Vortex identification in experimental velocity fields

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0.1 Abstract

Two methods for the identification of coherent structures in fluid flows are studied for possible combination in a hybrid method providing deeper insight and greater efficiency. One method uses wavelet transforms to partition the field into coherent and incoherent portions using a thresholding of the wavelet coefficients. The other method is based on the stability analysis of trajectories through the fluid, defining two type of Lagrangian coherent structures based on the degree hyperbolicity or ellipticity of the flow. It is hoped that computational efficiency of the wavelet method can be combined with the level of detail of the stability analysis to give a fast and insightful method for analyzing and identify coherent structures in a flow. Two types of tests are done to determine the feasibility of combining the two methods. The first test is used to determine the degree to which the coherent partition from the wavelet method agrees with the Lagrangian coherent structures of the stability analysis method. The second test is used to determine how suitable the coherent partition of the wavelet coefficients is for guiding the efficient placement of the tracers used by the stability analysis method. The results from the first test do not indicate a connection between the coherent wavelet partition and Lagrangian coherent structures, and also indicate that the wavelet method is susceptible to regions of high shear. The results from the second test show that basic methods for placing tracers based on the coherent wavelet partition do not perform better than a uniform distribution of particles throughout the flow. Overall, the results do not indicate that a combinations of the two methods will be beneficial, but that there are a number of possible future directions that warrant further pursuit of the topic.
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Chapter 1

Introduction

Turbulence is considered the last great unsolved problem in classical physics. The Navier-Stokes (NS) equations that govern fluid flows have been known for well over a hundred years, but these equations have only been solved analytically for a small number of highly simplified cases. Numerical approaches to solving the NS equations have not been exceptionally effective because the inherent complexity of turbulent flows leads to computational requirements that far exceed the capabilities of any system in the foreseeable future. This fundamental difficulty has led researchers to attempt to analyze and simulate turbulent flows by reducing the order and the complexity of the model problems.

While turbulent flows are often highly complex, they are not completely random. On the contrary, turbulent flows exhibit significant organization in the form of spatially localized coherent structures. Coherent structures are generally defined as bounded regions of flow with some common topological property, and the most common and well known type of coherent structure is the vortex. These localized regions of swirling flow are thought to be responsible for the active dynamics of the flow, with the non-coherent portions of the flow being passively stretched and advected by the coherent structures [11]. Coherent structures are also generally compact and often occupy small regions of a flow, with the majority of the flow being incoherent, though this in not always the case. Since these portions of the flow that are thought to drive the dynamics are localized in small spatial regions, it may be possible that the cost of simulating and analyzing turbulent flows can be significantly reduced by ignoring the incoherent regions of the flow and focusing on the coherent structures.
Surprisingly, while it is widely accepted that vortical coherent structures play a significant role in turbulence, a widely accepted, non-subjective definition for the term “vortex” has yet to be agreed upon by the fluid dynamics community. Many reasonable definitions have been proposed, each of which tends to consist of a combination of qualitative and quantitative properties of the flow. Robinson et al. [15] propose that: “A vortex exists when instantaneous streamlines mapped to a plane normal to the core exhibit a roughly circular or spiral pattern, when viewed in a reference frame moving with the center of the vortex core.” While there is no consistent definition of a “vortex core” either, they are generally considered to be the most central region of a vortex that exhibits sharp maxima in the vorticity field and sharp minima in the pressure field. In the region surrounding the vortex core, various flow field derivatives change more gradually.

Jeong and Hussain [8] suggest that vortices have the following properties:

1. A vortex core must have a net vorticity, and consequently a net circulation. Potential flow regions are excluded from vortex cores by this requirement, and a potential vortex is a vortex with zero cross-section;

2. The geometrical characteristics of the identified vortex must be Galilean invariant.

This definition goes beyond the requirement of having spiralling streamlines by requiring that the flow actually undergo a solid-body type rotation in the vortex. A flow can exhibit spiral streamlines without actually rotating the fluid, as in an irrotational potential flow. The additional criterion of Galilean invariance requires that the definition should not be affected by transformations between coordinate frames in constant relative motion, and is necessary for vortex definitions given in coordinates moving with the core of each vortex.

In the last few years, automatic numerical vortex identification methods based on several of these definitions have shown promising results when applied to turbulent velocity fields. These schemes use various approaches, such as topological analysis, signal processing, and stability analysis, and each scheme is generally tailored to a specific set of flow conditions (i.e. 2D versus 3D, discrete versus analytic, inviscid versus viscous). In recent years, methods grouped under two specific approaches have shown particularly promising results. The first approach is based on the analysis of the
velocity gradient tensor $\nabla \mathbf{u}$, and the second approach uses wavelet-based signal-processing techniques. Both of these approaches have formulations for both 2D and 3D velocity fields, apply to a wide range of turbulent flow conditions, and most importantly, have shown reliable results in identifying coherent structures in a number of traditionally difficult flows.

Each of these methods, though, has several drawbacks. The analytic methods tend to be either limited in applicability but fairly computationally efficient, or widely applicable but computationally inefficient. The wavelet-based methods are generally computationally efficient, but they merely partition the flow field, leaving a deeper understanding of the dynamics to further analysis. The velocity-gradient methods, on the other hand, can give much greater insight into the flow kinematics, but tend to require much more CPU time than the wavelet methods.

The goal of this project is to investigate the possible combination of these two types of methods in order to exploit the advantages of each, and, of course, minimize the disadvantages. Before proceeding with the specifics of the different hybrid approaches, some more in-depth background into the velocity gradient and wavelet methods will be given.

### 1.1 Velocity-Gradient-Based Methods

Vortex extraction techniques based on $\nabla \mathbf{u}$, the gradient of the velocity field, are basically mathematical formulations of the vortex descriptions in the previous sections, and each proposes a mathematical definition for a vortex based on calculations of analytic quantities from the flow field. They also share the common property of at least being Galilean invariant, meaning that they are invariant under transformations between coordinate systems in constant relative motion. These methods stem from initial attempts to define vortices using the magnitude of the vorticity vector $\omega = \nabla \times \mathbf{u}$, based on a user-selected threshold ([14], [6], [2]). In this early approach, regions where the vorticity magnitude exceeds this threshold are termed vortices. Such vorticity magnitude methods have been used fairly extensively to identify vortices, but they cannot distinguish between vortices and shear layers, because shear layers also exhibit high vorticity magnitudes. This results in a misclassification of shear layers as vortices, which can be a considerable problem because shear layers are quite common in fluid flows, especially along walls. Because of this problem, vorticity threshold methods are generally considered
The velocity gradient allows methods to effectively discriminate between vortices and shear layers. The first such method, proposed by Chong et al. [10], defines vortices as regions of flow where $\nabla \mathbf{u}$ has complex eigenvalues. Complex eigenvalues imply that the streamlines are spiral or closed in a frame moving with the vortex, and the definition is Galilean invariant. Two other gradient methods are based on the decomposition of the velocity gradient tensor into its symmetric and antisymmetric parts:

