Toward a general solution of the three-wave resonant interaction equations

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Toward a general solution of the three-wave resonant interaction equations

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The resonant interaction of three wavetrains is the simplest form of nonlinear interaction for dispersive waves of small amplitude. Such interactions arise frequently in applications ranging from nonlinear optics to internal waves in the ocean through the study of the weakly nonlinear limit of a dispersive system. The slowly varying amplitudes of the three waves satisfy a set of integrable nonlinear partial differential equations known as the three-wave equations. If we consider the special case of spatially uniform solutions, then we obtain the three-wave ODEs. The ODEs have been studied extensively, and their general solution is known in terms of elliptic functions. Conversely, the universally occurring PDEs have been solved in only a limited number of configurations. For example, Zakharov and Manakov (1973, 1976) and Kaup (1976) used inverse scattering to solve the three-wave equations in one spatial dimension on the real line. Similarly, solutions in two or three spatial dimensions on the whole space were worked out by Zakharov (1976), Kaup (1980), and others. These known methods of analytic solution fail in the case of periodic boundary conditions, although numerical simulations of the problem typically impose these conditions.

To find the general solution of an nth order system of ordinary differential equations, it is sufficient to find a function that satisfies the ODEs and has n constants of integration. The general solution of a PDE, however, is not well defined and is usually difficult, if not impossible, to attain. In fact, only a small number of PDEs have known general solutions. We seek a general solution of the three-wave equations, which has the advantage of being compatible with a wide variety of boundary conditions and any number of spatial dimensions. Our work indicates that the general solution of the three-wave equations can be constructed using the known general solution of the three-wave ODEs. In particular, we try to construct the general solution of the three-wave equations using a Painlevé-type analysis. For now, we consider a convergent Laurent series solution

(in time), which contains two real free constants and three real-valued functions (in space) that are arbitrary except for some differentiability constraints. In order to develop a full general solution of the problem, the two free constants must also be allowed to have spatial dependence, and one more function must be introduced. That is, a full general solution of the problem would involve six of these real-valued functions.

For my dad, who now knows more about three-wave resonance than he probably ever cared to.

And for my mum, who had to hear us talk about it.

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Chapter 1

Introduction

The resonant interaction of three wavetrains is one of the simplest forms of nonlinear interaction for dispersive waves of small amplitude. This behavior arises frequently in applications ranging from nonlinear optics to internal waves through the study of the weakly nonlinear limit of a dispersive system. The slowly varying amplitudes of the three waves satisfy a set of coupled, nonlinear partial differential equations known as the three-wave equations. We seek a general solution of the three-wave equations.

The three-wave equations describe the physical phenomenon of three-wave mixing, in which three wavetrains whose wavenumbers and frequencies satisfy a particular resonance condition interact nonlinearly. These resonant interactions can cause substantial energy transfer amongst wavetrains, and can have a significant impact on the evolution of the wavefield [20]. Since three-wave interactions are the simplest form of nonlinear wave interactions, it follows that these interactions have important consequences in almost all areas of physics where nonlinear wave phenomena can occur [33]. We restrict our attention to a single resonant triad, which means there are only three interacting waves. It is also possible to have multiple triads, but this is beyond the scope of this thesis.

The three-wave equations admit a great deal of special structure. For instance, they constitute an infinite dimensional Hamiltonian system that is completely integrable. Under certain conditions, a so-called explosive instability can occur in the model, which causes almost all nonzero solutions to blow up in finite time. The presence of this instability was originally discovered while modeling a problem in plasma physics [11]. The instability has also been found in a variety of other applications, including density-stratified shear flows [9], vorticity waves [39], and capillary-gravity waves [14]. For much of this thesis, we focus on the three-wave equations in the configuration that admits the explosive instability, although some attention is given to the case of bounded solutions.

If we consider the special case of spatially uniform solutions, then we obtain the three-wave ordinary differential equations. The ODEs have been studied extensively, and their general solution is known in terms of elliptic functions [5,30]. In this case, solutions of the ODEs can exchange energy periodically [20]. Conversely, the universally occurring PDEs have been solved in only a limited number of configurations, which are outlined in [14]. In particular, the equations have been solved for certain types of initial data using the inverse scattering transform. For example, Zakharov and Manakov (1973, 1976) and Kaup (1976) used inverse scattering to solve the three-wave equations in time and one spatial dimension on the real line, or in two spatial dimensions alone. Similarly, solutions in time and two spatial dimensions, or three spatial dimensions alone, on the whole space were worked out by Zakharov (1976), Kaup (1980), and Cornille (1979). These known methods of analytic solution fail in the case of periodic boundary conditions, although numerical simulations of the problem typically impose these conditions. The solution of the three-wave PDEs that we derive is compatible with any number of spatial dimensions and many types of boundary conditions.

For an nth order system of ordinary differential equations, finding the general solution amounts to finding a solution that satisfies the ODEs and contains n constants of integration [23]. However, it is significantly more difficult to find the general solution of a PDE, linear or nonlinear. Indeed, the list of PDEs whose general solutions are known is short. Common examples include d'Alembert's solution of the one-dimensional linear wave equation and the general solution of Liouville's equation,[1] which is obtained through Bäcklund transformations [24, §8.3]. Our motivation for studying the three-wave problem is two-fold. First, these equations are universal in the sense that they arise in countless physical applications. It is therefore useful if we can solve them in new

[1] Liouville's equation takes the form $\frac{\partial^2 u}{\partial x \partial y} = e^u$, and is solved by transforming it into the wave equation, $\frac{\partial^2 \phi}{\partial x \partial y} = 0$ [24, §8.3].

configurations. Second, and perhaps more importantly, the ability to add to the short list of PDEs with known general solutions could have significant implications.

The structure of this thesis is as follows. In Chapter 2, we provide extensive background on the structure of the three-wave equations. In particular, we detail the Hamiltonian structure and the integrability of the equations, both for the PDEs, and for the ODEs in the case of spatially uniform waves. We also explain what conditions lead to the occurrence of the explosive instability.

In Chapters 3 and 4, we focus on the three-wave ODEs. Specifically, we derive the general solution of the three-wave ODEs in two separate ways, both of which motivate the solution form that we pose later for the three-wave PDEs. In Chapter 3, we solve the three-wave ODEs by transforming the existing Hamiltonian system with three degrees of freedom into an equivalent system with only one degree of freedom. We then obtain the solution using Weierstrass elliptic functions. The well known structure of the Weierstrass functions allows us to determine which initial data lead to the explosive instability, and which lead to bounded solutions. In Chapter 4, we derive the general solution of the three-wave ODEs again, this time using the same method that we later extend to the three-wave PDEs. In particular, we use a Painlevé-type analysis, which allows us to construct the general solution of the ODEs in terms of a convergent Laurent series in time. We show that the solutions in Chapters 3 and 4 are equivalent. In either case, the solution involves six free real constants.

The main result of the thesis is in Chapter 5, where we use our knowledge of the three-wave ODEs in order to construct a "near-general" solution of the three-wave PDEs. Specifically, we use a Painlevé-type analysis once more in order to build the solution in terms of a convergent Laurent series in time, with coefficients that have spatial dependence. The solution we construct involves five free real functions in space and one free real constant. Primarily, however, we consider the case in which two of the free functions are constants, so that we are left with three free functions and three free constants. A full general solution of the problem would involve six free real-valued functions, which is why we refer to our solution as "near-general". In the latter half of Chapter 5, we derive the radius of convergence of the series solution of the three-wave PDEs in several cases, and show that the radius of convergence of the three-wave PDEs is smaller than that of the three-wave ODEs by a known factor.

Finally, in Chapter 6, we provide some numerical verification of our solution of the three-wave equations by comparing our solution to other known cases. In particular, we show that we can truncate the formal Laurent series solution of the three-wave PDEs after a small number of terms and still capture the behavior of the exact solution.

Chapter 2

Background for the Three-Wave Equations

This chapter begins with a brief explanation of which types of physical systems can lead to the three-wave partial differential equations. Following this, we derive the three-wave PDEs using the particular example of a two-layer water wave model, with the understanding that the equations can arise in many other contexts. We end the chapter with a discussion of some of the most significant properties of the three-wave equations, including their Hamiltonian structure and integrability, as well as the presence of an explosive instability.

2.1 Derivation

In order for the three-wave equations to arise, we must consider the time evolution of a physical system, the governing equations for which must be nonlinear. In addition, we require the following:

i. The system must be free of dissipation. That is, either there is no dissipation physically, or the effects of dissipation are so small that we can assume they are negligible.

ii. The system, linearized about the zero solution, must admit traveling wave solutions of the form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t)}$, where $\mathbf{k}$ is the wavenumber vector, ω is the wave frequency, and $\omega = \omega(\mathbf{k})$ is the linearized dispersion relation.

iii. The linearized system must be dispersive, so that $\omega(\mathbf{k})$ satisfies

$$\det\left(\frac{\partial^2\omega}{\partial k_m \partial k_n}\right) \neq 0,$$

where k_m and k_n are elements of the wavenumber vector $\mathbf{k}$. In this case, waves with different wave numbers $\mathbf{k}$ travel at different speeds [43, §11.1].

iv. The dispersion relation $\omega(\mathbf{k})$ admits three or more pairs $(\mathbf{k}, \omega(\mathbf{k}))$ such that

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 = \mathbf{0} \quad \text{and} \quad \omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \pm \omega(\mathbf{k}_3) = 0. \tag{2.1}$$

Any solution of (2.1) represents a resonant triad.

Assume i-iv hold. Let the nonlinear governing equations be described by

$$\mathcal{N}(\mathbf{u}) = 0, \tag{2.2}$$

where $\mathcal{N}$ can be a system of equations or a single equation, and $\mathbf{u}$ can be a scalar or vector quantity. Additionally, we assume that $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is a function of space $\mathbf{x}$ and time t, and that the rest state of the solution is represented by $\mathbf{u} = 0$. Assuming (2.2) cannot be solved exactly, we begin by linearizing about $\mathbf{u} = 0$. To that end, we introduce the small parameter ε and assume that $\mathbf{u}$ can be expanded in terms of a formal power series in ε. We substitute the expansion into (2.2) and collect terms that are linear in ε. If the resulting system has terms that have constant coefficients in $\mathbf{x}$ and τ, then $\mathbf{u}$ can be expanded as

$$\mathbf{u}(\mathbf{x}, t; \varepsilon) = \varepsilon \sum_{\mathbf{k}} U(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)} + \mathcal{O}\left(\varepsilon^2\right). \tag{2.3}$$

At $\mathcal{O}(\varepsilon)$, we obtain the linearized problem, which admits nontrivial solutions as long as the wave frequency ω and the wavenumber vector $\mathbf{k}$ satisfy the linearized dispersion relation, $\omega(\mathbf{k})$. At second order, if (2.1) is satisfied, then secular terms arise, so that the expansion (2.3) breaks down when $\varepsilon t = \mathcal{O}(1)$ or $\varepsilon \mathbf{x} = \mathcal{O}(1)$. This motivates us to introduce slow time and space scales. Then it can be shown that (2.3) can be superseded by

$$\mathbf{u}(\mathbf{x}, t; \varepsilon) = \varepsilon \sum_{m=1}^{3} A_m(\varepsilon \mathbf{x}, \varepsilon t) e^{i(\mathbf{k_m}\cdot\mathbf{x} - \omega(\mathbf{k_m})t)} + \mathcal{O}\left(\varepsilon^2\right).$$

[38]. In this case, secular terms arise at $\mathcal{O}(\varepsilon^2)$ unless the complex amplitudes $A_m(\varepsilon\mathbf{x}, \varepsilon t)$ satisfy

$$\frac{\partial A_1}{\partial \tau} + \mathbf{c}_1 \cdot \nabla A_1 = i\gamma_1 A_2^* A_3^*, \tag{2.4a}$$

$$\frac{\partial A_2}{\partial \tau} + \mathbf{c}_2 \cdot \nabla A_2 = i\gamma_2 A_1^* A_3^*, \tag{2.4b}$$

$$\frac{\partial A_3}{\partial \tau} + \mathbf{c}_3 \cdot \nabla A_3 = i\gamma_3 A_1^* A_2^*, \tag{2.4c}$$

where $\tau = \varepsilon t$, $\mathbf{X} = \varepsilon\mathbf{x}$, $\nabla = (\partial/\partial X_1, \ldots, \partial/\partial X_n)$, and for $m = 1, 2, 3$, A_m^* is the complex conjugate of A_m, γ_m is a real nonzero constant determined by the physical problem, and $\mathbf{c}_m$ is the group velocity corresponding to wavenumber $\mathbf{k}_m$, defined by

$$\mathbf{c}_m = \nabla\omega(\mathbf{k})\,|_{\mathbf{k}=\mathbf{k}_m},$$

Equations (2.4a)-(2.4c) are known as the three-wave equations.

For convenience, we introduce the change of variables

$$A_m = -\frac{i a_m}{\sqrt{|\gamma_k \gamma_\ell|}},$$

where $(k, \ell, m) = (1, 2, 3)$ cyclically. In this case, (2.4a)-(2.4c) can be written more succinctly as

$$\frac{\partial a_m}{\partial \tau} + \mathbf{c}_m \cdot \nabla a_m = \sigma_m a_k^* a_\ell^*, \tag{2.5}$$

where $\sigma_m = \operatorname{sign}(\gamma_m)$.

2.1.1 A two-layer water wave model

We now briefly explain how the three-wave equations in (2.4) can arise for a specific physical example. In particular, we consider internal waves in a two-layer fluid, where each layer is homogeneous. In general, internal waves can occur in a stably stratified fluid under the effects of gravity. For instance, usually both the atmosphere and the ocean are stably stratified, and can support many types of internal waves. Indeed, even water waves that occur at the interface between the ocean and the atmosphere can be considered an extreme case of internal waves, resulting from a large density gradient between the air and the water [2, §4.1.b].

A resonant triad involving a combination of surface waves and internal waves is one possible mechanism through which internal waves are produced [2, §4.2.b]. In our case, we consider the irrotational motion of two fluid layers, each containing an incompressible, inviscid fluid. We restrict our attention to the (x, z)-plane, with $-\infty < x < \infty$. Assume that the free surface is given by $z = \zeta(x, t)$, so that the undisturbed free surface lies at $z = 0$. Moreover, assume that the lower layer of fluid lies in a region defined by $-H \leq z \leq -h_1 + \eta(x, t)$, where $z = -h_1$ is the location of the undisturbed interface between the layers, and H is the total depth of the undisturbed layers. In fact, we define $H = h_1 + h_2$, so that h_1 is the undisturbed depth of the upper layer, and h_2 is the undisturbed depth of the lower layer. Additionally, ρ_1 and ρ_2 are the constant densities of the fluids in the upper and lower layers, respectively, with $\rho_1 < \rho_2$. See Figure 2.1 for a diagram of the two-layer fluid.

Let $\varphi_1(x, z, t)$ and $\varphi_2(x, z, t)$ be the velocity potentials in the upper and lower layers, respectively. We neglect surface tension at the free surface and at the interface. Additionally, we neglect the effects of the earth's rotation. This leads to the following system of equations

$$\text{In the upper layer:} \qquad \Delta\varphi_1 = 0, \tag{2.6a}$$

$$\text{In the lower layer:} \qquad \Delta\varphi_2 = 0, \tag{2.6b}$$

$$\text{On } z = \zeta(x,t): \qquad \frac{\partial \zeta}{\partial t} - \frac{\partial \varphi_1}{\partial z} = -\frac{\partial \varphi_1}{\partial x}\frac{\partial \zeta}{\partial x}, \tag{2.6c}$$

$$\frac{\partial \varphi_1}{\partial t} + g\zeta = -\frac{1}{2}\left|\nabla\varphi_1\right|^2, \tag{2.6d}$$

$$\text{On } z = \eta(x,t) - h_1: \qquad \frac{\partial \eta}{\partial t} - \frac{\partial \varphi_1}{\partial z} = -\frac{\partial \varphi_1}{\partial x}\frac{\partial \eta}{\partial x}, \tag{2.6e}$$

$$\frac{\partial \eta}{\partial t} - \frac{\partial \varphi_2}{\partial z} = -\frac{\partial \varphi_2}{\partial x}\frac{\partial \eta}{\partial x}, \tag{2.6f}$$

$$\rho_1\left(\frac{\partial \varphi_1}{\partial t} + g\eta\right) - \rho_2\left(\frac{\partial \varphi_2}{\partial t} + g\eta\right) = -\frac{\rho_1}{2}\left|\nabla\varphi_1\right|^2 + \frac{\rho_2}{2}\left|\nabla\varphi_2\right|^2, \tag{2.6g}$$

$$\text{On } z = -H: \qquad \frac{\partial \varphi_2}{\partial z} = 0, \tag{2.6h}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$, $\nabla = (\partial/\partial x, \partial/\partial z)$, and g is acceleration due to gravity. The system

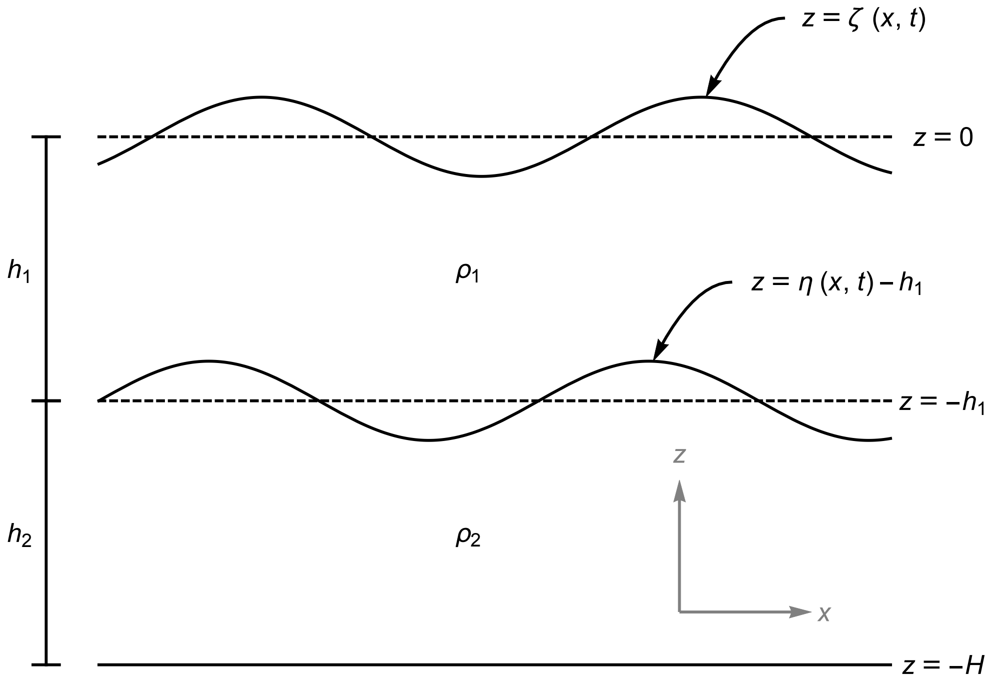


Figure 2.1: A depiction of the two-layer fluid model. The free surface is described by $z = \zeta(x,t)$, with the undisturbed free surface at $z = 0$. The undisturbed depth of the upper layer is h_1. The interface between the layers is described by $z = \eta(x,t) - h_1$, with the undisturbed interface at $z = -h_1$. The undisturbed depth of the lower layer is h_2, and the undisturbed total depth of both fluids is $H = h_1 + h_2$. The densities of the upper and lower layers are ρ_1 and ρ_2, respectively, with $\rho_1 < \rho_2$.

has the form $\mathcal{L}(\mathbf{u}) = \mathbf{v}$, where $\mathcal{L}$ is a linear operator, $\mathbf{u}(\mathbf{x}, t) = (\varphi_1, \varphi_2, \zeta, \eta)^T$, $\mathbf{x} = (x, z)$, and $\mathbf{v}$ includes all nonlinear terms on the right-hand side. In particular, we have

$$\mathcal{L} = \begin{pmatrix} M \\ G_1|_{z=\zeta} \\ G_2|_{z=\eta-h_1} \\ G_3|_{z=-H_1} \end{pmatrix},$$

where M is defined by

$$M = \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \end{pmatrix}, \tag{2.7}$$

and G_1, G_2, and G_3 are defined via

$$G_1 = \begin{pmatrix} -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial t} & 0 \\ \frac{\partial}{\partial t} & 0 & g & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} -\frac{\partial}{\partial z} & 0 & 0 & \frac{\partial}{\partial t} \\ 0 & -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial t} \\ \rho_1 \frac{\partial}{\partial t} & -\rho_2 \frac{\partial}{\partial t} & \rho_1 g & -\rho_2 g \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & \frac{\partial}{\partial z} & 0 & 0 \end{pmatrix}. \tag{2.8}$$

Note that if we rewrite (2.6) by collecting all terms on one side of the equation, in the form $\mathcal{L}(\mathbf{u}) - \mathbf{v} = 0$, then the system is of the form (2.2). Lastly, observe that (2.6c) and (2.6e)-(2.6f) are kinematic boundary conditions on the surface and the interface, respectively, while (2.6d) and (2.6g) are dynamic boundary conditions on the surface and the interface. Finally, (2.6h) enforces an impermeable bottom condition.

Next, following (2.3), we pose the following expansions

$$\varphi_1(x, z, t) = \varepsilon\varphi_{11}(x, z, t) + \varepsilon^2\varphi_{12}(x, z, t) + \mathcal{O}(\varepsilon^3), \tag{2.9}$$

$$\varphi_2(x, z, t) = \varepsilon\varphi_{21}(x, z, t) + \varepsilon^2\varphi_{22}(x, z, t) + \mathcal{O}(\varepsilon^3), \tag{2.10}$$

$$\eta(x, t) = \varepsilon\eta_1(x, t) + \varepsilon^2\eta_2(x, t) + \mathcal{O}(\varepsilon^3), \tag{2.11}$$

$$\zeta(x, t) = \varepsilon\zeta_1(x, t) + \varepsilon^2\zeta_2(x, t) + \mathcal{O}(\varepsilon^3), \tag{2.12}$$

or, more succinctly,

$$\mathbf{u}(x,z,t) = \varepsilon \mathbf{u}_1(x,z,t) + \varepsilon^2 \mathbf{u}_2(x,z,t) + \mathcal{O}(\varepsilon^3), \tag{2.13}$$

where $\varepsilon > 0$ is a small parameter. We substitute (2.13) into the system of equations (2.6) and solve the resulting equations order by order. Notice that we must take into account the boundary conditions at the free surface $z = \zeta(x,t)$ and at the interface $z = -h_1 + \eta(x,t)$. For instance, we expand $\varphi_{11}(x,z,t)$ at the free surface as

$$\begin{aligned}\varphi_{11}\left(x, z = \zeta(x,t), t\right) &= \varphi_{11}\left(x, \varepsilon\zeta_1 + \mathcal{O}(\varepsilon^2), t\right) \\ &= \varphi_{11}(x,0,t) + \varepsilon\zeta_1(x,t)\,\frac{\partial}{\partial z}\varphi_{11}(x,0,t) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

We expand $\varphi_{21}(x,z,t)$ similarly at $z = \zeta(x,t)$, and we use the same technique to expand both velocity potentials at the interface $z = \eta(x,t) - h_1$.

At $\mathcal{O}(\varepsilon)$, we obtain a linear system of equations of the form $\hat{\mathcal{L}}(\mathbf{u}_1) = 0$, where

$$\hat{\mathcal{L}} = \begin{pmatrix} M \\ G_1|_{z=0} \\ G_2|_{z=-h_1} \\ G_3|_{z=-H_1} \end{pmatrix}, \tag{2.14}$$

with M, G_1, G_2, and G_3 defined in (2.7)-(2.8). At $\mathcal{O}(\varepsilon^2)$, we obtain a system of equations of the form $\hat{\mathcal{L}}(\mathbf{u}_2) = F(\mathbf{u}_1)$, where $\hat{\mathcal{L}}$ is defined in (2.14) and F is a nonlinear operator.

Return to the first-order problem, $\hat{\mathcal{L}}(\mathbf{u}_1) = 0$, and assume that $\mathbf{u}_1(\mathbf{x},t)$ has an expansion of the form

$$\mathbf{u}_1(\mathbf{x},t) = \sum_{\mathbf{k}} U(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)},$$

where $\mathbf{k}$ is the wavenumber vector and ω is the wave frequency. Then nontrivial solutions of the first-order problem exist as long as ω and $\mathbf{k}$ satisfy the dispersion relation

$$\omega^4\left[1 + (1-D)T_1T_2\right] - \omega^2 gk\left[T_1 + T_2\right] + D(gk)^2 T_1T_2 = 0, \tag{2.15}$$

where

$$T_1 = \tanh kh_1, \quad T_2 = \tanh kh_2, \quad \text{and} \quad D = \frac{\rho_2 - \rho_1}{\rho_2}.$$

For a fixed k, there are four possible values of $\omega(k)$. In particular, for each k, there are two frequencies associated with internal waves at the interface, and two frequencies associated with surface waves. The dispersion relation in (2.15) is depicted by the solid black lines in Figure 2.2. The branches closest to the k-axis represent internal waves, while the branches further away from the k-axis represent surface waves.

A graphical technique exists to determine whether (2.15), or any other one-dimensional dispersion relation, admits resonant triad solutions that satisfy (2.1) [5]. First, any point A is chosen on the dispersion curve. The entire dispersion curve is then translated so that its new origin is at A. From here, we look for any intersection between the original dispersion curve and the translated curve. Any such intersection point represents a second wave that can form a resonant triad with A. In our case, let B be one such intersection point. Finally, we draw a vector from B that is equal to vector $\vec{AO}$, where O is the origin. This new vector ends at a point C that is, by construction, on the original dispersion curve. It follows that the waves described by points A, B, and C form a resonant triad and solve (2.1). The process can be seen in Figure 2.2 for the dispersion relation in (2.15). Moreover, we can see that the dispersion relation (2.15) admits more than one resonant triad. For example, we could have two surface waves and one internal wave, or two internal waves and one surface wave.

Figure 2.3 depicts more clearly the resonant triad of Figure 2.2. In particular, if point A has coordinates (k_1, ω_1) and point C has coordinates (k_2, ω_2), then by construction point B has coordinates $(k_1+k_2, \omega_1+\omega_2)$. If we define $k_3 = k_1+k_2$ and $\omega_3 = \omega_1+\omega_2$, then we have $k_1+k_2-k_3 = 0$ and $\omega_1 + \omega_2 - \omega_3 = 0$, both of which are of the form (2.1). Note that in this case, $\omega_1 > 0$ and $\omega_2, \omega_3 < 0$. Additionally, $k_m > 0$ for $m = 1, 2, 3$ in this case, so the three waves do not all travel in the same direction.

Now that we know (2.1) can be satisfied for our two-layer fluid problem, we restrict our attention to a single resonant triad. We then seek solutions to the linear problem of the form

$$\mathbf{u}_1(\mathbf{x}, t) = \sum_{m=1}^{3} \mathbf{A}_m(z) e^{i\theta_m} + c.c., \tag{2.16}$$

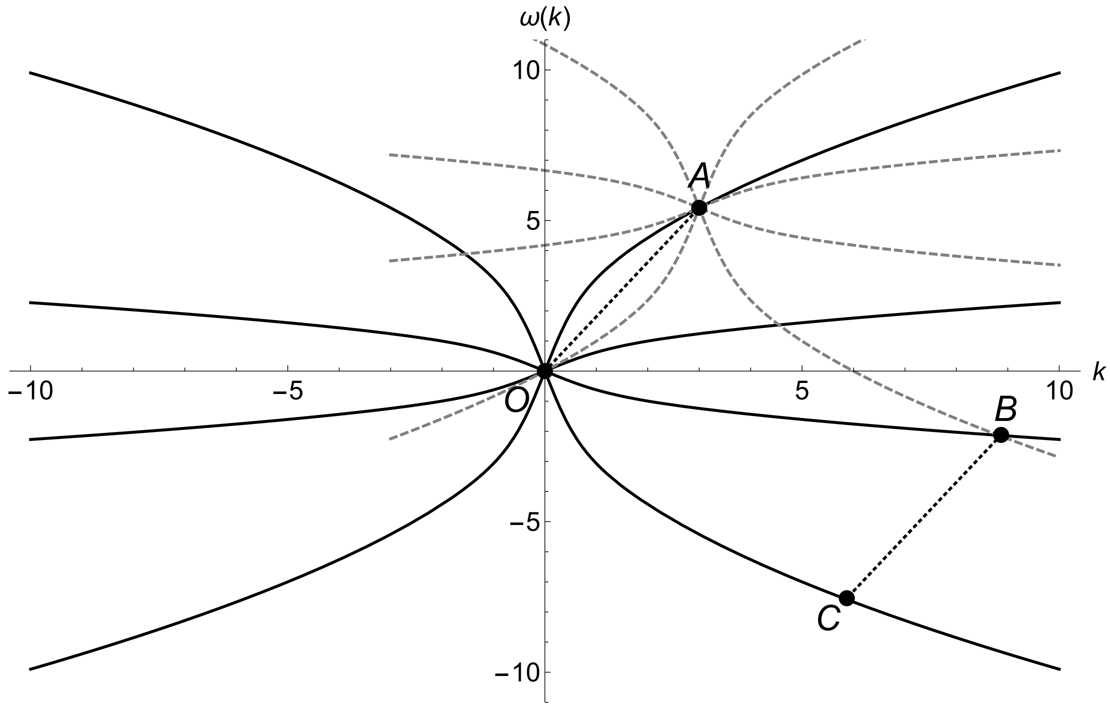


Figure 2.2: Construction of a resonant triad for the dispersion relation (2.15) for the two-layer fluid model described in (2.6). Figure generated with $h_1 = h_2 = 1$ and $\Delta = 0.01$. The process of constructing a resonant triad begins by choosing any point on the dispersion curve (the solid black curve), say point A. The entire dispersion curve is then translated to A (as depicted by the grey dashed curve). Any point of intersection between the original dispersion curve and the translated dispersion curve, like point B in the figure, represents a second wave that can form a resonant triad with A. Finally, we draw a vector parallel and equal to vector $\vec{AO}$ (the black dotted line). By construction, this vector meeets the dispersion curve at point C. The points A, B, and C represent solutions of (2.1).

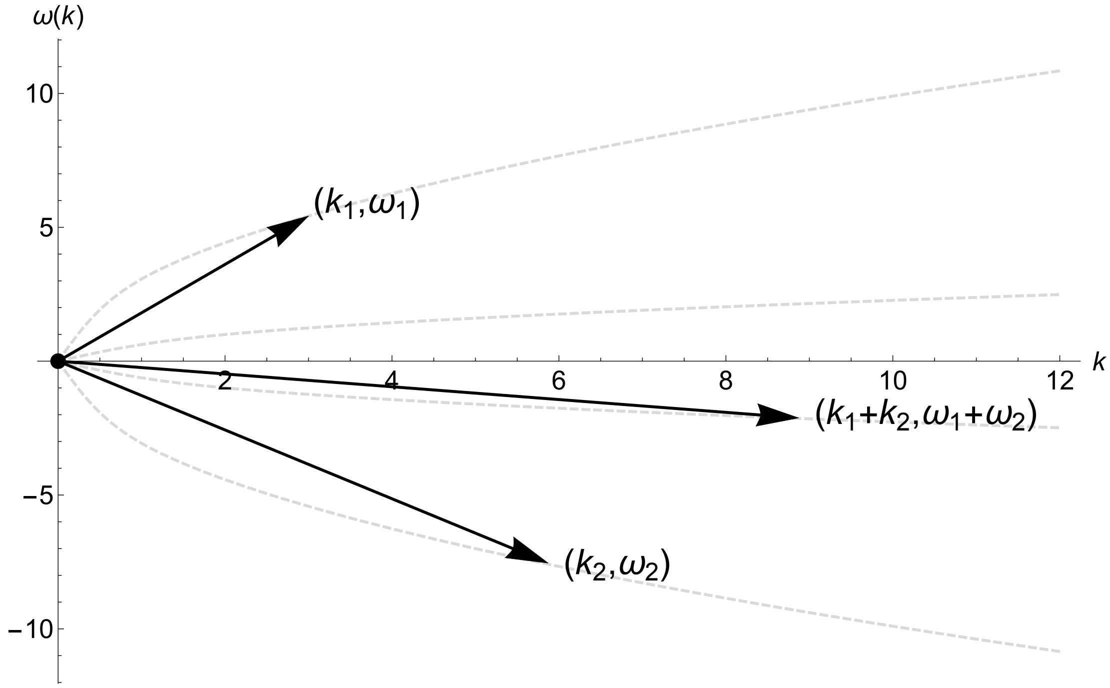


Figure 2.3: Resonant triad for the dispersion relation (2.15) for the two-layer fluid model described in (2.6). Figure generated with $h_1 = h_2 = 1$ and $\Delta = 0.01$.

where $c.c$ denotes the complex conjugate, $\mathbf{A}_m(z)$ is a 4×1 vector for each m whose last two elements are independent of z, and

$$\theta_m = k_m x - \omega_m t. \tag{2.17}$$

In this case, we defined $\omega_m = \omega(k_m)$. The solution of the linear problem is straightforward from here, although we omit the details since it is not the focus of this thesis.

