Lemma 5.8. *Let η be a nonnesting $\mathcal{P}$-set partition and let χ_η be as above. Define*

$$U_\eta = \left\{ g \in U_{\mathcal{P}} \;\middle|\; \begin{array}{l} g_{ij} = 0 \textit{ if there exists } k \overset{a}{\frown} l \in \eta \textit{ with} \\ (i,j) \neq (k,l) \textit{ and } k \preceq_{\mathcal{P}} i \prec_{\mathcal{P}} j \preceq_{\mathcal{P}} l \end{array} \right\},$$

and let $\mathfrak{u}_\eta = f(U_\eta)$. Then

$$\chi_\eta = \mathrm{Ind}_{U_\eta}^{U_{\mathcal{P}}} \mathrm{Res}_{U_\eta}^{U_{\mathcal{P}}} (\theta \circ \lambda \circ f),$$

where $\lambda \in \mathfrak{u}_{\mathcal{P}}^*$ is any functional with $\eta_\lambda = \eta$.

Proof. Note that U_η is a normal pattern subgroup of $U_{\mathcal{P}}$, and $\theta \circ \lambda \circ f$ is invariant under the conjugation action of $U_{\mathcal{P}}$. It follows that

$$\begin{aligned} \mathrm{Ind}_{U_\eta}^{U_{\mathcal{P}}} \mathrm{Res}_{U_\eta}^{U_{\mathcal{P}}} (\theta \circ \lambda \circ f)(g) &= \begin{cases} \frac{|U_{\mathcal{P}}|}{|U_\eta|} (\theta \circ \lambda \circ f)(g) & \text{if } g \in U_\eta, \\ 0 & \text{otherwise,} \end{cases} \\ &= \sum_{\substack{\mu \in \mathfrak{u}_{\mathcal{P}}^* \\ \mu|_{\mathfrak{u}_\eta} = \lambda|_{\mathfrak{u}_\eta}}} \theta \circ \mu \circ f \end{aligned}$$

by Lemma 3.6. At the same time,

$$\chi_\eta = \sum_{\substack{\mu \in \mathfrak{u}_{\mathcal{P}}^* \\ \eta_\mu = \eta}} \theta \circ \mu \circ f,$$

and

$$\{\mu \in \mathfrak{u}_{\mathcal{P}}^* \mid \eta_\mu = \eta\} = \{\mu \in \mathfrak{u}_{\mathcal{P}}^* \mid \mu|_{\mathfrak{u}_\eta} = \lambda|_{\mathfrak{u}_\eta}\}.$$

□

Lemma 5.8 allows us to calculate the dimensions of the characters χ_η.

Corollary 5.9. *Let η be a nonnesting $(\mathbb{F}_q, \mathcal{P})$-set partition. Then $\chi_\eta(1) = q^{|U:U_\eta|}$, where*

$$|U : U_\eta| = \left| \left\{ (i,j) \;\middle|\; \begin{array}{l} i \prec_{\mathcal{P}} j \text{ and there exists } k \overset{a}{\frown} l \in \eta \text{ with} \\ (i,j) \neq (k,l) \text{ and } k \preceq_{\mathcal{P}} i \prec_{\mathcal{P}} j \preceq_{\mathcal{P}} \end{array} \right\} \right|.$$

We now calculate the values of the characters χ_η on the elements of the sets K_ν.

Proposition 5.10. *Let η and ν be nonnesting $(\mathbb{F}_q, \mathcal{P})$-set partitions. If $g \in K_\nu$, then*

$$\chi_\eta(g) = \begin{cases} \chi_\eta(1) \displaystyle\prod_{\substack{i \overset{a}{\frown} j \in \eta \\ i \overset{b}{\frown} j \in \nu}} \theta(ab) & \begin{array}{l} \text{if there are no } i \overset{a}{\frown} j \in \eta \text{ and } k \overset{b}{\frown} l \in \nu \\ \text{with } (i,j) \neq (k,l) \text{ and } i \preceq_{\mathcal{P}} k \prec_{\mathcal{P}} l \preceq_{\mathcal{P}} j \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. There will be no $i \overset{a}{\frown} j \in \eta$ and $k \overset{b}{\frown} l \in \nu$ with $(i,j) \neq (k,l)$ and $i \preceq_{\mathcal{P}} k \prec_{\mathcal{P}} l \preceq_{\mathcal{P}} j$ exactly when $K_\nu \subseteq U_\eta$. Note that if $g \in U_\eta$, $i \overset{a}{\frown} j \in \eta$, and $g_{ij} \neq 0$, then $i \overset{g_{ij}}{\frown} j \in \nu$. Furthermore, if $\eta_\lambda = \eta$ and $g \in U_\eta$, we have

$$(\theta \circ \lambda \circ f)(g) = \prod_{i \overset{a}{\frown} j \in \eta} \theta(a g_{ij}).$$

The result follows from Lemma 5.8. $\square$

We can now prove that we have constructed a supercharacter theory of $U_{\mathcal{P}}$.

Theorem 5.11. *The sets*

$$\{K_\nu \mid \nu \text{ is a nonnesting } (\mathbb{F}_q, \mathcal{P})\text{-set partition}\} \text{ and}$$
$$\{\chi_\eta \mid \eta \text{ is a nonnesting } (\mathbb{F}_q, \mathcal{P})\text{-set partition}\}$$

are the superclasses and supercharacters for a supercharacter theory of $U_{\mathcal{P}}$.

Proof. As both sets are indexed by the nonnesting $(\mathbb{F}_q, \mathcal{P})$-set partitions, we have (SCT1). Proposition 5.10 gives us (SCT2), and (SCT3) follows from Corollary 5.7. $\square$

Remark. If $\mathcal{P}$ is the usual linear order on $[n]$, then $U_{\mathcal{P}} = UT_n(\mathbb{F}_q)$. In this case, the supercharacter theories of Theorems 5.4 and 5.11 coincide.