$$\nabla \mathbf{u} = \mathbf{S} + \mathbf{\Omega}$$  \hspace{1cm} (1.1)

where $\mathbf{S} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the rate-of-strain tensor and $\mathbf{\Omega} = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^T)$ is the vorticity tensor. Physically, the rate of strain represents the deformation of the fluid and the vorticity represents its solid-body rotation. The first method to use this decomposition is known as the $Q$-criterion of Hunt, Wray and Moin [7]. This method analyzes the second invariant $Q$ of the velocity gradient tensor, defining vortices as regions of space where

$$Q = \frac{1}{2} \left( |\mathbf{\Omega}|^2 - |\mathbf{S}|^2 \right) > 0$$  \hspace{1cm} (1.2)

The $Q$ term relates the relative magnitudes of the vorticity and the rate of strain; when $Q$ is greater than zero, the vorticity dominates the strain, thereby ruling out shear flows.

The $\lambda_2$-criterion of Jeong and Hussain uses $\mathbf{mathbf{\Omega}}$ and $\mathbf{S}$ in a different way, defining vortices as regions of flow where the intermediate eigenvalue of the symmetric tensor $\mathbf{S}^2 + \mathbf{\Omega}^2$ is less than zero. This definition is based on the fact that a negative intermediate eigenvalue of this tensor indicates a local pressure minimum, which is indicative of a vortex core [8].

These methods that are based on the local decomposition of $\nabla \mathbf{u}$ essentially compare the local rotation rate and the local shear rate, and define vortices as regions where the rotation rate dominates [13]. Of the methods, the $\lambda_2$ criterion, in particular, has been used widely in a number of different numerical and experimental applications, and is starting to become a standard for vortex identification. $\nabla \mathbf{u}$-based schemes have several shortcomings, however. For each method, at least one example has been given where the method provides ambiguous results. These methods have also been criticized for their use of local analysis, when vortices are, in reality, non-local.

Recently, Haller [5] has questioned whether Galilean invariance, alone, is sufficient to provide a general definition of a vortex, suggesting that the
definition should be objective as well. An objective definition remains invariant under any type of orthogonal transformation or translation, not just in frames in constant relative motion, as with Galilean invariance. He proposes a scheme that is both objective and Lagrangian (i.e. non-local), and this scheme appears to overcome the main shortcomings of the $\nabla \mathbf{u}$ methods. Haller’s method evaluates the dynamic stability of a passive tracer particle as it forms a trajectory through the flow field. At each point along the trajectory, the flow field is analyzed to determine whether the local stability is elliptic or hyperbolic. Haller defines both hyperbolic and elliptical Lagrangian coherent structure (LCS) based on the regions the tracer particle passes through. Trajectories that remain entirely in hyperbolic regions are within hyperbolic LCS, and particles that remain entirely within elliptic regions are within elliptic LCS. Hyperbolic LCS represent structures that stretch and fold the flow, promoting advective mixing. Elliptic LCS are regions of flow that do not stretch or fold, and vortex cores can be identified as closed sets of tracers that meet the criteria for being in elliptic LCS.

Haller’s method has several significant advantages over previous vortex identification techniques. First, the property of objectivity guarantees a consistent definition under all plausible coordinate changes because it implies invariance under coordinate changes of the form $\mathbf{x} = \mathbf{Q}(t) \mathbf{x} + \mathbf{b}(t)$, where $\mathbf{Q}(t)$ is a time-dependent proper orthogonal tensor, and $\mathbf{b}(t)$ is a time-dependent translation vector. In essence, this implies that the scheme will give the same results in any frame that does not physically distort the data in any non-uniform way. Also, by using a Lagrangian perspective, the definition accounts for non-local dynamics, thereby making use of significantly more information from throughout the flow and utilizing its inherent kinematics. An additional advantage of this method is that it does not rely on a subjective threshold to define a vortex, making it truly objective in more than one sense. The main disadvantage is that it can be computationally expensive. The evaluation requires the numerical time integration of tracer trajectories throughout the flow for each instantaneous flow snapshot. The resolution of the analysis depends on the number of tracers, and for time-dependent 3D turbulent flows, the computational cost of this method is prohibitively high, as such flows would require fine tracer resolution in a 3D arrangement over many time steps.
1.2 Wavelet-Based Methods

In recent years, wavelet-based methods have shown promise for coherent structure identification. These methods treat the flow in a fundamentally different way than the analytic definitions in previous sections, using signal processing methods rather than an approach based on the physics of the problem. Wavelet-based methods assume that the incoherent structures in a flow are essentially Gaussian white noise, and that the coherent structures are signal content embedded in that noisy field. Wavelets are classes of functions that transform a signal into a representation that is localized in both space and frequency, which also implies spatial and scale locality. Specifically, signals are decomposed into a series consisting of a mother wavelet function $\psi(x)$ and a set of daughter wavelets $\psi^{a,b}(x)$ that are formed by translation and scaling of the mother wavelet function. Signals are projected into this wavelet basis using a wavelet transform first proposed by Grossman and Morlet [3], and can be projected back to the original space using an inverse wavelet transform.

Because coherent structures in turbulence are essentially localized regions of high vorticity of different spatial scales, wavelets are a naturally well-suited basis for representing turbulent flows, and have been used to analyze and simulate coherent structures. In addition, wavelet analysis can be very computationally efficient. When using the forward and inverse fast wavelet transforms, the computational requirement is $O(n)$, where $n$ is the resolution of the transform. This is faster than the Fast Fourier Transform, which is $O(n \log_2 n)$. Because of this, relatively little computation is necessary to transform a signal to a wavelet basis, operate on it, and transform it back to the original basis.

Wavelet-based methods for vortex identification begin by transforming the vorticity field into a wavelet basis representation. Once in the wavelet basis, a number of different approaches can be employed to analyze and detect the coherent vortices in the data. Schram & Riethmuller [16] and Schram, Rambaud & Riethmuller [1], for instance, use a continuous wavelet transform on 2D experimental data to extract vortex statistics (among them the position, strength and core velocity of vortices) and have tested their method on a number of turbulent flows. For this method, the vortices are identified by performing a search for local maxima of the wavelet coefficient matrix, considering these maxima to represent individual vortex cores. The associated spatial scale of each maximum coefficient is then used to determine
the strength $\Gamma$ of the vortex and the size of the associated vortex core.

