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### A comparison of nonparametric efficiency estimators: DEA, FDH, DEAC, FDHC, order-m and quantile

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### Abstract

In this paper we compare six nonparametric estimators for technical efficiency and use them to evaluate the efficiency of the banking sector in Brazil. The estimators considered are data envelopment analysis (DEA), free disposal hull (FDH), bias corrected FDH (FDHC), bias corrected DEA (DEAC), order-m and alpha-conditional quantile. Their theoretical properties are discussed and their implementation is illustrated using a sample of 184 Brazilian banks that extends from 1995 to 2004. The results indicate that these estimators can lead to significant discrepancy in estimated efficiency scores. Order-m and alpha-conditional quantile estimators have proven to be useful tools in identifying extreme values and are shown to be rather robust relative to DEA and FDH. Bias correction for both DEA and FDH was problematic, producing significant changes in firms rankings and estimated efficiencies.

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# 1 Introduction

There now exists a voluminous literature that evaluates the efficiency of production units in various sectors of the economy, including, among others, the health, financial and educational sectors. Several empirical studies have evaluated the relative performance of hospitals, universities, schools, banks, insurance companies, government agencies, etc. See, *inter alia*, Abbott and Doucouliagos (2003), Canhoto and Dermine (2003), Steinmann et al. (2004) and Wheelock and Wilson (2011). See also Fried et al. (2008) for a recent comprehensive review of past theoretical and empirical developments.

In this context, nonparametric statistical models for production have gained popularity because they bypass potential misspecification problems that invariably result from tight parametric specifications, and because we now have a good understanding of the asymptotic properties of the estimators associated with these models. The two most popular nonparametric frontier estimators are Data Envelopment Analysis (DEA), introduced by Charnes et al. (1978) and Free Disposal Hull (FDH), introduced by Deprins et al. (1984). Park et al. (2000) were the first to establish consistency and weak convergence of an efficiency estimator based on the FDH estimator, whereas Kneip et al. (2008) established the same for the DEA estimators (see Simar and Wilson (2011) for a survey of recent extensions). However, the intrinsic bias, lack of robustness to extreme values and slow convergence rates have motivated a search for alternative estimators. Park et al. (2000) proposed a bias corrected free disposal hull estimator (FDHC) and Simar and Wilson (2000) have proposed interval estimators by bootstrapping DEA based efficiency scores. Aragon et al. (2005) and Martins-Filho and Yao (2008) proposed efficiency estimators based on conditional quantiles and Cazals et al. (2002) proposed more robust estimators for what they labeled  $m$ -frontiers. Conditional quantile estimators have the advantage of converging at parametric rates when the technology considered is for one output and multiple inputs. As a result, they do not suffer from the *curse of dimensionality* that plague DEA and FDH.

It is important to assess the robustness of estimated efficiency scores using different nonparametric estimation methods and modeling strategies. The use of different estimation procedures on the same sample may produce different conclusions regarding efficiency, and as a result may induce different regulatory policies or managerial decisions. There is, of course, a recognition that the methods mentioned above can be estimating different concepts, e.g., DEA and quantile based estimators are not estimating the same efficiency concept, however it remains important to quantify and contrast the magnitude of the differences in estimated efficiency scores. In this spirit, the main objective of this paper is to present and compare six nonparametric methods commonly use to estimate efficiency scores and to empirically evaluate how different are the estimated efficiency scores they produce when applied to the same sample of 184 Brazilian banks. As a result, this paper also presents the most complete set of results regarding the efficiency of the Brazilian banking sector, as previous studies in this industry have limited themselves to the use of DEA.

The paper is organized as follows. Section 2 introduces notation, describes production frontiers, technical efficiency and the nonparametric estimators considered in a unified notation. Section 3 describes the data and discusses the results of our empirical analysis, emphasizing the differences in estimated efficiency scores for our sample of Brazilian banks. Section 4 provides a brief conclusion.

## 2 Production Efficiency and Nonparametric Estimators

Let  $x \in \mathbb{R}_+^p$  be a non-negative input vector used to produce  $y \in \mathbb{R}_+^q$  a nonnegative output vector under the technology  $\Psi = \{(x, y) \in \mathbb{R}_+^p \times \mathbb{R}_+^q : x \text{ can produce } y\}$ . For a firm characterized by the production plan  $(x_0, y_0)$ , the output oriented efficiency score  $0 < \theta \leq 1$  is  $\theta(x_0, y_0) = \inf\{\theta : (x_0, \theta^{-1}y_0) \in \Psi\}$ . If  $\theta(x_0, y_0) = 1$  the firm is operating on the boundary of the technology  $\Psi$  and is labeled technically efficient. In this case, the optimal output level is  $y^\theta(x_0) = \theta^{-1}(x_0, y_0)y_0$ . In practical situations, where there is a desire to evaluate the technical efficiency of firms, the set  $\Psi$ , and consequently  $\theta$ , is not observed and must be estimated based on a random sample  $\chi_n = \{(x_i, y_i)\}_{i=1}^n$  of production plans.

**DEA and FDH estimators** - DEA estimates the technology  $\Psi$  by a linear convex boundary that envelops  $\chi_n$ , i.e.,  $\hat{\Psi}_{DEA} = \{(x, y) \in \mathbb{R}^{p+q} : y \leq \sum_{i=1}^n \lambda_i y_i, x \geq \sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0 \text{ for all } i\}$ , where for

any two vectors  $x$  and  $z$  of equal dimension, we write  $x \leq z$  to denote that the inequality holds element-wise. The output oriented efficiency score for a production plan  $(x, y)$  under variable returns to scale is given by

$$\hat{\theta}_{DEA}(x, y) = \left( \max_{\phi} \{ \phi : \phi y \leq Y\lambda, x \geq X'\lambda, \lambda' = (\lambda_1, \dots, \lambda_n), \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0 \} \right)^{-1}, \quad (1)$$

where  $X$  is an  $n \times p$  matrix containing all observations on the  $p$  inputs used in the production process and  $Y$  is an  $n \times q$  matrix containing all observations on the  $q$  outputs produced. If the restriction  $\sum_{i=1}^n \lambda_i = 1$  is excluded from the optimization defined in (1), then the resulting efficiency score  $\hat{\theta}_{DEA}(x, y)$  is associated with an estimated technology that exhibits constant returns to scale.