5.3 A Hopf monoid from the nonnesting supercharacter theories

In [2], Aguiar–Bergeron–Thiem construct a Hopf monoid from the groups of unitriangular matrices with entries in $\mathbb{F}_q$. In this section we will generalize their construction to pattern groups, and show that the nonnesting supercharacter theory defines a combinatorial Hopf monoid.

5.3.1 Vector species and Hopf monoids

The ideas of vector species and Hopf monoids are explored in great detail in [3]. We only present the necessary definitions and results for our construction.

If $\mathbf{Set}^{\times}$ denotes the category of finite sets with morphisms given by bijections and $\mathbf{Vec}$ denotes the category of vector spaces over the field $\mathbb{F}$, then a **vector species p** is a functor

$$\mathbf{p} : \mathbf{Set}^{\times} \to \mathbf{Vec}.$$

In other words, $\mathbf{p}$ is a collection of vector spaces $\mathbf{p}[I]$ indexed by finite sets I along with linear maps

$$\mathbf{p}[\sigma] : \mathbf{p}[I] \to \mathbf{p}[J]$$

for each bijection $\sigma : I \to J$. These maps must satisfy that

$$\mathbf{p}[\mathrm{id}] = \mathrm{id}$$

and

$$\mathbf{p}(\sigma \circ \tau) = \mathbf{p}(\sigma) \circ \mathbf{p}(\tau).$$

A **Hopf monoid** consists of a vector species $\mathbf{p}$ along with linear maps

$$\mu_{S,T} : \mathbf{p}[S] \otimes \mathbf{p}[T] \to \mathbf{p}[S \sqcup T]$$
$$\Delta_{S,T} : \mathbf{p}[S \sqcup T] \to \mathbf{p}[S] \otimes \mathbf{p}[T]$$

for each pair of disjoint sets S and T. These maps must respect bijections in the sense that if

$\sigma : S \to S'$ and $\tau : T \to T'$ are bijections, then the diagrams

$$
\begin{array}{ccc}
\mathbf{p}[S] \otimes \mathbf{p}[T] & \xrightarrow{\mu_{S,T}} & \mathbf{p}[S \sqcup T] \\
{\scriptstyle \mathbf{p}[\sigma] \otimes \mathbf{p}[\tau]} \downarrow & & \downarrow {\scriptstyle \mathbf{p}[\sigma \sqcup \tau]} \\
\mathbf{p}[S'] \otimes \mathbf{p}[T'] & \xrightarrow[\mu_{S',T'}]{} & \mathbf{p}[S' \sqcup T']
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{p}[S] \otimes \mathbf{p}[T] & \xleftarrow{\Delta_{S,T}} & \mathbf{p}[S \sqcup T] \\
{\scriptstyle \mathbf{p}[\sigma] \otimes \mathbf{p}[\tau]} \downarrow & & \downarrow {\scriptstyle \mathbf{p}[\sigma \sqcup \tau]} \\
\mathbf{p}[S'] \otimes \mathbf{p}[T'] & \xleftarrow[\Delta_{S',T'}]{} & \mathbf{p}[S' \sqcup T']
\end{array}
\tag{5.1}
$$

commute. If μ denotes the collection of $\mu_{S,T}$ and Δ the collection of $\Delta_{S,T}$, where each collection varies over all pairs of disjoint finite sets S and T, then μ is called the **product** and Δ is called the **coproduct**.

These collections must be **associative** and **coassociative** in that the diagrams

$$
\begin{array}{ccc}
\mathbf{p}[R] \otimes \mathbf{p}[S] \otimes \mathbf{p}[T] & \xrightarrow{\mathrm{id} \otimes \mu_{S,T}} & \mathbf{p}[R] \otimes \mathbf{p}[S \sqcup T] \\
{\scriptstyle \mu_{R,S} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mu_{R,S \sqcup T}} \\
\mathbf{p}[R \sqcup S] \otimes \mathbf{p}[T] & \xrightarrow[\mu_{R \sqcup S,T}]{} & \mathbf{p}[R \sqcup S \sqcup T]
\end{array}
\tag{5.2}
$$

and

$$
\begin{array}{ccc}
\mathbf{p}[R] \otimes \mathbf{p}[S] \otimes \mathbf{p}[T] & \xleftarrow{\mathrm{id} \otimes \Delta_{S,T}} & \mathbf{p}[R] \otimes \mathbf{p}[S \sqcup T] \\
{\scriptstyle \Delta_{R,S} \otimes \mathrm{id}} \uparrow & & \uparrow {\scriptstyle \Delta_{R,S \sqcup T}} \\
\mathbf{p}[R \sqcup S] \otimes \mathbf{p}[T] & \xleftarrow[\Delta_{R \sqcup S,T}]{} & \mathbf{p}[R \sqcup S \sqcup T]
\end{array}
\tag{5.3}
$$

commute for all pairwise disjoint finite sets S, R and T. They also must satisfy a compatibility property. Let (S_1, S_2) and (T_1, T_2) be pairs of disjoint finite sets such that $S_1 \sqcup S_2 = T_1 \sqcup T_2 = I$. Let

$$A = S_1 \cap T_1 \quad B = S_1 \cap T_2 \quad C = S_2 \cap T_1 \quad D = S_2 \cap T_2;$$

then the diagram

$$\begin{array}{ccccc}
\mathbf{p}[S_1] \otimes \mathbf{p}[S_2] & \xrightarrow{\mu_{S_1,S_2}} & \mathbf{p}[I] & \xrightarrow{\Delta_{T_1,T_2}} & \mathbf{p}[T_1] \otimes \mathbf{p}[T_2] \\
\Big\downarrow {\scriptstyle \Delta_{A,B} \otimes \Delta_{C,D}} & & & & \Big\uparrow {\scriptstyle \mu_{A,C} \otimes \mu_{B,D}} \\
\mathbf{p}[A] \otimes \mathbf{p}[B] \otimes \mathbf{p}[C] \otimes \mathbf{p}[D] & & \xrightarrow[\cong]{} & & \mathbf{p}[A] \otimes \mathbf{p}[C] \otimes \mathbf{p}[B] \otimes \mathbf{p}[D]
\end{array} \tag{5.4}$$

must commute.