The first nonlinear interactions appear at $\mathcal{O}(\varepsilon^2)$. Recall that at this order, we obtain the system $\hat{\mathcal{L}}\mathbf{u}_2 = F(\mathbf{u}_1)$, where $\hat{\mathcal{L}}$ is defined in (2.14) and $F(\mathbf{u}_1)$ is comprised of quadratic nonlinearities. In particular, due to (2.16), terms in $F(\mathbf{u}_1)$ have the form

$$e^{i(\theta_m - \theta_n)} + c.c., \quad \text{for } m = 1, 2, 3,$$

where n can vary between -3 and 3, and θ_m is defined in (2.17). For this notation, we also define $\theta_{-m} = -\theta_m$ and $\theta_0 = 0$.

Due to the presence of resonant triads defined in (2.1), it follows that $F(\mathbf{u}_1)$ produces terms that are in resonance with the linear left-hand side. For example, for the triad depicted in Figure 2.2, $\theta_1 + \theta_2 = \theta_3$. As a result, the system $\hat{\mathcal{L}}\mathbf{u}_2 = F(\mathbf{u}_1)$ acts like a resonantly forced, linear oscillator [20]. Consequently, $\mathbf{u}_2$ grows linearly in time. When $\varepsilon t = \mathcal{O}(1)$, the second term in the asymptotic expansion (2.13) has the same order of magnitude as the first term, and the underlying assumptions for our solution break down.

As an alternative, we introduce a slow time scale, $\tau = \varepsilon t$, and a slow spatial scale, $X = \varepsilon x$. Then we replace (2.13) with

$$\mathbf{u}(x, z, t, X, \tau) = \varepsilon \mathbf{u}_1(x, z, t, X, \tau) + \varepsilon^2 \mathbf{u}_2(x, z, t, X, \tau) + \mathcal{O}(\varepsilon^3), \tag{2.18}$$

With $\tau = \varepsilon t$ and $X = \varepsilon x$, we make the replacements $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X}$, and then we follow the same procedure as before, substituting (2.18) into (2.6) and solving the resulting equations order by order. At first order, we pose the solution expansions

$$\varphi_{11}(x, z, t, X, \tau) = \sum_{m=1}^{3} A_m(z, X, \tau) e^{i\theta_m} + c.c., \qquad \varphi_{21}(x, z, t, X, \tau) = \sum_{m=1}^{3} B_m(z, X, \tau) e^{i\theta_m} + c.c.$$
$$\eta_1(x, t, X, \tau) = \sum_{m=1}^{3} C_m(X, \tau) e^{i\theta_m} + c.c., \qquad \zeta_1(x, t, X, \tau) = \sum_{m=1}^{3} D_m(X, \tau) e^{i\theta_m} + c.c.$$

After a great deal of algebra, we find that in order to avoid secular terms, the following must hold

$$\frac{\partial C_1}{\partial \tau} + c_1 \frac{\partial C_1}{\partial X} = i\gamma_1 D_2^* D_3^*, \tag{2.19a}$$
$$\frac{\partial D_2}{\partial \tau} + c_2 \frac{\partial D_2}{\partial X} = i\gamma_2 C_1^* D_3^*, \tag{2.19b}$$
$$\frac{\partial D_3}{\partial \tau} + c_3 \frac{\partial D_3}{\partial X} = i\gamma_3 C_1^* D_2^*, \tag{2.19c}$$

where the interaction coefficients γ_m, $m = 1, 2, 3$, do not vanish identically and c_m is the group velocity associated with wavenumber k_m. We do not write down the form of the coefficients here because they are complicated and lengthy. Instead, we refer the reader to Appendix A in order to see the form of γ_1. The coefficients γ_2 and γ_3 are similar.

Notice that our resonant triad consists of two surface waves and an internal wave. A comparison of (2.19) with (2.4) shows that for the two-layer model, the slowly varying complex amplitudes of two surface waves and an internal wave satisfy the three-wave PDEs. Alternatively, the equations (2.19) could have been derived with two internal waves and one surface wave.

2.2 Properties

In this section, we investigate some of the properties of the three-wave ODEs and PDEs, focusing on their Hamiltonian structure, integrability, and the presence of an explosive instability. If we consider a resonant triad made up of three spatially uniform wavetrains, then (2.5) becomes

$$\frac{da_m}{d\tau} = \sigma_m a_k^* a_\ell^*, \tag{2.20}$$

where $a_m = a_m(\tau)$, $(k, \ell, m) = (1, 2, 3)$ cyclically, and we recall that $\sigma_m = \pm 1$. We refer to (2.20) as the three-wave ODEs.

2.2.1 Hamiltonian structure

A Hamiltonian is defined to be any C^1 function $H : M \to \mathbb{R}$, where M is a $2n$-dimensional manifold with coordinates denoted by

$$z = (q, p) = (q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n). \tag{2.21}$$

M is known as the phase space, while q and p are typically referred to as position and momentum variables, respectively. Furthermore, q, and p are each n-dimensional, and for this reason H is said to have n degrees of freedom. Hamiltonians are special for several reasons. A Hamiltonian defines motion on a $2n$-dimensional manifold. Moreover, Hamiltonians have many important properties, several of which we outline throughout the rest of this chapter [32, §9.3].

A Hamiltonian is associated with a system of ordinary differential equations defined by

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}(q, p), \tag{2.22a}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}(q, p), \tag{2.22b}$$

where the differentiation is performed componentwise. More explicitly, we have

$$\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}(q,p), \qquad i = 1,2,\ldots,n, \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}(q,p), \quad i = 1,2,\ldots,n. \end{aligned}$$

The system (2.22) is a Hamiltonian system of ODEs with n degrees of freedom. Finally, notice that we can write the system (2.22) more compactly as

$$\frac{dz}{d\tau} = J\nabla H, \quad \text{where} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2.23}$$

In this notation, I is the $n \times n$ identity matrix, 0 is the $n \times n$ zero matrix, z is a point in phase space defined in (2.21), and $\nabla = (\partial/\partial q_1, \ldots, \partial/\partial q_n, \partial/\partial p_1, \ldots, \partial/\partial p_n)^T$.

Next, let $(q(t), p(t))$ be an integral curve of the Hamiltonian system (2.22). We show next that if H is autonomous, then there is a constant E such that

$$H(q(t), p(t)) = E. \tag{2.24}$$

More succinctly, we say that energy is preserved along trajectories [7, 32]. Furthermore, we define a constant of the motion as a differentiable scalar function $F(q,p)$ with the property that for each integral curve $(q(t), p(t))$ in M, there exists a constant K such that

$$F(q(t), p(t)) = K.$$

It follows that if (2.24) holds, then the Hamiltonian H is a constant of the motion.

Now define the Poisson bracket of two scalar functions $F = F(q,p)$ and $G = G(q,p)$ as

$$\{F, G\} = \nabla F^T J \nabla G = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right). \tag{2.25}$$

In particular, notice that

$$\{q_m, q_n\} = 0, \quad \{p_m, p_n\} = 0, \quad \text{and} \quad \{q_m, p_n\} = \delta_{mn},$$

where δ_{mn} is the usual Kronecker delta.

Observe the following

$$\begin{aligned}\frac{d}{dt}F(q(t),p(t)) &= \sum_{j=1}^{n}\left(\frac{\partial F}{\partial q_j}\frac{dq_j}{dt}+\frac{\partial F}{\partial p_j}\frac{dp_j}{dt}\right)\\ &= \sum_{j=1}^{n}\left(\frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j}-\frac{\partial F}{\partial p_j}\frac{\partial H}{\partial q_j}\right)\\ &= \{F,H\},\end{aligned}$$

where we used (2.22) [7, §9.2]. As a result, a differentiable function $F(q,p)$ is a constant of the motion if

$$\{F,H\}=0.$$

Indeed, it is straightforward to see from the definition of the Poisson bracket that $\{H,H\}=0$, which verifies that H is a constant of the motion when H is independent of time.

We can interpret the constants of motion geometrically as follows. Each constant of the motion constrains the motion of the Hamiltonian system to lie on a level set in phase space. In particular, if F is a constant of the motion, then the associated level set is defined via

$$S_F^c=\{(q,p)\in M : F(q,p)=c\},$$

where c is in the range of F and S_F^c is a $(2n-1)$-dimensional submanifold. It follows that if a Hamiltonian system has k functionally independent constants of the motion, $F_1=H,F_2,\ldots,F_k$, then each integral curve $(q(t),p(t))$ must lie on the intersection

$$S_{F_1F_2\cdots F_k}^{c_1c_2\cdots c_k}\equiv S_{F_1}^{c_1}\cap\cdots\cap S_{F_k}^{c_k},$$

where c_j is in the range of F_j for $j=1,2,\ldots,k$. $S_{F_1F_2\cdots F_k}^{c_1c_2\cdots c_k}$ is a $(2n-k)$-dimensional submanifold, where $k\le n$. Note that functions $F_j(q,p)$ are called functionally independent at a point if the k gradient vectors $\nabla F_j(q,p)$, $j=1,2,\ldots,k$ are linearly independent. [7, §9.2].

We now return to the three-wave ODEs in (2.20). These equations constitute a Hamiltonian system with three degrees of freedom and with Hamiltonian

$$\tilde{H}=-iH=a_1a_2a_3-a_1^*a_2^*a_3^*, \tag{2.26}$$

where H is a real constant. Later, we return to the fact that $\tilde{H}$ in (2.26) is imaginary, as well as the proof that (2.20) is a Hamiltonian system. For now, we begin by deriving the constants of the motion.

Consider (2.20). Multiply both sides of the equation by a_m^* to obtain

$$a_m^* \frac{da_m}{d\tau} = \sigma_m a_k^* a_\ell^* a_m^*. \tag{2.27}$$

Similarly, multiply the conjugate of (2.20) by a_m to obtain

$$a_m \frac{da_m^*}{d\tau} = \sigma_m a_k a_\ell a_m. \tag{2.28}$$

Now add (2.27) and (2.28) to find

$$\frac{d}{d\tau} \left|a_m(\tau)\right|^2 = \sigma_m \left(a_k a_\ell a_m + a_k^* a_\ell^* a_m^*\right),$$

where we used the fact that $|a_m(\tau)|^2 = a_m a_m^*$. Dividing both sides of the equation by σ_m and using the fact that $\sigma_m = 1/\sigma_m$ (since $\sigma_m = \pm 1$), we have

$$\sigma_m \frac{d}{d\tau} \left|a_m(\tau)\right|^2 = a_k a_\ell a_m + a_k^* a_\ell^* a_m^*, \tag{2.29}$$

where $(k, l, m) = (1, 2, 3)$ cyclically. In particular, subtracting (2.29) with $m = 2$ and $m = 3$ from (2.29) with $m = 1$ yields

$$\frac{d}{d\tau}\left(\sigma_1 \left|a_1(\tau)\right|^2 - \sigma_2 \left|a_2(\tau)\right|^2\right) = 0,$$
$$\frac{d}{d\tau}\left(\sigma_1 \left|a_1(\tau)\right|^2 - \sigma_3 \left|a_3(\tau)\right|^2\right) = 0.$$

That is, the quantities in parentheses above must be constant for all time. As a result, we define the conserved quantities

$$K_2 = \sigma_1 \left|a_1(\tau)\right|^2 - \sigma_2 \left|a_2(\tau)\right|^2, \tag{2.30}$$

$$K_3 = \sigma_1 \left|a_1(\tau)\right|^2 - \sigma_3 \left|a_3(\tau)\right|^2, \tag{2.31}$$

where K_2 and K_3 are real constants. The expressions (2.30)-(2.31) are referred to as the Manley-Rowe relations, named after the authors of [28].

Next, return to (2.20), but this time multiply both sides of the equation by $a_k a_\ell$, and multiply both sides of the conjugate equation by $a_k^* a_\ell^*$. Now we have

$$a_k a_\ell \frac{da_m}{d\tau} = \sigma_m \left| a_k a_\ell \right|^2,$$
$$a_k^* a_\ell^* \frac{da_m^*}{d\tau} = \sigma_m \left| a_k a_\ell \right|^2.$$

Subtracting the two equations above yields

$$a_k a_\ell \frac{da_m}{d\tau} - a_k^* a_\ell^* \frac{da_m^*}{d\tau} = 0. \tag{2.32}$$

Finally, since (2.32) holds for $(k, \ell, m) = (1, 2, 3)$ cyclically, it follows that (2.32) actually constitutes three equations. Summing these three equations yields

$$\frac{d}{d\tau}\left(a_1 a_2 a_3 - a_1^* a_2^* a_3^*\right) = 0.$$

That is, the quantity in parentheses is constant for all time, which leads us to define a third conserved quantity,

$$\tilde{H} = a_1 a_2 a_3 - a_1^* a_2^* a_3^*, \tag{2.33}$$

where $\tilde{H}$ is a constant. Furthermore, if we write $a_m(\tau) = \left|a_m(\tau)\right| e^{i\varphi_m(\tau)}$ for some real function $\varphi_m(\tau)$ for $m = 1, 2, 3$, then it is straightforward to see that (2.33) becomes

$$\tilde{H} = \left|a_1 a_2 a_3\right| \left[e^{i(\varphi_1+\varphi_2+\varphi_2)} - e^{-i(\varphi_1+\varphi_2+\varphi_2)}\right].$$

Define $\Phi(\tau) = \varphi_1 + \varphi_2 + \varphi_3$. Then we have

$$\tilde{H} = 2i \left|a_1 a_2 a_3\right| \sin\Phi, \tag{2.34}$$

which shows that $\tilde{H} = iH$ for some real constant H.

Now we show that our system (2.20) is, in fact, Hamiltonian. To that end, define $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ as follows

$$p_m = \begin{cases} a_m, & \sigma_m = 1 \\ a_m^*, & \sigma_m = -1 \end{cases} \quad \text{and} \quad q_m = \begin{cases} a_m^*, & \sigma_m = 1 \\ a_m, & \sigma_m = -1. \end{cases} \tag{2.35}$$

Suppose, for instance, that $\sigma_1 = \sigma_2 = \sigma_3 = 1$. It follows from (2.26) that

$$\tilde{H} = -iH = p_1 p_2 p_3 - q_1 q_2 q_3,$$

so that

$$\frac{\partial \tilde{H}}{\partial p_m} = p_k p_\ell \quad \text{and} \quad \frac{\partial \tilde{H}}{\partial q_m} = -q_k q_\ell, \tag{2.36}$$

where as usual, $(k, \ell, m) = (1, 2, 3)$ cyclically. Additionally, in this case, (2.20) becomes

$$\frac{dp_m}{d\tau} = q_k q_\ell,$$

while the conjugate of (2.20) becomes

$$\frac{dq_m}{d\tau} = p_k p_\ell.$$

Comparing the equations for $\frac{dp_m}{d\tau}$ and $\frac{dq_m}{d\tau}$ with (2.36) shows that we have

$$\frac{dq_m}{d\tau} = \frac{\partial \tilde{H}}{\partial p_m} \quad \text{and} \quad \frac{dp_m}{d\tau} = -\frac{\partial \tilde{H}}{\partial q_m} \quad \text{for } m = 1, 2, 3. \tag{2.37}$$

The same procedure can be followed when $\sigma_1 = \sigma_2 = \sigma_3 = -1$, or when one of the σ_m's is different from the other two. In any case, (2.22) is obtained. As a result, we have a Hamiltonian system with three degrees of freedom, where the Hamiltonian is given by $\tilde{H}$ in (2.26). Moreover, we determined that there are three associated constants of the motion, given by the Hamiltonian (2.26), and (2.30)-(2.31). In the special case where the degree of the Hamiltonian system equals the number of constants of the motion, the system has the additional property of being *integrable*, which we discuss in the next section.

Certain systems of PDEs can be thought of as infinite dimensional Hamiltonian systems. For instance, consider the one-dimensional nonlinear wave equation with Dirichlet boundary conditions,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - f(u, x), \quad \text{and} \quad u(0, t) = u(L, t) = 0, \tag{2.38}$$

where $x \in (0, L)$, $u = u(x, t)$ is a scalar function, and $f(u, x)$ is a nonlinear term. We define the associated Hamiltonian as

$$H(u, p) = \int_0^L \left(\frac{1}{2} p^2 + \frac{1}{2} \left(u'\right)^2 + F(u, x) \right) dx, \tag{2.39}$$

where $u' = \frac{\partial u}{\partial x}$, $\frac{\partial F}{\partial u} = f$, and $p = \frac{\partial u}{\partial t}$ [13, p.67-83]. Then (2.38) can be written as

$$\begin{aligned}\frac{\partial u}{\partial t} &= p = \nabla_p H \\ \frac{\partial p}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - f = -\nabla_u H,\end{aligned}$$

where the gradients are functional derivatives taken with respect to the $L^2(\Omega)$ inner product. We omit the details of functional derivatives here. Suffice it to say that for a functional of the form

$$J[g] = \int_a^b L[x, g(x), g'(x)]\, dx,$$

it can be shown that the functional derivative is

$$\frac{\delta J}{\delta g} = \frac{\partial L}{\partial g} - \frac{d}{dx}\frac{\partial L}{\partial g'},$$

where $g' = dg/dx$ [19]. For instance, with $H(u, p)$ defined in (2.39), we have

$$\begin{aligned}\frac{\delta H}{\delta p} &= \frac{\partial}{\partial p}\left(\frac{1}{2}p^2 + \frac{1}{2}\left(u'\right)^2 + F(u, x)\right) - \frac{d}{dx}\left[\frac{\partial}{\partial p'}\left(\frac{1}{2}p^2 + \frac{1}{2}\left(u'\right)^2 + F(u, x)\right)\right] \\ &= p.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{\delta H}{\delta u} &= \frac{\partial}{\partial u}\left(\frac{1}{2}p^2 + \frac{1}{2}\left(u'\right)^2 + F(u, x)\right) - \frac{d}{dx}\left[\frac{\partial}{\partial u'}\left(\frac{1}{2}p^2 + \frac{1}{2}\left(u'\right)^2 + F(u, x)\right)\right] \\ &= \frac{\partial F}{\partial u} - \frac{d}{dx}u' \\ &= f - u'',\end{aligned}$$

where $f(u, x) = \frac{\partial F}{\partial u}$ and $u'' = \frac{\partial^2 u}{\partial x^2}$.

More succinctly, we have

$$\frac{\partial v}{\partial t} = J\nabla_v H,$$

where $v = (u, p)^T$ and J is defined in (2.23).

The three-wave PDEs given in (2.5) constitute a Hamiltonian system with Hamiltonian

$$-\int_\Omega \left[\frac{1}{2}\sum_{m=1}^{3} c_m\left(a_m \nabla a_m^* - a_m^* \nabla a_m\right) - \left(a_1 a_2 a_3 - a_1^* a_2^* a_3^*\right)\right] d\mathbf{x}, \qquad (2.40)$$

for $\mathbf{x} \in \Omega \in \mathbb{R}^d$. In particular, suppose that we define p_m and q_m for $m = 1, 2, 3$ using (2.35). Suppose further that we are in one spatial dimension, and that $\sigma_m = 1$ for $m = 1, 2, 3$. Other configurations of $\{\sigma_1, \sigma_2, \sigma_3\}$ can be treated similarly. In this case, we have

$$p_m = a_m \quad \text{and} \quad q_m = a_m^*, \quad m = 1, 2, 3.$$

The Hamiltonian is then defined via

$$H(q,p) = -\int \left[\frac{1}{2} \sum_{m=1}^{3} c_m \left(p_m q_m' - q_m p_m' \right) - \left(p_1 p_2 p_3 - q_1 q_2 q_3 \right) \right] dx,$$

where $p_m' = \frac{\partial p_m}{\partial x}$ and $q_m' = \frac{\partial q_m}{\partial x}$.

Observe the following

$$\begin{aligned}
\frac{\delta H}{\delta p_m} &= \frac{\partial}{\partial p_m} \left[-\frac{1}{2} \sum_{m=1}^{3} c_m \left(p_m q_m' - q_m p_m' \right) + \left(p_1 p_2 p_3 - q_1 q_2 q_3 \right) \right] \\
&\quad - \frac{d}{dx} \left\{ \frac{\partial}{\partial p_m'} \left[-\frac{1}{2} \sum_{m=1}^{3} c_m \left(p_m q_m' - q_m p_m' \right) + \left(p_1 p_2 p_3 - q_1 q_2 q_3 \right) \right] \right\} \\
&= -\frac{c_m}{2} q_m' + p_k p_\ell - \frac{d}{dx} \left[\frac{c_m}{2} q_m \right] \\
&= -\frac{c_m}{2} q_m' + p_k p_\ell - \frac{c_m}{2} q_m' \\
&= -c_m q_m' + p_k p_\ell \\
&= -c_m \frac{\partial a_m^*}{\partial x} + a_k a_\ell, \\
&= \frac{\partial a_m^*}{\partial \tau},
\end{aligned}$$

where we used $q_m' = \frac{\partial q_m}{\partial x} = \frac{\partial a_m^*}{\partial x}$, and $(k, \ell, m) = (1, 2, 3)$ cyclically. Consquently, we have

$$\frac{\delta H}{\delta p_m} = \frac{\partial a_m}{\partial \tau} = \frac{\partial q_m}{\partial \tau}, \tag{2.41}$$

for $m = 1, 2, 3$.

Next, we have

$$
\begin{aligned}
\frac{\delta H}{\delta q_m} &= \frac{\partial}{\partial q_m}\left[-\frac{1}{2}\sum_{m=1}^{3} c_m\left(p_m q_m' - q_m p_m'\right) + \left(p_1p_2p_3 - q_1q_2q_3\right)\right] \\
&\quad - \frac{d}{dx}\left\{\frac{\partial}{\partial q_m'}\left[-\frac{1}{2}\sum_{m=1}^{3} c_m\left(p_m q_m' - q_m p_m'\right) + \left(p_1p_2p_3 - q_1q_2q_3\right)\right]\right\} \\
&= \frac{c_m}{2}p_m' - q_k q_\ell - \frac{d}{dx}\left[-\frac{c_m}{2}p_m\right] \\
&= c_m p_m' - q_k q_\ell \\
&= c_m \frac{\partial a_m}{\partial x} - a_k^* a_\ell^*, \\
&= -\frac{\partial a_m}{\partial \tau},
\end{aligned}
$$

where we used $p_m' = \frac{\partial p_m}{\partial x} = \frac{\partial a_m}{\partial x}$, and $(k, \ell, m) = (1, 2, 3)$ cyclically. Consquently, we have

$$
\frac{\delta H}{\delta q_m} = -\frac{\partial a_m}{\partial \tau} = -\frac{\partial p_m}{\partial \tau}, \tag{2.42}
$$

for $m = 1, 2, 3$.

The combination of (2.41) and (2.42) demonstrate that the three-wave PDEs in (2.5) constitute a Hamiltonian system with Hamiltonian given by (2.40).

2.2.2 Integrability

There are many definitions of integrability for a system of ODEs, most of which have to do with being able to solve the system completely using quadratures. For our purposes, we work in the context of Liouville integrability.

First, we say that two scalar functions $F(q, p)$ and $G(q, p)$ are in involution if $\{F, G\} = 0$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket in (2.25). Then according to Liouville's Theorem, an n degree-of-freedom Hamiltonian system of ODEs is completely integrable if there are n constants of the motion F_j, $j = 1, \ldots, n$, that are functionally independent and in involution, $\{F_i, F_j\} = 0$ [32, §9.12]. It is trivially true that any Hamiltonian system with $n = 1$ is integrable.

For our system (2.20), it is straightforward to show that for the constants $\tilde{H}$, K_2, and K_3 defined in (2.26) and (2.30)-(2.31), we have $\{K_2, K_3\} = \{K_2, \tilde{H}\} = \{K_3, \tilde{H}\} = 0$. For instance, suppose $\sigma_1 = \sigma_2 = \sigma_3 = 1$. Then we can rewrite the conserved quantities (2.26) and (2.30)-(2.31) in terms of p and q as defined in (2.35),

$$\tilde{H} = p_1p_2p_3 - q_1q_2q_3$$

$$K_2 = p_1q_1 - p_2q_2$$

$$K_3 = p_1q_1 - p_3q_3.$$

Using the definition of the Poisson bracket with $n = 3$, we have

$$\begin{aligned}
\{K_2, \tilde{H}\} &= \sum_{j=1}^{n} \left(\frac{\partial K_2}{\partial q_j}\frac{\partial \tilde{H}}{\partial p_j} - \frac{\partial K_2}{\partial p_j}\frac{\partial \tilde{H}}{\partial q_j} \right) \\
&= \frac{\partial K_2}{\partial q_1}\frac{\partial \tilde{H}}{\partial p_1} - \frac{\partial K_2}{\partial p_1}\frac{\partial \tilde{H}}{\partial q_1} + \frac{\partial K_2}{\partial q_2}\frac{\partial \tilde{H}}{\partial p_2} - \frac{\partial K_2}{\partial p_2}\frac{\partial \tilde{H}}{\partial q_2} + \frac{\partial K_2}{\partial q_3}\frac{\partial \tilde{H}}{\partial p_3} - \frac{\partial K_2}{\partial p_3}\frac{\partial \tilde{H}}{\partial q_3} \\
&= p_1(p_2p_3) - q_1(-q_2q_3) + (-p_2)(p_1p_3) - (-q_2)(-q_1q_3) + 0 - 0 \\
&= p_1p_2p_3 + q_1q_2q_3 - p_1p_2p_3 - q_1q_2q_3 \\
&= 0.
\end{aligned}$$

The process for showing that $\{K_3, \tilde{H}\} = 0$ and $\{K_2, K_3\} = 0$ is similar. The same procedure can be followed for other configurations of the σ_m. Moreover, it is straightforward to show that $\tilde{H}$, K_2, and K_3 are functionally independent. It follows that (2.20) is completely integrable in the Liouville sense since it is a Hamiltonian system with three degrees of freedom and three constants of the motion that are functionally independent and in involution.

Next, recall that $z = (p, q)$ is a point in phase space and observe the following

$$\begin{aligned}
\{z, \tilde{H}\} &= \sum_{j=1}^{3} \left(\frac{\partial z}{\partial q_j}\frac{\partial \tilde{H}}{\partial p_j} - \frac{\partial z}{\partial p_j}\frac{\partial \tilde{H}}{\partial q_j} \right) \\
&= \frac{\partial z}{\partial q_1}\frac{\partial \tilde{H}}{\partial p_1} - \frac{\partial z}{\partial p_1}\frac{\partial \tilde{H}}{\partial q_1} + \frac{\partial z}{\partial q_2}\frac{\partial \tilde{H}}{\partial p_2} - \frac{\partial z}{\partial p_2}\frac{\partial \tilde{H}}{\partial q_2} + \frac{\partial z}{\partial q_3}\frac{\partial \tilde{H}}{\partial p_3} - \frac{\partial z}{\partial p_3}\frac{\partial \tilde{H}}{\partial q_3} \\
&= \frac{\partial z}{\partial q_1}\frac{dq_1}{d\tau} + \frac{\partial z}{\partial p_1}\frac{dp_1}{d\tau} + \frac{\partial z}{\partial q_2}\frac{dq_1}{d\tau} + \frac{\partial z}{\partial p_2}\frac{dp_2}{d\tau} + \frac{\partial z}{\partial q_3}\frac{dq_3}{d\tau} + \frac{\partial z}{\partial p_3}\frac{dp_3}{d\tau} \\
&= \frac{dz}{d\tau},
\end{aligned}$$

where we used (2.37) to eliminate $\tilde{H}$ from the equation. Consequently, a short-hand way of writing

(2.20) in Hamiltonian form is

$$\frac{dz}{d\tau} = \{z, \tilde{H}\}, \tag{2.43}$$

where $\tilde{H}$ is defined in (2.33), and the p and q components of z are defined in (2.35).

It is possible to transform a Hamiltonian system into another Hamiltonian system using a canonical transformation, a diffeomorphism that preserves the Hamiltonian structure of the system. In particular, we can use such a transformation to write (2.43) in terms of a new set of coordinates $Z = (Q, P)$, where $P = P(q, p)$ and $Q = Q(q, p)$. In doing so, (2.43) becomes

$$\frac{dZ}{d\tau} = \{Z, \mathcal{H}\},$$

where $\mathcal{H} = \mathcal{H}(Q, P)$ is the transformed Hamiltonian. In other words, we have

$$\frac{dQ_m}{d\tau} = \frac{\partial \mathcal{H}}{\partial P_m} \quad \text{and} \quad \frac{dP_m}{d\tau} = -\frac{\partial \mathcal{H}}{\partial Q_m},$$

for $m = 1, 2, 3$. Furthermore, canonical transformations preserve the Poisson bracket so that we have

$$\{P_m, P_n\} = 0, \quad \{Q_m, Q_n\} = 0, \quad \text{and} \quad \{Q_m, P_n\} = \delta_{mn}. \tag{2.44}$$

Conversely, if a transformation $Q = Q(q, p)$ and $P = P(q, p)$ is found such that (2.44) holds, then the transformation is canonical. We make use of canonical transformations in Section 3.1.

The system is said to be in "action-angle form" if $\mathcal{H}$ only depends on P, $\mathcal{H} = \mathcal{H}(P)$. In this case, it follows that

$$\frac{dQ_m}{d\tau} = \frac{\partial \mathcal{H}}{\partial P_m} \quad \text{and} \quad \frac{dP_m}{d\tau} = 0.$$

As a result, the variables P_m are invariant, and therefore must be functions of the constants of the motion [32]. The functions P_m for $m = 1, 2, 3$ are referred to as the action variables, while the functions Q_m are called the angle variables.

Next, we return to the three-wave PDEs given in (2.5). A Hamiltonian set of PDEs is completely integrable under the same conditions as the ODEs, but in an infinite dimensional phase space. That is, a Hamiltonian system of PDEs is integrable if there is a canonical transformation allowing the PDEs to be written in terms of an infinite set of action-angle variables. This theory was first developed by Zakharov and Faddeev in 1971 for the Korteweg de Vries equation [45]. In particular, they showed that solving the PDE using the inverse scattering transform (IST) amounts to finding a canonical transformation to action-angle variables. Later, it was shown that the three-wave equations (2.5) form an integrable Hamiltonian system [47].

As we discussed in the introduction, the full three-wave equations have been solved in various configurations in the past. For instance, Zakharov and Manakov (1973, 1976) and Kaup (1976) used IST to solve the three-wave equations in one spatial dimension on the real line. Similarly, solutions in two or three spatial dimensions and time were worked out by Zakharov (1976), Kaup (1980), and others. So far, little work has been done analytically on the problem with periodic boundary conditions, although some numerical simulations of the problem make use of such conditions. We hope to make some headway on this problem by constructing a near general solution of (2.5) that is consistent with periodic boundary conditions. Indeed, our approach should give solutions that are consistent with a wide variety of boundary conditions and any number of spatial dimensions (see Chapter 5).

2.2.3 The explosive instability

Another key property of the three-wave equations is that, under certain conditions, any nonzero solutions to (2.5) and (2.20) can blow up in finite time. We refer to this phenomenon as the explosive instability. In general, it signifies that the assumptions underlying our model have broken down. Moreover, the explosive instability indicates that the physical system has transitioned from one state (in which the model's underlying assumptions are valid) to another (in which they are not) [38].

As usual, we first consider the three-wave ODEs. If σ_1, σ_2, and σ_3 are not all equal, then the equations for the constants of the motion in (2.30)-(2.31) imply that $|a_m|^2$ is bounded for all τ for $m = 1, 2, 3$. It follows that $\sigma_1 = \sigma_2 = \sigma_3$ is a necessary condition for solutions to blow up in finite time. In particular, one can show that if $\sigma_1 = \sigma_2 = \sigma_3$, then $a_m(\tau)$ blows up in finite time as long as at least two of $\{a_1(0), a_2(0), a_3(0)\}$ are nonzero [11]. The proof that this condition is sufficient for blow up can be found in detail in [38].