The FDH estimator does not impose convexity of the estimated technology, only free disposability of inputs. The estimated technology is  $\hat{\Psi}_{FDH} = \{(x, y) \in \mathfrak{R}^{p+q} : y \leq y_i, x \geq x_i \text{ for all } (x_i, y_i) \in \chi_n\}$ . The output oriented efficiency score for a production plan  $(x, y)$  is  $\hat{\theta}_{FDH}(x, y) = \left( \max_{i: x_i \leq x} \min_{j=1, \dots, p} \left( \frac{y_i^j}{y^j} \right) \right)^{-1}$ , where  $y^j$  and  $y_i^j$  are respectively the  $j^{th}$  components of  $y$  and  $y_i$ .

By construction  $0 < \theta \leq \hat{\theta}_{DEA}(x, y) \leq \hat{\theta}_{FDH}(x, y) \leq 1$  for all  $(x, y) \in \Psi$  which implies that the DEA and FDH estimators are inherently biased. Kneip et al. (2008) and Park et al. (2000) have shown that under fairly mild conditions DEA and FDH estimated efficiency scores are consistent estimators of  $\theta(x, y)$  with converging rates which are respectively  $n^{-2/(p+q+1)}$  and  $n^{-1/(p+q)}$ . These convergence rates are slower than the parametric rate  $n^{-1/2}$  whenever  $p > 1$  or  $q > 1$  and become slower as  $p$  and  $q$  increase. The better rate of convergence for the DEA estimator is a consequence of the convexity assumption imposed on the technology.

The FDH estimator can be useful in identifying *dominant* or *dominated* firms. A firm  $i$  is dominated by firm  $j$  when  $x_j < x_i$  and  $y_j > y_i$ . Every firm that is inefficient based on an FDH estimator is dominated by one or more firms. Efficient firms are not dominated but are not necessarily dominant.

**The FDHC estimator** - Park et al. (2000) have proposed bias corrected FDH estimators which we label FDHC. They showed that  $P\left(n^{1/(p+q)}(\theta(x, y) - \hat{\theta}_{FDH}(x, y)) \leq z\right) = 1 - e^{-(\mu(x, y)z)^{p+q}} + o(1)$  where  $\mu(x, y)$  is a parameter. This distribution corresponds to that associated with a Weibull density with parameters  $n\mu(x, y)$  and  $p + q$ , i.e.,  $W(n\mu(x, y), p + q)$ . The  $r^{th}$  centered moment of the FDH efficiency estimator is given by  $E\left((\theta(x, y) - \hat{\theta}_{FDH}(x, y))^r\right) = -c_r \mu^{-r}(x, y) n^{-r/(p+q)} + o(n^{-r/(p+q)})$ , where  $c_r = \Gamma\left(\frac{p+q+r}{p+q}\right)$ . As a result the bias corrected FDH efficiency score estimator is given by

$$\tilde{\theta}_{FDHC}(x, y) = \hat{\theta}_{FDH}(x, y) - c_1 \mu^{-1}(x, y) n^{-1/(p+q)}. \quad (2)$$

Park et al. (2000) proposed a simple estimator  $\hat{\mu}(x, y)$  for  $\mu(x, y)$  based on the empirical distribution of the observations that dominate the point  $(x, y)$ .  $\hat{\mu}(x, y)$  allows for bias correction of bias and construction of confidence intervals for  $\theta(x, y)$ , i.e.,  $\left[\hat{\theta}_{FDH}(x, y), \hat{\theta}_{FDH}(x, y) + \hat{\mu}(x, y) n^{-1/(p+q)} z_{1-\alpha}\right]$  where  $z_{\alpha} = (-\log(1 - \alpha))^{1/(p+q)}$ . Park et al. (2000) show that as  $n \rightarrow \infty$  the FDHC estimator exhibits better performance relative to the uncorrected FDH estimator.

**The DEAC Estimator** - The asymptotic distribution of DEA estimators has been obtained by Gijbels et al. (1999) for the case where  $p = q = 1$ . For multiple input and output technologies Simar and Wilson (1998) and Simar and Wilson (2000) have proposed the use of the bootstrap to approximate distributions and confidence intervals. In the case of efficiency scores the idea is that, conditional on input usage, the observation  $(x_i, y_i)$  can be associated with the realization of a random variable  $\theta_i \in (0, 1]$  such that  $y_i = \theta_i y^{\partial}$  where  $y^{\partial}$  corresponds to the output level at the boundary of the technology given input usage  $x_i$ . Hence, the sample  $\chi_n$  can be represented by  $\chi_n = \{(x_i, y^{\partial} \theta_i)\}$ . Implementation of the bootstrap for the DEA efficiency score estimator can be done following these steps: 1. Obtain DEA efficiency scores as described by equation (1) for each firm  $(x_i, y_i)$  in the sample and label them  $\hat{\theta}_i$ ; 2. Obtain a sample (with replacement) of size  $n$  from  $\{\hat{\theta}_i\}_{i=1}^n$  using the reflexion method proposed by Simar and Wilson (1998) and label such set  $\{\theta_i^*\}_{i=1}^n$ ; 3. Using  $\{\theta_i^*\}_{i=1}^n$  produce a new sample  $\chi_n^* = \{(x_i, y_i^*)\}$  where  $y_i^* = \theta_i^* y_i / \hat{\theta}_i$ ; 4. Estimate  $\hat{\theta}_{i,b}$  using the

sample  $\chi_n^* = \{(x_i, y_i^*)\}$  in accordance to the procedure in equation (1). In step 4,  $b = 1, \dots, B$  represents the  $b^{\text{th}}$  iteration of the bootstrap procedure. Steps 2 through 4 are repeated  $B$  times to generate  $B$  sets of efficiency scores estimates for each firm. Finally, these  $B$  estimates can be used to construct empirical confidence intervals for the efficiency score for each firm and to produce bias corrected efficiency scores, since if this proposed bootstrap procedure is consistent, the distribution of  $\hat{\theta}_i^* - \hat{\theta}_i$  is approximately that of  $\theta_i - \hat{\theta}_i$  as  $B, n \rightarrow \infty$ .