We call a Hopf monoid **commutative** (respectively **cocommutative**) if the left (respectively right) diagram commutes for all S, T.

$$\begin{array}{ccc}
\mathbf{p}[S] \otimes \mathbf{p}[T] & \xrightarrow{\cong} & \mathbf{p}[T] \otimes \mathbf{p}[S] \\
{\scriptstyle \mu_{S,T}} \searrow & & \swarrow {\scriptstyle \mu_{T,S}} \\
& \mathbf{p}[S \sqcup T] &
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{p}[S] \otimes \mathbf{p}[T] & \xrightarrow{\cong} & \mathbf{p}[T] \otimes \mathbf{p}[S] \\
{\scriptstyle \Delta_{S,T}} \nwarrow & & \nearrow {\scriptstyle \Delta_{T,S}} \\
& \mathbf{p}[S \sqcup T] &
\end{array}$$

If $\mathbf{p}[\emptyset] = \mathbb{F}$ and the maps

$$\begin{aligned}
\mu_{S,\emptyset} &: \mathbf{p}[S] \otimes \mathbf{p}[\emptyset] \to \mathbf{p}[S], \\
\mu_{\emptyset,S} &: \mathbf{p}[\emptyset] \otimes \mathbf{p}[S] \to \mathbf{p}[S], \\
\Delta_{S,\emptyset} &: \mathbf{p}[S] \to \mathbf{p}[S] \otimes \mathbf{p}[\emptyset], \text{ and} \\
\Delta_{\emptyset,S} &: \mathbf{p}[S] \to \mathbf{p}[\emptyset] \otimes \mathbf{p}[S]
\end{aligned}$$

are the canonical identifications, we say that the Hopf monoid is **connected**. We will denote a Hopf monoid by the triple $(\mathbf{p}, \mu, \Delta)$.

5.3.2 The Hopf monoid on posets

For a finite set S, let $\mathbf{p}[S]$ be the $\mathbb{F}$-vector space with basis

$$\{x_{\mathcal{P}} \mid \mathcal{P} \text{ is a poset on } S\}$$

and $\mathbf{p}[\emptyset] = \mathbb{F}$. Given a bijection $\sigma : S \to T$, define

$$\begin{aligned}
\mathbf{p}[\sigma] : \mathbf{p}[S] &\to \mathbf{p}[T] \\
x_{\mathcal{P}} &\mapsto x_{\sigma\mathcal{P}},
\end{aligned}$$

where $\sigma\mathcal{P}$ is the poset on T with $t_1 \prec_{\sigma\mathcal{P}} t_2$ if and only if $\sigma^{-1}(t_1) \prec_{\mathcal{P}} \sigma^{-1}(t_2)$. This defines a vector species.

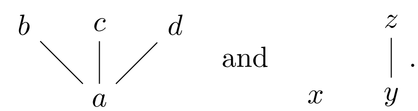
Let S and T be disjoint finite sets. If $\mathcal{P}$ is a poset on S and $\mathcal{Q}$ is a poset on T, define $\mathcal{P} \cdot \mathcal{Q}$ to be the poset on $S \sqcup T$ with $x \prec_{\mathcal{P}\cdot\mathcal{Q}} y$ if and only if

(1) $x, y \in S$ and $x \prec_{\mathcal{P}} y$,

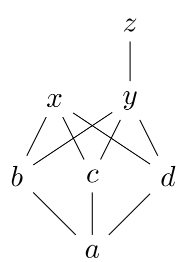
(2) $x, y \in T$ and $x \prec_{\mathcal{Q}} y$, or

(3) $x \in S$ and $y \in T$.

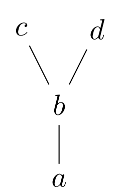
For example, let $S = \{a, b, c, d\}$ and $T = \{x, y, z\}$ with posets $\mathcal{P}$ and $\mathcal{Q}$ given by the Hasse diagrams



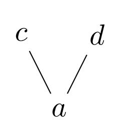
The Hasse diagram of $\mathcal{P} \cdot \mathcal{Q}$ is



For any subset $S \subseteq I$, we can define the restriction of a poset $\mathcal{P}$ of I to a poset $\mathcal{P}|_S$ of S by $x \leq_{\mathcal{P}|_S} y$ if and only if $x, y \in S$ and $x \leq_{\mathcal{P}} y$. For example, let $I = \{a, b, c, d\}$ and $\mathcal{P}$ be the poset with Hasse diagram



If $S = \{a, c, d\}$, then the Hasse diagram of $\mathcal{P}|_S$ is



For two disjoint finite sets S and T, define

$$\mu_{S,T}(x_{\mathcal{P}} \otimes x_{\mathcal{Q}}) = x_{\mathcal{P}\cdot\mathcal{Q}}$$

and

$$\Delta_{S,T}(x_{\mathcal{P}}) = x_{\mathcal{P}|_S} \otimes x_{\mathcal{P}|_T},$$

then extend by linearity.

Proposition 5.12 ([3, Section 13.1.1]). *The product and coproduct defined above make* $(\mathbf{p}, \mu, \Delta)$ *a Hopf monoid. This monoid is connected and cocommutative but not commutative.*

Proof. Conditions 5.1, 5.2, and 5.3, as well as connectedness and cocommutativity, are easy to check; we will only prove that 5.4 holds.