Consider the motivating example in which $a_1(\tau)$ is approximately constant, and $a_2(\tau)$ and $a_3(\tau)$ are much less than $a_1(\tau)$. The three-wave ODEs in (2.20) become

$$\frac{da_1}{d\tau} \approx 0, \quad \frac{da_2}{d\tau} = \sigma_2 a_1^* a_3^*, \quad \text{and} \quad \frac{da_3}{d\tau} = \sigma_3 a_1^* a_2^*.$$

Differentiating the evolution equation for $a_2(\tau)$ using the assumption that $a_1(\tau)$ is constant yields

$$\begin{aligned} \frac{d^2 a_2}{d\tau^2} &= \sigma_2 a_1^* \frac{da_3^*}{d\tau} \\ &= \sigma_2 \sigma_3 \left|a_1\right|^2 a_2, \end{aligned}$$

where we used the conjugate equation $\frac{da_3^*}{d\tau} = \sigma_3 a_1 a_2$ in the last equality. Similarly, we obtain

$$\frac{d^2 a_3}{d\tau^2} = \sigma_2 \sigma_3 \left|a_1\right|^2 a_3.$$

Now suppose that $\sigma_2 = \sigma_3$, so that $\sigma_2\sigma_3 = 1$. It follows that $a_2(\tau)$ and $a_3(\tau)$ behave exponentially, with growth terms $e^{|a_1|\tau}$ and decay terms $e^{-|a_1|\tau}$. Conversely, suppose that $\sigma_2 = -\sigma_3$, so that $\sigma_2\sigma_3 = -1$. In this case, $a_2(\tau)$ and $a_3(\tau)$ behave periodically with frequency $|a_1|$.

This behavior is known as the explosive instability. A problem in plasma physics led to the discovery that the same type of behavior is also observed when spatial dependence is reintroduced [11], although the conditions required for blow-up are slightly more complicated, and have only been worked out in certain cases [26].

Chapter 3

The Weierstrass Solution of the Three-Wave ODEs

Solutions of the three-wave ODEs have been well known for many decades in terms of elliptic functions [5, 8]. In this chapter, we derive the general solution of the three-wave ODEs in terms of Weierstrass elliptic functions. We then analyze the behavior of solutions, and describe how the explosive and nonexplosive cases can be contained within the same Weierstrass function.

3.1 The Hamiltonian system

In order to construct the general solution of the three wave ODEs in (2.20) using elliptic functions, we first convert our existing Hamiltonian system into an equivalent Hamiltonian system that has only one degree of freedom.

3.1.1 Change of variables

Recall that the three-wave ODEs in (2.20) constitute a Hamiltonian system with three degrees of freedom, as detailed in Section 2.2.1. In particular, we have

$$\frac{dq}{dt} = \frac{\partial \tilde{H}}{\partial p}(q,p), \tag{3.1a}$$

$$\frac{dp}{dt} = -\frac{\partial \tilde{H}}{\partial q}(q,p), \tag{3.1b}$$

where $\tilde{H}$ is given in (2.26), and $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ are defined via

$$p_m = \begin{cases} a_m, & \sigma_m = 1 \\ a_m^*, & \sigma_m = -1 \end{cases} \quad \text{and} \quad q_m = \begin{cases} a_m^*, & \sigma_m = 1 \\ a_m, & \sigma_m = -1. \end{cases} \tag{3.2}$$

To transform our Hamiltonian system into an equivalent system with one degree of freedom, we begin by breaking $a_m(\tau)$ into its magnitude and phase components as follows

$$a_m(\tau) = |a_m(\tau)|\, e^{i\varphi_m(\tau)}, \quad \text{for } m = 1, 2, 3,$$

where $\varphi_m(\tau)$ is a real function of τ for $m = 1, 2, 3$. Next, we define new variables $P = (P_1, P_2, P_3)$ and $Q = (Q_1, Q_2, Q_3)$ via

$$P_m = \sigma_m \left|a_m\right|^2, \tag{3.3a}$$

$$Q_m = i\varphi_m. \tag{3.3b}$$

We use (3.2) to write P and Q in terms of p and q as follows

$$P_m = \sigma_m p_m q_m, \tag{3.4a}$$

$$Q_m = \frac{\sigma_m}{2} \log\left(\frac{p_m}{q_m}\right). \tag{3.4b}$$

Then it is straightforward to see that

$$\{P_m(q,p), P_n(q,p)\} = \{Q_m(q,p), Q_n(q,p)\} = 0,$$

for all $m, n = 1, 2, 3$, where the Poisson bracket is defined in (2.25). Furthermore, if $m \neq n$, then

$$\{P_m(q,p), Q_n(q,p)\} = 0.$$

Finally, observe that

$$\begin{aligned}
\{P_m(q,p), Q_m(q,p)\} &= \sum_{j=1}^{3} \left(\frac{\partial P_m}{\partial q_j}\frac{\partial Q_n}{\partial p_j} - \frac{\partial P_m}{\partial p_j}\frac{\partial Q_n}{\partial q_j} \right) \\
&= \frac{\partial P_m}{\partial q_m}\frac{\partial Q_n}{\partial p_m} - \frac{\partial P_m}{\partial p_m}\frac{\partial Q_n}{\partial q_m} \\
&= (\sigma_m p_m)\left(\frac{\sigma_m}{2} \cdot \frac{1/q_m}{p_m/q_m}\right) - (\sigma_m q_m)\left(\frac{\sigma_m}{2} \cdot \frac{-p_m/q_m^2}{p_m/q_m}\right) \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

To summarize, we have

$$\{P_m, P_n\} = 0, \quad \{Q_m, Q_n\} = 0, \quad \text{and} \quad \{Q_m, P_n\} = \delta_{mn},$$

which we recall from (2.44) means that our transformation $P = P(q, p)$ and $Q = Q(q, p)$ is canonical. As a result, we know that the three-wave ODEs under this transformation retain their Hamiltonian structure.

3.1.2 Transformed Hamiltonian system

Recall from (2.26) and (2.34) that the Hamiltonian for the three-wave ODEs is given by

$$\begin{aligned} \tilde{H} = -iH &= a_1 a_2 a_3 - a_1^* a_2^* a_3^* \\ &= 2i\, |a_1 a_2 a_3|\, \sin \Phi, \end{aligned} \tag{3.5}$$

where H is real and $\Phi(\tau) = \varphi_1(\tau) + \varphi_2(\tau) + \varphi_3(\tau)$. Next, from the definition of P_m and Q_m in (3.3), we have that

$$|a_m| = \sqrt{\sigma_m P_m} \quad \text{and} \quad \varphi_m = -iQ_m,$$

where we note that $\operatorname{sign}(P_m) = \sigma_m$, so the quantity under the square root is real and positive. It follows from (3.5) that

$$\begin{aligned} \tilde{H} = -iH &= 2i\sqrt{\sigma P_1 P_2 P_3}\, \sin\left(-i\tilde{Q}\right) \\ &= 2\sqrt{\sigma P_1 P_2 P_3}\, \sinh \tilde{Q}, \end{aligned} \tag{3.6}$$

where $\sigma = \sigma_1\sigma_2\sigma_3$, $\tilde{Q} = Q_1 + Q_2 + Q_3$, and we used the fact that $\sin ix = i \sinh x$.

Next, we differentiate the equation for $\tilde{H}$ above with respect to P_m and Q_m to find

$$\frac{\partial \tilde{H}}{\partial Q_m} = 2\sqrt{\sigma P_1 P_2 P_3}\, \cosh \tilde{Q}, \tag{3.7}$$

and

$$\begin{aligned} \frac{\partial \tilde{H}}{\partial P_m} &= \frac{\sigma P_k P_\ell}{\sqrt{\sigma P_1 P_2 P_3}}\, \sinh \tilde{Q} \\ &= \sigma_m \sqrt{\frac{\sigma P_k P_\ell}{P_m}}\, \sinh \tilde{Q}, \end{aligned} \tag{3.8}$$

where $(k, \ell, m) = (1, 2, 3)$ are defined cyclically as usual.

Finally, we can differentiate (3.3), using the definitions of p and q in (3.2) and the three-wave ODEs in (2.20), to find

$$\begin{aligned}
\frac{dP_m}{d\tau} &= \frac{d}{d\tau}\left(\sigma_m p_m q_m\right) \\
&= \sigma_m \frac{dp_m}{d\tau} q_m + \sigma_m q_m \frac{dp_m}{d\tau} \\
&= a_1 a_2 a_3 + a_1^* a_2^* a_3^* \\
&= 2\left|a_1 a_2 a_3\right| \cosh\tilde{Q} \\
&= 2\sqrt{\sigma P_1 P_2 P_3}\,\cosh\tilde{Q}.
\end{aligned} \tag{3.9}$$

Additionally, we have

$$\begin{aligned}
\frac{dQ_m}{d\tau} &= \frac{d}{d\tau}\left[\frac{\sigma_m}{2}\log\left(\frac{p_m}{q_m}\right)\right] \\
&= \frac{d}{d\tau}\left[\frac{\sigma_m}{2}\left(\log p_m - \log q_m\right)\right] \\
&= \frac{\sigma_m}{2}\left[\frac{1}{p_m}\frac{dp_m}{d\tau} - \frac{1}{q_m}\frac{dq_m}{d\tau}\right] \\
&= -\sigma_m \left|\frac{a_k a_\ell}{a_m}\right| \sinh\tilde{Q} \\
&= -\sigma_m \sqrt{\frac{\sigma P_k P_\ell}{P_m}}\sinh\tilde{Q}.
\end{aligned} \tag{3.10}$$

From (3.7)-(3.10), we can see that $\frac{\partial H}{\partial Q} = \frac{dP}{d\tau}$ and $\frac{\partial H}{\partial P} = -\frac{dQ}{d\tau}$. If we define $\mathcal{H} = -\tilde{H}$, then we obtain a Hamiltonian system of the form (3.1),

$$\frac{dQ}{d\tau} = \frac{\partial \mathcal{H}}{\partial P} \quad \text{and} \quad \frac{dP}{d\tau} = -\frac{\partial \mathcal{H}}{\partial Q}, \tag{3.11}$$

where

$$\mathcal{H} = iH = -2\sqrt{\sigma P_1 P_2 P_3}\,\sinh\tilde{Q}. \tag{3.12}$$

3.1.3 Reduction to a second-order system

Next, we want to solve the system defined in (3.9)-(3.12). To do so, we convert our sixth-order system of ODEs to a second-order system of ODEs. That is, we convert the Hamiltonian system with three degrees of freedom in (3.9)-(3.12) to a Hamiltonian system with only one degree of freedom. In order to do this, we define new variables in terms of P and Q as follows

$$\rho(\tau) = P_1 = \sigma_1 \left|a_1(\tau)\right|^2 \quad \text{and} \quad \Phi(\tau) = \varphi_1(\tau) + \varphi_2(\tau) + \varphi_3(\tau) = -i\tilde{Q}(\tau), \tag{3.13}$$

where $\tilde{Q} = Q_1 + Q_2 + Q_3$ as before, and the definition of $\Phi(\tau)$ was previously used in (3.5).

Recall from (3.3a) that $P_m = \sigma_m |a_m|^2$. It follows that the conservation equations in (2.30)-(2.31) become

$$\rho - P_2 = K_2 \quad \text{and} \quad \rho - P_3 = K_3,$$

where we used $\rho = P_1$. Finally, solving for P_2 and P_3 in terms of ρ and using the definition of Φ in (3.13) allows us to transform (3.12) into

$$\begin{aligned} \mathcal{H} = iH &= -2\sqrt{\sigma\rho(\rho - K_2)(\rho - K_3)}\, \sinh i\Phi \\ &= -2i\sqrt{\sigma\rho(\rho - K_2)(\rho - K_3)}\, \sin\Phi. \end{aligned}$$

Dropping the i, we have the following definition of H,

$$H = -2\sqrt{\sigma\rho(\rho - K_2)(\rho - K_3)}\, \sin\Phi. \tag{3.14}$$

Note that H only depends on ρ and Φ.

Our Hamiltonian system with one degree of freedom is found by differentiating (3.14) with respect to ρ and Φ. First, we have

$$\frac{\partial H}{\partial \Phi} = -2\sqrt{\sigma\rho(\rho - K_2)(\rho - K_3)}\, \cos\Phi. \tag{3.15}$$

Differentiation of (3.14) with respect to ρ is slightly more complicated because we must keep track of signs when we differentiate the square root. Recall once more that due to (3.3a), we have $\operatorname{sign}(P_m) = \sigma_m$, assuming P_m is nonzero. It follows that $\operatorname{sign}(\rho) = \sigma_1$, $\operatorname{sign}(\rho - K_2) = \sigma_2$, and $\operatorname{sign}(\rho - K_3) = \sigma_3$; alternatively, $\sigma_1\rho \geq 0$, $\sigma_2(\rho - K_2) \geq 0$, and $\sigma_3(\rho - K_3) \geq 0$. With this in mind, we have

$$\begin{aligned}\frac{\partial H}{\partial \rho} &= -\frac{1}{\sqrt{\sigma\rho(\rho-K_2)(\rho-K_3)}}\Big[\sigma(\rho-K_2)(\rho-K_3)+\sigma\rho(\rho-K_3)+\sigma\rho(\rho-K_2)\Big]\sin\Phi \\ &= -\left(\sigma_1\sqrt{\frac{\sigma(\rho-K_2)(\rho-K_3)}{\rho}}+\sigma_2\sqrt{\frac{\sigma\rho(\rho-K_3)}{\rho-K_2}}+\sigma_3\sqrt{\frac{\sigma\rho(\rho-K_2)}{\rho-K_3}}\right)\sin\Phi. \qquad (3.16)\end{aligned}$$

Finally, to ensure that we have a Hamiltonian system, we must compute $d\rho/d\tau$ and $d\Phi/d\tau$. First of all, from the equation for $dP_1/d\tau$ in (3.9), we have

$$\frac{d\rho}{d\tau} = 2\sqrt{\sigma\rho(\rho-K_2)(\rho-K_3)}\,\cos\Phi, \qquad (3.17)$$

where we used the fact that $\cosh\tilde{Q} = \cosh i\Phi = \cos\Phi$. Summing the equations for $dQ_m/d\tau$ in (3.10) yields

$$\frac{d\tilde{Q}}{d\tau} = i\frac{d\Phi}{d\tau} = -\left(\sigma_1\sqrt{\frac{\sigma(\rho-K_2)(\rho-K_3)}{\rho}}+\sigma_2\sqrt{\frac{\sigma\rho(\rho-K_3)}{\rho-K_2}}+\sigma_3\sqrt{\frac{\sigma\rho(\rho-K_2)}{\rho-K_3}}\right)\sinh\tilde{Q}.$$

It follows that

$$\frac{d\Phi}{d\tau} = -\left(\sigma_1\sqrt{\frac{\sigma(\rho-K_2)(\rho-K_3)}{\rho}}+\sigma_2\sqrt{\frac{\sigma\rho(\rho-K_3)}{\rho-K_2}}+\sigma_3\sqrt{\frac{\sigma\rho(\rho-K_2)}{\rho-K_3}}\right)\sin\Phi, \qquad (3.18)$$

where we used $\sinh\tilde{Q} = \sinh i\Phi = i\sin\Phi$.

Equations (3.15)-(3.18) show that we have a Hamiltonian system with one degree of freedom, where the Hamiltonian H is defined in (3.14). To summarize, we must solve the system

$$H = -2\sqrt{\sigma\rho(\rho-K_2)(\rho-K_3)}\,\sin\Phi, \qquad (3.19a)$$

$$\frac{d\rho}{d\tau} = -\frac{\partial H}{\partial\Phi} = 2\sqrt{\sigma\rho(\rho-K_2)(\rho-K_3)}\,\cos\Phi, \qquad (3.19b)$$

$$\frac{d\Phi}{d\tau} = \frac{\partial H}{\partial\rho} = -\left(\sigma_1\sqrt{\frac{\sigma(\rho-K_2)(\rho-K_3)}{\rho}}+\sigma_2\sqrt{\frac{\sigma\rho(\rho-K_3)}{\rho-K_2}}+\sigma_3\sqrt{\frac{\sigma\rho(\rho-K_2)}{\rho-K_3}}\right)\sin\Phi. \quad (3.19c)$$

Note that the unknowns are $\rho(\tau)$ and $\Phi(\tau)$, while K_2, K_3, and H are assumed to be known; for instance, these constants can be determined using initial data for $a_m(\tau)$, $m = 1, 2, 3$, through equations (2.30)-(2.31) and (3.5).)

3.2 Phase plane analysis

Before we solve (3.19), we first investigate the behavior of the system using a phase plane analysis. First of all, consider the explosive regime, in which $\sigma_1 = \sigma_2 = \sigma_3$. Without loss of generality, assume that $\sigma_m = 1$ for $m = 1, 2, 3$. If instead $\sigma_m = -1$, then we can replace $K_2 \to -K_2$ and $K_3 \to -K_3$ in what follows.

With $\sigma_m = 1$, $m = 1, 2, 3$, the conservation laws (2.30)-(2.31) can be rearranged slightly to give

$$|a_2|^2 = \rho - K_2, \tag{3.20}$$

$$|a_3|^2 = \rho - K_3, \tag{3.21}$$

where we used the fact that $\rho = \sigma_1 |a_1|^2$. When $\sigma_1 = 1$, by definition ρ is nonnegative. Additionally, it follows from (3.20)-(3.21) that $\rho \geq K_2$ and $\rho \geq K_3$, respectively. As a result, we know that ρ lies in the unbounded domain,

$$\rho > \max\{K_2, K_3, 0\}, \tag{3.22}$$

where we use strict inequality since (3.19c) prohibits $\rho = 0$, $\rho = K_2$, and $\rho = K_3$. Note that the restriction $\rho \neq \{0, K_2, K_3\}$ is a result of the change of variables (3.4b), which forces $|a_m(\tau)|$ to be nonzero for $m = 1, 2, 3$. This is not true of the original three-wave ODEs in (2.20). That is, while ρ cannot take on the values $\{0, K_2, K_3\}$ in the system (3.19), the original system (2.20) does allow $|a_m(\tau)|$ to take on these values, as long as (2.30)-(2.31) still hold. There are no restrictions on the values of Φ in (3.19).

Figure 3.1 depicts the phase plane for (3.19) with $\sigma_1 = \sigma_2 = \sigma_3$. First of all, the figure is created using $K_2 = 2$ and $K_3 = 3$, so our domain is restricted to $\rho > 3$ in accordance with (3.22). It is evident that when $\Phi = n\pi$, $d\Phi/d\tau = 0$, which is consistent with (3.19c). Moreover, when $\Phi = (2n+1)\pi/2$, we see that $d\rho/d\tau = 0$, which is consistent with (3.19b). For large values of ρ (and if $\Phi \neq (2n+1)\pi/2$), the magnitude of $d\rho/d\tau$ grows, which is indicative of unbounded solutions. This is confirmed in Section 3.6.

Next, we investigate the behavior of the system (3.19) in the nonexplosive regime, when σ_1, σ_2, and σ_3 are not all equal. Without loss of generality, assume that $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. Other configurations of σ_1, σ_2, and σ_3 can be dealt with by changing the signs of K_2 and K_3 in what follows, or by a renumbering of the modes.

With $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$, the conservation laws (2.30)-(2.31) become, after rearranging,

$$|a_2|^2 = K_2 - \rho, \tag{3.23}$$

$$|a_3|^2 = K_3 - \rho. \tag{3.24}$$

It follows that $\rho \leq K_2$ and $\rho \leq K_3$. Moreover, we know that ρ is nonnegative since $\sigma_1 = 1$. As a result, we have that ρ is bounded by

$$0 < \rho < \min\{K_2, K_3\}, \tag{3.25}$$

where again, we use strict inequality due to (3.19c). Note that (3.25) implies that $K_2 \geq 0$ and $K_3 \geq 0$ for this configuration of σ_1, σ_2, and σ_3. If we consider another configuration of the σ_m's, say, $\sigma_1 = -\sigma_2 = \sigma_3 = 1$, then we find that our domain is restricted to $\max\{0, K_3\} \leq \rho \leq K_2$, where K_2 is necessarily nonnegative and $K_3 \leq K_2$. In fact, for any configuration of σ_1, σ_2, and σ_3 such that the three numbers are not equal, we find that ρ is bounded.

Figure 3.2 depicts the phase plane for (3.19) with $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. As with Figure 3.1, the phase plot in Figure 3.2 is created with $K_2 = 2$ and $K_3 = 3$, so in this case we restrict our domain to $0 < \rho < 2$, to be consistent with (3.25). As in the explosive case, we still have that $d\Phi/d\tau = 0$ when $\Phi = n\pi$, and $d\rho/d\tau = 0$ when $\Phi = (2n+1)\pi/2$. It is also clear from Figure 3.2 that the system (3.19) admits bounded, periodic solutions.

An alternative way of viewing the problem is to consider a plot of $(d\rho/d\tau)^2$ vs. ρ for a fixed Φ. In particular, let $\Phi = 2n\pi$. Then (3.19b) gives

$$\left(\frac{d\rho}{d\tau}\right)^2 = 4\sigma\rho(\rho - K_2)(\rho - K_3), \tag{3.26}$$

where we recall that $\sigma = \sigma_1\sigma_2\sigma_3$. In both the explosive case with $\sigma_1 = \sigma_2 = \sigma_3 = 1$, and the nonexplosive case with $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$, we have that $\sigma = 1$. Figure 3.3 shows a plot of (3.26) with $\sigma = 1$, $K_2 = 2$, and $K_3 = 3$. From the plot, we see that the ODE admits both bounded and unbounded solutions. The unbounded solutions are represented on the plot by the thick black line, where $\rho > \max\{0, K_2, K_3\}$. The bounded solutions are represented by the dashed red line, where ρ is restricted to $0 < \rho < \min\{K_2, K_3\}$.

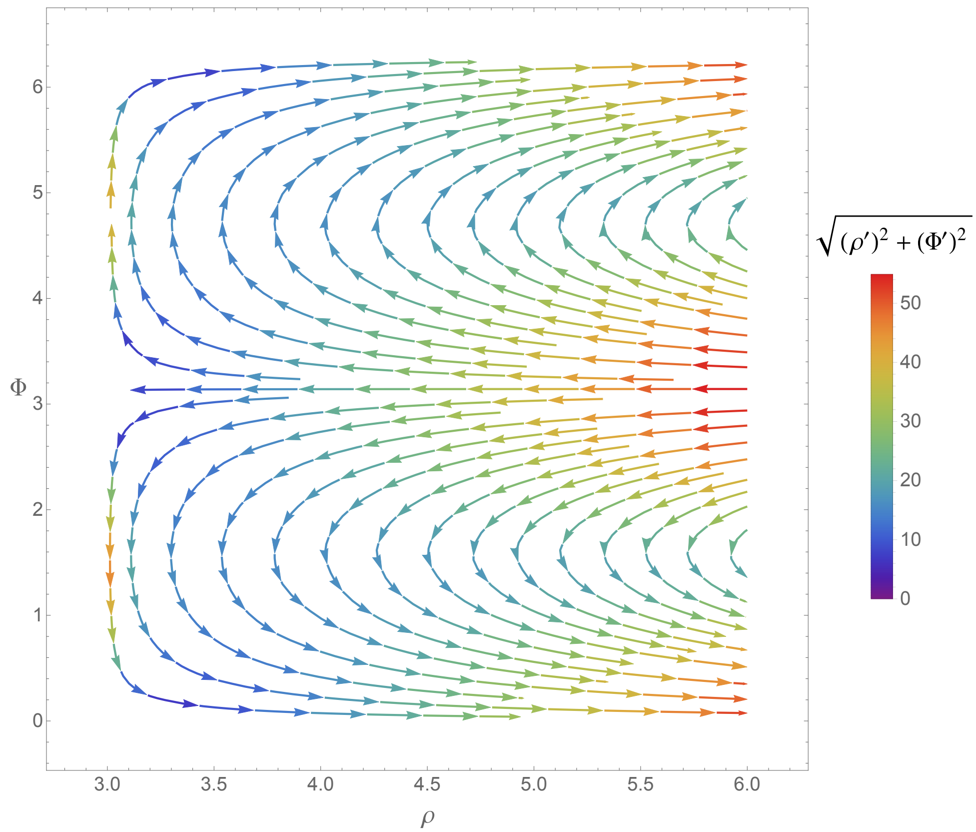


Figure 3.1: The phase plane of the system (3.19a)-(3.19c) with $\sigma_1 = \sigma_2 = \sigma_3 = 1$. This is the explosive regime, with $K_2 = 2$ and $K_3 = 3$. In this case, the domain is restricted to $\rho > \max\{K_2, K_3, 0\}$.

3.3 Solution in terms of elliptic functions (I)

In order to solve (3.19), and thus obtain a solution of the three-wave ODEs, we first convert (3.19b)-(3.19c) into a single second-order ODE. First of all, notice that we can rewrite (3.19b) using the definition of H in (3.19a) as

$$\frac{d\rho}{d\tau} = -H \cot \Phi. \tag{3.27}$$

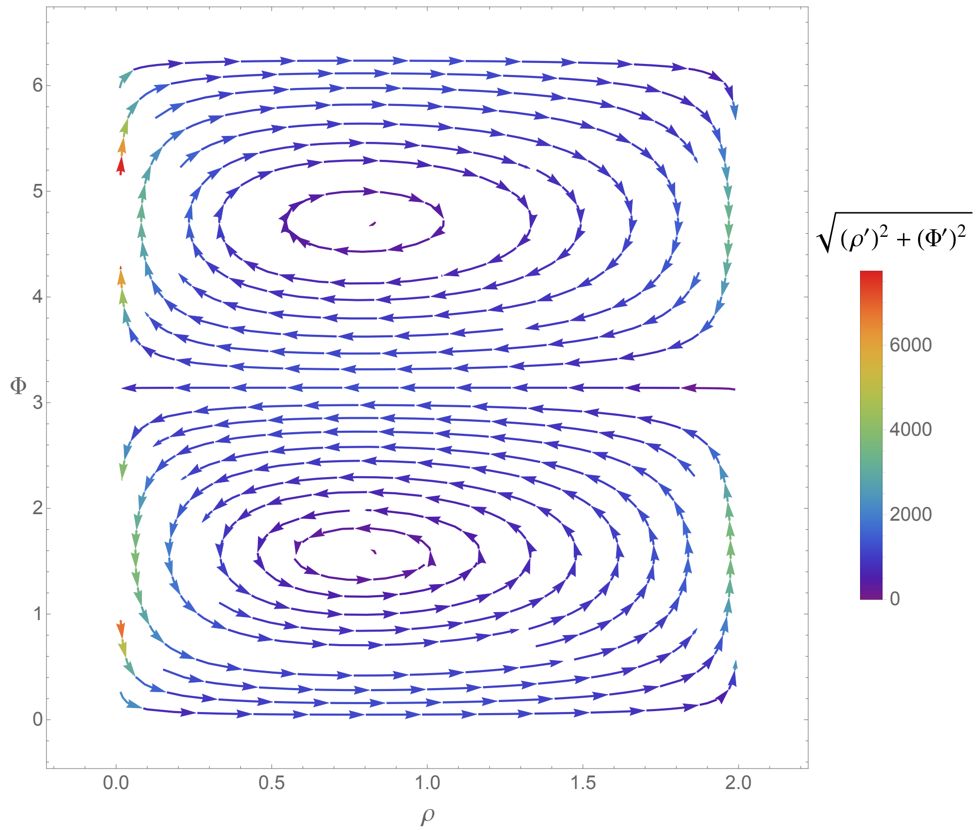


Figure 3.2: The phase plane of the system (3.19a)-(3.19c) with $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. This is the nonexplosive regime, with $K_2 = 2$ and $K_3 = 3$. In this case, the domain is restricted to $0 < \rho < \min\{K_2, K_3\}$. Bounded periodic solutions are seen.

Next, differentiate the equation above to obtain

$$\begin{aligned}\frac{d^2\rho}{d\tau^2} &= -\frac{\partial H}{\partial \tau}\cot\Phi - H\frac{\partial}{\partial \tau}(\cot\Phi)\\ &= -\left(\frac{\partial H}{\partial \rho}\frac{d\rho}{d\tau} + \frac{\partial H}{\partial \Phi}\frac{d\Phi}{d\tau}\right)\cot\Phi + H\csc^2\Phi\frac{d\Phi}{d\tau}\\ &= H\csc^2\Phi\frac{d\Phi}{d\tau},\end{aligned}$$

where we used the fact that $\partial H/\partial\rho = d\Phi/d\tau$ and $\partial H/\partial\Phi = -d\rho/d\tau$ to eliminate the first term. Now we substitute H and $d\Phi/d\tau$ from (3.19a) and (3.19c) to find

$$\begin{aligned}\frac{d^2\rho}{d\tau^2} &= 2\sqrt{\sigma\rho(\rho-K_2)(\rho-K_3)}\left(\sigma_1\sqrt{\frac{\sigma(\rho-K_2)(\rho-K_3)}{\rho}} + \sigma_2\sqrt{\frac{\sigma\rho(\rho-K_3)}{\rho-K_2}} + \sigma_3\sqrt{\frac{\sigma\rho(\rho-K_2)}{\rho-K_3}}\right)\\ &= 2\sigma\Big[(\rho-K_2)(\rho-K_3) + \rho(\rho-K_3) + \rho(\rho-K_2)\Big]. \qquad (3.28)\end{aligned}$$

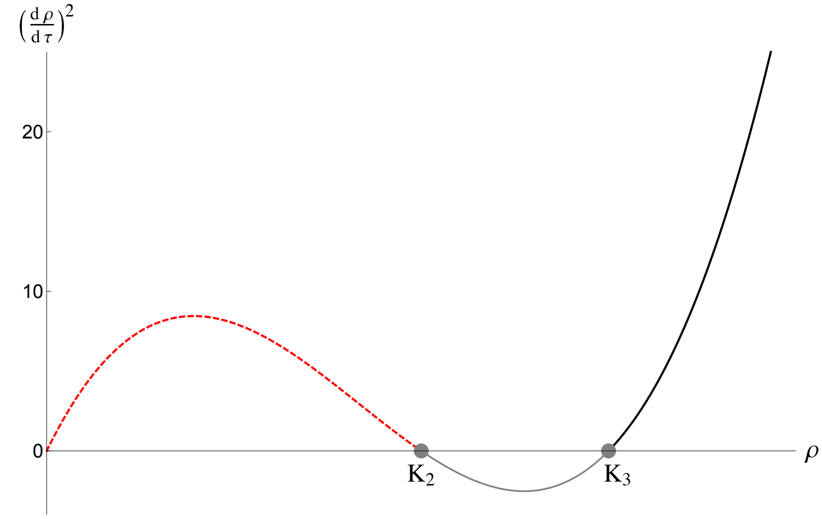


Figure 3.3: A plot of $(d\rho/d\tau)^2$ vs. ρ as given in (3.26) with $\sigma = 1$, $K_2 = 2$ and $K_3 = 3$. In particular, notice that the explosive regime with $\sigma_1 = \sigma_2 = \sigma_3 = 1$ is represented by the domain $\rho > \max\{0, K_2, K_3\}$, where $(d\rho/d\tau)^2$ grows without bound (the thick black curve). Conversely, the nonexplosive regime with $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$ is represented by the bounded domain $0 < \rho < \min\{K_2, K_3\}$, where $(d\rho/d\tau)^2$ is bounded (the dashed red curve).

Notice that (3.28) allows $\rho = 0$, $\rho = K_2$, and $\rho = K_3$, even though the system (3.19) does not. This is desirable since the original three-wave ODEs (2.20) allow $|a_m(\tau)|$ to take on values including $\{0, K_2, K_3\}$, as long as (2.30)-(2.31) still hold.

We solve (3.28) in the complex plane by transforming it into a differential equation whose solutions are known. To that end, we make the change of variables suggested by [23, Ch. 14],

$$\rho(\tau) = \sigma W(\tau) + \frac{1}{3}(K_2 + K_3). \tag{3.29}$$

Under this transformation, (3.28) becomes

$$\frac{d^2W}{d\tau^2} = 6W^2 - q, \tag{3.30}$$

where q is the constant defined by

$$q = \frac{2}{3}\left(K_2^2 + K_3^2 - K_2K_3\right), \tag{3.31}$$

which we note is nonnegative.

The general solution of (3.30) is known in terms of Weierstrass elliptic $\wp$-functions. In particular, we find that

$$W(\tau) = \wp(\tau - \tau_0; g_2 = 2q, g_3), \tag{3.32}$$

where $\wp$ is known as the Weierstrass elliptic p-function, with parameter values g_2 and g_3 [17, 23.3], and τ_0 is a complex valued constant. Using (3.29) to transform back to the ρ, we have

$$\rho(\tau) = \sigma\wp(\tau - \tau_0; g_2 = 2q, g_3) + \frac{K_2 + K_3}{3}, \tag{3.33}$$

where q is given in (3.31), g_3 is to be determined, and τ_0 is a complex-valued constant with some restrictions, namely that the imaginary part of τ_0 must be chosen so that $\rho(\tau)$ is real for real τ (a more in depth discussion of $\operatorname{Im}(\tau_0)$ is found in Section 3.6). We would like to verify (3.33) by substituting directly into (3.28). First, however, we require some background on Weierstrass elliptic functions.