As with the vorticity magnitude vortex criterion, the wavelet transform, alone, cannot distinguish between a vortex and a high shear region. The $\lambda_2$ criterion can applied to the region of the flow occupied by the vortex in order to separate the vortices from the shear flows to overcome this problem. This is done by evaluating each local maximum to see if the second eigenvalue of the tensor $S^2 + \Omega^2$ is less than zero [8]. This combination of methods reduces a flow to a set of point vortices, each with a position, strength, core diameter, and velocity. This represents a significant reduction in the information content from the velocity field, and hints at the potential of a combined method. The main drawbacks of this wavelet/$\lambda_2$ method are that it requires a user-defined threshold for the local maximum search, and that it assumes the vortices are isotropic and circular. This could obviously cause errors for elliptical or other irregularly shaped vortices, which can be fairly common in some flows. In addition, it is not clear how this method could be generalized to 3D flows because the point vortex definition does not have a simple analog in higher dimensions.

There are other ways to use wavelets in extracting vortices from a flow. Siegel and Weiss [17] suggest a similar approach to identifying vortices in 2D turbulence, employing a wavelet-packet based algorithm. Rather than searching for local maxima, their method separates the coherent and incoherent parts of the flow field using a filter-based approach. Their results show success in removing the incoherent parts of the flow in 2D, agree well with prior results, and show promise for more-complex flows in 2D and 3D. Pelligrino et al. [11] have proposed a similar method that removes the incoherent portion of the flow using what is effectively a non-linear noise filter using a standard wavelet transform. Whereas the previous wavelet methods use wavelets as a more-suitable basis for detection of localized signal maxima, these method directly exploits the assumptions that the incoherent portion of the flow field is Gaussian white noise, and that the coherent regions are embedded signal. The filtering process involves projecting the vorticity field onto an orthogonal wavelet basis using fast wavelet transforms, thresholding the wavelet coefficients in order to separate the data into coherent and incoherent sets, and then reconstructing the coherent and incoherent vorticity fields using inverse fast wavelet transforms. An aspect of particular value is that this method does not rely on any user-defined parameters. The choice of the threshold is based on theorems that indicate that optimal de-noising can be achieved using a wavelet basis, and is computed from the variance of the
total vorticity field. This method has been applied to numerically simulated turbulent flows, and the results give a coherent vorticity field containing less than five percent of the wavelet coefficients and containing 99% of the energy, and it exhibits the same $k^{-5/3}$ energy spectrum as the total flow [11]. These results not only show that the wavelet method can be extremely effective in compactly representing a turbulent flow, but also agrees with the hypothesis that the coherent structures are mainly responsible for the dynamics of the flow! One drawback to this method is that it has, to date, only been applied to high Reynolds number, complex turbulent flows, and the assumption of a Gaussian incoherent field could limit the effectiveness of the technique for identifying vortices in low Reynolds number and developing turbulent flows, or for non-turbulent flows. In addition, Pellegrino’s method only partitions the flow into the coherent and incoherent components, so additional analysis would be required to understand the underlying nature of the coherent structures.

1.3 Synthesized Methods

Both the velocity-gradient-based methods of section 1.1 and the wavelet methods of section 1.2 have inherent advantages and disadvantages when applied to practical turbulence data, and these advantages and disadvantages complement each other to a large extent. The main goal of this thesis is to investigate whether or not these two types of methods can be combined in some mutually beneficial way. It is not clear from the theoretical details and from existing results how the coherent set of wavelet coefficients of Pellegrino’s method relates to the hyperbolic and elliptic coherent structures of Haller’s method, if they relate at all. The first goal of this thesis is to investigate whether a correlation exists between the definition of “coherent” according to Pellegrino and the different types of coherent structures defined by Haller. Each wavelet coefficient is representative of a particular square region of points in the vorticity field. According to Pellegrino, if a particular wavelet coefficient is above the threshold that determines whether or not it is coherent, it means that the associated region of the vorticity field contains or is contained within a coherent structure. If there is a correlation between this definition and the Lagrangian coherent structures defined by Haller, then the spatial regions associated with each coherent wavelet coefficient should all display common results when analyzed using Haller’s method. In the first
set of results, this analysis is performed by placing a tracer in the center of each region corresponding to the coherent wavelet coefficients from the flow field and determining a “hyperbolicity time”, which is the number of steps in the course of the integration in which the tracer is within a locally hyperbolic region of the flow. Hyperbolic LCS, which physically are regions of stretching and folding in the flow, are indicated by highly hyperbolic regions with hyperbolicity times at or near the total number of timesteps. Elliptic LCS, which physically include vortex cores, are indicated by regions that exhibit hyperbolicity times of at or very near zero. For the purpose of this thesis, the existence of a correlation is established based on the following criteria: a correlation is evident if all of the tracer particles yield the same result, i.e. all hyperbolic or all elliptical; a correlation is evident if each region exhibits a particular type of consistent result, i.e. if tracers released in each region are either highly elliptic or highly hyperbolic; a correlation is not evident if the tracers are not consistently of one type, have a seemingly random distribution of types, or have a distribution that is roughly identical to the distribution from a uniform grid of tracer particles distributed across the field.

The logic behind these criteria is as follows. First, it is assumed that the velocity field being analyzed would have a fairly wide distribution of hyperbolicity times when sampled using a uniform grid of tracer particles. If all of the particles released from the spatial regions associated with the coherent coefficients are of a single type (i.e. being at or near the maximum or minimum hyperbolicity time value), then it is likely that the coherent wavelet coefficients are associated with a single type of Haller’s coherent structures, which are defined by extremes of the hyperbolicity time. If the particles have a split distribution in hyperbolicity time, then it is likely that the coefficients correlate with both types of Haller’s coherent structures. And if the distribution does not localize on the maximum or minimum extreme of the range of hyperbolicity times, then it is likely that the wavelet coefficients do not cluster in regions defined as coherent structures using Haller’s method. Essentially, this test will determine if the regions associated with the coherent wavelets are linked with regions that Haller’s method indicates are within LCS, thereby determining if a link exists between the definition of coherent given by each method.

Even if the above criteria imply a correlation, coherent wavelets and the LCS may not be fully correlated. The above criteria can only determine whether coherent wavelets imply LCS. To fully analyze any correlation between the two methods, it is also necessary to determine if LCS imply co-
herent wavelets. This is done here by plotting the regions associated with each coherent coefficient on top of the hyperbolicity time of the same field. The coefficient regions are compared against the LCS, which represent the regions of maximum and minimum hyperbolicity time on the plot. A correlation would be evident in this case if the coherent wavelet regions fully cover all examples of one or both of the types of LCS visible in a given plot. While this method has no quantitative basis, it provides a preliminary analysis that can be further backed up with statistical analysis if need be.