Let (S_1, S_2) and (T_1, T_2) be pairs of disjoint finite sets such that $S_1 \sqcup S_2 = T_1 \sqcup T_2 = I$. Let

$$A = S_1 \cap T_1 \quad B = S_1 \cap T_2 \quad C = S_2 \cap T_1 \quad D = S_2 \cap T_2,$$

and let $\mathcal{P}$ and $\mathcal{Q}$ be posets on S_1 and S_2, respectively. Then

$$\Delta_{T_1,T_2} \circ \mu_{S_1,S_2}(x_{\mathcal{P}} \otimes x_{\mathcal{Q}}) = x_{(\mathcal{P}\cdot\mathcal{Q})|_{T_1}} \otimes x_{(\mathcal{P}\cdot\mathcal{Q})|_{T_2}};$$

at the same time,

$$\begin{aligned}(\mu_{A,C} \otimes \mu_{B,D}) \circ \theta \circ (\Delta_{A,B} \otimes \Delta_{C,D})(x_{\mathcal{P}} \otimes x_{\mathcal{Q}}) &= (\mu_{A,C} \otimes \mu_{B,D}) \circ \theta(x_{\mathcal{P}|_A} \otimes x_{\mathcal{P}|_B} \otimes x_{\mathcal{Q}|_C} \otimes x_{\mathcal{Q}|_D}) \\ &= (\mu_{A,C} \otimes \mu_{B,D})(x_{\mathcal{P}|_A} \otimes x_{\mathcal{Q}|_C} \otimes x_{\mathcal{P}|_B} \otimes x_{\mathcal{Q}|_D}) \\ &= x_{\mathcal{P}|_A\cdot\mathcal{Q}|_C} \otimes x_{\mathcal{P}|_B\cdot\mathcal{Q}|_D},\end{aligned}$$

where θ is the isomorphism that switches the middle factors. It suffices to show that

$$(\mathcal{P}\cdot\mathcal{Q})|_{T_1} = \mathcal{P}|_A \cdot \mathcal{Q}|_C \quad \text{and} \quad (\mathcal{P}\cdot\mathcal{Q})|_{T_2} = \mathcal{P}|_B \cdot \mathcal{Q}|_D.$$

Let $x, y \in T_1$; then $x \prec_{\mathcal{P}\cdot\mathcal{Q}} y$ if and only if

(1) $x, y \in S_1$ and $x \prec_{\mathcal{P}} y$,

(2) $x, y \in S_2$ and $x \prec_{\mathcal{Q}} y$, or

(3) $x \in S_1$ and $y \in S_2$.

This is equivalent to saying that

(1) $x, y \in A$ and $x \prec_{\mathcal{P}} y$,

(2) $x, y \in C$ and $x \prec_{\mathcal{Q}} y$, or

(3) $x \in A$ and $y \in C$.

It follows that $(\mathcal{P} \cdot \mathcal{Q})|_{T_1} = \mathcal{P}|_A \cdot \mathcal{Q}|_C$; the second claim is identical. □

Remark. This Hopf monoid structure on $\mathbf{p}$ is the same as that of the dual Hopf monoid $\mathbf{P}^*$ constructed by Aguiar–Mahajan in [3, Section 13.1.1] with respect to a different basis.

5.4 The Hopf monoid on complex-valued functions of pattern groups

5.4.1 Construction

Our construction in this section mirrors that of [2] for unitriangular matrices. For the purposes of this section we will consider unitriangular matrices and pattern groups in a more general setting. For a finite set I, define an $\mathbb{F}_q$-algebra

$$M(I) = \{a = (a_{ij}) \mid i, j \in I \text{ and } a_{ij} \in \mathbb{F}_q\}$$

with addition and scalar multiplication defined pointwise and multiplication

$$(ab)_{ij} = \sum_{k \in I} a_{ik} b_{kj}.$$

If a total order is chosen on I, then $M(I)$ is canonically isomorphic to the algebra of matrices with entries indexed by the elements of I. Let $GL(I)$ denote the group of invertible elements of $M(I)$. If $\mathcal{P}$ is a poset on I, we define the pattern subgroup associated to $\mathcal{P}$ by

$$U_{\mathcal{P}} = \{g \in GL(I) \mid g_{ii} = 1 \text{ and } g_{ij} = 0 \text{ unless } i \preceq_{\mathcal{P}} j\}.$$

There is a contravariant functor

$$\mathbf{f} : \mathbf{Set} \to \mathbf{Vec}$$

from the category of sets to the category of complex vector spaces. This functor maps a set X to the space of functions from X to $\mathbb{C}$. There is a canonical isomorphism

$$\begin{aligned} \varphi : \mathbf{f}(X \times Y) &\to \mathbf{f}(X) \otimes \mathbf{f}(Y) \\ (\alpha, \beta) &\mapsto \alpha \otimes \beta. \end{aligned}$$

If $U_{\mathcal{P}}$ is a pattern group, then $\mathbf{f}(U_{\mathcal{P}})$ is the space of functions from $U_{\mathcal{P}}$ to $\mathbb{C}$. We use this to define a vector species as follows. For a finite set I, define

$$\mathbf{fp}[I] = \bigoplus_{\substack{\mathcal{P} \text{ is a} \\ \text{poset on } I}} \mathbf{f}(U_{\mathcal{P}}).$$

Any bijection $\sigma : I \to J$ induces an isomorphism $U_{\sigma\mathcal{P}} \cong U_{\mathcal{P}}$, which by functoriality gives an isomorphism $\mathbf{f}(U_{\mathcal{P}}) \cong \mathbf{f}(U_{\sigma\mathcal{P}})$. This means that $\mathbf{fp}$ is in fact a vector species.

Suppose that S and T are disjoint finite sets and that $\mathcal{P}$ and $\mathcal{Q}$ are posets on S and T, respectively. There is a homomorphism

$$\begin{aligned} \pi_{\mathcal{P},\mathcal{Q}} : U_{\mathcal{P}\cdot\mathcal{Q}} &\to U_{\mathcal{P}} \times U_{\mathcal{Q}} \\ g &\mapsto (g_S, g_T), \end{aligned}$$

where $(g_S)_{ij} = g_{ij}$ for all $i, j \in S$.