3.4 Weierstrass elliptic functions

An elliptic function is a single-valued doubly periodic function of a single complex variable which is analytic except at poles [3, p. 629]. We first introduce the concept of a lattice. In particular, if λ_1 and λ_2 are two nonzero real or complex numbers such that $\operatorname{Im}(\lambda_2/\lambda_1) > 0$, then the set of points $2m\lambda_1 + 2n\lambda_2$ with $m, n \in \mathbb{Z}$ constitutes a lattice Λ. We call $2\lambda_1$ and $2\lambda_2$ the lattice generators.

The Weierstrass $\wp$-function is defined by

$$\wp(z) = \wp(z; \lambda_1, \lambda_2) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda\backslash\{0\}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right], \tag{3.34}$$

where

$$\Lambda = \{\lambda : \lambda = 2m\lambda_1 + 2n\lambda_2 \text{ with } m, n \in \mathbb{Z}\}. \tag{3.35}$$

The function $\wp(z)$ is doubly periodic with periods $2\lambda_1$ and $2\lambda_2$ (that is, λ_1 and λ_2 are the half-periods of the function). It is an even function with double poles at the lattice points, all of which have residue zero. In particular, $\wp(z)$ has a double pole at $z = 0$. Note further that $\wp(z) - \frac{1}{z^2}$ is analytic in a neighborhood of the origin and vanishes at the origin.

The elliptic invariants of a Weierstrass $\wp$-function are defined by

$$g_2(\Lambda) = g_2(\lambda_1, \lambda_2) = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^4}, \tag{3.36}$$

$$g_3(\Lambda) = g_3(\lambda_1, \lambda_2) = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^6}. \tag{3.37}$$

The function $\wp(z)$ is often written $\wp(z) = \wp(z; g_2, g_3)$, so that the elliptic invariants are specified instead of the half-periods. In fact, for our purposes it is typically more useful to specify the invariants, rather than the half-periods.

The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the sense that if $g_j(\Lambda) = g_j(\Lambda')$ for $j = 2, 3$, then $\Lambda = \Lambda'$. Additionally, given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$, there exists a lattice Λ with $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants.

We define the discriminant of $\wp(z; g_2, g_3)$ as

$$\Delta = g_2^3 - 27g_3^2.$$

If g_2 and g_3 are real, then $\wp(z; g_2, g_3)$ is real along the real axis. Moreover, if $\Delta < 0$, then the lattice of poles Λ has a rhombic structure. In this case, the half-period λ_1 is real and positive, while the half-period λ_2 satisfies $\mathrm{Im}\,(\lambda_2) > 0$ and $\mathrm{Re}\,(\lambda_2) = \lambda_1/2$ [17, §23.5]. On the other hand, if $\Delta > 0$, then the lattice of poles Λ has a rectangular structure. In this case, λ_1 and λ_2/i are both real and positive. Note that a square is considered a special case of the rectangular lattice in which $\lambda_1 = \lambda_2/i$. Finally, note that the special case $\Delta = 0$ is a degenerate case for which the Weierstrass function is not defined. This case is discussed in more detail in Section 5.4.

The Weierstrass $\wp$-function is related to its derivative via

$$\left[\wp'(z)\right]^2 = 4\wp^3(z) - g_2\wp(z) - g_3. \tag{3.38}$$

Moreover, $\wp(z)$ is related to its second derivative $\wp''(z)$ through [17, 23.3.12],

$$\wp''(z) = 6\wp^2(z) - \frac{g_2}{2}. \tag{3.39}$$

Notice that (3.39) is equivalent to (3.30) with $q = g_2/2$, which is how we obtained (3.32).

3.5 Solution in terms of elliptic functions (II)

Now return to the three-wave ODEs in (2.20), where $a_m(\tau) = |a_m(\tau)|e^{i\varphi_m(\tau)}$ for $m = 1, 2, 3$. Recall that we defined $\rho(\tau) = \sigma_1 \left|a_1(\tau)\right|^2$ and $\Phi(\tau) = \varphi_1(\tau) + \varphi_2(\tau) + \varphi_3(\tau)$, and in Section 3.3, we found that

$$\rho(\tau) = \sigma\wp(\tau - \tau_0; g_2, g_3) + \frac{K_2 + K_3}{3}, \tag{3.40}$$

where

$$g_2 = \frac{4}{3}\left(K_2^2 + K_3^2 - K_2K_3\right). \tag{3.41}$$

We now know that $\wp(\tau - \tau_0; g_2, g_3)$ is a Weierstrass elliptic function, with elliptic invariants g_2 and g_3. Indeed, substituting $\rho(\tau)$ from (3.40) into (3.28) and using (3.41) yields the differential equation satisfied by $\wp(z)$, (3.39). As a result, we know that (3.40)-(3.41) satisfy (3.28). It remains to determine g_3, as well as $\Phi(\tau)$. Once we do this, we can then recover $a_1(\tau)$, $a_2(\tau)$, and $a_3(\tau)$.

In order to find g_3, we use the differential equation (3.38). First, rearrange (3.40) to give

$$\wp(\tau - \tau_0; g_2, g_3) = \sigma\left[\rho(\tau) - \frac{K_2 + K_3}{3}\right].$$

Additionally, differentiating yields

$$\wp'(\tau - \tau_0; g_2, g_3) = \sigma\rho'(\tau),$$

where the primes denote derivatives with respect to τ. Next, substitute $\wp$ and $\wp'$ into (3.38) to obtain

$$\left[\rho'(\tau)\right]^2 = 4\sigma\left[\rho(\tau) - \frac{K_2 + K_3}{3}\right]^3 - g_2\sigma\left[\rho(\tau) - \frac{K_2 + K_3}{3}\right] - g_3,$$

where we used the fact that $\sigma^2 = 1$. Expanding terms, substituting g_2 from (3.41), and simplifying yields

$$\left[\rho'(\tau)\right]^2 = 4\sigma\left[\rho^3 - \left(K_2 + K_3\right)\rho^2 + K_2K_3\rho\right] + \frac{4\sigma}{27}\left(K_2 - 2K_3\right)\left(2K_2 - K_3\right)\left(K_2 + K_3\right) - g_3. \tag{3.42}$$

Next, recall from (3.19b) that

$$\left(\frac{d\rho}{d\tau}\right)^2 = 4\sigma\rho(\rho - K_2)(\rho - K_3)\cos^2\Phi$$
$$= 4\sigma\left[\rho^3 - (K_2 + K_3)\,\rho^2 + K_2K_3\rho\right]\cos^2\Phi.$$

Substituting into (3.42) and rearranging gives

$$4\sigma\left(\cos^2\Phi - 1\right)\left[\rho^3 - (K_2 + K_3)\,\rho^2 + K_2K_3\rho\right] = \frac{4\sigma}{27}\,(K_2 - 2K_3)\,(2K_2 - K_3)\,(K_2 + K_3) - g_3. \tag{3.43}$$

Finally, observe from (3.19a) that

$$H^2 = 4\sigma\left[\rho^3 - (K_2 + K_3)\,\rho^2 + K_2K_3\rho\right]\sin^2\Phi.$$

Since $\sin^2\Phi = 1 - \cos^2\Phi$, (3.43) becomes

$$-H^2 = \frac{4\sigma}{27}\,(K_2 - 2K_3)\,(2K_2 - K_3)\,(K_2 + K_3) - g_3.$$

It follows that

$$g_3 = \frac{4\sigma}{27}\,(K_2 - 2K_3)\,(2K_2 - K_3)\,(K_2 + K_3) + H^2.$$

To summarize, so far we have fully determined $\rho(\tau)$, half of the solution of (3.19), as

$$\rho(\tau) = \sigma\wp(\tau - \tau_0; g_2, g_3) + \frac{K_2 + K_3}{3}, \tag{3.44}$$

where

$$g_2 = \frac{4}{3}\left(K_2^2 + K_3^2 - K_2K_3\right), \tag{3.45}$$
$$g_3 = \frac{4\sigma}{27}\,(K_2 - 2K_3)\,(2K_2 - K_3)\,(K_2 + K_3) + H^2. \tag{3.46}$$

It is important to notice that g_2 and g_3 are both real. As a result, we know from Section 3.4 that the Weierstrass function $\wp(\tau; g_2, g_3)$ is real-valued along the real-axis. It follows that $\wp(\tau - \tau_0; g_2, g_3)$ is also real along the real axis, as long as τ_0 is real. Indeed, we need $\wp(\tau - \tau_0; g_2, g_3)$ to be real along the real axis because $\rho(\tau)$ must, by definition, be real-valued for real τ (recall that $\rho(\tau) = \sigma_1|a_1(\tau)|^2$ when τ is real). For certain values of g_2 and g_3, it is possible for τ_0 to be complex, yet for

$\wp(\tau-\tau_0;g_2,g_3)$ to remain real along the real axis. However, the choice of the imaginary part of τ_0 is not arbitrary, and we discuss this matter further in Section 3.6.

Our next step is to use (3.19c) in order to recover $\Phi(\tau)$, given that we now know $\rho(\tau)$ is described by (3.44)-(3.46). The ODE (3.19c) can be separated as follows,

$$\frac{d\Phi}{\sin\Phi} = -\left(\sigma_1\sqrt{\frac{\sigma(\rho-K_2)(\rho-K_3)}{\rho}} + \sigma_2\sqrt{\frac{\sigma\rho(\rho-K_3)}{\rho-K_2}} + \sigma_3\sqrt{\frac{\sigma\rho(\rho-K_2)}{\rho-K_3}}\right)\, d\tau.$$

Integrating both sides we obtain

$$\log\left(\tan\frac{\Phi}{2}\right) + C = -\int_{\tau_i}^{\tau}\left(\sigma_1\sqrt{\frac{\sigma(\rho-K_2)(\rho-K_3)}{\rho}} + \sigma_2\sqrt{\frac{\sigma\rho(\rho-K_3)}{\rho-K_2}} + \sigma_3\sqrt{\frac{\sigma\rho(\rho-K_2)}{\rho-K_3}}\right)\, d\tau, \tag{3.47}$$

where τ_i is a constant.

In general, to satisfy the initial condition, we require

$$C = -\log\left(\tan\frac{\Phi_i}{2}\right), \tag{3.48}$$

where $\Phi_i = \Phi(\tau=\tau_i)$. Then the general solution for Φ is given by

$$\Phi(\tau) = 2\arctan\left\{\tan\left(\frac{\Phi_i}{2}\right)\,\exp\left[\int_{\tau_i}^{\tau} f(\rho(t))\,dt\right]\right\}, \tag{3.49}$$

where

$$f(\rho) = -\sigma_1\sqrt{\frac{\sigma(\rho-K_2)(\rho-K_3)}{\rho}} - \sigma_2\sqrt{\frac{\sigma\rho(\rho-K_3)}{\rho-K_2}} - \sigma_3\sqrt{\frac{\sigma\rho(\rho-K_2)}{\rho-K_3}} \tag{3.50}$$

with $\rho(\tau)$ given in (3.40).

Finally, we want to recover the original functions $a_m(\tau)$ for $m=1,2,3$ and for real τ. It is straightforward to recover the magnitude of each mode. First of all, since $\rho(\tau)=\sigma_1|a_1(\tau)|^2$, it follows that

$$|a_1(\tau)| = \sqrt{\sigma_1\rho(\tau)}.$$

Note that sign $(\rho)=\sigma_1$ (unless $\rho=0$), so the term under the square root is necessarily nonnegative. Next, consider (2.30)-(2.31). Rearranging slightly yields

$$|a_m(\tau)|^2 = \sigma_m\left(\rho-K_m\right), \quad m=2,3.$$

It follows that

$$|a_2(\tau)| = \sqrt{\sigma_2\,(\rho - K_2)} \quad \text{and} \quad |a_3(\tau)| = \sqrt{\sigma_3\,(\rho - K_3)}.$$

Again, notice that $\operatorname{sign}(\rho - K_m) = \sigma_m$ for $m = 2, 3$, so the quantities under the square roots are nonnegative. Since $\rho(\tau)$ is known, we can determine $|a_m(\tau)|$ for $m = 1, 2, 3$. In particular, using the definition of $\rho(\tau)$ in (3.44), we have

$$|a_1(\tau)|^2 = \sigma_1 \left[\sigma\wp(\tau - \tau_0; g_2, g_3) + \frac{K_2 + K_3}{3}\right], \tag{3.51}$$

$$|a_2(\tau)|^2 = \sigma_2 \left[\sigma\wp(\tau - \tau_0; g_2, g_3) + \frac{K_3 - 2K_2}{3}\right], \tag{3.52}$$

$$|a_3(\tau)|^2 = \sigma_3 \left[\sigma\wp(\tau - \tau_0; g_2, g_3) + \frac{K_2 - 2K_3}{3}\right]. \tag{3.53}$$

More succinctly, we have

$$|a_m(\tau)|^2 = \sigma_m \left[\sigma\wp(\tau - \tau_0; g_2, g_3) + C_m\right], \tag{3.54}$$

where C_m is a constant defined by

$$C_1 = \frac{K_2 + K_3}{3}, \tag{3.55}$$

$$C_2 = \frac{K_3 - 2K_2}{3}, \tag{3.56}$$

$$C_3 = \frac{K_2 - 2K_3}{3}. \tag{3.57}$$

We can take a square root of each side of (3.54) to obtain $|a_m(\tau)|$ for $m = 1, 2, 3$.

It remains to find the phases $\varphi_m(\tau)$ for $m = 1, 2, 3$. Recall from (3.3b) that $i\varphi_m = Q_m$, and $Q_m(\tau)$ satisfies the ODE in (3.10),

$$\frac{dQ_m}{d\tau} = -\sigma_m \sqrt{\frac{\sigma P_k P_\ell}{P_m}}\,\sinh\tilde{Q}, \tag{3.58}$$

where $(k, \ell, m) = (1, 2, 3)$ cyclically, $\tilde{Q} = Q_1 + Q_2 + Q_3$, and P_m is defined in (3.3a) for $m = 1, 2, 3$. In particular, we have that $P_m = \sigma_m |a_m|^2$, or $|a_m|^2 = \sigma_m P_m$. Consequently, the square root term in (3.58) can be written

$$\sqrt{\frac{|a_k(\tau)|^2\,|a_\ell(\tau)|^2}{|a_m(\tau)|^2}} = \frac{|a_k(\tau)|\,|a_\ell(\tau)|}{|a_m(\tau)|}.$$

Lastly, we can rewrite (3.58) in terms of φ_m as follows

$$\frac{d\varphi_m}{d\tau} = -\sigma_m \frac{|a_k(\tau)|\,|a_\ell(\tau)|}{|a_m(\tau)|} \sin\Phi,$$

where we used $\sinh\tilde{Q} = i\sin\Phi$. Integration yields

$$\varphi_m(\tau) = -\sigma_m \int_{\tau_i}^{\tau} \frac{|a_k(t)|\,|a_\ell(t)|}{|a_m(t)|} \sin\left[\Phi(t)\right] dt + \varphi_m(\tau_i), \tag{3.59}$$

where τ_i is a constant, $|a_m(\tau)|$ is given via (3.54) for $m = 1, 2, 3$, and $\Phi(\tau)$ is defined in (3.49). Finally, recall the definition of H in (3.5)

$$H = -2\,|a_1(\tau)a_2(\tau)a_3(\tau)| \sin\left[\Phi(t)\right].$$

As a result, we can rewrite (3.59) as

$$\begin{aligned}\varphi_m(\tau) &= -\sigma_m \int_{\tau_i}^{\tau} \frac{|a_k(t)|\,|a_\ell(t)|}{|a_m(t)|} \cdot \frac{H}{-2\,|a_1(t)a_2(t)a_3(t)|}\, dt + \varphi_m(\tau_i)\\ &= \frac{\sigma_m H}{2} \int_{\tau_i}^{\tau} \frac{1}{|a_m(t)|^2}\, dt + \varphi_m(\tau_i).\end{aligned} \tag{3.60}$$

Consequently, (3.54) and (3.60) define the general solution of the three-wave ODEs in (2.20) with $a_m(\tau) = |a_m(\tau)|\, e^{i\varphi_m(\tau)}$ for $m = 1, 2, 3$. We refer to this solution as the "Weierstrass solution." Note that while we derived the solution of the three-wave ODEs in terms of Weierstrass elliptic functions, we could have equivalently derived the general solution in terms of Jacobi elliptic functions. Some discussion of Jacobi elliptic functions is found in Section 4.4.

3.6 Analysis of solutions

In sections 3.3-3.5, we found the solution of the three-wave ODEs in terms of Weierstrass elliptic functions. We now investigate the behavior of these solutions.

An important aspect of our solution is that there are six free constants, since the three-wave ODEs constitute six real-valued equations. There are several ways in which to choose these six constants. One possibility is to prescribe initial data for $a_m(\tau)$ at some value $\tau = \tau_i$. In other words, we choose $|a_m(\tau_i)|$ and $\varphi_m(\tau_i)$ for $m = 1, 2, 3$ (or equivalently, we prescribe $P_m(\tau = \tau_i)$ and $Q_m(\tau = \tau_i)$, with P_m and Q_m defined in (3.3)). This amounts to six real constants.

Consider a specific example in which we prescribe initial data along $\tau = 0$. Let

$$a_1(0) = 3e^{i\frac{\pi}{3}}, \quad a_2(0) = 2e^{i\frac{\pi}{4}}, \quad \text{and} \quad a_3(0) = e^{-i\frac{\pi}{3}}. \tag{3.61}$$

We know from (3.5) that $H = -6\sqrt{2}$. Now we consider an explosive case and a nonexplosive case for the initial data in (3.61). In particular, we analyze the behavior of the magnitudes of $a_m(\tau)$, and then we analyze the behavior of the phases.

3.6.1 The explosive case

Suppose $\sigma_1 = \sigma_2 = \sigma_3 = 1$. In this case, (2.30)-(2.31) and (3.61) tell us that $K_2 = 5$ and $K_3 = 8$. Now that we have K_2, K_3, and H, we can determine the elliptic invariants, g_2 and g_3, of the Weierstrass function in (3.44) using (3.45)-(3.46). This determines how the poles of $\rho(\tau)$ are arranged in the complex τ-plane. In particular, we find that

$$\Delta = g_2^3 - 27g_3^2 > 0,$$

so we know that in the complex τ-plane, poles occur on a rectangular lattice (this structure is described in Section 3.4). Using Mathematica's built-in functionality,[1] we find that the half-periods of $\rho(\tau)$ for the known values of g_2 and g_3 are given by

$$\lambda_1 \approx 0.630 \quad \text{and} \quad \lambda_2 \approx 0.682i. \tag{3.62}$$

It remains to determine the value of τ_0 in (3.44). Since $\rho(0)$ is known (in fact, from (3.61), we have $\rho(0) = 9$), we can find τ_0 by inverting the following

$$\rho(0) = \sigma\wp(-\tau_0; g_2, g_3) + \frac{K_2 + K_3}{3}.$$

In our example, we find that

$$\tau_0 \approx 0.528. \tag{3.63}$$

[1] WeierstrassHalfPeriods[$\{g_2, g_3\}$] Support Article, Wolfram Research, Inc., `http://functions.wolfram.com/EllipticFunctions/WeierstrassHalfPeriods/introductions/WeierstrassUtilities/05/`, 1998-2015.

It follows that the poles of $\rho(\tau)$ occur at $\tau_0 \pm 2n\lambda_1 \pm 2p\lambda_2$, $n, p \in \mathbb{Z}$. Most significantly, we observe that τ_0 is real, which means that poles necessarily occur along the real axis, and hence $\rho(\tau)$ blows up in finite time for real τ. Note that since $\rho(\tau)$ is doubly periodic, we could replace τ_0 with $\tau_0 \pm 2p\lambda_2$, $p \in \mathbb{Z}$, and obtain the same solution. In this case, τ_0 has nonzero imaginary part, but the imaginary part is equal to a multiple of the function's period in the imaginary direction. As a result, $\rho(\tau)$ still has poles along the real axis.

Figure 3.4 shows a contour plot of $|\rho(\tau)|$ for complex τ, in which we see the periodic lattice of the Weierstrass elliptic function. The shading on the plot is darker at low values and lighter at high values, meaning the poles are at the center of the white spots in the plot. Notice that the poles are separated by $2\lambda_1 \approx 1.261$ in the real direction and $2\lambda_2 \approx 1.364i$ in the imaginary direction, and that poles lie along the real axis. In particular, there is a pole at $\tau = \tau_0$, with τ_0 given in (3.63).

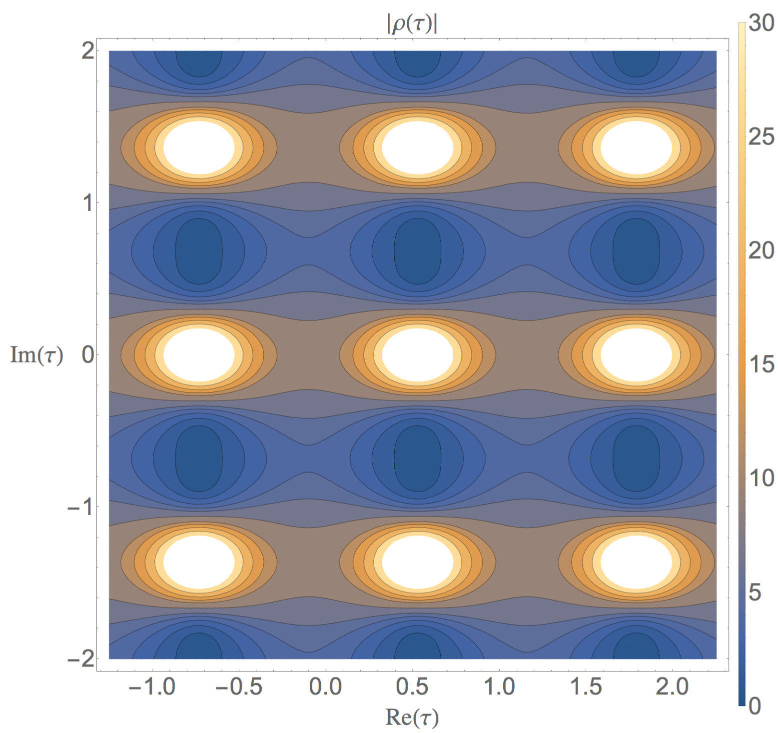


Figure 3.4: Contour plot of $|\rho(\tau)|$ in the explosive regime, where $\rho(\tau)$ is given in (3.44). We use $\sigma_1 = \sigma_2 = \sigma_3 = 1$ and the initial data in (3.61). This yields $K_2 = 5, K_3 = 8$, $H = -6\sqrt{2}$, and $\tau_0 \approx 0.528$. Notice that the poles occur along the real axis, which means solutions of the three-wave ODEs exhibit the explosive instability for real τ.

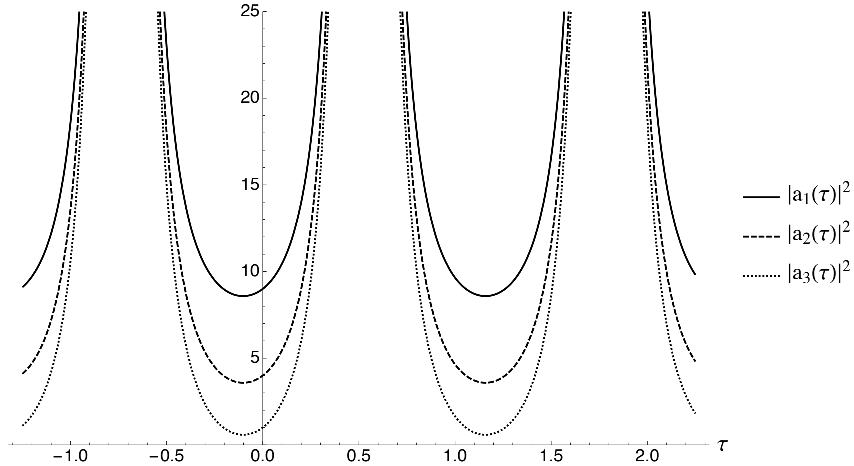


Figure 3.5: Plot of $\rho(\tau) = \sigma_1|a_1(\tau)|^2$, as well as $|a_2(\tau)|^2$ and $|a_3(\tau)|^2$ for real τ in the in the explosive regime, where $\rho(\tau)$ is given in (3.44). We use $\sigma_1 = \sigma_2 = \sigma_3 = 1$ and the initial data in (3.61). This yields $K_2 = 5, K_3 = 8$, $H = -6\sqrt{2}$, and $\tau_0 \approx 0.528$. Notice that solutions blow up in finite time, and that the half-period in the real direction is $\lambda_1 \approx 0.571$.

Figure 3.5 shows the unbounded behavior of $|a_m(\tau)|^2$ for $m = 1, 2, 3$ and for τ real. Since $\rho(\tau) = \sigma_1 |a_1(\tau)|^2$ when τ is real, the curve for $|a_1(\tau)|^2$ in Figure 3.5 can be compared to the behavior of $|\rho(\tau)|$ along the real axis in Figure 3.4. The poles along the real axis in Figure 3.4 correspond to the locations of the blow-ups seen in Figure 3.5.

Next, we consider the behavior of $\Phi(\tau)$ in the explosive case, and subsequently the behavior of the phases $\varphi_m(\tau)$ for $m = 1, 2, 3$. First of all, when considering the behavior of $\Phi(\tau)$, we restrict our attention to real values of τ that lie between two adjacent poles. Specifically, we restrict our attention to values of τ such that $\tau_0 - 2\lambda_1 < \tau < \tau_0$, where λ_1 and τ_0 are given in (3.62)-(3.63). In other words, we consider values of τ between the adjacent poles that occur on either side of $\tau = 0$. Now we can consider the limiting behavior of $\Phi(\tau)$ as $\tau \to \tau_0^-$ or $\tau \to (\tau_0 - 2\lambda_1)^+$.

Consider the solution for $\Phi(\tau)$ in (3.49). In order to determine how $\Phi(\tau)$ behaves as $\tau \to \tau_0^-$ or $\tau \to (\tau_0 - 2\lambda_1)^+$, we must first investigate the behavior of

$$\lim_{\tau \to \hat{\tau}} \exp\left[\int_0^\tau f(\rho(t))\, dt\right],$$

where $\hat{\tau} = \tau_0^-$ or $\hat{\tau} = (\tau_0 - 2\lambda_1)^+$, and $f(\rho(\tau))$ is defined in (3.50). As $\tau \to \tau_0^-$ or $\tau \to (\tau_0 - 2\lambda_1)^+$,

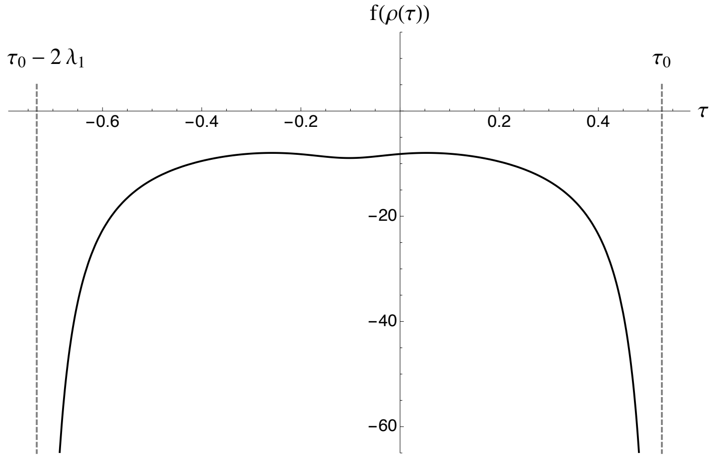


Figure 3.6: Plot of $f(\rho(\tau))$ in the explosive regime, where $\rho(\tau)$ is given in (3.44) and $f(\rho)$ is given in (3.50). We use $\sigma_1 = \sigma_2 = \sigma_3 = 1$ and the initial data in (3.61). This yields $K_2 = 5, K_3 = 8$, $H = -6\sqrt{2}$, and $\tau_0 \approx 0.528$. We plot $f(\rho(\tau))$ for $\tau_0 - 2\lambda_1 < \tau < \tau_0$, with the upper and lower bounds being the closest poles to $\tau = 0$ in $\rho(\tau)$.

we know that $\rho(\tau) \to +\infty$ (for $\sigma_1 = \sigma_2 = \sigma_3 = 1$). Then it is straightforward from the definition of $f(\rho)$ to see that $f(\rho) \to -\infty$ as $\tau \to \tau_0^-$ or $\tau \to (\tau_0 - 2\lambda_1)^+$. In fact, $f(\rho)$ is depicted in Figure 3.6, and it is clear that $f(\rho) \to -\infty$ when $\tau \to \tau_0$ or $\tau \to (\tau_0 - 2\lambda_1)^+$, with λ_1 and τ_0 given in (3.62) and (3.63), respectively.

It follows that

$$\lim_{\tau \to \tau_0^-} \int_0^\tau f(\rho(t))\, dt = -\infty,$$

and thus

$$\lim_{\tau \to \tau_0^-} \exp\left[\int_0^\tau f(\rho(t))\, dt\right] = 0.$$

As a result, we know from (3.49) that

$$\lim_{\tau \to \tau_0^-} \Phi(\tau) = 2 \arctan 0 = 0.$$

On the other hand, we have that

$$\lim_{\tau \to (\tau_0 - 2\lambda_1)^+} \int_0^\tau f(\rho(t))\, dt = -\lim_{\tau \to (\tau_0 - 2\lambda_1)^+} \int_\tau^0 f(\rho(t))\, dt = +\infty.$$

Consequently, we have that

$$\lim_{\tau \to (\tau_0 - 2\lambda_1)^+} \exp\left[\int_0^\tau f(\rho(t))\, dt\right] = +\infty.$$

As a result, we know from (3.49) that

$$\lim_{\tau \to (\tau_0 - 2\lambda_1)^+} \Phi(\tau) = \lim_{u \to +\infty} 2 \arctan u = 2 \cdot \frac{\pi}{2} = \pi.$$

Figure 3.7 depicts the behavior of $\Phi(\tau)$ when $\sigma_1 = \sigma_2 = \sigma_3 = 1$ for the initial condition in (3.61). It is clear from the figure that $\Phi(\tau) \to 0$ as $\tau \to \tau_0^-$, and $\Phi(\tau) \to \pi$ as $\tau \to (\tau_0 - 2\lambda_1)^+$ where λ_1 and τ_0 are given in (3.62) and (3.63), respectively.

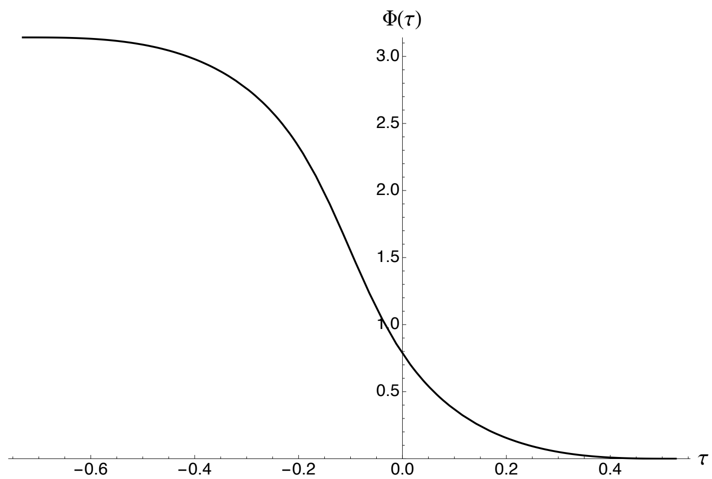


Figure 3.7: Plot of $\Phi(\tau)$ in the explosive regime, where $\Phi(\tau)$ is given in (3.49). We use $\sigma_1 = \sigma_2 = \sigma_3 = 1$ and the initial data in (3.61). This yields $K_2 = 5, K_3 = 8$, $H = -6\sqrt{2}$, and $\tau_0 \approx 0.528$. We plot $\Phi(\tau)$ for $\tau_0 - 2\lambda_1 < \tau < \tau_0$, with the upper and lower bounds being the closest poles to $\tau = 0$ in $\rho(\tau)$. Notice that $\Phi \to 0$ as $\tau \to \tau_0^-$, while $\tau \to \pi$ as $\tau \to (\tau_0 - 2\lambda_1)^+$.