**Order- $m$  Estimator** - Cazals et al. (2002) have proposed an efficiency concept and accompanying estimator that is more robust to extreme values as it does not use all sample values to conduct estimation. Their estimator is based on the concept of maximal expected frontier of order  $m$ . For a fixed value  $m \geq 1$  we select a random sample  $(y_1, \dots, y_m)$  of outputs from the distribution of the output vector given that input usage is less than  $x$  and define the boundary output of order  $m$  as

$$y_m^\partial(x) = E(\max\{y_1, \dots, y_m : x_1 \leq x, \dots, x_m \leq x\}) = \int_0^\infty (1 - F(y/x)^m) dy, \quad (3)$$

where  $F(y/x) = \frac{F(y,x)}{F_X(x)}$  is the conditional distribution of output given that input usage is below  $x$ . Here,  $F(y, x)$  and  $F_X(x)$  are the joint distribution of the output and input vectors and the marginal distribution of the input vector, respectively. As the order- $m$  frontier stipulates the expected maximum product, it is a direct consequence of the definition that  $y^\partial \geq y_m^\partial$  and  $\theta_m(x, y) \geq \theta(x, y)$ . In addition, since not all data are used to produce the  $m$ -frontier,  $\theta_m(x, y)$  may take values outside the interval  $(0,1]$  and as  $m \rightarrow \infty$  we have  $y_m^\partial \rightarrow y^\partial$  and  $\theta_m(x, y) \rightarrow \theta(x, y)$ . Hence, as  $m \rightarrow \infty$  the  $m$ -frontier approaches the true frontier and, as a consequence, the associated efficiency score of order  $m$  approaches the true efficiency. Estimation of the  $m$ -frontier is accomplished by substituting the  $F(y/x)$  in equation (3) by its empirical version  $\hat{F}(x, y) = \frac{\sum_{i=1}^n I(y_i \leq y, x_i \leq x)}{\sum_{i=1}^n I(x_i \leq x)}$

where  $I(A)$  is the indicator function for the set  $A$ , i.e.,  $\hat{y}_m^\partial(x) = \int_0^\infty (1 - \hat{F}(y/x)^m) dy$ . Cazals et al. (2002) suggested obtaining an estimator for the efficiency score following these steps: 1. For a given input usage  $x$ , select all observations such that  $x_i \leq x$  and obtain a sample of size  $m$  with replacement given by  $\{y_{1,b}, \dots, y_{m,b}\}$ ; 2. Obtain  $\theta_m^b(x, y) = \left( \max_{1 \leq i \leq m} \min_{1 \leq j \leq p} \frac{y_{i,b}^j}{y^j} \right)^{-1}$ ; 3. Repeat steps 1 and 2 for  $b = 1, \dots, B$ .

**Order- $\alpha$  Quantile Estimator** - Aragon et al. (2005) proposed an alternative robust nonparametric estimator of frontiers based on conditional quantiles of  $F(y/x)$ . Let  $\Psi$  be the support of  $F(x, y)$ , then the production function can be written as  $y^\partial(x) = q_1(x) = \sup\{y \geq 0 : F(y/x) < 1\}$  where  $q_1(x)$  denotes the quantile of order 1 for  $F(y/x)$ . This suggests the definition of production functions of order  $\alpha \in [0, 1]$  as  $q_\alpha(x) = y_\alpha^\partial(x) = \sup\{y \geq 0 : F(y/x) < \alpha\}$ . By definition of a quantile,  $q_\alpha(x) < q_1(x)$  for all  $\alpha \in (0, 1)$  and, as a result, efficiency indices calculated based on quantiles of order less than 1 are smaller than those calculated based on the frontier with  $\alpha = 1$ . It is easy to verify that  $\lim_{\alpha \rightarrow 1} q_\alpha(x) = q_1(x)$  and  $\lim_{\alpha \rightarrow 1} \theta_\alpha(x, y) = \theta(x, y)$ . Implementation of  $\alpha$ -quantile estimators is straightforward. First, let  $N_x = \sum_{i=1}^n I(x_i \leq x)$  be the number of observations for which  $x_i \leq x$  and order the outputs in this relevant subsample as  $y_{i_1} \leq \dots \leq y_{i_{N_x}}$ . Then,

for all  $\alpha$   $\hat{q}_\alpha(x) = \begin{cases} y_{i_{\alpha N_x}} & \text{if } \alpha N_x \in N \\ y_{i_{\alpha N_x + 1}} & \text{if } \alpha N_x \notin N \end{cases}$  where  $N$  represents the natural numbers. Aragon et al. (2005) have shown that  $\hat{q}_\alpha(x)$  is a consistent estimator for  $q_\alpha(x)$  and that  $\sqrt{n}(\hat{q}_\alpha(x) - q_\alpha(x))$  is asymptotically normal. It is important to note that  $\hat{q}_\alpha(x)$  is defined only for the case where  $q = 1$  and  $p \geq 1$ .

### 3 Estimating Banking Efficiency - Data and Results

**Data.** We define inputs and output for the banking sector following the intermediation approach. Banks are considered financial intermediaries that attract financial resources using capital and labor and invest these resources in the form of loans or other financial instruments. All variables are measured in monetary units.

As in Nakane and Weintraub (2005), we define a technology with a single output  $y$  which is the sum of three components: a) all loans provided to clients minus reserves for unpaid credit lines (net credit lines); b) all investments made in stocks, bonds and other financial instruments; c) all other other credit lines

or financial instruments not included in traditional banking products. When relevant, we investigate the performance of estimators and the degree to which they are affected by the dimensionality of the output vector, using three outputs  $y_1$ ,  $y_2$ ,  $y_3$  representing items a), b) and c). As in Berger and Mester (2003), we consider four inputs in the production process: labor, physical capital, own capital and financial inputs: labor ( $x_1$ ) is the total number of employees by the end of each semester; physical capital ( $x_2$ ) is the total value of all fixed or immobilized assets; own capital ( $x_3$ ) is all other assets owned by the banks and ( $x_4$ ) denotes all financial assets secured from third parties, i.e., the total amount of deposits, financial resources that have been captured through open market operations, loans and transfers from government, and from abroad.