While it is not clear how the coherent structures defined by these two methods are related, the analysis of a flow field by Haller’s method does give significant insight into the hidden structures. The problem with Haller’s method is that it is computationally inefficient to the point of being impractical for large problems, and so it is important to find ways in which the performance of this method can be improved. At a superficial level, the wavelet method also shows promise for this because the coherent regions that it extracts from the flow have represented on the order of one percent of the total number of wavelet coefficients in the flows they analyze. This essentially means that the wavelet method can compress the coherent information in the flow by a factor of 100:1. The key observation on which this thesis is based is that if a similar reduction in the number of required tracer particles can be achieved with Haller’s method, it could become a much more practical technique for analyzing flows. In the set of tests done here to determine the correlation between the different types of coherent structures, the analysis was concerned with the correlation between the spatial regions represented by the wavelet coefficients and the maxima and minima points of the hyperbolicity time field. In this case, the analysis is more concerned with the correlation between the spatial regions of the wavelet coefficients and the regions of the flow that require closely spaced tracers because of greater detail and sharper gradients in the hyperbolicity time field. The difficulty here is in how to get from a set of coherent wavelet coefficients to a more optimal set of tracer particles, and again there does not appear to be any single, obvious technique for this.

For this thesis, the decision was made to attempt to exploit the spatial position and spatial scale of each of the coherent wavelet coefficients as a way to explore the different possibilities for tracer placement and assess the effectiveness at a very general level. Three methods for placing tracer particles are investigated. The first simply places a single tracer at the center of the spatial region that corresponds to each coherent wavelet coefficient. This is
the most simplistic approach, and a good starting point for the analysis. The next approach uses the spatial scale of each coherent wavelet coefficient to place a uniform grid of particles across the entire spatial region represented by each coefficient. This method is a bit coarse due to the arbitrary nature of the uniform distribution within the region, but it is a good starting point for analyzing the effectiveness of adding in spatial scale information. The third method places a random distribution of points with a Gaussian profile that is centered on the point corresponding to each coherent coefficient and scaled in proportion to the wavelet scale. This method produces much less structured tracers, and is used here to determine whether the analysis benefits from a more-relaxed distribution of particles that is still most dense within the region associated with the coefficient. For each test case, the results are analyzed against a uniform distribution to determine whether the resolution improves for a similar number of tracer particles. The overall goal of this portion of the research is not to necessarily find an optimal method for tracer distribution, but rather to do a preliminary assessment of the potential capability of the coherent wavelet coefficient partitioning.
Chapter 2

Background

2.1 Haller’s Method

The vortex identification algorithm used in this thesis is based on the objective vortex criterion developed by Haller [4], [5]. Haller’s method describes vortices based on the stability of fluid trajectories in 2D or 3D incompressible flow. The process for identifying a vortex using this method consists of two steps. Starting with the discrete velocity field (either steady or time-dependent), tracer particles are placed at initial coordinates on a grid chosen as described in section 1.3, and numerically integrated in time through the field to produce particle trajectories. For each tracer trajectory, a mathematical vortex criterion is then applied at each point in the path to evaluate whether the particle remains in an elliptic region of the flow or strays into a hyperbolic region. For this thesis, the number of points along the trajectory that are locally hyperbolic will be tracked, giving a “hyperbolicity time” for each tracer path. Coherent structures are considered to be the sets of tracers that have locally maximum or minimum values of hyperbolicity time. The specific details of each of these steps are given in the following sections.

2.1.1 Tracer Trajectory Generation

The creation of a set of tracer trajectories is the first step in Haller’s vortex identification process. The algorithm itself does not require any specific structure to the initial placement of the tracers because they are evaluated independently. The resolution of the vortex identification process does, however, depend on the tracer placement, so it is important to place enough
tracers in the flow to capture enough detail about the vortices. The most straight-forward approach is to place the tracers on an evenly spaced square or rectangular grid. This method was used in [4] and all results here are compared against it.

The tracers are integrated forward over a finite number of time-steps through the velocity field, resulting in a trajectory path for each tracer. Currently, the integration interval and timestep are chosen by trial and error. The integration procedure on a discrete field requires a solver to perform the time integration and an interpolation method to compute the velocity vector at an arbitrary point inside the field. For integration schemes, research has shown that predictor-corrector methods such as fourth-order Runge-Kutta tend to be the best choice for integrating over discrete velocity fields [9]. Euler schemes have also been investigated; although they are less computationally costly, these schemes tend to give larger errors in regions of curved, vortical flow [12]. For this thesis, a fourth order Runge-Kutta method is used to generate the tracer trajectories. Various interpolation schemes have been investigated in this context, though there is less consensus on what the best method is [9]. The most commonly used interpolation function in 2D is the bilinear function. This method is relatively fast and works well for structured velocity fields, but it is not mass-conservative, which can cause significant errors in regions of high flow curvature [9]. Bicubic and cubic spline interpolation methods are mass conservative, but are also much more computationally costly, with cubic spline being the most expensive. In general, bicubic interpolation appears to be the best compromise between accuracy and efficiency, and was hence chose for use in this thesis.

2.1.2 Tracer Trajectory Evaluation

Haller’s method describes vortices based on the stability of fluid trajectories in 2D, incompressible flow. The stability analysis is performed on the strain acceleration tensor $M$, which is given by

$$ M = \frac{\partial}{\partial t} S + \nabla S v + S \nabla v + \nabla v^T S $$

where $S$ is the rate-of-strain tensor given by $\frac{1}{2} \left( \nabla v + \nabla v^T \right)$ and $v$ is the 2D or 3D velocity field. For this project, only 2D flows are analyzed, but the steps required to extend the algorithm to 3D flows are minimal.
The stability evaluation of the strain acceleration tensor is based on determining whether the field exhibits hyperbolic or elliptical behavior at the specified point. The local hyperbolicity of a point in a trajectory is determined by restricting \( M \) to a local cone of zero strain, expressed by \( M_Z \), and evaluating whether \( M_Z \) is positive definite or indefinite. The elliptic region is the set of points where either \( M_Z \) is indefinite or \( S \) vanishes, and the hyperbolic region is the set of points where \( M_Z \) is positive definite.