Finally, we investigate the behavior of $\varphi_m(\tau)$, $m = 1, 2, 3$, where $\varphi_m(\tau)$ is given in (3.60). The behavior of $\varphi_m(\tau)$ is best understood by looking at its leading order behavior. We show in the next section that

$$\varphi_m(\tau) = \frac{H}{2} \int_0^\tau \left[(t - \tau_0)^2 - C_m (t - \tau_0)^4 + \mathcal{O}\left((t - \tau_0)^6 \right) \right] \, dt + \varphi_m(0), \tag{3.64}$$

where we set $\tau_i = 0$ in (3.60), and C_m is defined in (3.55)-(3.57). Integrating, we find that

$$\varphi_m(\tau) = \varphi_m(0) + \frac{H}{6} \left[(\tau - \tau_0)^3 + \tau_0^3 \right] - \frac{HC_m}{10} \left[(\tau - \tau_0)^5 + \tau_0^5 \right] + \mathcal{O}\left((\tau - \tau_0)^7 \right) + D,$$

where D is a constant that incorporates all the remaining terms that come from evaluating the integral's antiderivative at $\tau = 0$. As $\tau \to \tau_0^-$, every $(\tau - \tau_0)$ term approaches zero, and we are left with

$$\lim_{\tau\to\tau_0^-} \varphi_m(\tau) = \varphi_m(0) + \frac{H}{6}\tau_0^3 - \frac{HC_m}{10}\tau_0^5 + D.$$

Substituting $\varphi_m(0)$ and τ_0 from (3.61) and (3.63), respectively, we obtain

$$\lim_{\tau\to\tau_0^-} \varphi_1(\tau) \approx 0.99 + D, \quad \lim_{\tau\to\tau_0^-} \varphi_2(\tau) \approx 0.55 + D, \quad \text{and} \quad \lim_{\tau\to\tau_0^-} \varphi_3(\tau) \approx -1.38 + D. \tag{3.65}$$

Figure 3.8 depicts the behavior of $\varphi_m(\tau)$ for $m = 1, 2, 3$ on $\tau_0 - 2\lambda_1 < \tau < \tau_0$. We can see that while (3.65) is not exact (since D is unknown), it does give a good approximation of the behavior of the phases as τ approaches τ_0, as long as D is small.

In order to determine the behavior of $\varphi_m(\tau)$ when τ approaches $\tau_0 - 2\lambda_1$, we follow the same procedure, but we expand about $\tau_0 - 2\lambda_1$ instead of τ_0 in (3.64). In this case, we find that $\varphi_m(\tau)$ increases away from $\varphi_m(0)$ as $\tau \to (\tau_0 - 2\lambda_1)^+$, and again approaches a constant. We omit the details, but the behavior is seen in Figure 3.8.

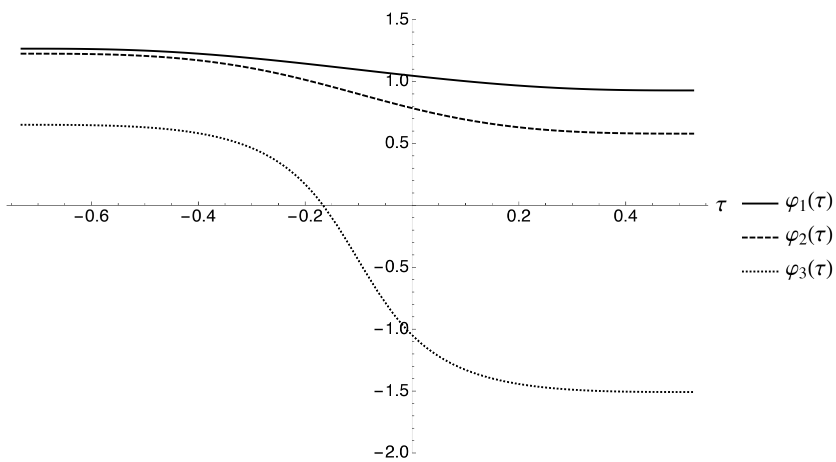


Figure 3.8: Plot of $\varphi_m(\tau)$ for $m = 1, 2, 3$ in the nonexplosive regime, where $\varphi_m(\tau)$ is given in (3.60). We use $\sigma_1 = \sigma_2 = \sigma_3 = 1$ and the initial data in (3.61). This yields $K_2 = 5, K_3 = 8,\ H = -6\sqrt{2}$, and $\tau_0 \approx 0.528$.

3.6.2 The nonexplosive case

Suppose $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. This time (2.30)-(2.31) and (3.61) tell us that $K_2 = 13$ and $K_3 = 10$. Then we determine the elliptic invariants, g_2 and g_3, of the Weierstrass function in (3.44) using (3.45)-(3.46). In this case, we find that

$$\Delta = g_2^3 - 27g_3^2 > 0,$$

so we know that in the complex τ-plane, poles occur on a rectangular lattice. We find that the half-periods of $\rho(\tau)$ are given by

$$\lambda_1 \approx 0.571 \quad \text{and} \quad \lambda_2 \approx 0.470i. \tag{3.66}$$

As in the explosive case, it remains to determine the value of τ_0 in (3.44), which we determine by applying the initial condition. In our example, we find that

$$\tau_0 \approx -0.460 + 0.470i. \tag{3.67}$$

Again, the poles of $\rho(\tau)$ occur at $\tau_0 \pm 2n\lambda_1 \pm 2p\lambda_2$, $n, p \in \mathbb{Z}$. Most significantly, we observe that τ_0 has a nonzero imaginary part. If the nonzero imaginary part is equal to a whole period in the imaginary direction (that is, if $\mathrm{Im}\,(\tau_0) = 2n\lambda_2$, $n \in \mathbb{Z}$), then the poles of $\rho(\tau)$ lie along the real axis, since the Weierstrass elliptic function naturally has poles along the real axis. This was the true of the explosive case we considered previously. In this nonexplosive case, however, we have that $\mathrm{Im}\,(\tau_0) = \lambda_2$. As a result, the poles of the Weierstrass function are shifted away from the real axis by one half period in the imaginary direction. Consequently, $\rho(\tau)$ has no poles along the real axis. More importantly, since $\rho(\tau)$ is doubly periodic with a rectangular lattice, the poles are aligned in such a way that for a given pole above the real axis, there is another pole below and equidistant from the real axis. Because the poles above the real axis align with the poles below the real axis in this way, $\rho(\tau)$ remains real-valued and bounded along the real axis.

Figure 3.9 shows a contour plot of $|\rho(\tau)|$ for complex τ, in which we see the rectangular periodic lattice of the Weierstrass elliptic function. As with Figure 3.4, the shading on the plot is darker at low values and lighter at high values, meaning the poles are at the center of the white spots in the plot. Notice that the poles are separated by $2\lambda_1 \approx 1.142$ in the real direction and $2\lambda_2 \approx -0.941i$ in the imaginary direction, and that poles do not lie along the real axis. Instead, the

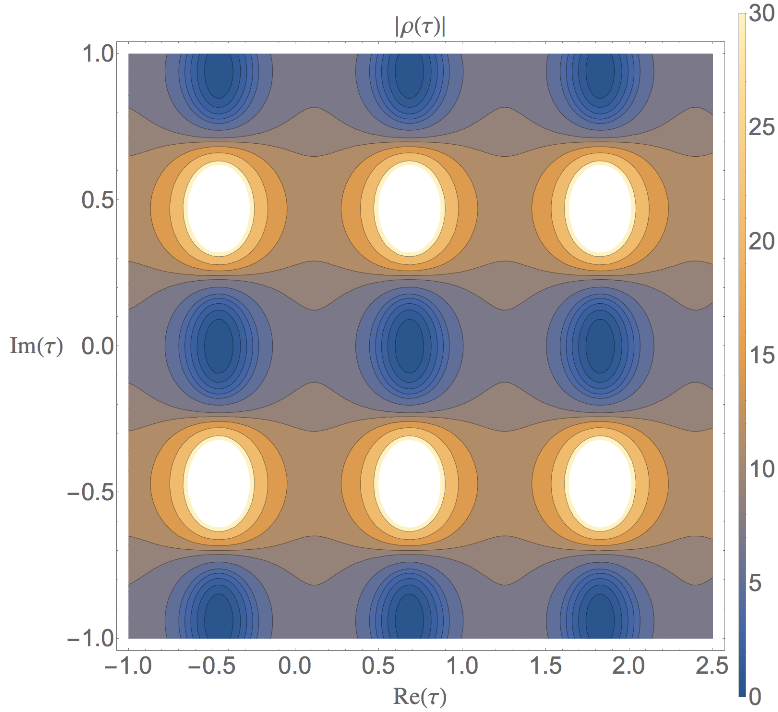


Figure 3.9: Contour plot of $|\rho(\tau)|$ in the nonexplosive regime, where $\rho(\tau)$ is given in (3.44). We use $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$ and the initial data in (3.61). This yields $K_2 = 13, K_3 = 10$, $H = -6\sqrt{2}$, and $\tau_0 \approx -0.460 + 0.470i$. Notice that the poles occur off the real axis, which means solutions of the three-wave ODES are bounded for real τ.

poles above the real axis are reflected across the real axis. In particular, there is a pole at $\tau = \tau_0$, with τ_0 given in (3.67).

Figure 3.10 shows the bounded periodic behavior of $|a_m(\tau)|^2$ for $m = 1, 2, 3$ and for τ real. Since $\rho(\tau) = \sigma_1 |a_1(\tau)|^2$ when τ is real, the curve for $|a_1(\tau)|^2$ in Figure 3.10 can be compared to the behavior of $|\rho(\tau)|$ along the real axis in Figure 3.9.

Next we consider the behavior of $\Phi(\tau)$ for real τ in the nonexplosive case. There is no need to restrict our attention to a specific domain of τ, since there are no poles along the real axis. First, consider the behavior of $f(\rho(\tau))$ as defined in (3.50). $f(\rho(\tau))$ is bounded and periodic with period $2\lambda_1$, and minimum values at $\tau = \mathrm{Re}\,(\tau_0) \pm 2n\lambda_1$, $n \in \mathbb{Z}$. Additionally, $f(\rho(\tau))$ has zero mean over a period. As a result, it follows from the definition of $\Phi(\tau)$ in (3.49) that $\Phi(\tau)$ is bounded and periodic as well.

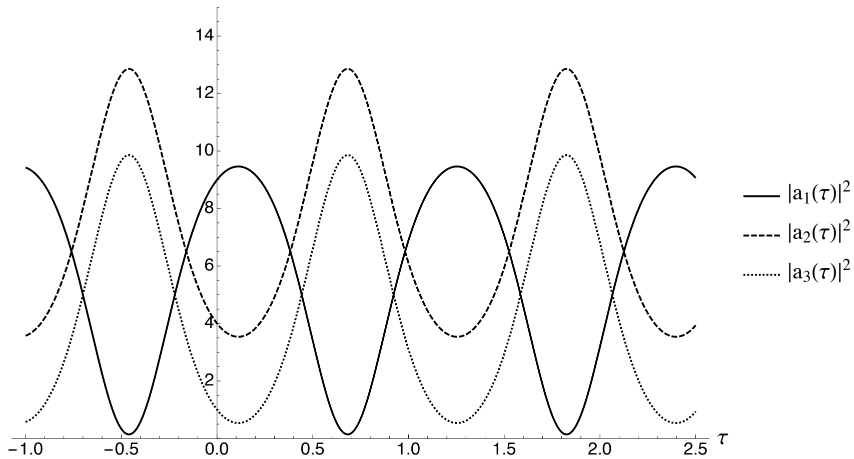


Figure 3.10: Plot of $\rho(\tau) = \sigma_1|a_1(\tau)|^2$, as well as $|a_2(\tau)|^2$ and $|a_3(\tau)|^2$ for real τ in the nonexplosive regime, where $\rho(\tau)$ is given in (3.44). We use $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$ and the initial data in (3.61). This yields $K_2 = 13, K_3 = 10$, $H = -6\sqrt{2}$, and $\tau_0 \approx -0.460 + 0.470i$. Notice that solutions are bounded and periodic along the real τ-axis.

Figure 3.11 depicts $f(\rho(\tau))$ when $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$ for the initial condition in (3.61). Figure 3.12 depicts the behavior of $\Phi(\tau)$ under the same conditions. It is clear from the figures that both $f(\rho(\tau))$ and $\Phi(\tau)$ are bounded and periodic.

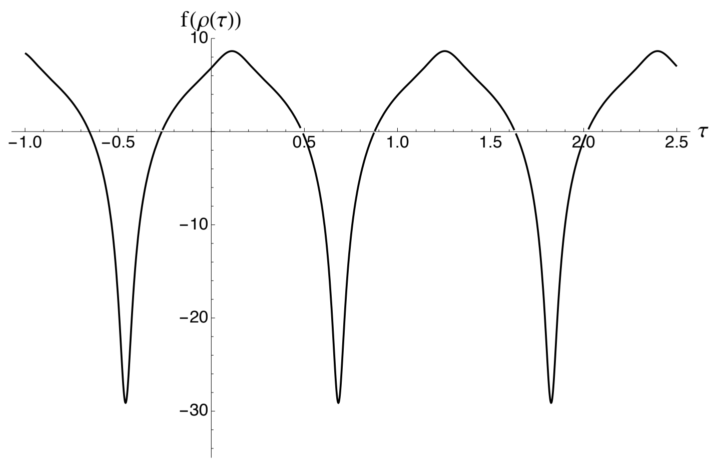


Figure 3.11: Plot of $f(\rho(\tau))$ in the nonexplosive regime, where $\rho(\tau)$ is given in (3.44) and $f(\rho)$ is given in (3.50). We use $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$ and the initial data in (3.61). This yields $K_2 = 13, K_3 = 10$, $H = -6\sqrt{2}$, and $\tau_0 \approx -0.460 + 0.470i$.

Finally, Figure 3.13 depicts the behavior of $\varphi_m(\tau)$ for $m = 1, 2, 3$. Recall the solution for

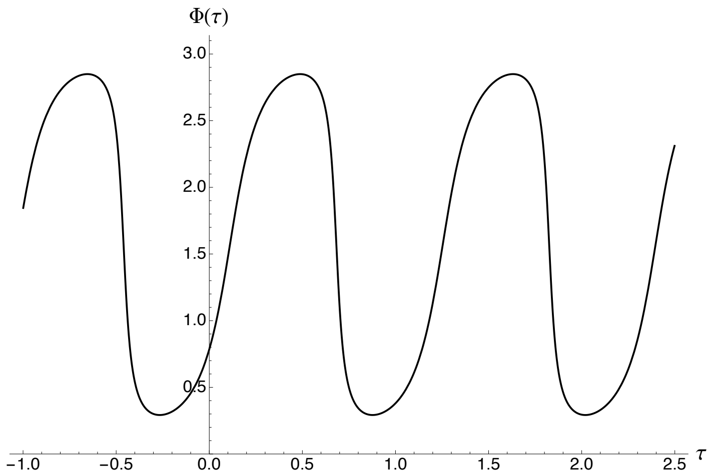


Figure 3.12: Plot of $\Phi(\tau)$ in the nonexplosive regime, where $\Phi(\tau)$ is given in (3.49). We use $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$ and the initial data in (3.61). This yields $K_2 = 13, K_3 = 10$, $H = -6\sqrt{2}$, and $\tau_0 \approx -0.460 + 0.470i$. Notice the bounded periodic behavior of $\Phi(\tau)$.

$\varphi_m(\tau)$ in (3.60). We see in Figure 3.10 that $|a_m(\tau)|$ is bounded, periodic, and positive for $m = 1, 2, 3$. As a result, the integrand in (3.60) is also bounded, periodic, and positive. The integral in (3.60) amounts to adding up the area under the curve of $1/|a_m(\tau)|^2$. Consequently, for each m, $\varphi_m(\tau)$ has a mean value that is either monotonically increasing or decreasing with τ, depending on σ_m, and $\varphi_m(\tau)$ oscillates periodically about this uniformly changing mean.

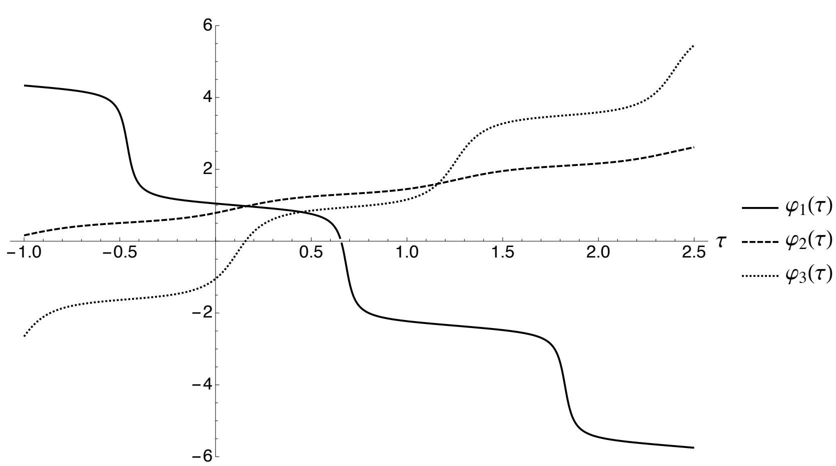


Figure 3.13: Plot of $\varphi_m(\tau)$ for $m = 1, 2, 3$ in the nonexplosive regime, where $\varphi_m(\tau)$ is given in (3.60). We use $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$ and the initial data in (3.61). This yields $K_2 = 13, K_3 = 10$, $H = -6\sqrt{2}$, and $\tau_0 \approx -0.460 + 0.470i$.

We just investigated a particular explosive and nonexplosive case of the three-wave ODEs, given the initial conditions in (3.61). At the beginning of this section, we explained that the Weierstrass general solution of the three-wave ODEs outlined in (3.54) and (3.60) has six free constants, since the three-wave ODEs constitute six real-valued equations. In our examples, we assumed that the six free constants were specified using the initial data $|a_m(\tau)|$ and $\varphi_m(\tau)$ for $m = 1, 2, 3$. This is just one possibility of how to choose the six constants.

An alternative choice is to pick the six constants to be K_2, K_3, H, two of $\{\varphi_1(\tau_i), \varphi_2(\tau_i), \varphi_3(\tau_i)\}$, and the real part of τ_0. First of all, the choice of K_2, K_3, and H determines how the poles of $\rho(\tau)$ are arranged in the complex plane. From (3.44), we know that the locations of the poles of $\rho(\tau)$ are determined by

$$\wp(\tau - \tau_0; g_2, g_3),$$

where g_2 and g_3 are given in (3.45)-(3.46). In particular, $g_2 = g_2(K_2, K_3)$ and $g_3 = g_3(K_2, K_3, H)$. Recall from Section 3.4 that if g_2 and g_3 are real (which is always true for us since K_2, K_3, and H are real), then $\wp(\tau; g_2, g_3)$ is real for real τ. As a result, we know if τ_0 is real and arbitrary, then $\wp(\tau - \tau_0; g_2, g_3)$ (and thus $\rho(\tau)$) is still real along the real axis. It remains to determine the allowed values for the imaginary part of τ_0.

When g_2 and g_3 are real, we know that the poles of $\wp(\tau; g_2, g_3)$ occur in either a rhombic periodic lattice or a rectangular periodic lattice. We define the discriminant of $\wp(\tau - \tau_0; g_2, g_3)$ as

$$\Delta = g_2^3 - 27g_3^2. \tag{3.68}$$

Then the poles occur in a rhombic lattice if $\Delta < 0$, and in a rectangular lattice if $\Delta > 0$. In both of the examples we considered above, we found that $\Delta > 0$. As a result, the poles of $\rho(\tau)$ occurred in a rectangular lattice, as seen in Figures 3.4 and 3.9. The key feature that distinguished the explosive case from the nonexplosive case was the choice of the imaginary part of τ_0. Indeed, $\mathrm{Im}\,(\tau_0)$ was responsible for determining whether the poles of $\rho(\tau)$ occurred along the real axis (the explosive case) or off the real axis (the nonexplosive case).

Recall that for a rectangular lattice, the half-periods λ_1 and λ_2 of the Weierstrass function are such that both λ_1 and λ_2/i are real and positive. When $\Delta > 0$ there are two possibilities for choosing the imaginary part of τ_0:

(1) The explosive case, $\sigma_1 = \sigma_2 = \sigma_3$: $\mathrm{Im}\,(\tau_0) = 2n|\lambda_2|$, $n \in \mathbb{Z}$ (the simplest example of which is $\mathrm{Im}\,(\tau_0) = 0$).

(2) The nonexplosive case, $\sigma_1 = -\sigma_2 = -\sigma_3$: $\mathrm{Im}\,(\tau_0) = (2n+1)|\lambda_2|$, $n \in \mathbb{Z}$.

These are the only possible choices of $\mathrm{Im}\,(\tau_0)$ that keep $\wp(\tau - \tau_0; g_2, g_3)$, and $\rho(\tau)$, real along the real axis. In the first case, the poles necessarily occur along the real axis. In the second case, the poles are shifted off the real axis by exactly one half period in the imaginary direction. Since the lattice is rectangular, the poles above the real axis are then reflected below the real axis, which forces $\wp(\tau - \tau_0; g_2, g_3)$ to be real along the real axis.

If the lattice associated with $\wp(\tau - \tau_0; g_2, g_3)$ is rhombic (i.e. $\Delta < 0$), then the half-period λ_1 is real and positive, while the half-period λ_2 satisfies $\mathrm{Im}\,(\lambda_2) > 0$ and $\mathrm{Re}\,(\lambda_2) = \lambda_1/2$. In this case, we are restricted to the explosive regime only. If τ_0 is real or $\tau_0 = 2n\lambda_2$, then the poles occur along the real axis, and $\rho(\tau)$ is real along the real axis. However, if $\mathrm{Im}\,(\tau_0) \neq 2n\lambda_2$, then $\rho(\tau)$ is not real along the real axis, which violates the definition $\rho(\tau) = \sigma_1\,|a_1(\tau)|^2$. Thus we cannot move the poles away from the real axis in such a way that $\wp(\tau - \tau_0; g_2, g_3)$ is both bounded and real for real τ.

In summary, we have the following breakdown:

(1) The explosive case, $\sigma_1 = \sigma_2 = \sigma_3$:

 (a) Rectangular lattice, $\Delta > 0$: $\mathrm{Im}\,(\tau_0) = 2n|\lambda_2|$

 (b) Rhombic lattice, $\Delta < 0$: $\mathrm{Im}\,(\tau_0) = 2n|\lambda_2|$

(2) The nonexplosive case, $\sigma_1 = -\sigma_2 = -\sigma_3$:

 (a) Rectangular lattice, $\Delta > 0$: $\mathrm{Im}\,(\tau_0) = (2n+1)|\lambda_2|$

 (b) No possibility of a rhombic lattice.

Once K_2, K_3, H, and τ_0 have been chosen, we see from (3.54) that $|a_m(\tau)|$ is determined for $m = 1, 2, 3$. Moreover, from (3.19a), $\sin\Phi$ is determined. In particular, $\Phi(\tau_i)$ is determined (to within a multiple of $2n\pi$). Since $\Phi(\tau_i) = \varphi_1(\tau_i) + \varphi_2(\tau_i) + \varphi_3(\tau_i)$, it follows that we can only choose two of $\{\varphi_1(\tau_i), \varphi_2(\tau_i), \varphi_3(\tau_i)\}$ independently. Then the solution of the three-wave ODEs is fully determined.

There are other ways to choose the six free constants in the general Weierstrass solution, but from now on, we work in the context of the six free real-valued constants

$$\{K_2, K_3, H, \mathrm{Re}\,(\tau_0), \varphi_1(\tau_i), \varphi_2(\tau_i)\}\,. \tag{3.69}$$

Chapter 4

The Laurent Series Solution of the Three-Wave ODEs

In Chapter 3, we determined the general solution of the three-wave ODEs (2.20) in terms of Weierstrass elliptic functions. This solution is useful to us for many reasons throughout this thesis, as we see in the next two chapters. In this chapter, we derive the general solution of the three-wave ODEs in the explosive regime using an alternative method in order to motivate the method of solution for the three-wave PDEs in Chapter 5. In particular, we build a convergent Laurent series solution of the three-wave ODEs. We show that the Laurent series solution derived in this chapter is equivalent to the Weierstrass solution of the previous chapter, and discuss where the two solutions are known to converge. Finally, we end the chapter with a discussion of how our Laurent series solution can be extended to the nonexplosive regime.

4.1 The Painlevé conjecture

In this section we give some motivation for our construction of the general solution of the three-wave ODEs in terms of a Laurent series. We begin with some definitions and some history. First, a *critical point* is understood to be any singularity of a solution of an ODE that is not a pole of any order (thus critical points include branch points and essential singularities). A *movable* critical point is a critical point whose location depends on the constants of integration. At the beginning of the twentieth century, Painlevé and others investigated second-order nonlinear ODEs of the form

$$w'' = F(w', w, z), \tag{4.1}$$

where F is rational in w' and w, and locally analytic in z [23, Ch.14]. They determined that there are fifty canonical equations with the property of having no movable critical points. This property is now known as the Painlevé property. More specifically, a system of ODEs of any order is said to possess the Painlevé property if for every solution, every movable singularity is a pole [2, §3.7].

According to the Painlevé conjecture, proposed in 1978, a system of nonlinear PDEs is solvable by an inverse scattering transform (i.e., is integrable) only if every nonlinear ODE obtained from the PDE by an exact reduction has the Painlevé property, perhaps after a transformation of variables [2, §3.7.b]. There is no systematic way to find all reductions of a PDE, so the test is not definitive in determining which equations can be solved by IST. However, it is an effective tool for finding PDEs that cannot be solved by IST.

The Painlevé conjecture suggests a connection between the integrability of a nonlinear PDE and the Painlevé property. We propose to use a Painlevé-type singularity analysis in order to construct the general solution of a particular system of integrable nonlinear PDEs, namely the three-wave equations in (2.5). Indeed, the structure of the full three-wave equations (2.5) turns out to be so strongly linked to the structure of the three-wave ODEs, it is not surprising to discover that the Painlevé property, a property of ODEs, is useful in motivating the general solution of this set of PDEs.

The strategy is as follows. We start by constructing the general solution of the three-wave ODEs in (2.20) using a series solution. Since (2.20) is a six-dimensional system of equations, the general solution is one which satisfies the ODEs and has six constants of integration. Once we have the general solution of the ODEs, we use its structure in order to guess a solution form of the three-wave PDEs in (2.5).

To be precise, in order to find the general solution of the three-wave ODEs in (2.20), we use the following procedure:

(1) Find the dominant behavior of $a_m(\tau)$ in (2.20). Since (2.5) is completely integrable, the Painlevé conjecture dictates that the only allowable movable singularities are poles.

(2) Carry on the expansion to higher order until all six of the constants of integration appear. This is the beginning of the Laurent series of the general solution of (2.20).

(3) Finally, find all remaining coefficients in the expansion in terms of the free constants, and show that the series has a nonzero radius of convergence.

Subsequently, we use the same basic approach to construct the general solution of the three-wave PDEs in (2.5). This is outlined more thoroughly in Chapter 5.

4.2 Construction of the Laurent series solution

Recall that the three-wave PDEs are known to be integrable [47]. It follows from the Painlevé conjecture in the previous section that every set of ODEs obtained from the three-wave PDEs by an exact reduction has the Painlevé property. Since the three-wave ODEs can be obtained from the PDEs in a number of ways (see Chapter 6), it follows that the three-wave ODEs must possess the Painlevé property. That is, for every solution of the ODEs, every movable singularity is a pole. Indeed, the only singularities of the Weierstrass general solution in the previous section are movable poles.

Recall that the three-wave ODEs are given by

$$\frac{da_m}{d\tau} = \sigma_m a_k^* a_\ell^*, \tag{4.2}$$

where $(k, \ell, m) = (1, 2, 3)$ are defined cyclically. To derive our alternative general solution of the three-wave ODEs, we first want to find the dominant behavior of $a_m(\tau)$ near a movable singularity. We assume that there is a movable singularity at $\tau = \tau_0$, and that as $\tau \to \tau_0$,

$$a_m(\tau) \sim \frac{\alpha_m}{(\tau - \tau_0)^p}, \tag{4.3}$$

where α_m is a complex constant for $m = 1, 2, 3$, and p is the same positive number for $m = 1, 2, 3$. Additionally, assume that τ and τ_0 are real, so that $(\tau - \tau_0)^* = \tau - \tau_0$. Substituting (4.3) into (4.2), we obtain

$$-\frac{p\alpha_m}{(\tau - \tau_0)^{p+1}} \sim \sigma_m \frac{\alpha_k^* \alpha_\ell^*}{(\tau - \tau_0)^{2p}}. \tag{4.4}$$

It follows that the only choice is $p = 1$. As a result, we know that $a_m(\tau)$ behaves like a simple pole at leading order near $\tau = \tau_0$.

Next, we continue the expansion beyond leading order. If the singularity is a pole, then the expansion has the form

$$a_m(\tau) = \frac{1}{\tau - \tau_0}\left[\alpha_m + \beta_m(\tau-\tau_0) + \gamma_m(\tau-\tau_0)^2 + \delta_m(\tau-\tau_0)^3 + \cdots\right], \tag{4.5}$$

where $\alpha_m, \beta_m, \gamma_m$, and δ_m are complex constants for $m = 1, 2, 3$. Substituting (4.5) into (4.2) yields

$$\begin{aligned}
&\frac{1}{(\tau-\tau_0)^2}\left\{-\alpha_m + \gamma_m(\tau-\tau_0)^2 + 2\delta_m(\tau-\tau_0)^3 + \cdots\right\} \\
&\quad = \frac{\sigma_m}{(\tau-\tau_0)^2}\Big\{\alpha_k^*\alpha_\ell^* + (\alpha_k^*\beta_\ell^* + \alpha_\ell^*\beta_k^*)(\tau-\tau_0) + (\beta_k^*\beta_\ell^* + \alpha_k^*\gamma_\ell^* + \alpha_\ell^*\gamma_k^*)(\tau-\tau_0)^2 \\
&\qquad\qquad + (\alpha_k^*\delta_\ell^* + \alpha_\ell^*\delta_k^* + \beta_k^*\gamma_\ell^* + \beta_\ell^*\gamma_k^*)(\tau-\tau_0)^3 + \cdots\Big\}.
\end{aligned}$$

At lowest order, we find that

$$-\alpha_1 = \sigma_1\alpha_2^*\alpha_3^*, \quad -\alpha_2 = \sigma_2\alpha_1^*\alpha_3^* \quad \text{and} \quad -\alpha_3 = \sigma_3\alpha_1^*\alpha_2^*. \tag{4.6}$$

Assuming $\alpha_m \neq 0$ for $m = 1, 2, 3$, this implies that

$$|\alpha_m|^2 = \sigma_k\sigma_\ell,$$

where $(k, \ell, m) = (1, 2, 3)$ are defined cyclically as usual. This is only possible if $\sigma_1 = \sigma_2 = \sigma_3$, which means we are restricted to the explosive domain. This is consistent with our assumption that τ_0 is real, meaning that a pole lies along the real axis, and solutions blow up in finite time. For the remainder of this chapter, assume that we are in the explosive regime unless otherwise stated. In this case, we have

$$|\alpha_1| = |\alpha_2| = |\alpha_3| = 1.$$

Consequently, we write

$$\alpha_m = e^{i\psi_m}, \quad m = 1, 2, 3,$$

where ψ_m is real for $m = 1, 2, 3$. If we define the sum of the phases to be $\Psi = \psi_1 + \psi_2 + \psi_3$, then each equation in (4.6) reduces to

$$e^{i\Psi} = -\sigma,$$

$\sigma = \sigma_m = \sigma_1\sigma_2\sigma_3$ for $m = 1, 2, 3$. Consequently, (4.6) is satisfied as long as

$$\Psi \equiv \psi_1 + \psi_2 + \psi_3 = \begin{cases} 2n\pi, & \sigma = -1 \\ (2n+1)\pi, & \sigma = 1. \end{cases} \tag{4.7}$$

For simplicity, we now rewrite (4.5) as

$$a_m(\tau) = \frac{e^{i\psi_m}}{\tau - \tau_0}\left[1 + \beta_m(\tau - \tau_0) + \gamma_m(\tau - \tau_0)^2 + \delta_m(\tau - \tau_0)^3 + \cdots\right], \tag{4.8}$$

where the constants β_m, γ_m, and δ_m differ from those in (4.5) by a factor of $e^{i\psi_m}$.

We know by now that the three-wave ODEs constitute a sixth-order system, and its general solution has six real constants of integration. In the Laurent series solution posed in (4.8), the first of these constants is τ_0. Furthermore, two of $\{\psi_1, \psi_2, \psi_3\}$ are free constants, while the third is determined by (4.7). Thus, we have determined three real free constants so far (τ_0, ψ_1, and ψ_2). We must carry out our expansion (4.8) until we obtain the remaining three constants of integration.