The data were obtained from accounting statements filed every semester and made available by the Consulting firm Austin-Asis. The number of employees was provided by the Central Bank of Brazil. The sample includes a total of 184 banks for the period that extends from June 1995 to June 2004. We proceeded with estimation using three sub-periods: June 1995 - December 1997; June 1998 - December 2000; June 2001 - June 2004. We assume that within sub-periods the technology is the same, but it can vary across sub-periods.<sup>1</sup> The sample is representative of the Brazilian banking sector, with more than 90 percent of total assets in the sector, 92.3 percent of all credit operations and more than 94 percent of the deposits in December of 2003. All monetary values we obtained are in nominal terms and were (in)deflated to June 2004 values using the price series IGP-DI. In what follows we present results obtained for the last sample sub-period. In this period, there are a total of 734 observations on 122 banks. 109 observations are on domestically owned public banks, 378 are on privately owned domestic banks and 247 are own foreign banks. **DEA and FDH Estimation.** Table 1 shows the number of efficient banks (observations) using DEA and FDH. When we assume a one-output technology, according to the FDH estimator, 390 observations - 53 percent of the sample - are efficient, a number significantly larger than the 57 efficient observations obtained using the DEA estimator (7.7 percent of the sample). When three outputs are used, the disparity between the estimators is even larger. In this case, FDH pointed to 628 efficient observations (approximately 86 percent of the sample). For DEA, the number of efficient observations increases to 113 (15 percent of the sample). These results are a consequence of two main factors: a) the assumption of convexity implicit in DEA and b) the slow convergence rate of FDH. The flexibility of FDH, which imposes little restriction on the technology, produces regions of the estimated technology set where many banks can be compared only to each other, and as a result generate a great number of firms that are efficient and non-dominant. For the one output model, 120 observations (16 percent of the sample) are efficient and non-dominant, and in the case of three outputs, 72 percent of the observations are efficient and non-dominant. In the case of DEA, these results are less pronounced as observations are compared to hypothetical production plans that result from the convex combination of observed firms. The left panel in Figure 1 plots estimated efficiency scores for DEA and FDH under one-output technology, and the right panel shows the same plot under a three-output technology. The left panel shows a clear pattern of positive correlation between the two estimates, but in the right panel the number inefficient observations according to both estimators is quite small, making relative statements regarding firm efficiency difficult.

Table 2 shows descriptive statistics for estimated efficiency by DEA and FDH. According to FDH estimation, average efficiency was approximately 85 percent, whereas for DEA the average efficiency was below 60 percent. For the three-output technology the average efficiency using DEA was close to 72 percent and for FDH it was above 97 percent. As can be seen from this table, the differences between estimators is not restricted to average efficiency. Spearman's correlation coefficient between DEA and FDH estimated efficiencies (Table 3) is only 0.59, suggesting that the estimators differ significantly in ranking firms. When the three output technology is considered the correlation is even smaller at 0.35.

**DEAC and FDHC Estimation.** We have computed DEAC and FDHC estimators and constructed 95

<sup>1</sup>In an attempt to avoid the inclusion of banks that might operate with distinct technologies, we have excluded those with less than 20 employees, those that have a zero value for any of the inputs or outputs characterizing the technology and those with no checking accounts or commercial operations. These exclusions also eliminated banks that are going through any sort of operational anomaly, restructuring due to regulatory intervention, liquidation or are being merged with, or acquired by, other banks. We have also excluded observations with values for output that appear unrealistic or misreported.

percent confidence intervals.<sup>2</sup> Table 2 shows that average estimated efficiencies for FDHC are very different from those for FDH. The average and median efficiency scores fell to values close to 40 percent. Whereas for FDH more than 50 percent of observations were at the frontier, for FDHC approximately 75 percent of the observations showed efficiency scores below 60 percent. Results are also significantly different for firm rankings. The largest value for Spearman's correlation between FDH and FDHC (Table 3) is 0.44.

The significant discrepancy between the two estimators calls for a more detailed analysis of the advantages and disadvantages for each method. When we examined FDHC estimated efficiencies, we verified that a great number of observations shared efficiency score. Table 4 provides six different scores obtained via FDHC and the number of observations associated with each of them. All of these observations were efficient under FDH. The results are a direct consequence of how the FDH bias is corrected (equation (2)). The correction depends on the estimated parameter  $\mu(x, y)$  which depends directly on the proportion of observations dominating the production plan  $(x_i, y_i)$ , which is being evaluated. Hence, if this proportion is the same for any two firms that were efficient under FDH then the value for FDHC will also be the same. In addition, when this proportion is small (as is the case when the number of observations is 108) the drop in efficiency is significant. The left panel on Figure 2, where efficiency scores were listed in increasing order from the lower bound of the confidence interval (FDHL), illustrates this scenario. The right panel on Figure 2 plots FDH and FDHC efficiency scores. The points that appear with FDH efficiency scores equal to one represent several observations. As shown on Table 4, several of the production plans which are efficient under FDH have the same efficiency under FDHC since the number of observations that dominate these production plans is the same, and as a result they overlap in the graph. The right panel on Figure 2 also presents quasi parallel and nearly vertical curves. The closest of these curves to the vertical axis represents units dominated by only one production plan ( $NW = 1$ ), the next corresponds to firms dominated by two production plans ( $NW = 2$ ), etc. For a fixed FDH efficiency score, the greatest the estimated  $\mu(x, y)$  the greatest will be the estimated FDHC. Intuitively, the greatest the estimated value of  $\mu(x, y)$  the more observations will dominate the production plan that is being evaluated and the greatest the number of observations used to correct the bias of the FDH estimator, therefore improving the performance of FDHC. As a result, when we move from left to right on the panel the less dramatic is the bias correction. It should be noted that Park et al. (2000) argued that a sample size of  $n = 500$  should be sufficient to obtain satisfactory results in the case where  $p + q = 5$ . However, these authors considered only observations in the interior of the technology. The results above seem to indicate that even with more than 700 observations the estimates for units that are close to the frontier can be quite deficient. Once again, the slow convergence rate (dimensionality problem) of nonparametric estimators can have some serious practical consequences. Our simulations suggest that in some cases the bias of the FDHC estimator is larger than that of FDH. This is a direct consequence of the correction procedure. Since the true frontier is not known, the FDH frontier is used to estimate the parameter  $\mu(x, y)$  and correct the bias, introducing additional stochasticity into the correction equation. Our results indicate that the additional noise introduced by this estimation can have significant empirical consequences in evaluating efficiency.