Hyperbolic points exhibit saddle-type instability, which lead to exponential stretching and folding of the nearby fluid. Elliptic regions do not exhibit this type of behavior, instead tending to remain stable in a localized region such as a vortex core. Haller defines LCS based on the which regions a tracer trajectory passes through. Hyperbolic LCS are defined as sets of tracer particles that remain in hyperbolic regions throughout their paths. Similarly, elliptic LCS are defined as regions of tracer particles that remain within elliptic regions. The hyperbolicity and ellipticity are mutually exclusive, so the existence of one can be inferred from an analysis of the other. For example, both hyperbolic and elliptic LCS can be detected by finding the portion of time interval that each tracer spends in the hyperbolic region. Particles that spend all of their time in the hyperbolic region are considered to be within hyperbolic LCS. Particles that spend none of their time in hyperbolic regions must spend all of their time in elliptical regions, and are thus considered to be within elliptical LCS. Finally, LCS are defined as connected regions of tracers that exhibit the appropriate characteristic behavior. This is the method used in this thesis to detect LCS.

### 2.2 Wavelet-Based Flow Field Decomposition

The wavelet-based vortex-extraction scheme used here follows the method proposed by Pellegrino et al. [11]. As described in section 1.2, this method decomposes the vorticity field into coherent and incoherent parts using a wavelet-based denoising technique, based on the multiresolution analysis of the signal. The thesis of this project is that information about the spatial location and spatial scale of the coherent coefficients can then be used to place the tracers used in Haller’s stability analysis of the flow field.

The wavelet analysis starts with a discrete velocity field. The field can be either 2D or 3D, but only the 2D case is being considered here. The first step is to calculate the vorticity field \( \omega(x) \). The vorticity is the local curl
of the velocity field and can be found by \( \omega(x) = \nabla \times u(x) \), where \( u(x) \) is the velocity field. Note that the vorticity vector \( \omega(x) \) is different than the vorticity tensor \( \Omega \) used in a number of the analytic criteria, although they quantify the same physical flow property.

The vorticity field is then projected into the wavelet basis by performing a 2D fast wavelet transform on the vorticity field. This operation produces a set of wavelet coefficients \( \tilde{\omega}_{j, i_x, i_y}^{\mu} \), where \( j \) is the wavelet scale, \( i_x \) and \( i_y \) are the spatial indices, and \( \mu \) is the index for the three combinations of scaling and wavelet function in 2D space that results from the combinations of the 1D transforms applied to the vorticity field. Note that this transform requires that the analysis grid be square and have a number of points that is a power of 2, so vorticity fields not meeting this requirement must be resampled.

After the wavelet coefficients are computed, a threshold is applied to separate the coherent and incoherent coefficients. The threshold is given by

\[
\tau = \sigma(\omega) \sqrt{2 \log N}
\]

where \( \sigma(\omega) \) is the standard deviation of the vorticity field and \( N \) is the total number of points in the field. This non-linear threshold, based on the statistical variance of the field, has been shown to optimally separate the wavelet coefficients into Gaussian and non-Gaussian components, which correspond to the coherent and incoherent portions of the flow, respectively [11]. The coherent and incoherent wavelet fields can then transformed back into vorticity fields by taking the 2D inverse fast wavelet transforms of the coherent and incoherent sets of wavelet coefficients to obtain a coherent vorticity field and an incoherent vorticity field.

Pellegrino et al. have used wavelet transforms based on the Coifman 12 wavelet family, shown in figure 2.1, with good success, and the initial implementation here also uses this wavelet family. The wavelet family can have a significant impact on the quality of the results, and so different wavelet families could be implemented as an additional avenue of exploration, as described in section 5 of this thesis.
Figure 2.1: The Coifman 12 wavelet and scaling function
Chapter 3

Implementation

The code for this project is written in Matlab and has three main components. One component analyzes a given discrete vorticity field using the wavelet transform technique, returning the set of coherent wavelet coefficients. Another component integrates a set of tracer particles forward in time through a given discrete, instantaneous velocity vector field to generate a set of tracer trajectories. The final component analyzes a set of tracer trajectories using Haller’s method to determine the stability characteristics of the flow.

These three components are combined in different ways to explore the different research issues of this thesis. To perform the evaluation of the stability characteristics of the regions that each wavelet coefficient represents, a single tracer particle is integrated through the velocity field starting at the location corresponding to each coefficient. Each trajectory is then analyzed using Haller’s method, giving the hyperbolicity time for the trajectory.

To perform the wavelet-based tracer placement, the wavelet coefficients are passed to a function that returns a set of non-uniformly distributed tracer particle positions, which are then integrated and analyzed using Haller’s method. The results are then interpolated to a regular grid to be analyzed and compared to results from a uniform grid. Further details about each of the components are given in the subsequent sections.
3.1 Wavelet Coefficient Generation

The wavelet coefficient generation routine works by transforming the vorticity field into the wavelet basis, then partitioning the wavelet coefficients into coherent and incoherent sets. These coherent and incoherent sets can be transformed back to the original basis using an inverse wavelet transform to obtain coherent and incoherent vorticity fields, but these fields have not been used in the present research.

For this project, the wavelet coefficients are obtained by performing a 2D fast wavelet transform with Coifman 12 wavelets. It is certainly possible to use other classes of wavelets, such as Daubechies or Haar wavelets, but Coifman wavelets have been used successfully in several similar cases of vorticity field analysis ([11], [16]), and the fundamental shape of the Coifman-series wavelets is similar to the typically Gaussian-like appearance that vortices take on in the vorticity field. From the original vorticity field, the denoising threshold $\tau$ is found using equation 2.2. All coefficients greater than the threshold are partitioned into the set of coherent coefficients. The rest form the set of incoherent coefficients, with the remaining empty coefficients in each partition set to zero. The set of coherent wavelets can then be passed on to other sections of code.

3.2 Tracer Generation from Wavelet Coefficients

For both the wavelet coefficient analysis and the wavelet-based tracer generation methods, each coefficient in wavelet space is used to create one or more tracer particles in physical space. Various function are used to return, for a given wavelet coefficient, the corresponding spatial coordinate and spatial scale of the wavelet. Coordinates of the particles are then passed on to the tracer integration routine as the initial positions for tracer integration. For the coefficient analysis – and for one of the cases of the tracer generation – this function places a single tracer at the center of the region covered by the coherent wavelet coefficient. In the second tracer placement approach tested here, a uniform grid of particles that fill the entire spatial region associated with the coefficient are created using both the location and the scale of the corresponding wavelet. In the third tracer generation case, a Gaussian random distribution of particles is created that is centered on the position of the
coherent wavelet and scaled according to the spatial scale of the coefficient.

3.3 Tracer Integration

Tracer integration is fairly straightforward. Starting from a set of initial coordinate value at arbitrary locations (typically generated from the wavelet coefficients), the position of each passive particle is integrated through the velocity field. As described in section 2.1.1. The integration uses a standard fourth-order Runge-Kutta scheme and off-grid velocity values are obtained using bicubic interpolation. The new tracer positions of the are written to a file after each update, to be used later in the analysis. For all cases, the integration was carried out over 200 intervals with a timestep of $7.5 \times 10^{-3}$, which allowed all of the tracers to sufficiently propagate and highlight the coherent structures in the field used for the test cases.