To that end, substituting (4.8) into (4.2) yields

$$\begin{aligned}
&\frac{e^{i\psi_m}}{(\tau-\tau_0)^2}\left\{-1 + \gamma_m(\tau-\tau_0)^2 + 2\delta_m(\tau-\tau_0)^3 + \cdots\right\} \\
&\qquad = \frac{\sigma e^{-i(\psi_k+\psi_\ell)}}{(\tau-\tau_0)^2}\Big\{1 + (\beta_\ell^* + \beta_k^*)(\tau-\tau_0) + (\beta_k^*\beta_\ell^* + \gamma_\ell^* + \gamma_k^*)(\tau-\tau_0)^2 \\
&\qquad\qquad + (\delta_\ell^* + \delta_k^* + \beta_k^*\gamma_\ell^* + \beta_\ell^*\gamma_k^*)(\tau-\tau_0)^3 + \cdots\Big\},
\end{aligned}$$

where we used $\sigma_m = \sigma$, which holds if $\sigma_1 = \sigma_2 = \sigma_3$. Next, we multiply both sides by $(\tau-\tau_0)^2 e^{i(\psi_k+\psi_\ell)}$ and use $e^{i\Psi} = -\sigma$ to find that

$$\begin{aligned}
&-\gamma_m(\tau-\tau_0)^2 - 2\delta_m(\tau-\tau_0)^3 + \cdots \\
&= (\beta_\ell^* + \beta_k^*)(\tau-\tau_0) + (\beta_k^*\beta_\ell^* + \gamma_\ell^* + \gamma_k^*)(\tau-\tau_0)^2 + (\delta_\ell^* + \delta_k^* + \beta_k^*\gamma_\ell^* + \beta_\ell^*\gamma_k^*)(\tau-\tau_0)^3 + \cdots.
\end{aligned}$$

At order $(\tau - \tau_0)$, we must solve

$$0 = \beta_k^* + \beta_\ell^*, \tag{4.9}$$

and the conjugate equations. The solution is given by

$$\beta_1 = \beta_2 = \beta_3 = 0. \tag{4.10}$$

At order $(\tau-\tau_0)^2$ with $\beta_m = 0$ for $m = 1, 2, 3$, we have

$$-\gamma_m = \gamma_k^* + \gamma_\ell^*. \tag{4.11}$$

If we write $\gamma_m = u_m + iv_m$, where u_m and v_m are real constants for $m = 1, 2, 3$, then we obtain

$$u_1 + u_2 + u_3 = 0 \quad \text{and} \quad v_1 = v_2 = v_3 = 0.$$

We conclude that

$$\gamma_1 + \gamma_2 + \gamma_3 = 0 \quad \text{and} \quad \mathrm{Im}\,(\gamma_m) = 0. \tag{4.12}$$

Finally, at order $(\tau-\tau_0)^3$ with $\beta_m = 0$ for $m = 1, 2, 3$, we have

$$-2\delta_m = \delta_k^* + \delta_\ell^*, \tag{4.13}$$

We write $\delta_m = s_m + it_m$, where s_m and t_m are real constants for $m = 1, 2, 3$, and we obtain

$$s_1 = s_2 = s_3 = 0 \quad \text{and} \quad t_1 = t_2 = t_3.$$

We conclude that

$$\delta_1 = \delta_2 = \delta_3 \equiv \delta \quad \text{and} \quad \mathrm{Re}\,(\delta_m) = 0. \tag{4.14}$$

At this point, we rewrite (4.8) as

$$a_m(\tau) = \frac{e^{i\psi_m}}{\tau-\tau_0}\left[1 + \gamma_m(\tau-\tau_0)^2 + i\delta(\tau-\tau_0)^3 + \cdots\right], \tag{4.15}$$

where γ_m and δ are both real real.

Next, recall that the three-wave ODEs admit the conserved quantities in (2.26) and (2.30)-(2.31), restated below

$$\tilde{H} = -iH = a_1a_2a_3 - a_1^*a_2^*a_3^*, \tag{4.16}$$

$$K_2 = \sigma_1 \left|a_1(\tau)\right|^2 - \sigma_2 \left|a_2(\tau)\right|^2, \tag{4.17}$$

$$K_3 = \sigma_1 \left|a_1(\tau)\right|^2 - \sigma_3 \left|a_3(\tau)\right|^2, \tag{4.18}$$

where K_2, K_3, and H are real. Substituting (4.15) into (4.16) yields

$$-iH = \frac{e^{i\Psi}}{(\tau-\tau_0)^3}\left\{1+(\gamma_1+\gamma_2+\gamma_3)(\tau-\tau_0)^2+3i\delta(\tau-\tau_0)^3+\cdots\right\} \\ -\frac{e^{-i\Psi}}{(\tau-\tau_0)^3}\left\{1+(\gamma_1+\gamma_2+\gamma_3)(\tau-\tau_0)^2-3i\delta(\tau-\tau_0)^3+\cdots\right\}. \tag{4.19}$$

Since $e^{i\Psi} = e^{-i\Psi} = -\sigma$, the equation reduces to

$$-iH = -3i\sigma\delta - 3i\sigma\delta + \mathcal{O}(\tau-\tau_0),$$

which implies that

$$\delta = \frac{\sigma H}{6}.$$

This holds exactly; we discuss in the following two sections that the Laurent series of $a_m(\tau)$ converges in some deleted neighborhood of the pole at τ_0, and that the series for $a_m(\tau)$ must equal $a_m(\tau)$ inside this region. Since $a_m(\tau)$ satisfies (4.16), it follows that the Laurent series of $a_m(\tau)$ must also satisfy (4.16) exactly wherever the series converges. As a result, all nonconstant terms on the right-hand side of (4.19) are zero.

Lastly, substituting (4.15) into (4.17) and (4.18) gives

$$K_2 = \frac{\sigma_1}{(\tau-\tau_0)^2}\left\{1+2\gamma_1(\tau-\tau_0)^2+\cdots\right\} - \frac{\sigma_2}{(\tau-\tau_0)^2}\left\{1+2\gamma_2(\tau-\tau_0)^2+\cdots\right\}$$
$$K_3 = \frac{\sigma_1}{(\tau-\tau_0)^2}\left\{1+2\gamma_1(\tau-\tau_0)^2+\cdots\right\} - \frac{\sigma_3}{(\tau-\tau_0)^2}\left\{1+2\gamma_3(\tau-\tau_0)^2+\cdots\right\}.$$

In the explosive case, where $\sigma_1 = \sigma_2 = \sigma_3$, we obtain

$$K_2 = 2\sigma(\gamma_1-\gamma_2)+\mathcal{O}(\tau-\tau_0), \quad \text{and} \quad K_3 = 2\sigma(\gamma_1-\gamma_3)+\mathcal{O}(\tau-\tau_0),$$

where $\sigma = \sigma_m = \sigma_1\sigma_2\sigma_3$. Using the fact that $\gamma_1+\gamma_2+\gamma_3 = 0$ from (4.12), we determine that

$$\gamma_1 = \frac{\sigma}{6}(K_2+K_3), \tag{4.20}$$

$$\gamma_2 = \frac{\sigma}{6}(K_3-2K_2), \tag{4.21}$$

$$\gamma_3 = \frac{\sigma}{6}(K_2-2K_3). \tag{4.22}$$

In summary, the first terms of the Laurent series solution of (4.2) for the explosive case are given by

$$a_m(\tau) = \frac{e^{i\psi_m}}{\tau - \tau_0}\left[1 + \gamma_m(\tau-\tau_0)^2 + \frac{i\sigma H}{6}(\tau-\tau_0)^3 + \cdots\right], \tag{4.23}$$

where the real constants γ_m are given in (4.20)-(4.22).

At this point, we have found all six of the free constants in the series, namely K_2, K_3, H, τ_0, ψ_1, and ψ_2. Each of these constants is real, although later we discuss the possibility of allowing τ_0 to have a nonzero imaginary part, and thus extending our Laurent series solution to the nonexplosive case. These constants are equivalent to the constants (3.69) that arise in the Weierstrass solution. The series solution (4.23) is complete once we determine all remaining terms in the series.

We write the full series solution $a_m(\tau)$ more succinctly as

$$a_m(\tau) = \frac{e^{i\psi_m}}{\xi}\sum_{n=0}^{\infty} A_n^m \xi^n, \tag{4.24}$$

where $\xi = \tau - \tau_0$, and we observe that $d/d\tau = d/d\xi$. Moreover, a comparison with (4.23) tells us that

$$A_0^m = 1, \quad A_1^m = 0, \quad A_2^m = \gamma_m, \quad \text{and} \quad A_3^m = \frac{i\sigma H}{6}, \tag{4.25}$$

for $m = 1, 2, 3$, and with γ_m defined in (4.20)-(4.22).

In order to determine A_n^m for $n \geq 4$, we start as usual by substituting (4.24) into (4.2). The left-hand side becomes

$$\begin{aligned}\frac{da_m}{d\tau} &= \frac{e^{i\psi_m}}{\xi^2}\left[-A_0^m + A_2^m\xi^2 + 2A_3^m\xi^3 + 3A_4^m\xi^4 + 4A_5^m\xi^5 + \cdots\right]\\ &= \frac{e^{i\psi_m}}{\xi^2}\sum_{n=0}^{\infty}(n-1)A_n^m\xi^n.\end{aligned}$$

Likewise, the product $a_k^* a_\ell^*$ on the right-hand side is given by

$$\begin{aligned}a_k^* a_\ell^* &= \frac{e^{-i(\psi_k+\psi_\ell)}}{\xi^2}\left[A_0^{k*}A_0^{\ell*} + \left(A_0^{k*}A_1^{\ell*} + A_1^{k*}A_0^{\ell*}\right)\xi + \left(A_0^{k*}A_2^{\ell*} + A_1^{k*}A_1^{\ell*} + A_2^{k*}A_0^{\ell*}\right)\xi^2 + \cdots\right]\\ &= \frac{e^{-i(\psi_k+\psi_\ell)}}{\xi^2}\sum_{n=0}^{\infty}\sum_{p=0}^{n} A_p^{k*}A_{n-p}^{\ell*}\,\xi^n.\end{aligned}$$

We substitute the expansions of the right and left-hand sides into (4.2), multiply both sides by ξ^2, and use the fact that $e^{i\Psi} = -\sigma$ (where in the explosive case, $\sigma = \sigma_m$ for $m = 1, 2, 3$). It follows that at $\mathcal{O}(\xi^n)$, we obtain

$$(n-1)A_n^m = -\sum_{p=0}^{n} A_p^{k*} A_{n-p}^{\ell*}.$$

Note that if $n = 0, 1, 2$ or 3, then we obtain equations (4.6), (4.9), (4.11), and (4.13), respectively. Now if we also use the fact that $A_0^m = 1$ for $m = 1, 2, 3$, then we are left with

$$(n-1)A_n^m + A_n^{k*} + A_n^{\ell*} = -\sum_{p=1}^{n-1} A_p^{k*} A_{n-p}^{\ell*}. \tag{4.26}$$

We can use (4.26) to determine all remaining terms in the series (4.23). In particular, we can write (4.26) as the following linear system

$$\begin{pmatrix} n-1 & 0 & 0 & 0 & 1 & 1 \\ 0 & n-1 & 0 & 1 & 0 & 1 \\ 0 & 0 & n-1 & 1 & 1 & 0 \\ 0 & 1 & 1 & n-1 & 0 & 0 \\ 1 & 0 & 1 & 0 & n-1 & 0 \\ 1 & 1 & 0 & 0 & 0 & n-1 \end{pmatrix} \begin{pmatrix} A_n^1 \\ A_n^2 \\ A_n^3 \\ A_n^{1*} \\ A_n^{2*} \\ A_n^{3*} \end{pmatrix} = \begin{pmatrix} b_n^1 \\ b_n^2 \\ b_n^3 \\ b_n^{1*} \\ b_n^{2*} \\ b_n^{3*} \end{pmatrix}, \tag{4.27}$$

where

$$b_n^m = -\sum_{p=1}^{n-1} A_p^{k*} A_{n-p}^{\ell*}. \tag{4.28}$$

The determinant of the matrix in (4.27) is given by

$$D = n^2(n+1)(n-2)^2(n-3).$$

Thus, the system has a unique solution if $n \neq -1, 0, 2, 3$, which explains why we obtained free constants for $n = 0, 2, 3$. We already know A_n^m for $n = 0, 1, 2, 3$ from (4.25), so we use (4.27) to determine A_n^m for $n \geq 4$ and for $m = 1, 2, 3$.

To summarize, we have determined that the general solution of the three-wave ODEs in the explosive regime is given by

$$a_m(\tau) = \frac{e^{i\psi_m}}{\xi} \sum_{n=0}^{\infty} A_n^m \xi^n, \tag{4.29}$$

where $\xi = \tau - \tau_0$ and the series coefficients are given by

$$A_0^m = 1, \quad A_1^m = 0, \quad A_2^m = \gamma_m, \quad A_3^m = \frac{i\sigma H}{6}, \tag{4.30}$$

where γ_m is defined in (4.20)-(4.22), and

$$(n-1)A_n^m + A_n^{k*} + A_n^{\ell*} = -\sum_{p=1}^{n-1} A_p^{k*} A_{n-p}^{\ell*}, \quad n \geq 4. \tag{4.31}$$

We refer to (4.29)-(4.31) as the formal Laurent series solution of the three-wave ODEs, where the modifier "formal" is removed once convergence of the series is proven in Chapter 5. Note that the solution contains six real free constants,

$$\{K_2, K_3, H, \psi_1, \psi_2, \tau_0\}. \tag{4.32}$$

These constants are equivalent to those in (3.69).

4.2.1 Equivalence of solutions

Since both the Laurent series solution in (4.29)-(4.31) and the Weierstrass solution in (3.54) and (3.60) satisfy the three-wave ODEs (4.2), and each contains six free constants, it follows that they both describe the general solution of the ODEs. A standard result in complex variables states that a meromorphic function has a Laurent series in a deleted neighborhood of each of its poles. The radius of convergence of the series is the distance between the pole at the center of the expansion, and the nearest singularity. Moreover, within the radius of convergence of the series, the function is defined by its Laurent series [1]. As a result, if we can show that the Laurent series solution in (4.29)-(4.31) has a nonzero radius of convergence, then it follows that the Laurent series solution equals the Weierstrass solution inside that region. We show that the Laurent series has a nonzero radius of convergence in Section 4.3. For now, however, we give some motivation for why the two general solutions are equivalent. To that end, we expand the Weierstrass solution about $\tau = \tau_0$ and show that we obtain the first few terms of the Laurent series solution.

We require the Laurent series expansion of $\wp(\xi; g_2, g_3)$, where $\xi = \tau - \tau_0$. From [17, §23.9], we have

$$\wp(\xi; g_2, g_3) = \frac{1}{\xi^2} + \sum_{n=2}^{\infty} c_n \xi^{2n-2}, \tag{4.33}$$

where $c_2 = g_2/20$, $c_3 = g_3/28$, and c_n is determined by

$$c_n = \frac{3}{(2n+1)(n-3)} \sum_{j=2}^{n-2} c_j c_{n-j}, \quad n \geq 4.$$

We restrict our attention to the explosive case, so that $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ and $\tau_0 \in \mathbb{R}$. It follows from (3.54) that $|a_m(\tau)|^2$ can be expanded as

$$\begin{aligned} |a_m(\tau)|^2 &= \wp(\xi; g_2, g_3) + \sigma C_m \\ &= \frac{1}{\xi^2} + \sum_{n=2}^{\infty} c_n \xi^{2n-2} + \sigma C_m \\ &= \frac{1}{\xi^2} \left[1 + \sigma C_m \xi^2 + c_2 \xi^4 + c_3 \xi^6 + c_4 \xi^8 + \cdots \right], \end{aligned} \tag{4.34}$$

where C_m is defined in (3.55)-(3.57).

In order to determine $|a_m(\tau)|$, we take a square root of the series above using the binomial expansion, $(1+z)^{1/2} = 1 + \frac{z}{2} - \frac{z^2}{8} + \cdots$. This yields

$$|a_m(\tau)| = \frac{1}{|\xi|} \left[1 + \frac{C_m}{2} \xi^2 + \left(-\frac{C_m^2}{8} + \frac{c_2}{8}\right) \xi^4 + \left(\frac{C_m^3}{16} - \frac{C_m c_2}{4} + \frac{c_3}{2}\right) \xi^6 + \cdots \right]. \tag{4.35}$$

Next, we consider $\varphi_m(\tau)$ in (3.60). First, we need to expand $1/|a_m(\tau)|^2$. From (4.34), we obtain

$$\frac{1}{|a_m(\tau)|^2} = \xi^2 \left[1 - C_m z^2 + (C_m^2 - c_2) \xi^4 + (2 C_m c_2 - c_3 - c_1^3) z^6 + \cdots \right],$$

where we used $(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$. Then from (3.60), we have

$$\begin{aligned} \varphi_m(\tau) &= \frac{\sigma H}{2} \int_{\tau_i}^{\tau} \frac{1}{|a_m(\tau)|^2} \, dt + \varphi_m(\tau_i) \\ &= \frac{\sigma H}{2} \int_{\tau_i}^{\tau} (t - \tau_0)^2 \left[1 - C_m (t - \tau_0)^2 + (C_m^2 - c_2)(t - \tau_0)^4 + \cdots \right] \, dt + \varphi_m(\tau_i) \\ &= \frac{\sigma H}{2} \left[\frac{1}{3} \xi^3 - \frac{C_m}{5} \xi^5 + \frac{C_m^2 - c_2}{7} \xi^7 + \cdots \right] + F_m, \end{aligned}$$

where F_m incorporates all the constant terms, namely

$$F_m = \varphi_m(\tau_i) - \frac{\sigma H}{2} \left[\frac{1}{3} (\tau_i - \tau_0)^3 - \frac{C_m}{5} (\tau_i - \tau_0)^5 + \frac{C_m^2 - c_2}{7} (\tau_i - \tau_0)^7 + \cdots \right]$$

for $m = 1, 2, 3$. Note that since F_m involves the free constants $\varphi_1(\tau_i)$ and $\varphi_2(\tau_i)$ for $m = 1$ and 2, respectively, we can assume that F_1 and F_2 are free constants, and F_3 is determined.

Since $a_m(\tau) = |a_m(\tau)|e^{i\varphi_m(\tau)}$, we also need the expansion of $e^{i\varphi_m(\tau)}$. To that end, we write

$$e^{i\varphi_m(\tau)} = \exp\left(iF_m\right)\exp\left(\frac{i\sigma H}{2}\left[\frac{1}{3}\,\xi^3 - \frac{C_m}{5}\,\xi^5 + \frac{C_m^2 - c_2}{7}\,\xi^7 + \cdots\right]\right).$$

Then since $e^{iz} = 1 + iz - \frac{z^2}{2} - \frac{iz^3}{6} + \cdots$, we have

$$e^{i\varphi_m(\tau)} = e^{iF_m}\left[1 + \frac{i\sigma H}{6}\,\xi^3 - \frac{i\sigma H C_m}{10}\,\xi^5 - \frac{H^2}{72}\,\xi^6 + \cdots\right]. \tag{4.36}$$

Finally, we put (4.35) and (4.36) together to obtain

$$\begin{aligned} a_m(\tau) &= |a_m(\tau)|\, e^{i\varphi_m(\tau)} \\ &= \frac{1}{|\xi|}\left[1 + \frac{C_m}{2}\xi^2 + \frac{i\sigma H}{6}\xi^3 + \left(-\frac{C_m^2}{8} + \frac{c_2}{2}\right)\xi^4 + \cdots\right]. \end{aligned}$$

Substituting from the definition of C_m in (3.55)-(3.57) and the definition of g_2 in (3.45), we find that

$$a_m(\tau) = \frac{e^{iF_m}}{|\xi|}\left[1 + \gamma_m\xi^2 + \frac{i\sigma H}{6}\xi^3 + \left(-\frac{\gamma_m^2}{2} + \frac{g_2}{40}\right)\xi^4 + \cdots\right]. \tag{4.37}$$

A quick comparison of (4.37) with (4.23) shows that the first three terms of the series agree if we let $F_m = \psi_m$ (this is allowed since both are free constants). Indeed, if we compute further terms in (4.23) using (4.29)-(4.30), we find that the ξ^4 term also agrees. This is a convincing argument, though not a proof, that the Laurent series solution and the Weierstrass solution are equivalent. Next, we show that the Laurent series converges, and therefore it must equal the Weierstrass solution within its radius of convergence.

4.3 Convergence of solutions

In this section, we briefly discuss where the general solution of the three-wave ODEs converges. The computation is trivial for the ODEs, but we discuss it here since it becomes important when determining where the general solution of the three-wave PDEs converges.

First, consider the Weierstrass solution described in (3.54) and (3.60). We know where the Weierstrass solution converges because its behavior is well understood. In particular, we know that the radius of convergence of the Weierstrass solution is the smallest distance between any two poles. Let λ_1 and λ_2 be the half-periods of $\wp(\tau - \tau_0; g_2, g_3)$, where g_2 and g_3 are defined in (3.45)-(3.46). Then from (3.54), the radius of convergence of $|a_m(\tau)|^2$ for $m = 1, 2, 3$ is given by

$$R_{\text{ODE}} = \min\{2|\lambda_1|, 2|\lambda_2|\}. \tag{4.38}$$

We showed in Section 4.2.1 that $e^{i\varphi_m(\tau)}$ does not have any poles, thus $a_m(\tau)$ converges whenever $|a_m(\tau)|$ converges. As a result, (4.38) defines the radius of convergence of the general solution of the three-wave ODEs.

An alternative approach is to compute the radius of convergence using the Laurent series solution. In this case, we simply apply the ratio test to determine where the solution converges. That is, we can see from (4.29) that the series for $a_m(\tau)$ converges when

$$\lim_{n\to\infty} \left| \frac{A_{n+1}^m \, \xi^{n+1}}{A_n^m \, \xi^n} \right| < 1,$$

where $\xi = \tau - \tau_0$. It follows that the radius of convergence of the Laurent series solution is given by

$$R_{\text{ODE}} = \left(\lim_{n\to\infty} \left| \frac{A_{n+1}^m}{A_n^m} \right| \right)^{-1}, \tag{4.39}$$

as long as the limit exists. In this case, (4.38) and (4.39) produce the same number. Moreover, we know a priori that the radius of convergence of the Laurent series is the distance between the pole at the center of the expansion and the nearest singularity, so the radius must be equal to the radius in (4.38), which is the smallest distance between any two poles. Since the Laurent series has a nonzero radius of convergence, it equals the Weierstrass solution in the region where the series converges. This is interesting because the Weierstrass solution incorporates both the explosive and the nonexplosive cases, while the Laurent series solution was derived for solutions in the explosive regime. This suggests that we can extend the Laurent series solution in (4.29)-(4.31) to allow for bounded, periodic solutions. In particular, we can choose the imaginary part of τ_0 to be nonzero in order to force the Laurent series solutions along the real axis to be bounded. The choice of $\text{Im}\,(\tau_0)$, however, must be chosen by looking at the discriminant of the corresponding Weierstrass solution, as outlined in Section 3.6. We discuss the restrictions of the nonexplosive case in more detail in Section 4.4.

4.4 The nonexplosive case

Constructing the Laurent series general solution of the three-wave ODEs in the nonexplosive regime poses some difficulties, which we touched upon in Section 4.2. Here, we go into more detail. In particular, we construct some solutions of the three-wave ODEs in the nonexplosive case, although our analysis is not rigorous.

Typically, we pose a series solution of the three-wave ODEs of the form (4.5),

$$a_m(\tau) = \frac{1}{\tau - \tau_0}\left[\alpha_m + \beta_m(\tau-\tau_0) + \gamma_m(\tau-\tau_0)^2 + \delta_m(\tau-\tau_0)^3 + \cdots\right], \tag{4.40}$$

where $\alpha_m, \beta_m, \gamma_m$, and δ_m are complex constants for $m = 1, 2, 3$, and τ is real. In the nonexplosive regime, it is assumed that the imaginary part of τ_0 is nonzero, so that the pole lies off the real axis. As a result, substituting (4.40) into the three-wave ODEs (4.2) yields

$$\begin{aligned}
&\frac{1}{(\tau-\tau_0)^2}\left\{-\alpha_m + \gamma_m(\tau-\tau_0)^2 + 2\delta_m(\tau-\tau_0)^3 + \cdots\right\} \\
&\quad = \frac{\sigma_m}{(\tau-\tau_0^*)^2}\Big\{\alpha_k^*\alpha_\ell^* + (\alpha_k^*\beta_\ell^* + \alpha_\ell^*\beta_k^*)(\tau-\tau_0^*) + (\beta_k^*\beta_\ell^* + \alpha_k^*\gamma_\ell^* + \alpha_\ell^*\gamma_k^*)(\tau-\tau_0^*)^2 \\
&\qquad\qquad + (\alpha_k^*\delta_\ell^* + \alpha_\ell^*\delta_k^* + \beta_k^*\gamma_\ell^* + \beta_\ell^*\gamma_k^*)(\tau-\tau_0^*)^3 + \cdots\Big\}.
\end{aligned}$$

It is not possible to solve the equation order by order for the unknown coefficients because the $(\tau-\tau_0)$ terms on the left-hand side do not match the $(\tau-\tau_0^*)$ terms on the right-hand side. Note that this problem does not exist in the explosive case, in which τ_0 is real and $\tau_0^* = \tau_0$.

In order to motivate a series solution in the nonexplosive case, we consider the Jacobian elliptic functions. Elliptic functions in general are described at the start of Section 3.4. In particular, Jacobi elliptic functions have simple poles at lattice points, while Weierstrass elliptic functions have double poles at lattice points. Jacobi elliptic functions are defined as inverses of the elliptic integral of the first kind,

$$u = F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{4.41}$$

There are twelve Jacobi elliptic functions, but we deal primarily with the three defined via

$$\mathrm{sn}\,(u;k) = \sin\phi, \quad \mathrm{cn}\,(u;k) = \cos\phi, \quad \text{and} \quad \mathrm{dn}\,(u;k) = \sqrt{1 - k^2 \sin^2\phi}. \tag{4.42}$$

[16, Ch. 6]. The parameter k is known as the elliptic modulus.

The functions in (4.42) satisfy the identities

$$\mathrm{sn}^2(u;k) + \mathrm{cn}^2(u;k) = 1 \quad \text{and} \quad k^2\mathrm{sn}^2(u;k) + \mathrm{dn}^2(u;k) = 1. \tag{4.43}$$

The functions also satisfy the following coupled nonlinear ODEs,

$$\frac{d}{du}\mathrm{sn}\,u = \mathrm{cn}\,u\,\mathrm{dn}\,u, \tag{4.44}$$

$$\frac{d}{du}\mathrm{cn}\,u = -\mathrm{sn}\,u\,\mathrm{dn}\,u, \tag{4.45}$$

$$\frac{d}{du}\mathrm{dn}\,u = -k^2\mathrm{sn}\,u\,\mathrm{cn}\,u. \tag{4.46}$$

These ODEs look somewhat similar to the three-wave ODEs due to the nonlinear products on the right-hand side. Indeed, define

$$S(u;k) = k\,\mathrm{sn}\,(u;k), \quad C(u;k) = k\,\mathrm{cn}\,(u;k), \quad \text{and} \quad D(u;k) = \mathrm{dn}\,(u;k). \tag{4.47}$$

Then (4.44)-(4.46) become

$$\frac{d}{du}S(u) = C(u)\,D(u), \tag{4.48}$$

$$\frac{d}{du}C(u) = -S(u)\,D(u), \tag{4.49}$$

$$\frac{d}{du}D(u) = -S(u)\,C(u). \tag{4.50}$$

At this point, (4.48)-(4.50) look like the three-wave ODEs when $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$, but with no conjugates on the right-hand side. However, if $k \in [0,1]$, then the Jacobi elliptic functions are real-valued along the real u-axis [17, §22.2]. That is, if $k \in [0,1]$, then $S(u) = S^*(u)$, $C(u) = C^*(u)$, and $D(u) = D^*(u)$ for real u. As a result, along the real u-axis, we have

$$\frac{d}{du}S(u) = C^*(u)\,D^*(u), \tag{4.51}$$

$$\frac{d}{du}C(u) = -S^*(u)\,D^*(u), \tag{4.52}$$

$$\frac{d}{du}D(u) = -S^*(u)\,C^*(u). \tag{4.53}$$

A comparison with the three-wave ODEs in (4.2) shows that that $S(u)$, $C(u)$, and $D(u)$ satisfy the three-wave equations for real u in the nonexplosive regime, where $\{\sigma_1, \sigma_2, \sigma_3\}$ are not all equal. Moreover, the identities (4.43) in terms of $S(u), C(u)$, and $D(u)$ become

$$S^2(u) + C^2(u) = k^2 \quad \text{and} \quad S^2(u) + D^2(u) = 1. \tag{4.54}$$

Let $\tau = u$ and define

$$a_1(\tau) = S(\tau), \quad a_2(\tau) = C(\tau), \quad \text{and} \quad a_3(\tau) = D(\tau).$$

Then (4.51)-(4.53) constitute the three-wave ODEs for real τ with $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. Furthermore, with these values, the identities in (4.54) are equivalent to the Manley-Rowe relations in (2.30)-(2.31) with $K_2 = k^2$ and $K_3 = 1$.

We have now found a family of solutions of the three-wave ODEs in the nonexplosive regime. Next, we want to use the Laurent series expansions of $S(u)$, $C(u)$, and $D(u)$ in order to predict the form we should pick for $a_m(\tau)$, $m = 1, 2, 3$, in the nonexplosive case. It is known that $\mathrm{sn}\,(u; k)$, $\mathrm{cn}\,(u; k)$, and $\mathrm{dn}\,(u; k)$ have poles at $u = \pm iK'$ and $u = 2K \pm iK'$, where K is the complete elliptic integral of the first kind, defined using (4.41) as

$$K = K(k) = F(\pi/2, k)$$

[17, §22.4]. Furthermore, K' is defined by

$$K' = K'(k) = K(k'), \quad \text{where} \quad k' = \sqrt{1 - k^2}.$$

Note that $K'(k) \neq dK/dk$. Additionally, notice that other poles occur on a rectangular periodic lattice outside of $\pm iK'$ and $2K \pm iK'$. This is consistent with our findings in Section 3.6; in particular, we determined in Section 3.6 that the Weierstrass general solution of the three-wave ODEs must possess a rectangular lattice of poles in order for the solution to apply to the nonexplosive regime.

Suppose u_0 is one of the poles of $\mathrm{sn}\,(u; k), \mathrm{cn}\,(u; k)$, and $\mathrm{dn}\,(u; k)$. For example, suppose that $u_0 = iK'$. It is straightforward to determine the first few terms of the Laurent series of $\mathrm{sn}\,(u; k)$,

$\mathrm{cn}\,(u;k)$, and $\mathrm{dn}\,(u;k)$ about $u = u_0$ by subsituting an expression of the form (4.5) into (4.44)-(4.46) and solving the resulting equations order by order. Following this procedure, we find that the beginning of the Laurent series of $\mathrm{sn}\,(u;k)$, $\mathrm{cn}\,(u;k)$, and $\mathrm{dn}\,(u;k)$ about $u = u_0$ are given by

$$\mathrm{sn}\,(u;k) = \frac{1}{\zeta}\left[\frac{1}{k} + \frac{1+k^2}{6k}\,\zeta^2 + \frac{1}{360k}\left(7 - 22k^2 + 7k^4\right)\,\zeta^4 + \mathcal{O}(\zeta^6)\right], \tag{4.55}$$

$$\mathrm{cn}\,(u;k) = \frac{1}{\zeta}\left[-\frac{i}{k} + \frac{i(2k^2-1)}{6k}\,\zeta^2 + \frac{i}{360k}\left(-7 - 8k^2 + 8k^4\right)\,\zeta^4 + \mathcal{O}(\zeta^6)\right], \tag{4.56}$$

$$\mathrm{dn}\,(u;k) = \frac{1}{\zeta}\left[-i - \frac{i(k^2-2)}{6}\,\zeta^2 - \frac{i}{360}\left(-8 + 8k^2 + 7k^4\right)\,\zeta^4 + \mathcal{O}(\zeta^6)\right], \tag{4.57}$$

where $\zeta = u - u_0$. It follows from (4.47) that the series expansions for $S(u;k), C(u;k)$, and $D(u;k)$ are given by

$$S(u;k) = \frac{1}{\zeta}\left[1 + \frac{1+k^2}{6}\,\zeta^2 + \frac{1}{360}\left(7 - 22k^2 + 7k^4\right)\,\zeta^4 + \mathcal{O}(\zeta^6)\right], \tag{4.58}$$

$$C(u;k) = \frac{1}{\zeta}\left[-i + \frac{i(2k^2-1)}{6}\,\zeta^2 + \frac{i}{360}\left(-7 - 8k^2 + 8k^4\right)\,\zeta^4 + \mathcal{O}(\zeta^6)\right], \tag{4.59}$$

$$D(u;k) = \frac{1}{\zeta}\left[-i - \frac{i(k^2-2)}{6}\,\zeta^2 - \frac{i}{360}\left(-8 + 8k^2 + 7k^4\right)\,\zeta^4 + \mathcal{O}(\zeta^6)\right]. \tag{4.60}$$

We know that $S(u)$, $C(u)$, and $D(u)$ satisfy (4.51)-(4.53) along the real axis by construction. Consider, for example, the evolution equation for $S(u)$ in (4.51). If we substitute (4.58)-(4.60) into (4.51) and simplify, we obtain

$$\begin{aligned}
&\frac{1}{\zeta^2}\left[-1 + \frac{1+k^2}{6}\,\zeta^2 + \frac{1}{120}\left(7 - 22k^2 + 7k^4\right)\,\zeta^4 + \mathcal{O}(\zeta^6)\right] \\
&\qquad = \frac{1}{(\zeta^*)^2}\left[-1 + \frac{1+k^2}{6}\,(\zeta^*)^2 + \frac{1}{120}\left(7 - 22k^2 + 7k^4\right)\,(\zeta^*)^4 + \mathcal{O}((\zeta^*)^6)\right],
\end{aligned} \tag{4.61}$$

where $\zeta^* = u - u_0^*$. Observe that the right and left-hand sides are only equal term by term if $\zeta = \zeta^*$. This is impossible, however, since $\mathrm{Im}\,(u_0) \neq 0$. Similarly, we know that $S(u), C(u)$, and $D(u)$ are real along the real axis by construction, yet (4.58)-(4.60) show that the Laurent series expansions of these functions are clearly not real along the real axis on a term by term basis. We need to resolve this issue before we can proceed.