The results obtained using DEAC are closer to those produced by DEA, specially in contrast to the FDH case. The average estimated efficiency score for DEAC was 45 percent, for DEA was 56 percent, and Spearman's correlation coefficient was 0.9. The left panel on Figure 3 provides a plot of estimated efficiency scores obtained by DEA and DEAC. Bootstrap confidence intervals for DEA efficiency scores are quite wide making it difficult to make significant efficiency comparisons across banks. In general, large and medium size banks, as well as those close to the frontier are associated with larger confidence intervals. In addition, several of the corrected DEA estimates fell outside of the associated confidence interval. The bias correction procedure used in DEA introduces additional stochasticity, and although the bootstrap is a valid alternative to asymptotic confidence intervals, in our case, point-wise bootstrap intervals are somewhat unsatisfactory. The right panel on Figure 3, where estimates were listed in increasing order from the lower limit of the confidence interval (DEAL) illustrate the results.<sup>3</sup>

<sup>2</sup>Due to slow convergence rates for the three-output technology estimators, we focus our discussion on the one-output case.

<sup>3</sup>Note that the suitability of the bootstrap procedure proposed by Simar and Wilson (1998) has not been established formally, although Monte Carlo simulations seem to support the procedure.

Although the results described above show great variability, the list of inefficient observations is generally quite similar. Among the 30 observations deemed more inefficient using FDH, 26 of them are also inefficient according to FDHC. In the case of DEA, all of the 30 more inefficient observations remain the most inefficient DEAC. Of these 30 most inefficient firms according to DEA/DEAC, 14 are also inefficient according to FDHC. This seems to indicate that the use of a number of estimators might be a good practical strategy to weed out the firms that have serious operational problems leading to sustained levels of inefficiency.

**$\alpha$ -Quantile and Order- $m$  Estimation.** Efficiency estimates are highly dependent on the choice of  $m$  and  $\alpha$ . These parameters define the position of the frontier relative to the data, and the larger their values the greater the number of data used in estimation. Large values of  $m$ , and values of  $\alpha$  close to 1, make both estimators less robust to outliers. In our analysis of the Brazilian banking sector we used  $m = 75, 150, 300, 1500$  and  $\alpha = 0.98, 0.985, 0.99, 0.995$ .

Table 5 shows that 15 percent of all observations are efficient when  $m = 150$  and 31 percent are efficient when  $\alpha = 0.985$ . Relative to FDH, where 53 percent of observations were efficient,  $\alpha$  and  $m$  frontiers allow for a finer efficiency ranking of firms. The  $m$  and  $\alpha$  frontiers define new reference sets for efficiency statements which allow for comparisons among firms that are not comparable under DEA and FDH. The maximum values for efficiency of order  $m = 150$  and  $\alpha = 0.985$  are at times larger than 2.5 and 3.5, respectively. This suggests that one or more observations are well above the mass of data and can be considered extreme values in the sample. In the case of  $\alpha$ -frontiers, even when  $\alpha = 0.995$  the maximum index is still very large at 3.14.

The left top panel in Figure 4 plots estimated efficiency scores using FDH and those from an order- $m = 150$  frontier. The right top panel provides the same for a  $\alpha = 0.985$  frontier. There is a clear positive correlation between these estimated efficiency scores. The vertical line of production plans appearing on the right end of both panels results from the capacity that these more robust methods have to discriminate and categorize efficiencies in a finer manner relative to FDH. A large number firms which are efficient according to FDH are not under  $m$  and  $\alpha$  frontier estimation. There are production plans which were efficient under FDH estimation and have  $m$  and  $\alpha$  estimated efficiencies of order 3 and 4, respectively. The occurrence of such production plans emphasizes the importance of using robust estimation methods to evaluate the efficiency of plans in the interior of the technology.

The robustness of  $\alpha$  and  $m$  frontier estimators is also useful in unveiling banks which have severe efficiency problems. If the observations used as references for the evaluation of the efficiency score of a particular bank are extreme values, the efficiency score estimated based on DEA and FDH will be impacted downwardly.

The ranking correlation of efficiency scores from  $\alpha$  and  $m$  estimators is 0.9, and the same correlation between  $m$  and FDH estimators is 0.8. Clearly, as  $\alpha \rightarrow 1$  and as  $m \rightarrow \infty$  the correlation approaches 1. Figure 4 also provides plots of estimated efficiency scores based on FDH,  $m$  and  $\alpha$  estimators of various orders. The lower right hand panel in Figure 4 shows that the correlation between the  $m = 300$  and  $m = 150$  estimates is very high, indicating the robustness of the efficiency rankings for a wide range of  $m$ . Also, there is strong correlation (middle right hand panel) between FDH scores and those from the  $m$ -estimator with  $m = 1500$ . As observed by Aragon et al. (2005), Figure 4 also shows greater robustness of  $\alpha$  estimators relative to  $m$  estimators. Even when  $\alpha = 0.99$  there are a number of observations where the estimated efficiency scores is 1 for FDH but much larger than 1 for the  $\alpha$ -estimator (lower left hand panel). If these observations are extreme values, the estimated efficiencies obtained for the remaining firms based on the  $\alpha$ -estimator are more robust than those produced based on FDH and  $m$  estimators. Lastly,  $\alpha$  and  $m$  estimators impose no restriction on the shape of the technology. As such, these estimators can be used as a check on the results obtained via DEA and FDH, and in particular can be used to assess the convexity assumption imposed by DEA.

**Analysis of Brazilian Banks.** For the purpose of investigating how our estimated efficiencies can inform public policy for the Brazilian banking sector, we classified all banks into two broad categories: size (large, medium, small and micro)<sup>4</sup> and capital ownership (domestic public, domestic private and foreign).

<sup>4</sup>In Brazil, there is no universally adopted classification of banks according to size. Here, we adopt the following classification: a) large - banks with assets greater than R\$ 25 billion reais; b) medium - banks with assets between R\$ 5 and R\$ 25 billion reais; c) small - banks with assets between R\$ 1 and R\$ 5 billion reais; d) micro - banks with assets less than R\$ 1 billion reais. For the period of our analysis the annual average exchange rate for US\$ 1 oscillated between R\$2.37 and R\$3.11.