We use this homomorphism to define a product on $\mathbf{fp}$. Given posets $\mathcal{P}$ and $\mathcal{Q}$ on disjoint finite sets S and T, define

$$\begin{aligned} \mu_{\mathcal{P},\mathcal{Q}} : \mathbf{f}(U_{\mathcal{P}}) \otimes \mathbf{f}(U_{\mathcal{Q}}) &\to \mathbf{f}(U_{\mathcal{P}\cdot\mathcal{Q}}) \\ \alpha \otimes \beta &\mapsto (\alpha, \beta) \circ \pi. \end{aligned}$$

In other words, $\mu_{\mathcal{P},\mathcal{Q}} = \mathbf{f}(\pi_{\mathcal{P},\mathcal{Q}}) \circ \varphi^{-1}$. Extending by linearity over all posets on S and T, we get a map

$$\mu_{S,T} : \mathbf{fp}[S] \otimes \mathbf{fp}[T] \to \mathbf{fp}[S \sqcup T].$$

Remark. The map $\mu_{\mathcal{P},\mathcal{Q}}$ is the inflation map with respect to the projection π. That is,

$$\mu_{\mathcal{P},\mathcal{Q}} = \mathrm{Inf}_{U_\mathcal{P} \times U_\mathcal{Q}}^{U_{\mathcal{P}\cdot\mathcal{Q}}} \circ \varphi^{-1}.$$

We can also use a homomorphism to define the coproduct. Suppose that S and T are disjoint and that $\mathcal{P}$ is a poset on $S \sqcup T$. Define a homomorphism

$$\sigma_{\mathcal{P}|_S,\mathcal{P}|_T} : U_{\mathcal{P}|_S} \times U_{\mathcal{P}|_T} \to U_\mathcal{P}$$

by

$$(\sigma_{\mathcal{P}|_S,\mathcal{P}|_T}(g,h))_{i,j} = \begin{cases} g_{ij} & \text{if } i,j \in S, \\ h_{ij} & \text{if } i,j \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} \Delta_{\mathcal{P}|_S,\mathcal{P}|_T} : \mathbf{f}(U_\mathcal{P}) &\to \mathbf{f}(U_{\mathcal{P}_S}) \otimes \mathbf{f}(U_{\mathcal{P}_T}) \\ \alpha &\mapsto \varphi(\alpha \circ \sigma_{\mathcal{P}|_S,\mathcal{P}|_T}); \end{aligned}$$

in other words, $\Delta_{\mathcal{P}|_S,\mathcal{P}|_T} = \varphi \circ \mathbf{f}(\sigma_{\mathcal{P}|_S,\mathcal{P}|_T})$. Extending by linearity over all posets on $S \sqcup T$, we obtain a map

$$\Delta_{S,T} : \mathbf{fp}[S \sqcup T] \to \mathbf{fp}[S] \otimes \mathbf{fp}[T].$$

Remark. The map $\Delta_{\mathcal{P}|_S,\mathcal{P}|_T}$ is the restriction map; that is,

$$\Delta_{\mathcal{P}|_S,\mathcal{P}|_T} = \varphi \circ \mathrm{Res}_{U_{\mathcal{P}_S} \times U_{\mathcal{P}_T}}^{U_\mathcal{P}}.$$

Theorem 5.13. *Let μ and Δ be the collections of all $\mu_{S,T}$ and $\Delta_{S,T}$; then the triple $(\mathbf{fp}, \mu, \Delta)$ is a connected, cocommutative Hopf monoid.*

Proof. We will only prove Condition 5.4 as the proofs of the other conditions are analogous. Let (S_1, S_2) and (T_1, T_2) be pairs of disjoint finite sets such that $S_1 \sqcup S_2 = T_1 \sqcup T_2 = I$. Let

$$A = S_1 \cap T_1, \quad B = S_1 \cap T_2, \quad C = S_2 \cap T_1, \text{ and } D = S_2 \cap T_2.$$

Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are posets on S_1 and S_2, and $\mathcal{P} = \mathcal{P}_1 \cdot \mathcal{P}_2$. We need to show that the diagram

$$\begin{CD}
\mathbf{f}[U_{\mathcal{P}_1}] \otimes \mathbf{f}[U_{\mathcal{P}_2}] @>{\mu_{\mathcal{P}_1,\mathcal{P}_2}}>> \mathbf{f}[U_{\mathcal{P}}] @>{\Delta_{\mathcal{P}|_{T_1},\mathcal{P}|_{T_2}}}>> \mathbf{f}[U_{\mathcal{P}|_{T_1}}] \otimes \mathbf{f}[U_{\mathcal{P}|_{T_2}}] \\
@V{\Delta_{\mathcal{P}_1|_A,\mathcal{P}_1|_B} \otimes \Delta_{\mathcal{P}_2|_C,\mathcal{P}_2|_D}}VV @. @AA{\mu_{\mathcal{P}_1|_A,\mathcal{P}_2|_C} \otimes \mu_{\mathcal{P}_1|_B,\mathcal{P}_2|_D}}A \\
\mathbf{f}[U_{\mathcal{P}_1|_A}] \otimes \mathbf{f}[U_{\mathcal{P}_1|_B}] \otimes \mathbf{f}[U_{\mathcal{P}_2|_C}] \otimes \mathbf{f}[U_{\mathcal{P}_2|_D}] @>>{\cong}> @. \mathbf{f}[U_{\mathcal{P}_1|_A}] \otimes \mathbf{f}[U_{\mathcal{P}_2|_C}] \otimes \mathbf{f}[U_{\mathcal{P}_1|_B}] \otimes \mathbf{f}[U_{\mathcal{P}_2|_D}]
\end{CD}$$