Recall that a function $f(z)$ equals its Laurent series within the radius of convergence of the series, except at the singularity itself. In the case of $S(u)$, $C(u)$, and $D(u)$, it can be shown that the region of convergence of their Laurent series intersects the real axis for many k-values, although this is beyond the scope of the thesis. Consequently, for at least some k and some portion of the real axis, the infinite series represented by (4.58)-(4.60) must be real-valued. Although the series are not real term-by-term, it turns out that as the number of terms in the series approaches infinity, the imaginary part of the series goes to zero. In particular, numerical evidence suggests that the number of zeros of the imaginary part of the series increases approximately linearly with the number of terms in the series, although we have not proved this result.

This motivates us to look for solutions of the three-wave ODEs in the nonexplosive regime of the form

$$a_m(\tau) = e^{i\psi_m} T_m(\tau), \tag{4.62}$$

where

$$T_m(\tau) = \frac{1}{\tau - \tau_0}\left[\alpha_m + \beta_m(\tau-\tau_0) + \gamma_m(\tau-\tau_0)^2 + \delta_m(\tau-\tau_0)^3 + \cdots\right].$$

In particular, we assume that τ_0 is complex, and that $T_m(\tau)$ is a real-valued function for real τ.

First, suppose that $\mathrm{Im}\,(\tau_0) = K' \pm 2nK'$, $n \in \mathbb{Z}$, so that τ_0 is a pole of $\mathrm{sn}\,(\tau; k)$, $\mathrm{cn}\,(\tau; k)$, and $\mathrm{dn}\,(\tau; k)$, and that $T_1(\tau) = S(\tau)$, $T_2(\tau) = C(\tau)$, and $T_3(\tau) = D(\tau)$. In this case, substituting into the three-wave ODEs and using the fact that $T_m(\tau) = T_m^*(\tau)$ for $m = 1, 2, 3$ yields

$$e^{i\psi_m}\frac{dT_m}{d\tau} = \sigma_m e^{-i(\psi_j+\psi_\ell)} T_j T_\ell,$$

where $(j, \ell, m) = (1, 2, 3)$ cyclically. If $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$, then the three-wave ODEs are satisfied by $a_m(\tau)$ in (4.62) as long as

$$\psi_1 + \psi_2 + \psi_3 = 2n\pi, \tag{4.63}$$

where n is an integer. Recall that the Manley-Rowe constants are $K_2 = k^2$ and $K_3 = 1$ in this case, where k is the elliptic modulus. Notice that with $a_m(\tau)$ defined in (4.62), we are allowing $a_m(\tau)$, $m = 1, 2, 3$, to be complex-valued along the real axis due to the phase $e^{i\psi_m}$. However, the series portion of $a_m(\tau)$ is still assumed to be real. Finally, note that we can add any real number to τ_0, and $a_m(\tau)$ will still satisfy the three-wave ODEs for $m = 1, 2, 3$.

To summarize, we have

$$a_m(\tau) = e^{i\psi_m} T_m(\tau), \tag{4.64}$$

where

$$T_m(\tau) = \frac{1}{\xi} \sum_{n=0}^{\infty} A_{2n}^m \xi^{2n}, \tag{4.65}$$

$\xi = \tau - \tau_0$, and $T_m(\tau)$ is real (though any truncated version of $T_m(\tau)$ is not real). The first two coefficients for each m are given by

$$\begin{aligned} A_0^1 &= -1 \qquad & A_2^1 &= \frac{1+k^2}{6}, \\ A_0^2 &= -i \qquad & A_2^2 &= \frac{i(2k^2-1)}{6}, \\ A_0^3 &= -i \qquad & A_2^3 &= -\frac{i(k^2-2)}{6}, \end{aligned}$$

and the remaining coefficients are determined by

$$(2n-1)A_{2n}^m + A_0^\ell A_{2n}^k + A_0^k A_{2n}^\ell = \sigma_m \sum_{p=1}^{n-1} A_{2p}^k A_{2(n-p)}^\ell, n \geq 3, \tag{4.66}$$

where $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. Note that the definition of H in (3.5) implies that $H = 0$ in this case. Additionally, note that although the imaginary part of τ_0 is fixed at $\mathrm{Im}\,(\tau_0) = K' \pm 2nK'$, $n \in \mathbb{Z}$, we are still free to choose $\mathrm{Re}\,(\tau_0)$. This leaves us with four free real constants,

$$\{k, \psi_1, \psi_2, \mathrm{Re}\,(\tau_0)\},$$

with ψ_3 determined via (4.63). Equations (4.64)-(4.66) constitute a four parameter family of solutions of the three-wave ODEs in the nonexplosive regime. It is not a fully general solution, which would require six free constants, but it is close.

So far in this section, we have used the Jacobi elliptic functions to derive a family of solutions of the three-wave ODEs in the nonexplosive regime. Alternatively, suppose we formally pose the expansion in (4.62) with the assumption that τ_0 is complex-valued (but so far unknown). Again, we assume that $T_m(\tau)$ is real-valued for real τ. Finally, without loss of generality, assume that we are in the nonexplosive regime with $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. Substituting (4.62) into the three-wave

ODEs and the Manley-Rowe relations yields the following,

$$a_m(\tau) = e^{i\psi_m} T_m(\tau), \quad \text{where} \quad T_m(\tau) = \frac{\alpha_m}{\xi} \sum_{n=0}^{\infty} A_{2n}^m \xi^{2n}. \tag{4.67}$$

In (4.67), we have that $\psi_1 + \psi_2 + \psi_3 = 2n\pi$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = -i$, $A_0^m = 1$ for $m = 1, 2, 3$, and

$$A_2^1 = \frac{K_2 + K_3}{6}, \quad A_2^2 = \frac{K_3 - 2K_2}{6}, \quad \text{and} \quad A_3^2 = \frac{K_2 - 2K_3}{6}. \tag{4.68}$$

The remaining coefficients in (4.67) are determined by the recursion

$$(2n-1)A_{2n}^m + A_{2n}^k + A_{2n}^\ell = -\sum_{p=1}^{n-1} A_{2p}^k A_{2(n-p)}^\ell, \tag{4.69}$$

for $n \geq 3$. Again, note that the definition of H in (3.5) implies that $H = 0$ in this case.

The coefficients A_2^m are the same as (4.20)-(4.22). Additionally, the coefficients A_{2n}^m in (4.67) are the same as those in (5.24) in the case where $H = 0$ (see, for example, equations (5.146)-(5.147), page 131). Furthermore, if we write α_m in exponential form for $m = 1, 2, 3$, then we can rewrite (4.67) as

$$a_1(\tau) = \frac{e^{i\psi_1}}{\xi} \sum_{n=0}^{\infty} A_{2n}^1 \xi^{2n}, \quad a_j(\tau) = \frac{e^{i(\psi_j - \pi/2)}}{\xi} \sum_{n=0}^{\infty} A_{2n}^j \xi^{2n},$$

where $j = 2, 3$. Then the sum of the phases becomes $\psi_1 + \psi_2 + \psi_3 - \pi = (2n-1)\pi$, using the fact that $\psi_1 + \psi_2 + \psi_3 = 2n\pi$. This recovers (4.7), since $\sigma = 1$ when $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. It follows that (4.67)-(4.69) are equivalent to the Laurent series expansion we derived in (4.29)-(4.31) when $H = 0$. Moreover, if we set $K_2 = k^2$ and $K_3 = 1$, it is straightforward to show that we recover $T_1(\tau) = S(\tau), T_2(\tau) = C(\tau)$, and $T_3(\tau) = D(\tau)$, where the expansions for $S(\tau), C(\tau)$, and $D(\tau)$ are given in (4.58)-(4.60), respectively.

Next, notice that when $H = 0$, the discriminant in (3.68) becomes

$$\Delta = K_2^2 K_3^2 (K_2 - K_3)^2. \tag{4.70}$$

Recall from Section 3.6 that when $\Delta < 0$, we are restricted to the nonexplosive regime, while when $\Delta > 0$, we can obtain both the explosive and the nonexplosive cases. When $H = 0$, it is clear from (4.70) that the discriminant is nonnegative. As a result, as long as $K_2, K_3 \neq 0$ and $K_2 \neq K_3$, we can always obtain a nonexplosive solution of the three-wave ODEs when $H = 0$. Note that it is possible to have $\Delta > 0$ when $H \neq 0$, but this case is not covered by (4.67)-(4.69).

Finally, we need to know whether $\mathrm{Im}\,(\tau_0)$ can be chosen arbitrarily. Unsurprisingly, it turns out that $\mathrm{Im}\,(\tau_0)$ must be chosen carefully. In particular, given K_2 and K_3, the elliptic invariants g_2 and g_3 must be computed using (3.45)-(3.46). Then the half-periods λ_1 and λ_2 of the associated Weierstrass function can be calculated. For the rectangular lattice generated by K_2 and K_3, we know that λ_1 is real, while λ_2 is pure imaginary. Consequently, we must choose

$$\mathrm{Im}\,(\tau_0) = |\lambda_2| \pm 2n|\lambda_2|, \quad n \in \mathbb{Z}.$$

We are then able to choose $\mathrm{Re}\,(\tau_0)$ freely.

To summarize again, we hypothesize that the Laurent series solution of the three-wave ODEs defined in (4.29)-(4.31) applies to the nonexplosive case at least under the following conditions

(1) $H = 0$

(2) $\mathrm{Im}\,(\tau_0) = |\lambda_2| \pm 2n|\lambda_2|$, $n \in \mathbb{Z}$, where λ_2 is the imaginary half period of the Weierstrass function associated with a given choice of K_2 and K_3.

We also assume that $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$, although this restriction can be relaxed to allow for any nonexplosive configuration of $\{\sigma_1, \sigma_2, \sigma_3\}$. Note that this nonexplosive solution constitutes a five parameter family of solutions, with free constants

$$\{K_2, K_3, \psi_1, \psi_2, \mathrm{Re}\,(\tau_0)\}.$$

There are some small additional restrictions on $\mathrm{Im}\,(\tau_0)$ that have to do with the radius of convergence of the series solution. Specifically, if K_2 and K_3 (and the value of $\mathrm{Im}\,(\tau_0)$ that they generate) are such that the radius of convergence of the solution does not include any part of the real axis, then we are not guaranteed that the solution is real anywhere along the real axis.

Chapter 5

The Three-Wave PDEs

In the previous chapter, we showed how to construct the general solution of the three-wave ODEs in terms of a Laurent series using Painlevé analysis. In this chapter, we extend the solution techniques of the previous chapter in order to construct a near-general solution of the three-wave PDEs. In particular, we construct the solution of the PDEs using a formal Laurent series in time. We then make the solution rigorous under certain conditions by showing that the series is convergent in some region of the complex τ-plane.

5.1 A near-general solution

Recall the three-wave PDEs in (2.5). For simplicity, we restrict our attention in what follows to a single spatial dimension, although this restriction can be easily lifted. Our method of solution does not change if we increase the number of spatial dimensions.

The three-wave PDEs in one spatial dimension are given below,

$$\frac{\partial a_m}{\partial \tau} + c_m \frac{\partial a_m}{\partial x} = \sigma_m a_k^* a_\ell^*, \tag{5.1}$$

where $(k, \ell, m) = (1, 2, 3)$ cyclically, $a_m = a_m(x, \tau)$, c_m is the group velocity corresponding to mode a_m, and $\sigma_m = \pm 1$ for $m = 1, 2, 3$. In order to construct the general solution of the three-wave PDEs in (5.1), we take advantage of the structure of the general solution of the three-wave ODEs. In particular, we attempt to construct a solution of the PDEs that is similar to (4.29)-(4.31), the Laurent series general solution of the ODEs. To do so, we follow a procedure similar to that outlined in Section 4.2.

Recall that for the series solution of the ODEs, we looked for six real, free constants since the ODEs constitute six real-valued equations. For the series solution of the PDEs, however, we look for free functions of x, rather than constants. A fully general solution of the PDEs would contain six such functions, but we find five free functions of x and one free constant. For this reason, we refer to our solution of the three-wave PDEs as a "near-general" solution. In particular, currently we do not allow the center of our expansion, the pole $\tau = \tau_0$, to depend on x. Whether this constraint can be eliminated is an open question.

First of all, we assume that there is a simple pole at $\tau = \tau_0$, with $\tau_0 \in \mathbb{R}$, and that as $\tau \to \tau_0$,

$$a_m(x, \tau) \sim \frac{\alpha_m(x)}{\tau - \tau_0}, \tag{5.2}$$

where $\alpha_m(x)$ is to be determined. The postulation above is analagous to (4.3). Substituting into (5.1) yields, for real τ,

$$-\frac{\alpha_m(x)}{(\tau - \tau_0)^2} + c_m \frac{\alpha_m'(x)}{\tau - \tau_0} = \sigma_m \frac{\alpha_k^*(x)\alpha_\ell^*(x)}{(\tau - \tau_0)^2}, \tag{5.3}$$

where the prime denotes a derivative with respect to x. Notice that the spatial derivative term is less singular than the other terms in (5.3). As a result, the leading order behavior of the three-wave PDEs is the same as the leading order behavior of the three-wave ODEs; this motivates us to use the structure of the ODEs to build a solution of the PDEs. At leading order we have

$$-\frac{\alpha_m(x)}{(\tau - \tau_0)^2} = \sigma_m \frac{\alpha_k^*(x)\alpha_\ell^*(x)}{(\tau - \tau_0)^2},$$

which is equivalent to (4.4) with $p = 1$.

On the other hand, suppose we try to introduce spatial dependence into τ_0 by posing a solution of the form

$$a_m(x, \tau) \sim \frac{\alpha_m(x)}{\tau - \tau_0(x)}$$

for τ near $\tau_0(x)$. Then substituting into (5.1) yields

$$-\frac{\alpha_m(x)}{(\tau - \tau_0(x))^2} + c_m \left(\frac{\alpha_m'(x)}{\tau - \tau_0(x)} + \frac{\alpha_m(x)\tau_0'(x)}{(\tau - \tau_0(x))^2} \right) = \sigma_m \frac{\alpha_k^*(x)\alpha_\ell^*(x)}{(\tau - \tau_0(x))^2}.$$

We can see that part of the spatial derivative term is as singular as the temporal derivative term and the nonlinear product on the right-hand side. In this case, the dominant behavior of the three-wave

PDEs does not match the dominant behavior of the three-wave ODEs. This means it no longer makes sense to use the structure of the three-wave ODEs in order to predict the solution of the PDEs. For this reason, we assume the dominant behavior of solutions of the three-wave PDEs is given by (5.2). In particular, we assume that τ_0 is independent of x, and accept the restriction that the solution we derive is not fully general.

We now pose the following series solution of the three-wave PDEs, comparable to (4.5),

$$a_m(x,\tau) = \frac{1}{\tau-\tau_0}\left[\alpha_m(x) + \beta_m(x)(\tau-\tau_0) + \gamma_m(x)(\tau-\tau_0)^2 + \delta_m(x)(\tau-\tau_0)^3 + \cdots\right], \tag{5.4}$$

where $\alpha_m(x), \beta_m(x), \gamma_m(x)$, and $\delta_m(x)$ are complex-valued functions for $m = 1, 2, 3$, and τ_0 is real. Substituting (5.4) into (5.1) yields

$$\begin{aligned}
&\frac{1}{(\tau-\tau_0)^2}\Big\{-\alpha_m(x) + \gamma_m(x)\,(\tau-\tau_0)^2 + 2\delta_m(x)\,(\tau-\tau_0)^3 + \cdots\Big\} \\
&+ \frac{c_m}{(\tau-\tau_0)^2}\Big\{\alpha_m'(x)(\tau-\tau_0) + \beta_m'(x)(\tau-\tau_0)^2 + \gamma_m'(x)(\tau-\tau_0)^3 + \cdots\Big\} \\
&= \frac{\sigma_m}{(\tau-\tau_0)^2}\Big\{\alpha_k^*(x)\alpha_\ell^*(x) + \left[\alpha_k^*(x)\beta_\ell^*(x) + \alpha_\ell^*(x)\beta_k^*(x)\right](\tau-\tau_0) \\
&\qquad + \left[\beta_k^*(x)\beta_\ell^*(x) + \alpha_k^*(x)\gamma_\ell^*(x) + \alpha_\ell^*(x)\gamma_k^*(x)\right](\tau-\tau_0)^2 \\
&\qquad + \left[\alpha_k^*\delta_\ell^*(x) + \alpha_\ell^*(x)\delta_k^*(x) + \beta_k^*(x)\gamma_\ell^*(x) + \beta_\ell^*(x)\gamma_k^*(x)\right](\tau-\tau_0)^3 + \cdots\Big\},
\end{aligned}$$

where we assume that $\tau \in \mathbb{R}$.

At lowest order, we find that

$$-\alpha_m(x) = \sigma_m\alpha_k^*(x)\alpha_\ell^*(x), \tag{5.5}$$

which is the same as (4.6) since the three-wave ODEs and PDEs have the same dominant behavior at leading order. As a result, we know that (5.5) has a solution when $\sigma_1 = \sigma_2 = \sigma_3$, the explosive regime. This is consistent with our assumption that $\tau_0 \in \mathbb{R}$. In fact, for the remainder of this chapter, assume that we are restricted to the explosive regime unless otherwise stated.

The solution to (5.5) is given by

$$\alpha_m(x) = e^{i\psi_m(x)},$$

where $\psi_m(x)$ is a real function of x for $m = 1, 2, 3$, satisfying

$$\Psi(x) \equiv \psi_1(x) + \psi_2(x) + \psi_3(x) = \begin{cases} 2n\pi, & \sigma = -1 \\ (2n+1)\pi, & \sigma = 1. \end{cases} \tag{5.6}$$

In particular, we observe that two of $\{\psi_1(x), \psi_2(x), \psi_3(x)\}$ are real, free functions of x, while the third is determined by (5.6). It remains to find the last three free functions.

Next, for simplicity, we rewrite (5.4) as follows

$$a_m(x,\tau) = \frac{e^{i\psi_m(x)}}{\tau - \tau_0}\left[1 + \beta_m(x)(\tau-\tau_0) + \gamma_m(x)(\tau-\tau_0)^2 + \delta_m(x)(\tau-\tau_0)^3 + \cdots\right], \tag{5.7}$$

where $\beta_m(x)$, $\gamma_m(x)$, and $\delta_m(x)$ are complex-valued functions of x that differ from those in (5.4) by a factor of $e^{i\psi_m(x)}$. When we substitute (5.7) into (5.1), we obtain

$$\begin{aligned}
&\frac{e^{i\psi_m(x)}}{(\tau-\tau_0)^2}\Big\{-1 + \gamma_m(x)\,(\tau-\tau_0)^2 + 2\delta_m(x)\,(\tau-\tau_0)^3 + \cdots\Big\} \\
&+ \frac{c_m\, e^{i\psi_m(x)}}{(\tau-\tau_0)^2}\Big\{i\psi_m'(\tau-\tau_0) + (i\psi_m'\,\beta_m + \beta_m')(\tau-\tau_0)^2 + (i\psi_m'\,\gamma_m + \gamma_m')(\tau-\tau_0)^3 + \cdots\Big\} \\
&= \frac{\sigma\, e^{-i(\psi_k(x)+\psi_\ell(x))}}{(\tau-\tau_0)^2}\Big\{1 + [\beta_\ell^*(x) + \beta_k^*(x)]\,(\tau-\tau_0) + [\beta_k^*(x)\beta_\ell^*(x) + \gamma_\ell^*(x) + \gamma_k^*(x)]\,(\tau-\tau_0)^2 \\
&\qquad + [\delta_\ell^*(x) + \delta_k^*(x) + \beta_k^*(x)\gamma_\ell^*(x) + \beta_\ell^*(x)\gamma_k^*(x)]\,(\tau-\tau_0)^3 + \cdots\Big\},
\end{aligned}$$

where the primes denote derivatives with respect to x, and we used the fact that we are now in the explosive regime, so that σ_m can be replaced with σ. Finally, we multiply both sides by $(\tau - \tau_0)^2 e^{i(\psi_k(x)+\psi_\ell(x))}$ and use $e^{i\Psi(x)} = -\sigma$ to find that

$$\begin{aligned}
&-\gamma_m(x)\,(\tau-\tau_0)^2 - 2\delta_m(x)\,(\tau-\tau_0)^3 + \cdots \\
&- c_m\Big\{i\psi_m'(\tau-\tau_0) + (i\psi_m'\,\beta_m + \beta_m')(\tau-\tau_0)^2 + (i\psi_m'\,\gamma_m + \gamma_m')(\tau-\tau_0)^3 + \cdots\Big\} \\
&= [\beta_\ell^*(x) + \beta_k^*(x)]\,(\tau-\tau_0) + [\beta_k^*(x)\beta_\ell^*(x) + \gamma_\ell^*(x) + \gamma_k^*(x)]\,(\tau-\tau_0)^2 \\
&\qquad + [\delta_\ell^*(x) + \delta_k^*(x) + \beta_k^*(x)\gamma_\ell^*(x) + \beta_\ell^*(x)\gamma_k^*(x)]\,(\tau-\tau_0)^3 + \cdots .
\end{aligned}$$

At order $(\tau - \tau_0)$, we must solve

$$-ic_m\psi_m'(x) = \beta_k^*(x) + \beta_\ell^*(x), \tag{5.8}$$

The solution for $\beta_m(x)$ is unique, and is given by

$$\beta_m(x) = \frac{i}{2}\Big[c_k\psi_k'(x) + c_\ell\psi_\ell'(x) - c_m\psi_m'(x)\Big], \tag{5.9}$$

where $(k,\ell,m) = (1,2,3)$ are defined cyclically. Note that $\beta_m(x)$ is pure imaginary for $m = 1,2,3$. Additionally, notice that if the phases $\psi_m(x)$ are independent of x, then $\beta_m(x) = 0$. This is comparable to what we found at first order for the series solution of the three-wave ODEs in (4.10).

At order $(\tau-\tau_0)^2$, we have

$$-\gamma_m(x) - c_m\left[i\psi_m'(x)\beta_m(x) + \beta_m'(x)\right] = \beta_k^*(x)\beta_\ell^*(x) + \gamma_\ell^*(x) + \gamma_k^*(x). \tag{5.10}$$

The solution of (5.10) for $\gamma_m(x)$ is not unique, so free constants arise at this order. In order to solve (5.10), we break the equation into its real and imaginary parts. We find that the imaginary part of $\gamma_m(x)$ is uniquely determined to be

$$\mathrm{Im}\left(\gamma_m(x)\right) = \frac{1}{4}\Big[c_m(c_k+c_\ell)\,\psi_m''(x) - (c_k-c_\ell)\left(c_k\,\psi_k''(x) - c_\ell\,\psi_\ell''(x)\right)\Big], \tag{5.11}$$

where we substituted $\beta_m(x)$ from (5.9).

Conversely, the real part of $\gamma_m(x)$ is not uniquely determined. However, we find that the sum of the real parts of $\gamma_m(x)$ for $m = 1,2,3$ satisfy

$$\begin{aligned}\mathrm{Re}\left(\gamma_1(x)\right) &+ \mathrm{Re}\left(\gamma_2(x)\right) + \mathrm{Re}\left(\gamma_3(x)\right)\\ &= -\frac{1}{4}\Big[c_1^2(\psi_1')^2 + c_2^2(\psi_2')^2 + c_3^2(\psi_3')^2 - 2c_1c_2\psi_1'\psi_2' - 2c_1c_3\psi_1'\psi_3' - 2c_2c_3\psi_2'\psi_3'\Big].\end{aligned} \tag{5.12}$$

As a result, we can choose two of $\{\mathrm{Re}\left(\gamma_1(x)\right), \mathrm{Re}\left(\gamma_2(x)\right), \mathrm{Re}\left(\gamma_3(x)\right)\}$ independently, and the third is determined by (5.12). Additionally, notice that if $\psi_m(x)$ is independent of x for $m = 1,2,3$, then (5.11)-(5.12) reduce to (4.12). In particular, if the phases have no x dependence, then $\mathrm{Im}\left(\gamma_m(x)\right) = 0$ for $m = 1,2,3$, and (5.12) becomes

$$\mathrm{Re}\left(\gamma_1(x)\right) + \mathrm{Re}\left(\gamma_2(x)\right) + \mathrm{Re}\left(\gamma_3(x)\right) = 0.$$

Finally, at order $(\tau-\tau_0)^3$, we have

$$-2\delta_m(x) - c_m\left[i\psi_m'(x)\,\gamma_m(x) + \gamma_m'(x)\right] = \delta_\ell^*(x) + \delta_k^*(x) + \beta_k^*(x)\gamma_\ell^*(x) + \beta_\ell^*(x)\gamma_k^*(x). \tag{5.13}$$

Again, (5.13) does not admit unique solutions. We split $\delta_m(x)$ into its real and imaginary parts and discover that the real part of $\delta_m(x)$ is uniquely determined for $m = 1, 2, 3$. In particular, we have

$$\begin{aligned}\mathrm{Re}\,(\delta_m(x))(x) = \frac{1}{8}\Big\{ & 2\left[c_k \mathrm{Re}\,(\gamma_k'(x)) + c_\ell \mathrm{Re}\,(\gamma_\ell'(x))\right] - 6c_m \mathrm{Re}\,(\gamma_m'(x)) \\ & + c_m\psi_m''\left[2c_m(-c_m + c_k + c_\ell)\psi_m' + (c_m - c_k)c_k\psi_k' + (c_m - c_\ell)c_\ell\psi_\ell'\right] \\ & + c_k\psi_k''\left[2c_m(c_m - c_k + c_\ell)\psi_m' + c_k(c_k - c_m)\psi_k' - c_\ell(c_m + c_\ell)\psi_\ell'\right] \\ & + c_\ell\psi_\ell''\left[2c_m(c_m + c_k - c_\ell)\psi_m' - c_k(c_m + c_k)\psi_k' + c_\ell(c_\ell - c_m)\psi_\ell'\right]\Big\},\end{aligned} \tag{5.14}$$

for $(k, \ell, m) = (1, 2, 3)$ cyclically, where we reiterate that only two of $\{\mathrm{Re}\,(\gamma_1(x)), \mathrm{Re}\,(\gamma_2(x)), \mathrm{Re}\,(\gamma_3(x))\}$ are free functions, and the third is determined by (5.12).

The imaginary part of $\delta_k(x)$ for $k = 1, 2$ is given by

$$\begin{aligned}\mathrm{Im}\,(\delta_k(x))(x) = \mathrm{Im}\,(\delta_3(x)) + \frac{1}{12}\Big\{ & \mathrm{Re}\,(\gamma_k(x))\left[-6c_k\psi_1'(x) + 2c_\ell\psi_2'(x) - 2c_3\psi_3'(x)\right] \\ & + \mathrm{Re}\,(\gamma_\ell(x))\left[4c_k\psi_1'(x) - 4c_3\psi_3'(x)\right] \\ & + \mathrm{Re}\,(\gamma_3(x))\left[2c_k\psi_1'(x) - 2c_\ell\psi_2'(x) + 6c_3\psi_3'(x)\right] \\ & - c_1\psi_1^{(3)}\left[c_\ell c_3 + c_k(c_\ell + 2c_3)\right] + c_2\psi_2^{(3)}\left[c_k c_\ell - c_3(2c_k + c_\ell)\right] \\ & + c_3\psi_3^{(3)}\left[c_3(2c_k + c_\ell) - 3c_k c_\ell\right]\Big\},\end{aligned} \tag{5.15}$$

where $(k, \ell) = (1, 2)$ cyclically, and $\mathrm{Im}\,(\delta_3(x))$ is a real, free function of x. Notice that if the phases are independent of x, and if $\mathrm{Re}\,(\gamma_2(x))$ and $\mathrm{Re}\,(\gamma_3(x))$ are chosen to be constant, then (5.14)-(5.15) reduce to (4.14).

At this point, we have found five free real functions in the formal Laurent series solution (5.7) of the three-wave PDEs, as well as one free real constant. The free functions and constant are

$$\{\psi_1(x), \psi_2(x), \mathrm{Re}\,(\gamma_1(x)), \mathrm{Re}\,(\gamma_2(x)), \mathrm{Im}\,(\delta_3(x))\} \quad \text{and} \quad \mathrm{Re}\,(\tau_0). \tag{5.16}$$

The series (5.7) is fully determined once we find all remaining terms in the series. To that end, we write the series solution $a_m(x, \tau)$ more succinctly as

$$a_m(x, \tau) = \frac{e^{i\psi_m(x)}}{\xi}\sum_{n=0}^{\infty}\mathcal{A}_n^m(x)\xi^n, \tag{5.17}$$

where $\xi = \tau - \tau_0$, and we observe that $\partial/\partial\tau = \partial/\partial\xi$. Moreover, a comparison with (4.23) tells us that

$$\mathcal{A}_0^m(x) = 1, \quad \mathcal{A}_1^m(x) = \beta_m(x), \quad \mathcal{A}_2^m(x) = \gamma_m(x), \quad \text{and} \quad \mathcal{A}_3^m(x) = \delta_m(x), \tag{5.18}$$

for $m = 1, 2, 3$, where $\beta_m(x)$ is given in (5.9), the imaginary and real parts of $\gamma_m(x)$ are given in (5.11) and (5.12), respectively, and the real and imaginary parts of $\delta_m(x)$ are given in (5.14) and (5.15), respectively.

In order to determine $\mathcal{A}_n^m(x)$ for $n \geq 4$, we start as usual by substituting (5.17) into (5.1). The left-hand side becomes

$$\begin{aligned}
\frac{\partial a_m}{\partial \tau} + c_m \frac{\partial a_m}{\partial x} &= \frac{e^{i\psi_m}}{\xi^2}\left[-\mathcal{A}_0^m + \mathcal{A}_2^m\xi^2 + 2\mathcal{A}_3^m\xi^3 + 3\mathcal{A}_4^m\xi^4 + 4\mathcal{A}_5^m\xi^5 + \cdots\right] \\
&\quad + c_m\frac{e^{i\psi_m}}{\xi^2}\left[(\mathcal{A}_0^{m'} + i\psi_m'\mathcal{A}_0^m)\xi + (\mathcal{A}_1^{m'} + i\psi_m'\mathcal{A}_1^m)\xi^2 + (\mathcal{A}_2^{m'} + i\psi_m'\mathcal{A}_2^m)\xi^3 + \cdots\right] \\
&= \frac{e^{i\psi_m}}{\xi^2}\left[\sum_{n=0}^{\infty}(n-1)\mathcal{A}_n^m\xi^n + c_m\sum_{n=1}^{\infty}\left(\mathcal{A}_{n-1}^{m'} + i\psi_m'\mathcal{A}_{n-1}^m\right)\right],
\end{aligned} \tag{5.19}$$

while the product $a_k^* a_\ell^*$ on the right-hand side is given by

$$\begin{aligned}
a_k^* a_\ell^* &= \frac{e^{-i(\psi_k+\psi_\ell)}}{\xi^2}\left[\mathcal{A}_0^{k*}\mathcal{A}_0^{\ell*} + \left(\mathcal{A}_0^{k*}\mathcal{A}_1^{\ell*} + \mathcal{A}_1^{k*}\mathcal{A}_0^{\ell*}\right)\xi + \left(\mathcal{A}_0^{k*}\mathcal{A}_2^{\ell*} + \mathcal{A}_1^{k*}\mathcal{A}_1^{\ell*} + \mathcal{A}_2^{k*}\mathcal{A}_0^{\ell*}\right)\xi^2 + \cdots\right] \\
&= \frac{e^{-i(\psi_k+\psi_\ell)}}{\xi^2}\sum_{n=0}^{\infty}\sum_{p=0}^{n}\mathcal{A}_p^{k*}\mathcal{A}_{n-p}^{\ell*}\xi^n.
\end{aligned} \tag{5.20}$$

Note that we dropped the explicit x dependence of $\psi_m(x)$ and $\mathcal{A}_n^m(x)$ for notational simplicity.