The entries in Table 6, obtained from a model that uses a single output, show that when using DEA and FDH estimators the percentage of efficient units among domestic public banks is significantly smaller than that associated with domestic private and foreign banks. This is also verified on Table 7 where average estimated efficiency is smallest for the group of domestic public banks. The pattern of variation of estimated efficiency as a function of bank size, shows that large and micro banks show a larger proportion of observations along the FDH frontier. It is worth mentioning that in the case of large banks, a considerable percentage of observations corresponds to efficient banks, but non-dominant. The average estimated efficiencies obtained using DEA and FDH are closer for large banks than for small or micro banks. These estimates indicate that the DEA frontier is located significantly above that obtained from FDH for small banks. However, they become much closer for the large bank segment of the data. Given the less restrictive nature of the FDH frontier, these results point to a non-convex technology for the Brazilian banking sector.<sup>5</sup>

To assess the robustness of our analysis based on DEA and FDH, we analyze estimated efficiency patterns based on order- $m$  and  $\alpha$ -quantile estimators. The numbers in Table 7 show that for small values of  $m$  and  $\alpha$ , the average efficiency estimates are closer to results obtained under FDH estimation. For example, for  $m = 150$ , the ratio of average estimated efficiency for domestic private banks and domestic public banks is 1.14 (0.98/0.86). whereas this ratio is 1.34 (0.87/0.65) under FDH. However, as the group of publicly owned banks exhibits a much larger percentage of units below the frontier (63%) as compared to domestic banks that are privately owned (10%), this result can be indicative that average estimate efficiencies are being influenced by extreme values. As such, we show on Table 8 average estimated efficiency based on order- $m$  and  $\alpha$ -quantile estimators that exclude extreme values. The figures we obtain support previous results that indicate that foreign and privately owned domestic banks are more efficient than those publicly owned. We note that the relative poor performance of publicly owned banks are due in large part to the influence of regional and local state banks.

When we assess average estimated efficiency based on bank size, the estimates show an even larger discrepancy between large and small banks than we had observed under DEA and FDH. These results are highly influenced by the efficiency scores associated with large retail banks. However, even when such banks are excluded, we continue to observe a significant positive impact of size on efficiency scores.

Our results seem to indicate that the process of privatization of publicly owned state banks that has been implemented by the Brazilian Federal government, and the process of mergers and acquisitions that has characterized the recent history of the Brazilian banking sector may have contributed to an increase in the banking sector efficiency. However, we should emphasize that the desirability of such privatization and consolidation process should be measured taking a broader view that includes its impact on market concentration, prices and quality of banking services.

## 4 Conclusion

We have computed and compared various nonparametric estimators for technical efficiency using data on Brazilian banks. Our analysis demonstrates that when undergoing an analysis of sectoral efficiency, conclusions may vary significantly depending on the estimation method used for efficiency scores. Our analysis suggests that different estimators should be used as complements, not as substitutes. The possibility of identifying a group of firms that is inefficient by various methods of estimation, as well as a group of firms that can serve as a reference to the sector may be the more robust empirical result that emerges from a study of efficiency. The assumption of convexity (inherent in DEA) had a significant impact on performance. Much inefficiency suggested by DEA estimation may be the result of such assumption. The use of  $\alpha$  and  $m$  frontiers helped identify a number of production plans that could be viewed as extreme cases. These production plans have a significant impact on the estimated efficiency scores produced by DEA and FDH. Efficiency scores for firms that are dominated by these extreme values are downwardly estimated producing efficiency scores that may be distorted.

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<sup>5</sup>The results obtained when we estimate efficiencies based on DEAC and FDHC lead to a qualitatively similar analysis. For this reason the specific results are omitted and available from the authors upon request.

## Appendix - Tables and Figures

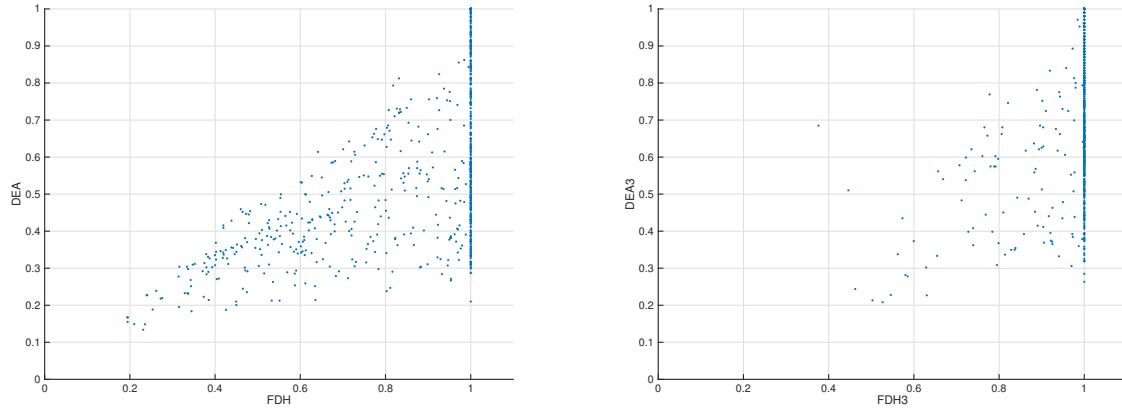


Figure 1: DEA and FDH Efficiency Estimates - Left panel: one output technology; Right panel: three output technology

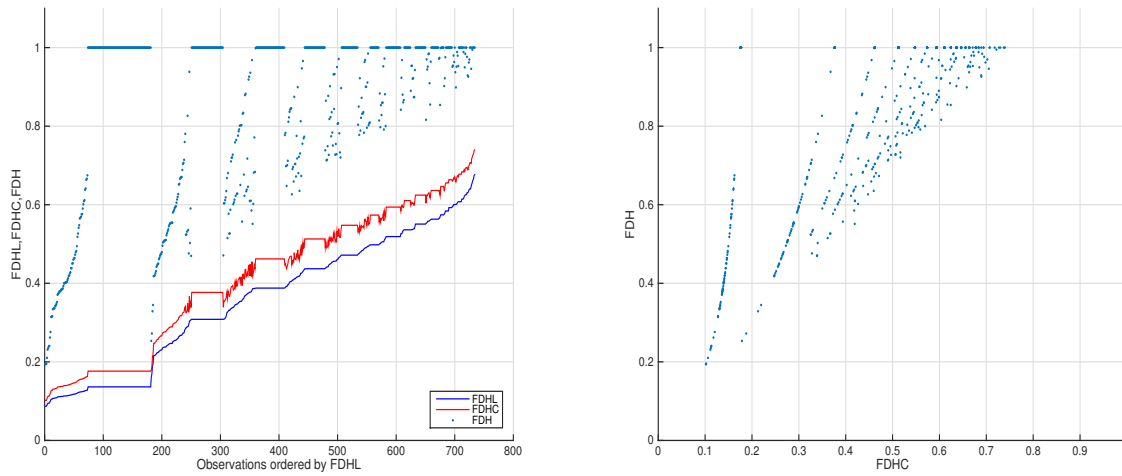


Figure 2: Left panel: FDHL, FDHC, FDH; Right panel: FDH and FDHC Efficiency Estimates