commutes. By functoriality, it is enough to show that the diagram

$$\begin{CD}
U_{\mathcal{P}_1} \times U_{\mathcal{P}_2} @<{\pi_{\mathcal{P}_1,\mathcal{P}_2}}<< U_{\mathcal{P}} @<{\sigma_{\mathcal{P}|_{T_1},\mathcal{P}|_{T_2}}}<< U_{\mathcal{P}|_{T_1}} \times U_{\mathcal{P}|_{T_2}} \\
@A{\sigma_{\mathcal{P}_1|_A,\mathcal{P}_1|_B} \times \sigma_{\mathcal{P}_2|_C,\mathcal{P}_2|_D}}AA @. @VV{\pi_{\mathcal{P}_1|_A,\mathcal{P}_2|_C} \times \pi_{\mathcal{P}_1|_B,\mathcal{P}_2|_D}}V \\
U_{\mathcal{P}_1|_A} \times U_{\mathcal{P}_1|_B} \times U_{\mathcal{P}_2|_C} \times U_{\mathcal{P}_2|_D} @<<{\cong}< @. U_{\mathcal{P}_1|_A} \times U_{\mathcal{P}_2|_C} \times U_{\mathcal{P}_1|_B} \times U_{\mathcal{P}_2|_D}
\end{CD}$$

commutes. Suppose that $(g,h) \in U_{\mathcal{P}|_{T_1}} \times U_{\mathcal{P}|_{T_2}}$; then

$$\pi_{\mathcal{P}_1,\mathcal{P}_2} \circ \sigma_{\mathcal{P}|_{T_1},\mathcal{P}|_{T_2}}(g,h) = (x,y),$$

where

$$(x)_{ij} = \begin{cases} g_{ij} & \text{if } i,j \in S_1 \cap T_1, \\ h_{ij} & \text{if } i,j \in S_1 \cap T_2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(y)_{ij} = \begin{cases} g_{ij} & \text{if } i,j \in S_2 \cap T_1, \\ h_{ij} & \text{if } i,j \in S_2 \cap T_2, \\ 0 & \text{otherwise.} \end{cases}$$

At the same time, we have

$$(\sigma_{\mathcal{P}_1|_A,\mathcal{P}_1|_B} \times \sigma_{\mathcal{P}_2|_C,\mathcal{P}_2|_D}) \circ \theta \circ (\pi_{\mathcal{P}_1|_A,\mathcal{P}_2|_C} \times \pi_{\mathcal{P}_1|_B,\mathcal{P}_2|_D})(g,h) = (u,v),$$

where θ is the isomorphism that switches the middle factors,

$$(u)_{ij} = \begin{cases} g_{ij} & \text{if } i,j \in A, \\ h_{ij} & \text{if } i,j \in B, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(y)_{ij} = \begin{cases} g_{ij} & \text{if } i, j \in C, \\ h_{ij} & \text{if } i, j \in D, \\ 0 & \text{otherwise,} \end{cases}$$

thus $(x, y) = (u, v)$. □

5.4.2 Submonoids

For a poset $\mathcal{P}$ on I, define

$$\begin{aligned} \mathbf{cf}(U_{\mathcal{P}}) &= \{\text{class functions on } U_{\mathcal{P}}\}, \\ \mathbf{scf}(U_{\mathcal{P}}) &= \left\{ \begin{array}{l} \text{superclass functions on } U_{\mathcal{P}} \text{ in the} \\ \text{algebra group supercharacter theory} \end{array} \right\}, \text{ and} \\ \mathbf{nnf}(U_{\mathcal{P}}) &= \left\{ \begin{array}{l} \text{superclass functions on } U_{\mathcal{P}} \text{ in the} \\ \text{nonnesting supercharacter theory} \end{array} \right\}. \end{aligned}$$

Note that

$$\mathbf{nnf}(U_{\mathcal{P}}) \subseteq \mathbf{scf}(U_{\mathcal{P}}) \subseteq \mathbf{cf}(U_{\mathcal{P}}) \subseteq \mathbf{f}(U_{\mathcal{P}}).$$

Define vector species **cfp**, **scfp**, and **nnfp** by

$$\begin{aligned} \mathbf{cfp}[I] &= \bigoplus_{\substack{\mathcal{P} \text{ is a} \\ \text{poset on } I}} \mathbf{cf}(U_{\mathcal{P}}), \\ \mathbf{scfp}[I] &= \bigoplus_{\substack{\mathcal{P} \text{ is a} \\ \text{poset on } I}} \mathbf{scf}(U_{\mathcal{P}}), \text{ and} \\ \mathbf{nnfp}[I] &= \bigoplus_{\substack{\mathcal{P} \text{ is a} \\ \text{poset on } I}} \mathbf{nnf}(U_{\mathcal{P}}). \end{aligned}$$

These vector species are all subspecies of **fp**, and in fact we can define submonoids of $(\mathbf{fp}, \mu, \Delta)$.

Proposition 5.14. *The triple* $(\mathbf{cfp}, \mu, \Delta)$ *defines a Hopf submonoid of* $(\mathbf{fp}, \mu, \Delta)$.

Proof. We only need to show that μ and Δ restrict to a product and a coproduct on **cfp**. This is equivalent to the maps $\sigma_{\mathcal{P},\mathcal{Q}}$ and $\pi_{\mathcal{P},\mathcal{Q}}$ sending conjugate elements to conjugate elements. As $\sigma_{\mathcal{P},\mathcal{Q}}$ and $\pi_{\mathcal{P},\mathcal{Q}}$ are group homomorphisms, this is in fact the case. □

Proposition 5.15. *The triple* $(\mathbf{scfp}, \mu, \Delta)$ *defines a Hopf submonoid of* $(\mathbf{cfp}, \mu, \Delta)$.