3.4 Tracer Analysis

The tracer analysis uses Haller’s method to evaluate the stability and dynamics of each tracer trajectory, and is based on code originally written by Dr. Haller. For each tracer, the code calculates a hyperbolicity time, which is the number of timesteps in which the tracer met the criterion for hyperbolicity ($M_2$ is positive definite) in the course of its trajectory. The analysis requires the velocity gradient tensor, which is computed on the discrete grid of the velocity field using first-order central differences, with first-order backward and forward differences at the edges. The remaining flow field properties, including the rate of strain tensor and the strain acceleration tensor, are calculated by interpolating the value of the velocity gradient tensor using bicubic interpolation, then algebraically calculating the remaining quantities.
Chapter 4

Results

4.1 Numerical Flow Field

Figure 4.1: Lid-driven cavity flow data – streamlines and vorticity contour plot

The test case used for the majority of the results in this chapter is a numerical solution of the lid-driven cavity flow: a flow with three solid no-slip walls and one moving wall that drives the flow. It is one of the most thoroughly studied cases in computational fluid dynamics, and a significant amount of data exists on the flow. The flow also has a number of features that make it well suited to this project. At low Reynolds numbers, the
flow is steady and develops a large vortex in the center and two smaller vortices in the corners opposite the moving wall. The large vortex rotates at a significantly faster rate than the smaller vortices, which rotate much more slowly and are thus weaker. In addition, significant vorticity peaks form in the corners adjacent to the moving wall due to shear flow. This vorticity is due to the sharp turn that the fluid takes immediately before and after attaching to the moving wall, and has a much higher value than the vorticity of any of the true vortices. A final convenient property is that the flow is bounded on all sides, so that no tracers will escape the grid, which would require special handling or extrapolation of some sort.

The data set was generated using Fluent for a Reynolds number of 1000 and a resolution of $256 \times 256$ grid points, and the solution was iterated until the continuity residual dropped below $10^{-5}$. The velocity contour plots with streamlines and vorticity contour plot are given in figure 4.1. The wavelet transform of the vorticity field is given in figure 4.2, also with a resolution of $256 \times 256$. The hyperbolicity time plot from the analysis of the field using Haller’s method is given in figure 4.3. The hyperbolicity time plot was created using 200 $\times$ 200 tracer particles on a uniform grid over 200 integration time steps with a $\Delta t$ of $0.75 \times 10^{-4}$ s (This $\Delta t$ has been used throughout unless otherwise stated).

4.2 Correlation Between Coherent Structure Definitions

Analysis of the correlation between the wavelet-diagnosed coherent structures and Haller’s LCS is accomplished using the techniques and criteria described in section 1.3 with the denoising threshold calculated using equation 2.2.

For the $256 \times 256$ vorticity field, the thresholding gives a total of 346 coefficients in the coherent field, and the integration took place for 200 timesteps. The hyperbolicity times for these coefficients are shown in histogram form in figure 4.4. This figure contains the histograms for the full $256 \times 256$ field, as well as downsampled fields from $16 \times 16$ through $128 \times 128$, increasing by powers of two. Below $16 \times 16$, only two coefficients are present, both with zero hyperbolicity time values. Downsampling by a factor of two is equivalent to removing the set of wavelet coefficients with the smallest spatial scales since. This is done to visualize any trends that exist between hyperbolicity count
and wavelet scale, and to determine whether any variation exists across the smaller scales (which could be due to noise or other sampling problems), and to determine the effects of the shear features in the flow, which are on much smaller scales than the vortices.

The histograms at the resolutions of $64 \times 64$ and above show a strong bimodal distribution, with two peaks at the extremes of hyperbolicity times of 0 and 200, representing tracers that either never entered into a hyperbolic region or remained entirely within a hyperbolic region. This trend is not present below $64 \times 64$ resolution, indicating that the associated wavelet regions with spatial scales of $8 \times 8$ grid points or smaller are not strongly hyperbolic. The coefficients with zero hyperbolicity time appear to dominate at the smaller resolutions, indicating that they correspond to wavelet

Figure 4.2: Wavelet coefficients for lid-driven cavity vorticity field – resolution is $256 \times 256$
coefficients with large spatial scales. Below $16 \times 16$ resolution, there are only two coefficients, both with a zero hyperbolicity time value. Upon further investigation, it became apparent that all of the elliptic points lie on the stationary walls of the box, and do not move due to the boundary conditions, which is interpreted as non-hyperbolic behavior by Haller’s method. Since these particles are not really part of the flow, they should not be considered in the analysis, and so all of the valid points that the wavelet coefficient method returns are highly hyperbolic. Furthermore, the most highly hyperbolic regions associated with wavelet coefficients correspond to coefficients with small spatial scale, and very few coefficients at the large scales.

The histogram of hyperbolicity times for the lid-driven cavity computed using a uniform distribution of $100 \times 100$ particles is given in figure 4.5.
Figure 4.4: Hyperbolicity time histogram for wavelet coefficients – Resolutions from $16 \times 16$ through $256 \times 256$ for the analyzed signal
with the wavelet coefficient-based histograms, this histogram also shows a significant number of tracer particles with high hyperbolicity count, though there are more particles in the mid and lower ranges. When the corner points are disregarded, the uniform distributions appear to be nearly identical to the $256 \times 256$ resolution histogram in figure 4.4.

![Hyperbolicity Time Histogram](image)

**Figure 4.5:** Hyperbolicity time histogram for a uniform distribution of $100 \times 100$ tracers

There does not appear to be any unique trend in the histograms pertaining to the regions of coherent wavelet coefficients when compared with the uniform distribution. While the coherent wavelet coefficient regions have a high occurrence of particles with high and maximum hyperbolicity, the uniform field also displays this peak at maximum hyperbolicity. Because the peak is not unique to the wavelet coefficients, the existence of a correlation cannot be drawn here because a random distribution of 346 tracers would
presumably give the same results. If the regions associated with the coherent wavelet coefficients were uniformly of maximum hyperbolicity time, or if the points were much more closely concentrated about the maximum, then good evidence would exist for a correlation between the Haller’s hyperbolic LCS and the coherent regions detected by the wavelet method.

Next, the lid-driven cavity flow is studied to establish whether the coherent wavelet coefficient regions appear to correspond with any particular features of the flow field, or if their distribution were more random in nature. This is done by overlaying the positions of the tracer particles that correspond to the regions of space associated with the coherent wavelet coefficients on the hyperbolicity time plot in order to determine if any sort of topological organization of particles exists. This is shown in figure 4.6.