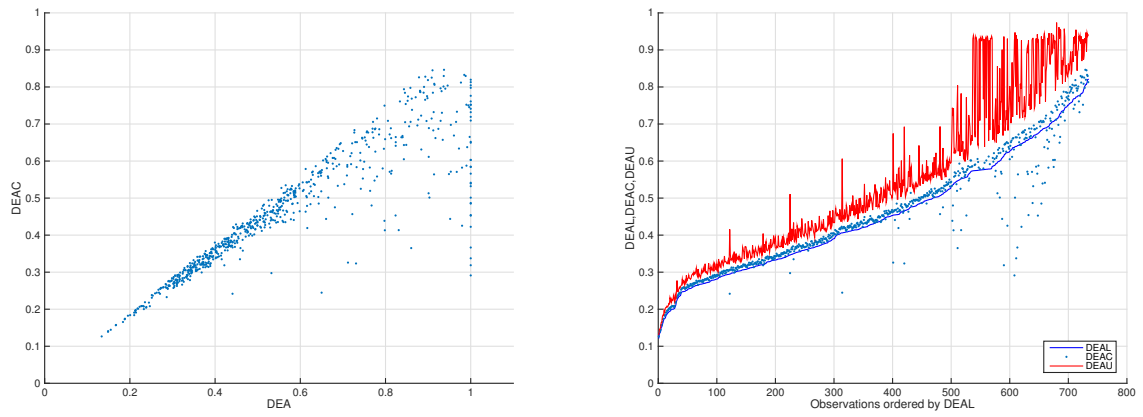


Figure 3: Left panel: DEAC and DEA Efficiency Estimates; Right panel: DEAL, DEAC, DEAU

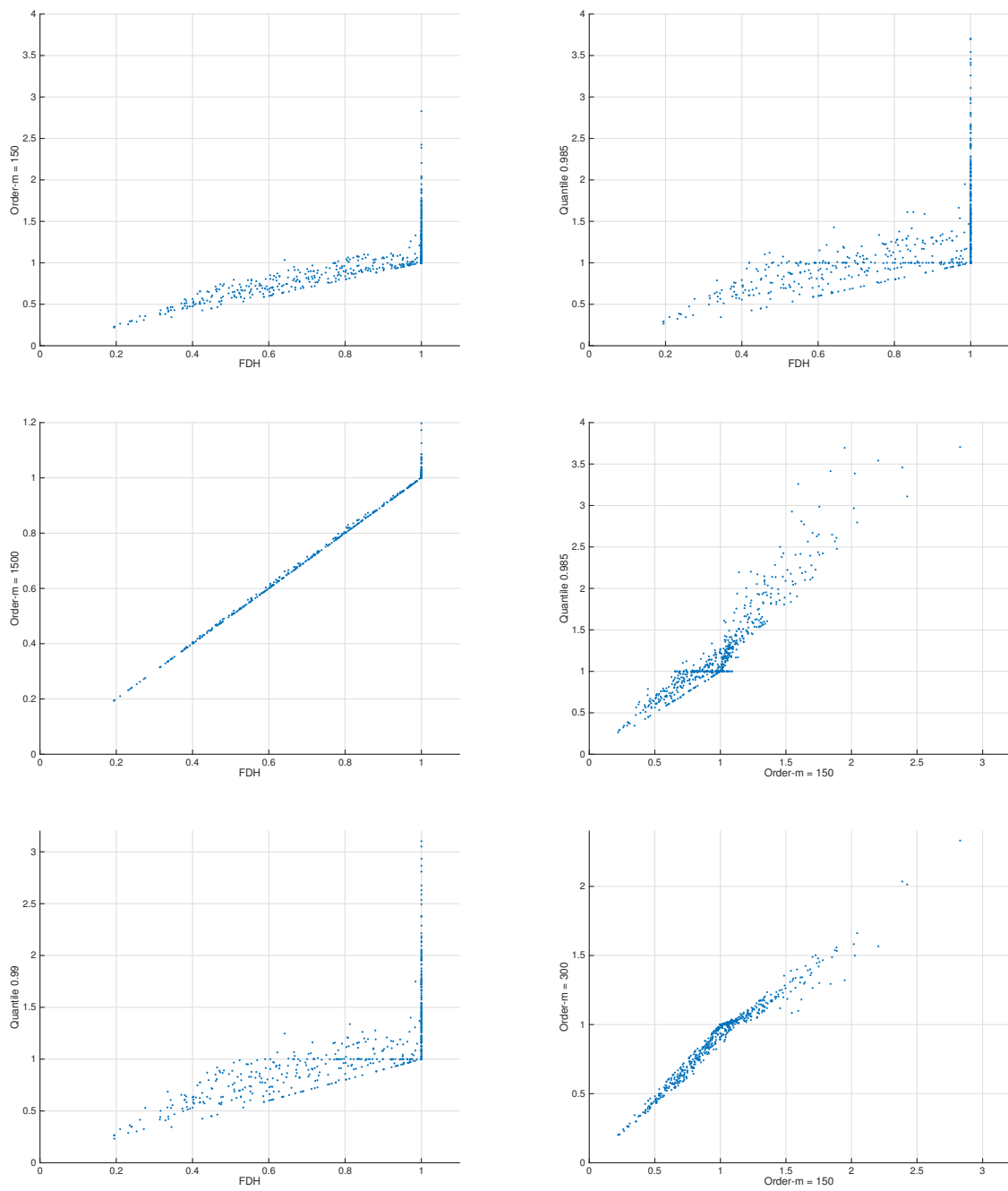


Figure 4: Efficiency Estimates: FDH, Order  $m$  and Quantile Estimators

Table 1: Number ( $n$ ) and Percentage (%) of Efficient, Inefficient, Dominant and Non-dominant Firms

Variable	FDH	%	DEA	%	FDH3	%	DEA3	%
Efficient Firms	390	53	57	7.77	628	85.56	113	15.40
Inefficient Firms	344	47	677	92.23	106	14.44	621	84.60
Dominant Firms	241	32.83	68		68	9.26		
Non-dominant Firms	120	16.35	531		531	72.34		
Total number of Firms	734	100	734	100	734	100	734	100

Note: FDH3 e DEA3 refer to the results of model with 3 outputs.