Proof. Once again, we only need show that, for posets $\mathcal{P}$ and $\mathcal{Q}$, the maps $\sigma_{\mathcal{P},\mathcal{Q}}$ and $\pi_{\mathcal{P},\mathcal{Q}}$ send elements in the same superclass to elements in the same superclass. First suppose that g and h are in same superclass of $U_{\mathcal{P}\cdot\mathcal{Q}}$; then there exist $u, v \in U_{\mathcal{P}\cdot\mathcal{Q}}$ such that $uf(g)v = f(h)$. We have that

$$\pi_{\mathcal{P},\mathcal{Q}}(u)f(\pi_{\mathcal{P},\mathcal{Q}}(g))\pi_{\mathcal{P},\mathcal{Q}}(v) = f(\pi_{\mathcal{P},\mathcal{Q}}(h)),$$

hence $\pi_{\mathcal{P},\mathcal{Q}}(g)$ and $\pi_{\mathcal{P},\mathcal{Q}}(h)$ are in the same superclass of $U_{\mathcal{P}} \times U_{\mathcal{Q}}$. Similarly, if $(g_1, g_2), (h_1, h_2) \in U_{\mathcal{P}} \times U_{\mathcal{Q}}$ are in the same superclass, then there exist $(u_1, u_2), (v_1, v_2) \in U_{\mathcal{P}} \times U_{\mathcal{Q}}$ such that $(u_1, u_2)f((g_1, g_2))(v_1, v_2) = f((h_1, h_2))$, and

$$\sigma_{\mathcal{P},\mathcal{Q}}((u_1, u_2))f(\sigma_{\mathcal{P},\mathcal{Q}}((g_1, g_2)))\sigma_{\mathcal{P},\mathcal{Q}}((v_1, v_2)) = f(\sigma_{\mathcal{P},\mathcal{Q}}((h_1, h_2))).$$

□

Proposition 5.16. *The triple* $(\mathbf{nnfp}, \mu, \Delta)$ *defines a Hopf submonoid of* $(\mathbf{scfp}, \mu, \Delta)$.

Proof. Once again, we only need show that, for posets $\mathcal{P}$ and $\mathcal{Q}$ on disjoint sets S and T, the maps $\sigma_{\mathcal{P},\mathcal{Q}}$ and $\pi_{\mathcal{P},\mathcal{Q}}$ send elements in the same superclass to elements in the same superclass. Let $g, h \in U_{\mathcal{P}\cdot\mathcal{Q}}$ be in the same superclass; in other words, $\eta_g = \eta_h$, where

$$\eta_g = \left\{ i \overset{g_{ij}}{\frown} j \;\middle|\; \begin{array}{l} g_{ij} \neq 0 \text{ and if } (k,l) \neq (i,j) \text{ and} \\ i \preceq_{\mathcal{P}} k \prec_{\mathcal{P}} l \preceq_{\mathcal{P}} j, \text{ then } g_{kl} = 0 \end{array} \right\}.$$

Note that

$$\eta_{\pi_{\mathcal{P},\mathcal{Q}}(g)} = \{ i \overset{a}{\frown} j \mid i \overset{a}{\frown} j \in \eta_g \text{ and } i, j \in S \text{ or } i, j \in T\},$$

hence $\eta_{\pi_{\mathcal{P},\mathcal{Q}}(g)} = \eta_{\pi_{\mathcal{P},\mathcal{Q}}(h)}$, and $\pi_{\mathcal{P},\mathcal{Q}}(g)$ and $\pi_{\mathcal{P},\mathcal{Q}}(h)$ are in the same superclass of $U_{\mathcal{P}} \times U_{\mathcal{Q}}$.

Similarly, if $(g_1, g_2), (h_1, h_2) \in U_{\mathcal{P}} \times U_{\mathcal{Q}}$ are in the same superclass, then $\eta_{(g_1,g_2)} = \eta_{(h_1,h_2)}$. We have

$$\eta_{\sigma_{\mathcal{P},\mathcal{Q}}((g_1,g_2))} = \eta_{(g_1,g_2)},$$

hence $\sigma_{\mathcal{P},\mathcal{Q}}((g_1, g_2))$ and $\sigma_{\mathcal{P},\mathcal{Q}}((h_1, h_2))$ are in the same superclass of $U_{\mathcal{P}\cdot\mathcal{Q}}$. □

We mention several other submonoids of interest; in [2], Aguiar et al. construct Hopf monoids from the groups of unitriangular matrices with entries in $\mathbb{F}_q$. For a finite set I, let $L[I]$ be the set of linear orders on I. These are all posets on I; let

$$\begin{aligned}
\mathbf{f}(U)[I] &= \bigoplus_{\mathcal{P}\in L[I]} \mathbf{f}(U_{\mathcal{P}}), \\
\mathbf{cf}(U)[I] &= \bigoplus_{\mathcal{P}\in L[I]} \mathbf{cf}(U_{\mathcal{P}}), \\
\mathbf{scf}(U)[I] &= \bigoplus_{\mathcal{P}\in L[I]} \mathbf{scf}(U_{\mathcal{P}}), \text{ and} \\
\mathbf{nnf}(U)[I] &= \bigoplus_{\mathcal{P}\in L[I]} \mathbf{nnf}(U_{\mathcal{P}}).
\end{aligned}$$

In [2], Aguiar et al. show that the triple $(\mathbf{f}(U), \mu, \Delta)$ is a Hopf monoid with submonoids $(\mathbf{cf}(U), \mu, \Delta)$ and $(\mathbf{scf}(U), \mu, \Delta)$.

Corollary 5.17. *We have the following inclusions of Hopf monoids:*

$$\begin{aligned}
(\mathbf{f}(U), \mu, \Delta) &\subseteq (\mathbf{fp}, \mu, \Delta), \\
(\mathbf{cf}(U), \mu, \Delta) &\subseteq (\mathbf{cfp}, \mu, \Delta), \\
(\mathbf{scf}(U), \mu, \Delta) &\subseteq (\mathbf{scfp}, \mu, \Delta), \\
(\mathbf{nnf}(U), \mu, \Delta) &\subseteq (\mathbf{nnfp}, \mu, \Delta), \ \textit{and} \\
(\mathbf{nnf}(U), \mu, \Delta) &\subseteq (\mathbf{scf}(U), \mu, \Delta).
\end{aligned}$$

5.4.3 Combinatorics of the product and coproduct

In [2], the product and coproduct on $(\mathbf{scf}(U), \mu, \Delta)$ are described in terms of $\mathbb{F}_q$-set partitions. We cannot do the same for $(\mathbf{scfp}, \mu, \Delta)$ as there is no known combinatorial description of the supercharacters and superclasses. We can, however, do so for $(\mathbf{nnfp}, \mu, \Delta)$, and this description will reduce to a combinatorial description of the product and coproduct on $(\mathbf{nnf}(U), \mu, \Delta)$. First, we present the results from [2].