![Figure 4.6: Wavelet coefficient coordinates overlaid on hyperbolicity time plot](image)

Figure 4.6: Wavelet coefficient coordinates overlaid on hyperbolicity time plot
Note that the majority of the particles in figure 4.6 are concentrated in the top portion of the flow, adjacent to the moving lid. The highest concentration of tracers overall is in the upper right corner, which is where the highly localized vorticity peak occurs due to shear at the interface between the moving and solid walls. While the majority of the tracers do lie in hyperbolic regions, the coefficient locations show almost no correlation with the regions of high hyperbolicity, (the darker red and maroon regions of the plot). This indicates that while the regions associated with coherent wavelet coefficients have high hyperbolicity and thus lie within hyperbolic LCS, those coefficients do not represent a significant portion of the total signal of the entire portion of the flow that is highly hyperbolic. The selected coherent wavelet coefficients do not represent the elliptic coherent structures because none of the regions associated with these coefficients are elliptic (indicated by tracers with zero hyperbolicity time), and the coefficients do not represent the hyperbolic coherent structures because they do not appear to correlate well with the full topological structure of the highly hyperbolic regions of flow field. Based on this, there appears to be no connection between coherent wavelet coefficients and the Lagrangian coherent structures of Haller’s method.

Part of the problem here is that the dominant peak in the vorticity field in the upper right corner of the domain, along with the high-vorticity region in the shear layers along the walls, is being detected as the main signal in the vorticity field, with the coefficients associated with the vortices and mixing regions in the flow being taken as incoherent for the most part. Haller’s method, on the other hand (figure 4.3), does not appear to be affected by these shear zones, with the results clearly highlighting the main center vortex and two smaller vortices in the bottom corners. The fact that the wavelet method essentially failed to separate the coherent portions of the flow is a bit surprising, considering the success of the method in past research. One thing to consider is that past experiments were done at high Reynolds numbers in flows with many obvious vortices and no strong shear layers or walls. The lid-driven cavity, on the other hand, is a notoriously difficult problem in general, is at lower Reynolds number, and has a strong shear zone along its walls that creates high vorticity values. Moreover, the wavelet method defines the incoherent portions of the flow as Gaussian white noise. Since the shear-layer vorticity is larger than the vorticity in the vortices, the wavelet method determines that the noise floor is above the level of the vortices, and so the shear essentially drowns out any portion of the meaningful signal.
More coefficients associated with the vortices could presumably be selected as coherent by using a different threshold, but this will still admit coefficients associated with the shear peaks instead of ignoring them altogether, which would be ideal. While the behavior is consistent with how the wavelet noise filter should perform, it indicates that there are some limitations to the use of wavelet filtering for detection of coherent structures in flows with dominant shear layers.

4.3 Tracer Positioning using Wavelet Coefficients

Three methods for tracer placement were investigated, providing a preliminary assessment of the suitability for using the coherent wavelet coefficients to improve the efficiency of Haller’s method. The first method is the most simplistic and obvious: placing a single tracer at the spatial coordinate of each coherent wavelet coefficient. To evaluate this, the driven cavity flow was again analyzed at a resolution of $256 \times 256$, yielding 346 coefficients. Tracers at the spatial positions associated with these 346 coherent coefficients were integrated for 200 timesteps and analyzed using Haller’s method. The process for this is identical to the analysis of section 4.2 and yields the same set of tracer positions as in figure 4.6. This test merely served as an initial starting point to assess the spatial distribution of the wavelet coefficients. Due to the small number and sparse distribution of these tracers with respect to the full resolution of the velocity field, no meaningful results can be attained for the hyperbolicity time plot. The point of the method is simply to give some insight into the general distribution of particles that could be constructed using the spatial content of the thresholded wavelet coefficients. As described in section 4.2, the coherent wavelets appear to cluster predominantly around the shear regions along the walls, with much more sparse coverage in the central region. Based on the hyperbolicity time plot shown in figure 4.3, an ideal distribution of the wavelet coefficients, then, would be dense in the regions that have the greatest variation hyperbolicity time, such as in the ringed regions around the central and corner vortices, and sparse in the more-uniform regions, such as the region in the middle of the center vortex. While this would cause some clustering of particles in the vicinity of the mixing regions around the center vortex, there would virtually no parti-
cles in or around the two corner vortices, indicating that these vortices may not be well resolved.

The second method evaluated here uses both the spatial location and the spatial scale of the wavelet coefficients to choose the tracer positions. Instead of creating a single point for each coherent wavelet coefficient, a uniformly distributed grid of particles is created so that the particles cover the entire spatial region represented by the particular coefficient. Results have been calculated for grids of $4 \times 4$, $8 \times 8$ and $16 \times 16$ particle grids for each coefficient. The tracers are integrated through the flow and the paths are analyzed to obtain hyperbolicity time values; the results are then interpolated to a uniform grid for analysis. The tracer particle placement for each case can be seen in figure 4.7. The resulting interpolated hyperbolicity field are shown, along with plots of the hyperbolicity time for an an equivalent number of tracers placed on a uniform grid, in figures 4.8, 4.9 and 4.10.

Figure 4.7: Tracer Distribution for $4 \times 4$, $8 \times 8$ and $16 \times 16$ particles per coherent wavelet coefficient
The general increase in the number of particles for this method compared to placing a single tracer makes it possible to obtain enough coverage for interpolation across the full spatial domain, so the results can be meaningfully compared with a uniform grid. The initial tracer distribution also gives further insight into the associated spatial scales of the wavelet coefficients represented in the different regions of the flow, since the size of the region determines the particle distribution. Smaller scale wavelet coefficients are common in the upper corners and larger scales are more common in the lower and middle regions. This is again probably due to the shear peaks, which are small in scale with respect to the rest of the flow, but high in vorticity. The distribution of particles, however, is quite coarse, with regions of high density and little to no density closely interspersed, and the distribution does not appear to correspond well with the underlying hyperbolicity field. The tracers, again, are densely packed near the shear regions, with more sparse coverage in the center of the cavity and especially near the corner vortices. For all three levels of resolution, the uniform distribution produces visibly better results, which is a strong indication that simply distributing a uniform grid of particles in the region corresponding to each coherent wavelet coefficient is not an efficient method for analyzing the flow field. The plots show particular weakness in resolving the corner vortices, with significant error in the lower right vortex in all cases. While the results do not indicate that this method is more efficient, they do motivate further investigation.
The final method employed to assess the suitability of using wavelet coefficients to guide tracer placement distributes a random, Gaussian array of particles at each associated coherent coefficient location, scaled according to the spatial scale of the coefficient. Tests were run for 16, 64 and 256 particles per coefficient, as in the test for the uniform grid at each coefficient. A $256 \times 256$ wavelet coefficient field was again used, with a total of 346 coherent coefficients being used to generate tracer particle positions, and the tracers were advanced through 200 timesteps. The final results were again interpolated to a regular grid in order to plot and compare the results. These interpolated hyperbolicity time plots are given in figure 4.11, with the tracer particle positions shown overlaid in figure 4.12.