Table 2: Summary Statistics for Estimated Efficiencies - FDH, DEA, FDHC and DEAC

Statistic	DEA	FDH	DEA3	FDH3	FDHC	FDHL	DEAL	DEAC	DEAU
Minimum	0.134	0.194	0.208	0.376	0.101	0.087	0.122	0.126	0.131
Median	0.516	1	0.709	1	0.455	0.395	0.420	0.430	0.459
Mean	0.565	0.847	0.722	0.976	0.411	0.355	0.451	0.453	0.529
Maximum	1	1	1	1	0.739	0.676	0.854	0.804	0.918

Note: DEAL and DEAU refer to the lower and upper limits of the confidence interval.

Table 3: Spearman's Correlation

Estimator	DEA	DEAC	FDH	FDHC	Order- $m$	$\alpha$ -Quantile
DEA	1	0.93	0.59	0.14	0.74	0.73
DEAC		1	0.54	0.32	0.78	0.79
FDH			1	0.36	0.82	0.63
FDHC				1	0.45	0.42
Order- $m$					1	0.90
$\alpha$ -Quantile						1

Table 4: FDHC Efficiency Score for FDH Efficient Observations

Efficiency level under FDHC	0.176	0.376	0.462	0.512	0.547	0.594
Number of observations	108	54	49	35	28	25

Table 5: Summary Statistics for Estimated Efficiencies - Order- $m$  and  $\alpha$ -Quantile

Statistic	Order- $m$				$\alpha$ -Quantile			
	$m=75$	$m=150$	$m=300$	$m=1500$	$\alpha=0.98$	$\alpha=0.985$	$\alpha=0.99$	$\alpha=0.995$
Minimum	0.26	0.21	0.20	0.19	0.29	0.27	0.23	0.21
1st Quartile	0.92	0.82	0.76	0.71	1	0.99	0.91	0.79
Median	1.02	1	1	1	1.04	1	1	1
Mean	1.12	0.98	0.91	0.85	1.27	1.18	1.07	0.95
3rd Quartile	1.25	1.07	1.01	1	1.42	1.29	1.12	1
Maximum	3.24	2.82	2.33	1.19	4.04	3.71	3.42	3.14
% of Observations ( $\theta = 1$ )	0.12	0.15	0.20	0.37	0.25	0.31	0.35	0.42
% of Observations ( $\theta > 1$ )	0.55	0.44	0.34	0.15	0.54	0.44	0.32	0.16

Table 6: Number ( $n$ ) and Percentage (%) of Efficient, Inefficient, Dominant and Non-dominant Firms

	Total	FDH						DEA			
		Efficient		Dominant		Non-Dominant		Inefficient		Efficient	
Ownership		n	%	n	%	n	%	n	%	n	%
Domestic Public	109	23	21	5	14.29	18	85.71	86	79	3	2.75
Domestic Private	378	220	58	148	67.28	72	32.72	158	42	29	7.67
Foreign	247	147	60	117	79.60	30	20.40	100	40	25	10.12
Size	Obs										
Large	84	54	64.28	19	35.19	35	64.81	30	35.72	9	10.71
Medium	153	75	49	55	73.33	20	26.67	78	51	21	13.72
Small	234	88	37.60	68	77.28	20	22.72	146	62.40	6	2.56
Micro	263	173	65.77	128	74	45	26	90	34.23	21	7.98

Table 7: Average Estimated Efficiencies under DEA, FDH, Order- $m$  and  $\alpha$ -Quantile Estimation by Size and Ownership Structure

Ownership	Order- $m$			$\alpha$ -Quantile				FDH	DEA
	$m=150$	$m=300$	$m=1500$	$\alpha=0.98$	$\alpha=0.985$	$\alpha=0.99$	$\alpha=0.995$		
Domestic Public	0.86	0.77	0.66	1.17	1.08	0.97	0.86	0.65	0.45
Domestic Private	0.98	0.92	0.87	1.21	1.13	1.04	0.95	0.87	0.55
Foreign	1.05	0.95	0.89	1.40	1.30	1.15	1.00	0.89	0.62
Size									
Large	1.38	1.19	0.96	2.01	1.81	1.58	1.37	0.94	0.83
Medium	1.08	0.95	0.84	1.58	1.43	1.24	1.02	0.83	0.69
Small	0.89	0.82	0.79	1.15	1.07	0.96	0.84	0.79	0.49
Micro	0.90	0.89	0.88	0.97	0.95	0.91	0.89	0.881	0.47

Note: Nonparametric Kruskal-Wallis tests indicate statistically significant differences in average estimated efficiencies among size and ownership bank groups. A linear regression, where estimated efficiency scores serve as regressand and dummy variables for ownership structure, size, whether or not the bank operates in retail and credit markets, serve as regressors indicates that publicly owned banks are inefficient relative to privately owned ones, but also significant differences in efficiency between domestic privately owned banks and foreign banks. As noted by ? results from this “two-step” regression analysis should be interpret cautiously due to inherent mis-specification of such models. The results from these tests are available from the authors upon request.

Table 8: Order- $m$  and  $\alpha$ -Quantile Average Efficiency Estimates

	$m=150$	$\alpha=0.985$
Ownership		
Domestic Public	0.71	0.86
Domestic Private	0.94	1.06
Foreign	1.02	1.23
Size		
Large	1.19	1.52
Medium	1.00	1.28
Small	0.88	1.04
Micro	0.90	0.95

Note: All observations that have order- $m = 150$  estimated efficiencies greater than 1.5 have been excluded.

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