Let $\mathcal{P}$ be a linear order on I; then $U_{\mathcal{P}} \cong UT_n(\mathbb{F}_q)$ (where $|I| = n$). The supercharacters and superclasses of $U_{\mathcal{P}}$ in the algebra group supercharacter theory are indexed by the set of $(\mathbb{F}_q, \mathcal{P})$-set partitions, which we will denote $\Pi(\mathcal{P})$. For $\eta \in \Pi(\mathcal{P})$, let K_η be the superclass of $U_{\mathcal{P}}$ associated to η; define

$$\kappa_\eta(g) = \begin{cases} 1 & \text{if } g \in K_\eta, \\ 0 & \text{otherwise.} \end{cases}$$

The κ_η form a basis of $\mathbf{scf}(U_\mathcal{P})$, on which the product and coproduct of $(\mathbf{scf}(U), \mu, \Delta)$ have a nice combinatorial description.

Proposition 5.18 ([2, Equation 42])**.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be linear orders on disjoint sets S and T, respectively. If $\eta \in \Pi(\mathcal{P})$ and $\nu \in \Pi(\mathcal{Q})$, we have*

$$\mu_{\mathcal{P},\mathcal{Q}}(\kappa_\eta \otimes \kappa_\nu) = \sum_{\substack{\rho \in \Pi(\mathcal{P}\cdot\mathcal{Q}) \\ \rho|_S = \eta, \rho|_T = \nu}} \kappa_\rho.$$

Proposition 5.19 ([2, Equation 43])**.** *Let $\mathcal{P}$ be a linear order on I and let $I = S \sqcup T$. If $\eta \in \Pi(\mathcal{P})$, we have*

$$\Delta_{\mathcal{P}|_S, \mathcal{P}|_T}(\kappa_\eta) = \begin{cases} \kappa_{\eta|_S} \otimes \kappa_{\eta|_T} & \textit{if } \eta = \eta|_S \sqcup \eta|_T, \\ 0 & \textit{otherwise.} \end{cases}$$

Now let $\mathcal{P}$ be any poset on I; the supercharacters and superclasses of $U_\mathcal{P}$ in the nonnesting supercharacter theory are indexed by the set of nonnesting $(\mathbb{F}_q, \mathcal{P})$-set partitions, which we will denote by $NN(\mathcal{P})$. For $\eta \in NN(\mathcal{P})$, let K_η denote the superclass associated to η; define

$$\kappa_\eta(g) = \begin{cases} 1 & \text{if } g \in K_\eta, \\ 0 & \text{otherwise.} \end{cases}$$

The κ_η form a basis of $\mathbf{nnfp}(U_\mathcal{P})$, on which the product and coproduct of $(\mathbf{nnfp}, \mu, \Delta)$ have a nice combinatorial description.

Proposition 5.20. *Let $\mathcal{P}$ and $\mathcal{Q}$ be posets on disjoint sets S and T, respectively. If $\eta \in NN(\mathcal{P})$ and $\nu \in NN(\mathcal{Q})$, then*

$$\mu_{\mathcal{P},\mathcal{Q}}(\kappa_\eta \otimes \kappa_\nu) = \sum_{\substack{\rho \in NN(\mathcal{P}\cdot\mathcal{Q}) \\ \rho|_S = \eta, \rho|_T = \nu}} \kappa_\rho.$$

Proof. Let $g \in U_{\mathcal{P}\cdot\mathcal{Q}}$, and let $\rho \in NN(\mathcal{P} \cdot \mathcal{Q})$ be such that $g \in K_\rho$. Then we have

$$\pi_{\mathcal{P},\mathcal{Q}}(g) \in K_{\rho|_S} \times K_{\rho|_T}.$$

It follows that

$$\mu_{\mathcal{P},\mathcal{Q}}(\kappa_\eta \otimes \kappa_\nu)(g) = \begin{cases} 1 & \text{if } \rho|_S = \eta, \rho|_T = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

□

Proposition 5.21. *Let $\mathcal{P}$ be a poset on I and let $I = S \sqcup T$. If $\eta \in NN(\mathcal{P})$, then*

$$\Delta_{\mathcal{P}|_S,\mathcal{P}|_T}(\kappa_\eta) = \begin{cases} \kappa_{\eta|_S} \otimes \kappa_{\eta|_T} & \textit{if } \eta = \eta|_S \sqcup \eta|_T, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $(g, h) \in U_{\mathcal{P}|_S} \times U_{\mathcal{P}|_T}$, with $g \in K_\nu$ and $h \in K_\rho$; then

$$\sigma_{\mathcal{P}|_S,\mathcal{P}|_T}(g, h) \in K_{\nu \sqcup \rho}.$$

It follows that

$$\varphi(\Delta_{\mathcal{P}|_S,\mathcal{P}|_T}(\kappa_\eta))((g, h)) = \begin{cases} 1 & \text{if } \eta = \nu \sqcup \rho, \\ 0 & \text{otherwise,} \end{cases}$$

where once again φ is the canonical isomorphism from $\mathbf{f}(U_{\mathcal{P}|_S}) \otimes \mathbf{f}(U_{\mathcal{P}|_T})$ to $\mathbf{f}(U_{\mathcal{P}|_S} \times U_{\mathcal{P}|_T})$. □

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