Compared with the uniform distribution, there is virtually no structure or organization apparent in the Gaussian tracer distribution, which can be seen by the general improvement in resolution of the center and corner vortex structures, and along the top of the box. The particles concentrate around the top corners as in the previous cases, and the most sparse concentration of particles is found along the bottom edge. The resulting hyperbolicity plots are much closer to the uniform grid plots than in the previous test. The improvement in quality is likely due to the wider distribution of particles across the whole field, and also likely due to the absence of the large regions that are empty of tracer particles seen in figure 4.7. Even visually, the results still do not show any improvement over the uniform distribution, however. As in
the previous results, the resolution around the corner vortices is particularly bad, though the resolution around the upper portions of the flow is visually fairly close to that in the uniform distributions.

Overall, using thresholded wavelet coefficients to select tracer positions do not yield better results than a simple, uniform distribution of tracers. The problem appears to lie in the concentration of the wavelet coefficients around the shear regions, and their failure to cover the center and corner vortices and their regions of interaction. From this, it does not appear that wavelet noise thresholding is a suitable technique for speeding up the computation time of Haller’s method without sacrificing significant accuracy in the solution.

4.4 Additional Details of the Wavelet Coefficients

One interesting note on combining wavelets and Haller’s method is that the full wavelet coefficient field exhibits features that are similar to those in the hyperbolicity time field. These two fields are shown in figure 4.13. Note that in this figure, the wavelet coefficients field is upside-down with respect to the hyperbolicity time field. In the highest resolution subspaces of the wavelet coefficients (the largest of the recursively nested squares), darker regions are visible that are very similar to the mixing regions shown in the hyperbolicity.
Figure 4.11: Hyperbolicity time field for 16, 64 and 256 particles per coefficient using Gaussian distribution

time plot. To more clearly illustrate this, the bottom right space (the highest resolution window) has been isolated and plotted in 4.14 using a color map similar to the hyperbolicity time field. This portion of the wavelet field clearly shows structures similar to those shown in the hyperbolicity time field, and the structures correspond to the local maxima and minima within the wavelet coefficients. The shear regions are still visible in the corners, and are still the largest peaks in the field. The coherent wavelet partition appears to have failed to detect these peaks because it was overwhelmed by the strength of the shear peak. This result, however, indicates that there is still potential for using thresholding of the maxima and minima of the wavelet coefficients in order to generate tracer positions. One possible work-around for this type of problem in the future would be to modify the vorticity field in some way to reduce the highest peaks, for example by analyzing the log of the absolute value of vorticity field. This type of technique, however, would likely admit
more coefficients as coherent, and thus reduce the effectiveness of the data reduction in this method.

Since the similarity between the wavelet coefficient values and the hyperbolicity time is most evident at the higher resolutions, it may be more beneficial to use the continuous wavelet transform to aid the tracer placement. This technique would be much more similar to the method of Schram & Riethmuller [16], who define vortices by the local maxima of the continuous wavelet transform coefficients. The continuous wavelet transform analyzes each scale at full resolution, rather than with the recursively downsampled scales of the fast wavelet transform, which could give more detailed information about the flow field. While this method would still be likely to detect the shear peaks, it may obtain better coefficient distribution around the flow features of interest as well. The downside to the continuous wavelet transform is that it does not share the efficiency of the linear-runtime fast wavelet
Figure 4.13: Wavelet coefficients for lid-driven cavity vorticity field compared with the hyperbolicity time plot of the same field.

transform, and it requires more storage space. The overhead would thus be higher, but it may still prove beneficial.
Figure 4.14: Highest resolution wavelet coefficients for lid-driven cavity vorticity field
Chapter 5

Conclusions and Future Work

The results of this thesis show that the shortcomings and the dissimilarities of the wavelet method and Haller’s method rule out a mutually beneficial combination of these methods as they are stated. The two methods each define types of coherent structures that they are able to identify, but based on the results presented here, the definitions of coherent do not agree for flows with vorticity peaks due to shear flow. Without a consistent definition of what is coherent and what is incoherent, it is very difficult to see any way to combine the two methods for the purpose of performing coherent structure analysis or extraction. The weakness appears to be in the wavelet-based method, and particularly with the use of the vorticity field, which gives peaks for both vortices and high shear regions. The results certainly do not discount the use of wavelets in general, but it does not appear that the particular method of Pellegrino et al. [11] gives a robust-enough definition of coherent structures for use in general fluid flows.

The sensitivity of the wavelet method to shear also appears to fundamentally limit the possibility of creating a method for more optimal tracer placement for Haller’s method. One of the main strengths of Haller’s method is that it is not susceptible to shear peaks in the flow. All of the placement methods employed tended to concentrate the tracers near the regions of high shear, causing loss of resolution in the portions of the flow furthest from the shear peaks. A uniform distribution of particles across the flow gave visibly better resolution of the structures in the flow than the wavelet-based methods employed in every case.

There are a number of interesting future paths suggested by this research. The wavelet method used here was very specific, and there are many other
possible ways to analyze a flow using wavelets. It would be interesting to try other types of wavelet transforms, such as the wavelet-packet transform or a continuous wavelet transform. The continuous wavelet transform in particular allows for analysis across the full range of possible scales, giving much more potential information to use in the detection of coherent structures. While these methods are not as computationally efficient as the fast wavelet transform, they present a different possibilities for the analysis of the field. Also, it would be interesting to try the wavelet method using other wavelet bases. The method used here used only one in the myriad of known wavelet families. It may be that other types of wavelet families can suppress the effects of the shear peak and better detect the coherent structures in general flows. It would also be interesting to attempt the wavelet filtering on something other than the vorticity. The velocity gradient, rate-of-strain and strain acceleration tensor fields are all more closely related to Haller’s method, which may lead to better results, and these fields could also be less sensitive to shear peaks.

A final possible future direction for this work is in extending the first set of tests on what it means to be ‘coherent’. Numerous studies of and references to coherent structures are given in the literature, but the term itself has no apparent consistent definition. A possible future project would be to attempt to classify and analyze the range of definitions in order to see the ways in which each are similar and different. This may eventually lead to a formal classification system for coherent structures that can be used as a basis for deeper study.
Bibliography