We substitute (5.19)-(5.20) into the three-wave PDEs in (5.1), multiply by $\xi^2 e^{i(\psi_k+\psi_\ell)}$, and use the fact that $e^{i\Psi} = -\sigma$. Then at order ξ^n, we obtain

$$(n-1)\mathcal{A}_n^m + c_m\left(\mathcal{A}_{n-1}^{m'} + i\psi_m'\mathcal{A}_{n-1}^m\right) = -\sum_{p=0}^{n}\mathcal{A}_p^{k*}\mathcal{A}_{n-p}^{\ell*}, \quad n \geq 1.$$

Finally, we use the fact that $\mathcal{A}_0^m(x) = 1$ for $m = 1, 2, 3$ in order to rewrite the above equation as

$$(n-1)\mathcal{A}_n^m(x) + \mathcal{A}_n^{k*}(x) + \mathcal{A}_n^{\ell*}(x) = -c_m\left[\mathcal{A}_{n-1}^{m'}(x) + i\psi_m'(x)\mathcal{A}_{n-1}^m(x)\right] - \sum_{p=1}^{n-1}\mathcal{A}_p^{k*}(x)\mathcal{A}_{n-p}^{\ell*}(x). \tag{5.21}$$

Notice that for $n = 1, 2, 3$, we obtain equations of the form (5.8), (5.10), and (5.13), respectively. Additionally, equation (5.21) is analogous to the recursion relation (4.26) that defines the coefficients in the ODE series solution. In particular, (5.21) has an extra term on the right-hand side of the recursion due to the spatial derivative term in the three-wave PDEs.

Finally, we can write (5.21) as a matrix equation of the form (4.27). We have

$$\begin{pmatrix} n-1 & 0 & 0 & 0 & 1 & 1 \\ 0 & n-1 & 0 & 1 & 0 & 1 \\ 0 & 0 & n-1 & 1 & 1 & 0 \\ 0 & 1 & 1 & n-1 & 0 & 0 \\ 1 & 0 & 1 & 0 & n-1 & 0 \\ 1 & 1 & 0 & 0 & 0 & n-1 \end{pmatrix} \begin{pmatrix} \mathcal{A}_n^1(x) \\ \mathcal{A}_n^2(x) \\ \mathcal{A}_n^3(x) \\ \mathcal{A}_n^{1*}(x) \\ \mathcal{A}_n^{2*}(x) \\ \mathcal{A}_n^{3*}(x) \end{pmatrix} = \begin{pmatrix} b_n^1(x) \\ b_n^2(x) \\ b_n^3(x) \\ b_n^{1*}(x) \\ b_n^{2*}(x) \\ b_n^{3*}(x) \end{pmatrix}, \tag{5.22}$$

where

$$b_n^m(x) = -c_m \left[\mathcal{A}_{n-1}^{m'}(x) + i\psi_m'(x)\mathcal{A}_{n-1}^m(x)\right] - \sum_{p=1}^{n-1} \mathcal{A}_p^{k*}(x)\mathcal{A}_{n-p}^{\ell*}(x). \tag{5.23}$$

Recall from Section 4.2 that the determinant of the matrix above is

$$D = n^2(n+1)(n-2)^2(n-3),$$

which indicates that (5.22) does not admit unique solutions when $n = 0, 2, 3$. This is consistent with the five free constants we found in (5.16) that arose at orders $n = 0$, 2, and 3.

In summary, we have determined that the near-general solution of the three-wave PDEs in the explosive regime is given by the formal Laurent series

$$a_m(x,\tau) = \frac{e^{i\psi_m}}{\xi} \sum_{n=0}^{\infty} \mathcal{A}_n^m(x)\xi^n, \tag{5.24}$$

where $\xi = \tau - \tau_0$, and the series coefficients are given by

$$\mathcal{A}_0^m(x) = 1, \quad \mathcal{A}_1^m(x) = \beta_m(x), \quad \mathcal{A}_2^m(x) = \gamma_m(x), \quad \mathcal{A}_3^m(x) = \delta_m(x), \tag{5.25}$$

and for $n \geq 4$,

$$(n-1)\mathcal{A}_n^m(x) + \mathcal{A}_n^{k*}(x) + \mathcal{A}_n^{\ell*}(x) = -c_m \left[\mathcal{A}_{n-1}^{m'}(x) + i\psi_m'(x)\mathcal{A}_{n-1}^m(x)\right] - \sum_{p=1}^{n-1} \mathcal{A}_p^{k*}(x)\mathcal{A}_{n-p}^{\ell*}(x). \tag{5.26}$$

The functions $\psi_m(x)$, $\beta_m(x)$, $\gamma_m(x)$, and $\delta_m(x)$ are defined via (5.6), (5.9), (5.11)-(5.12), and (5.14)-(5.15), respectively.

Again, note that the solution contains five real, free functions of x, and one real constant,

$$\{\psi_1(x), \psi_2(x), \mathrm{Re}\,(\gamma_1(x)), \mathrm{Re}\,(\gamma_2(x)), \mathrm{Im}\,(\delta_3(x))\} \quad \text{and} \quad \mathrm{Re}\,(\tau_0). \tag{5.27}$$

It is useful later to be able to relate the functions in (5.27) to the free constants in (4.32). To that end, we define the following

$$\mathcal{K}_2(x) = \mathrm{Re}\,(\gamma_1(x)), \quad \mathcal{K}_3(x) = \mathrm{Re}\,(\gamma_2(x)), \quad \text{and} \quad \mathcal{H}(x) = 6\,\mathrm{Im}\,(\delta_3(x)),$$

where the factor of 6 in $\mathcal{H}(x)$ is introduced for convenience. With these definitions, (5.27) becomes

$$\{\psi_1(x), \psi_2(x), \mathcal{K}_2(x), \mathcal{K}_3(x), \mathcal{H}(x)\} \quad \text{and} \quad \mathrm{Re}\,(\tau_0), \tag{5.28}$$

which looks similar to (4.32).

We reiterate that the solution of the three-wave PDEs formed by (5.24)-(5.26) with the free functions and constants in (5.27) is not fully general. In order for the solution to be general, we must introduce x dependence into τ_0, which is difficult for the reasons outlined at the start of this section. Nonetheless, our formal solution is more general than existing solutions for several reasons. First of all, our derivation could be easily repeated in more than one spatial dimension with no change. Indeed, our method of solution holds for an arbitrary number of spatial dimensions. Additionally, we have imposed no boundary data in constructing our solution, which means the solution should be compatible with any type of boundary conditions.

5.2 Convergence of solutions

In the previous section, we used Painlevé analysis to construct a formal Laurent series solution of the three-wave PDEs with five free functions of x, and one free constant. In this section, we want to make our solution rigorous by determining under what conditions the series in (5.24)-(5.26) converges, so that we can determine where our solution is valid.

In this section, we show where the series solution of the three-wave PDEs converges for several special cases. In particular, we consider the case where the phases, $\psi_m(x)$, are constant for $m = 1, 2, 3$. We then consider several subcases. First of all, we consider what happens when $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$ and $\mathcal{H}(x) \neq 0$. This appears to be the simplest case. Next, we consider what happens when $\mathcal{H}(x) = 0$, and at least one of $\{\mathcal{K}_2(x), \mathcal{K}_3(x)\}$ is nonzero. Within this case, we consider three situations: (i) $\mathcal{K}_2(x) = \mathcal{K}_3(x)$, (ii) one of $\{\mathcal{K}_2(x), \mathcal{K}_3(x)\}$ is zero, and (iii) $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$ are nonzero and unrelated. The results of our analysis are summarized in the last column of Table 5.1. The rest of the contents of Table 5.1 are explained as we progress through the chapter.

In order to gain some traction with our convergence proofs, we enforce certain smoothness restrictions on the functions $\{\mathcal{K}_2(x), \mathcal{K}_3(x), \mathcal{H}(x)\}$. These restrictions are detailed later.

Before we outline the form of the convergence proofs, we first examine the relation between the free constants (4.32) that appear in the series solution of the three-wave ODEs, and the free functions (5.28) that appear in the series solution of the three-wave PDEs. In particular, suppose that the functions in (5.28) are independent of x. Then the series solution of the ODEs in (4.29)-(4.31) is equivalent to the series solution of the PDEs in (5.24)-(5.26) under the following conditions

$$\mathcal{K}_2(x) = \frac{\sigma}{6}\left(K_2 + K_3\right), \quad \mathcal{K}_3(x) = \frac{\sigma}{6}(K_3 - 2K_2), \quad \text{and} \quad \mathcal{H}(x) = \sigma H. \tag{5.29}$$

Indeed, under these conditions, we have that $A_n^m = \mathcal{A}_n^m(x)$ for $n \geq 0$, and for $m = 1, 2, 3$, where A_n^m are the constants that appear in the ODE series solution in (4.29), and $\mathcal{A}_n^m(x)$ are the functions that appear in the PDE series solution in (5.24). Moreover, if the functions in (5.28) are independent of x, then $\psi_m(x) = \psi_m$, for $m = 1, 2, 3$, where ψ_m is the phase that appears in (4.29). Then the series solutions of the three-wave PDEs and ODEs are equivalent.

5.2.1 Outline of Proof

The structure of the convergence proofs for each of the cases in Table 5.1 is similar, although Case 2(iii) deviates slightly from the usual procedure. In all cases, the goal is to find the radius of convergence of the series solution of the three-wave PDEs by writing the series in a way that admits the use of the ratio test. We outline the structure of the proofs below.

	ODE Vals $\{K_2, K_3, H\}$	PDE Vals $\{\mathcal{K}_2, \mathcal{K}_3, \mathcal{H}\}$	ODE Radius	PDE Radius
Case 1	$\{0, 0, H\}$	$\left\{0, 0, \frac{\mathcal{H}(x)}{6}\right\}$	$\lvert\xi\rvert < \frac{3.06}{\lvert H\rvert^{1/3}}$	$\lvert\xi\rvert e^{\frac{1}{3}ck\lvert\xi\rvert} < \frac{3.06}{\lVert\mathcal{H}\rVert^{1/3}}$
Case 2(i)	$\{0, K, 0\}$	$\left\{\frac{\mathcal{K}(x)}{6}, \frac{\mathcal{K}(x)}{6}, 0\right\}$	$\lvert\xi\rvert < \frac{\pi}{\lvert K\rvert^{1/2}}$	$\lvert\xi\rvert e^{\frac{3}{2}ck\lvert\xi\rvert} < \frac{\pi}{\lVert\mathcal{K}\rVert^{1/2}}$
Case 2(ii)	$\{K, 2K, 0\}$	$\left\{\frac{\mathcal{K}(x)}{2}, 0, 0\right\}$	$\lvert\xi\rvert < \frac{2.62}{\lvert K\rvert^{1/2}}$	$\lvert\xi\rvert e^{\frac{3}{2}ck\lvert\xi\rvert} < \frac{2.62}{\lVert\mathcal{K}\rVert^{1/2}}$
Case 2(iii)*	$\{K_2, K_3, 0\}$	$\{\mathcal{K}_2(x), \mathcal{K}_3(x), 0\}$	$\lvert\xi\rvert < \min\left\{\frac{2.57}{\lvert K_2\rvert^{1/2}}, \frac{\pi}{\lvert K_3\rvert^{1/2}}\right\}$	$\lvert\xi\rvert e^{\frac{3}{2}ck\lvert\xi\rvert} < \frac{1.28}{[\max\{\lVert\mathcal{K}_2\rVert, \lVert\mathcal{K}_3\rVert\}]^{1/2}}$

Table 5.1: The radius of convergence of the Laurent series solution of the three-wave ODEs and PDEs for some configurations of $\{K_2, K_3, H\}$ (in the ODEs) and $\{\mathcal{K}_2(x), \mathcal{K}_3(x), \mathcal{H}(x)\}$ (in the PDEs). In the table, $\|\cdot\| = \|\cdot\|_\infty$.
*The proof of Case 2(iii) involves a numerical observation that is not needed in the other cases.

5.2.1.1 The convergence proof

To begin, choose one of the following cases from Table 5.1. In each case, assume the phases $\psi_m(x)$ are constant for the PDE series solution.

Case 1: $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$, which corresponds to the ODE case in which $K_2 = K_3 = 0$.

Case 2: $\mathcal{H}(x) = 0$, which corresponds to the ODE case in which $H = 0$. Within this case, we consider three possible subcases.

(i) $\mathcal{K}_2(x) = \mathcal{K}_3(x) \equiv \mathcal{K}(x)/6$, where $\mathcal{K}(x)$ is a real function of x and the factor of $1/6$ is introduced for convenience. From (5.29), we see that this case corresponds to the assumption in the ODEs that $K_2 = 0$ and $K_3 = K$, where K is some real constant.

(ii) $\mathcal{K}_2(x) \equiv \mathcal{K}(x)/2$ and $\mathcal{K}_3(x) = 0$, where $\mathcal{K}(x)$ is a real function of x and the factor of $1/2$ is introduced for convenience. From (5.29), we see that this corresponds to the

ODE case where $K_2 = K$ and $K_3 = 2K$ for some real constant K.

(iii) $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$ are nonzero and unrelated. Similarly, in the ODEs, this corresponds to the case where K_2 and K_3 are real constants without any special relation between them.

Next, follow Steps 1-6 below in order to find the radius of convergence of the formal series solution of the three-wave PDEs.

Step 1: For the chosen case, find the series solution of the three-wave ODEs, $a_m(\tau)$, for the given values of $\{K_2, K_3, H\}$ using (4.29)-(4.31). In particular, for our special cases, we show that

$$a_m(\tau) = \frac{e^{i\psi_m}}{\xi} \sum_{n=0}^{\infty} A^m_{\mu n} \xi^{\mu n}, \tag{5.30}$$

where $\xi = \tau - \tau_0$, and where $\mu = 3$ for Case 1 and $\mu = 2$ for Case 2. We then find the radius of convergence of the series using one of the methods outline in Section 4.3. Denote the radius of convergence by R_{ODE}. (Note: There is a slight caveat involved with finding the radius of convergence for Case 2(i) which we explain in detail in Section 5.4.)

Step 2: Next, find the series solution of the three-wave PDEs, $a_m(x, \tau)$, for the chosen values of $\{\mathcal{K}_2(x), \mathcal{K}_3(x), \mathcal{H}(x)\}$ using (5.24)-(5.26).

Step 3: Define

$$c = \max\{|c_1|, |c_2|, |c_3|\}. \tag{5.31}$$

Furthermore, assume there is a finite positive k such that one of the following is true

$$\text{Case 1:} \qquad \left\| \frac{d^n}{dx^n} \mathcal{H}(x) \right\|_\infty \le k^n \|\mathcal{H}\|_\infty \tag{5.32}$$

$$\text{Case 2(i), 2(ii):} \qquad \left\| \frac{d^n}{dx^n} \mathcal{K}(x) \right\|_\infty \le k^n \|\mathcal{K}\|_\infty, \tag{5.33}$$

$$\text{Case 2(iii):} \qquad \left\| \frac{d^n}{dx^n} \mathcal{K}_j(x) \right\|_\infty \le k_j^n \|\mathcal{K}_j\|_\infty, \quad j = 2, 3. \tag{5.34}$$

Step 4: Bound the series for $a_m(x, \tau)$ from Step 3 using (5.31) and one of (5.32)-(5.34). This amounts to repeated application of the triangle inequality. The bound on $a_m(x, \tau)$ should be of the form

$$|a_m(x,\tau)| \leq \frac{1}{|\xi|}\sum_{p=0}^{\infty}\sum_{n=\mu p}^{\infty} q_{n,p}^{m}(ck)^{n-\mu p}\,|\xi|^{n}\,, \tag{5.35}$$

where $\xi = \tau - \tau_0$ as usual, $q_{n,p}^m$ are real nonnegative constants, and $\mu = 3$ or $\mu = 2$ for Case 1 or 2, respectively.

Step 5: The goal in this step is to reduce the double sum in (5.35) to a single sum, so that we can then easily apply the ratio test and find the radius of convergence of the series for $a_m(x,\tau)$. Case 1 and 2 must be treated slightly differently here. Note that the procedure for Case 1 turns out to be easier than the procedure for Case 2, although the principle for each case is the same. Additionally, note that the procedure for Case 2(iii) is slightly different from that used for Case 2(i)-(ii). This is explained further in Section 5.6.

Case 1: It is possible to find an exact formula for the constants $q_{n,p}^m$ in (5.35). In particular, it can be shown that

$$q_{n,p}^1 = q_{n,p}^2 = q_{n,p}^3 = \frac{p^{n-\mu p}}{(n-\mu p)!}\,q_{\mu p,p}, \quad n \geq \mu p, \tag{5.36}$$

where $\mu = 3$ (See Appendix B). Furthermore, it turns out that $q_{\mu p,p} = |A_{\mu p}|$, where $A_{\mu p}$ is the coefficient from the ODE series in (5.30) when $H = 0$. We omit the dependence of $A_{\mu p}$ on m since for Case 1, $A_{\mu p}^1 = A_{\mu p}^2 = A_{\mu p}^3$ (see Section 5.3).

We substitute (5.36) into (5.35). After some simplification, we obtain a single infinite sum of the form

$$|a_m(x,\tau)| \leq \frac{1}{|\xi|}\sum_{p=0}^{\infty} q_{\mu p,p}|\xi|^{\mu p}e^{ckp|\xi|}. \tag{5.37}$$

Case 2: Here we define

$$A(x,\tau) = \sum_{m=1}^{3}|a_m(x,\tau)|\,.$$

We can show that $A(x,\tau)$ is bounded as follows

$$A(x,\tau) \leq \frac{1}{|\xi|}\sum_{p=0}^{\infty}\sum_{n=\mu p}^{\infty} r_{n,p}(ck)^{n-\mu p}\,|\xi|^{n}\,, \tag{5.38}$$

where $\mu = 2$ and $r_{n,p}$ are real constants defined via

$$r_{n,p} = \frac{n+4}{(n+1)(n-2)}\left[p r_{n-1,p} + \frac{1}{2}\sum_{\ell=1}^{p-1}\sum_{j=\mu\ell}^{n-\mu(p-\ell)} r_{j,\ell}\, r_{n-j,p-\ell}\right], \quad n > \mu p. \tag{5.39}$$

It can then be shown that

$$r_{n,p} \le \frac{(\gamma\, p)^{n-\mu p}}{(n-\mu p)!}\,(\alpha p + \beta)\; r_{\mu p,p}, \quad n > \mu p, \tag{5.40}$$

where α, β, and γ are real constants to be determined, with α and γ positive, and

$$r_{\mu p,p} = q^1_{\mu p,p} + q^2_{\mu p,p} + q^3_{\mu p,p} = |A^1_{\mu p}| + |A^2_{\mu p}| + |A^3_{\mu p}|.$$

Then substitute the bound on $r_{n,p}$ in (5.40) into (5.38). Simplify to obtain a single infinite sum of the form

$$A(x,\tau) \le \frac{1}{|\xi|}\sum_{p=0}^{\infty} r_{\mu p,p}|\xi|^{\mu p}(\alpha p + \beta)e^{\gamma c k p|\xi|}. \tag{5.41}$$

Note that the proof of (5.40) has two main parts:

(1) Prove that

$$r_{n,p} \le \frac{(\gamma\, p)^{n-\mu p}}{(n-\mu p)!}\sum_{\ell=1}^{p-1} r_{\mu\ell,\ell}\, r_{\mu(p-\ell),p-\ell}. \tag{5.42}$$

(See Appendix C.)

(2) Prove that

$$\sum_{\ell=1}^{p-1} r_{\mu\ell,\ell}\, r_{\mu(p-\ell),p-\ell} \le (\alpha p + \beta)\; r_{\mu p,p}. \tag{5.43}$$

Step 6: Find the radius of convergence for $a_m(x,\tau)$ using the ratio test.

Case 1: Apply the ratio test to the bounded series in (5.37). Then the series converges if

$$|\xi|^{\mu} e^{ck|\xi|} \lim_{p\to\infty}\left|\frac{q_{\mu(p+1),p+1}}{q_{\mu p,p}}\right| < 1,$$

where $\mu = 3$. Rearranging and using the fact that $q_{\mu p,p} = |A_{\mu p}|$ yields

$$e^{\frac{ck|\xi|}{\mu}}|\xi| < R_{\mathrm{ODE}} \quad \text{or} \quad e^{\frac{ck|\tau-\tau_0|}{\mu}}|\tau-\tau_0| < R_{\mathrm{ODE}},$$

where R_{ODE} is the radius of convergence from Step 2, with $|H|$ replaced by $\|\mathcal{H}\|_\infty$.

Case 2: Apply the ratio test to the bounded series in (5.41). Then the series converges if

$$|\xi|^{\mu} e^{\gamma ck|\xi|} \lim_{p\to\infty} \left| \frac{r_{\mu(p+1),p+1}}{r_{\mu p,p}} \right| < 1,$$

where $\mu = 2$. Rearranging and using the fact that $r_{\mu p,p} = |A^1_{\mu p}| + |A^2_{\mu p}| + |A^3_{\mu p}|$ yields

$$e^{\frac{\gamma ck|\xi|}{\mu}} |\xi| < \tilde{R} \quad \text{or} \quad e^{\frac{\gamma ck|\tau-\tau_0|}{\mu}} |\tau - \tau_0| < \tilde{R},$$

where

$$\tilde{R} = \left(\lim_{p\to\infty} \left| \frac{|A^1_{\mu(p+1)}| + |A^2_{\mu(p+1)}| + |A^3_{\mu(p+1)}|}{|A^1_{\mu p}| + |A^2_{\mu p}| + |A^3_{\mu p}|} \right| \right)^{-1/\mu}.$$

Notice that $\tilde{R}$ can be determined from the ODE series solution in Step 2.

Steps 1-6 give a general outline for how to go about finding the radius of convergence of the formal Laurent series solution of the three-wave PDEs for all of the cases in Table 5.1, although the details vary with each case. In the following sections, we give more details for each of the cases. The bulk of the work for all cases is in Step 5, namely in proving equations like (5.36), and in establishing bounds like (5.42)-(5.43). The proofs of these results in some cases are lengthy and extremely technical. At times, the specifics of theses proofs are placed in the appendices for the sake of readability.

5.3 Radius of convergence: Case 1

Consider Case 1, in which $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$ in the solution of the three-wave PDEs, and $K_2 = K_3 = 0$ in the solution of the three-wave ODEs. In order to find the radius of convergence of the Laurent series solution of the three-wave PDEs in this case, we follow the steps outlined in the previous section. Recall that the phases $\psi_m(x)$ in the PDE solution are assumed to be constant.

Step 1: First, we want to find the series solution of the three-wave ODEs when $K_2 = K_3 = 0$ using (4.29)-(4.31). The first few terms of the series are given by

$$a_m(\tau) = \frac{e^{i\psi_m}}{\tau - \tau_0} \Big[1 + \frac{i\sigma H}{6}(\tau-\tau_0)^3 + \frac{H^2}{252}(\tau-\tau_0)^6 + \frac{i\sigma H^3}{4536}(\tau-\tau_0)^9 + \frac{11H^4}{2476656}(\tau-\tau_0)^{12} + \cdots \Big]. \quad (5.44)$$

It is straightforward to show by induction that when $K_2 = K_3 = 0$, then (4.29) reduces to

$$a_m(\tau) = \frac{e^{i\psi_m}}{\xi} \sum_{n=0}^{\infty} A_{3n}\, \xi^{3n}, \tag{5.45}$$

where $\xi = \tau - \tau_0$, $A_0 = 1$, $A_3 = i\sigma H/6$, and (4.31) becomes

$$(3n-1)A_{3n} + 2A_{3n}^* = -\sum_{p=1}^{n-1} A_{3p}^* A_{3(n-p)}^*, \quad \text{for } n \geq 2.$$

Notice that the dependence on the mode m in (5.44) and (5.45) only appears in the phase, so $A_{3n}^m \equiv A_{3n}$ for $m = 1, 2, 3$.

To determine the radius of convergence of (5.45), recall Section 4.3. The radius of convergence of the solution of the three-wave ODEs can be determined in one of two ways: using the Weierstrass solution in (3.54) and (3.60), or using the Laurent series solution in (4.29)-(4.31).

First, consider the Weierstrass solution. When $K_2 = K_3 = 0$, the elliptic invariants in (3.45)-(3.46) are given by $g_2 = 0$ and $g_3 = H^2$. Further, if $g_2 = 0$ and g_3 is real, then the half-periods λ_1 and λ_2 are explicitly defined as follows [17, §23.5]

$$\lambda_1 = e^{-i\pi/3}\lambda_2 = \frac{\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi g_3^{1/6}}, \tag{5.46}$$

and the lattice generated by λ_1 and λ_2 is rhombic. As a result, we know from (4.38) that the general solution of the three-wave ODEs converges when

$$|\tau - \tau_0| < \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\, |H|^{1/3}} \approx \frac{3.06}{|H|^{1/3}}, \tag{5.47}$$

where we substituted $g_3 = H^2$ into (5.46). Note that since $\lambda_1 = e^{i\pi/3}\lambda_2$, the lattice of poles is rhombic, and thus we are restricted to the explosive regime only.

Alternatively, we can apply the ratio test to (5.45) to determine the radius of convergence. In this case, the ratio test indicates that the series (5.45) converges if

$$\lim_{n\to\infty} \left| \frac{A_{3(n+1)}\xi^{3(n+1)}}{A_{3n}\xi^{3n}} \right| < 1.$$

However, in practice, it turns out that this limit does not exist; for each increase in n, the value of $|A_{3(n+1)}/A_{3n}|$ oscillates between two numbers. In order to deal with this, we split the series in

(5.45) into its even and odd parts. To that end, we define

$$S_{\text{even}} = \frac{e^{i\psi_m}}{\xi}\sum_{n=0}^{\infty} A_{6n}\,\xi^{6n} \quad \text{and} \quad S_{\text{odd}} = \frac{e^{i\psi_m}}{\xi}\sum_{n=0}^{\infty} A_{3(2n+1)}\,\xi^{3(2n+1)}, \tag{5.48}$$

so that $a_m(\tau) = S_{\text{even}} + S_{\text{odd}}$. The even and odd sums converge, respectively, under the conditions

$$\lim_{n\to\infty}\left|\frac{A_{6(n+1)}\xi^{6(n+1)}}{A_{6n}\xi^{6n}}\right| < 1 \quad \text{and} \quad \lim_{n\to\infty}\left|\frac{A_{3(2(n+1)+1)}\xi^{3(2(n+1)+1)}}{A_{3(2n+1)}\xi^{3(2n+1)}}\right| < 1.$$

Rearranging, and using $\xi = \tau - \tau_0$ yields the following convergence criteria for the even and odd series in (5.48)

$$|\tau - \tau_0| < \left(\lim_{n\to\infty}\left|\frac{A_{6(n+1)}}{A_{6n}}\right|\right)^{-1/6} \quad \text{and} \quad |\tau - \tau_0| < \left(\lim_{n\to\infty}\left|\frac{A_{3(2(n+1)+1)}}{A_{3(2n+1)}}\right|\right)^{-1/6}. \tag{5.49}$$

Each of the limits in 5.49 yields the same value. In particular, we find that both the even and odd series in (5.48) converge under the condition

$$|\tau - \tau_0| < \frac{3.05991}{|H|^{1/3}}.$$

It follows that the full series (5.45) converges with radius of convergence

$$R_{\text{ODE}} \approx \frac{3.06}{|H|^{1/3}},$$

which is equivalent to (5.47). That is, we find the same radius of convergence using both the Weierstrass solution and the Laurent series solution of the three-wave ODEs. This is reassuring, as the two solutions should be interchangeable.

Note that (5.47) and (5.49) combined imply that

$$\left(\lim_{n\to\infty}\left|\frac{A_{6(n+1)}}{A_{6n}}\right|\right)^{-1/6} = \left(\lim_{n\to\infty}\left|\frac{A_{3(2(n+1)+1)}}{A_{3(2n+1)}}\right|\right)^{-1/6} = \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\,|H|^{1/3}}, \tag{5.50}$$

which becomes useful later.

Step 2: Next, we find the series solution of the three-wave PDEs when $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$ using

(5.24)-(5.26). The first few terms of (5.24) are given by

$$
\begin{aligned}
a_m(x,\tau) = \frac{e^{i\psi_m}}{\xi}\bigg\{ 1 &+ \frac{i}{6}\mathcal{H}\xi^3 - \frac{i}{24}\left(2c_m + c_k + c_\ell\right)\mathcal{H}'\,\xi^4 \\
&+ \frac{i}{120}\big(3c_m^2 + c_k^2 + c_k c_\ell + c_\ell^2 + 2c_m\left(c_k + c_\ell\right)\big)\,\mathcal{H}''\,\xi^5 \\
&+ \frac{1}{5040}\Big[20\mathcal{H}^2 - 7i\big(4c_m^3 + c_k^3 + c_k^2 c_\ell + c_k c_\ell^2 + c_\ell^3 + 3c_m^2\left(c_k + c_\ell\right) \\
&\qquad\qquad + 2c_m\left(c_k^2 + c_k c_\ell + c_\ell^2\right)\big)\mathcal{H}^{(3)}\Big]\xi^6 \\
&+ \Big[(\cdots)\mathcal{H}\mathcal{H}' + (\cdots)\mathcal{H}^{(4)}\Big]\xi^7 + \cdots \bigg\},
\end{aligned}
\tag{5.51}
$$

where $(k,\ell,m) = (1,2,3)$ cyclically, and where $\mathcal{H}^{(3)}$ and $\mathcal{H}^{(4)}$ denote the third and fourth derivative of $\mathcal{H}$ with respect to x, respectively. Notice that increasingly higher derivatives of $\mathcal{H}(x)$ appear as the number of terms in the series increases, as well as more nonlinear terms in $\mathcal{H}$ and its derivatives. Also, notice that the dependence on the mode m in (5.51) only appears through the group velocities and the phase.

From (5.51), we can see that the series solution of the three-wave PDEs quickly becomes complicated as the number of terms increases, even in this simple case. In order to determine under what conditions (5.51) converges, we restrict our attention to a particular family of functions $\mathcal{H}(x)$ in the next step.

Step 3: In accordance with (5.32), we assume that

$$
\left\| \frac{d^n}{dx^n}\mathcal{H}(x) \right\|_\infty \leq k^n \|\mathcal{H}\|_\infty, \tag{5.52}
$$

where k is a real positive constant. This is relatively general since it allows $\mathcal{H}(x)$ to be, for example, any trigonometric polynomial.

Step 4: Using (5.52) and the definition of c in (5.31), we bound (5.51) using the triangle inequality.

This first few terms in the bounded series are given by

$$
\begin{aligned}
|a_m(x,\tau)| \le \frac{1}{|\xi|}\Bigg\{ 1 &+ \frac{\|\mathcal{H}\|}{6}|\xi|^3 + \frac{\|\mathcal{H}\|}{6}ck|\xi|^4 + \frac{\|\mathcal{H}\|}{12}(ck)^2|\xi|^5 + \left[\frac{\|\mathcal{H}\|^2}{252} + \frac{\|\mathcal{H}\|}{36}(ck)^3\right]|\xi|^6 \\
&+ \left[\frac{\|\mathcal{H}\|^2}{126}ck + \frac{\|\mathcal{H}\|}{144}(ck)^4\right]|\xi|^7 + \left[\frac{\|\mathcal{H}\|^2}{126}(ck)^2 + \frac{\|\mathcal{H}\|}{720}(ck)^5\right]|\xi|^8 \\
&+ \left[\frac{\|\mathcal{H}\|^3}{4536} + \frac{\|\mathcal{H}\|^2}{189}(ck)^3 + \frac{\|\mathcal{H}\|}{4320}(ck)^6\right]|\xi|^9 \\
&+ \left[\frac{\|\mathcal{H}\|^3}{1512}ck + \frac{\|\mathcal{H}\|^2}{378}(ck)^4 + \frac{\|\mathcal{H}\|}{30240}(ck)^7\right]|\xi|^{10} \\
&+ \left[\frac{\|\mathcal{H}\|^3}{1008}(ck)^2 + \frac{\|\mathcal{H}\|^2}{945}(ck)^5 + \frac{\|\mathcal{H}\|}{241920}(ck)^8\right]|\xi|^{11} \\
&+ \left[\frac{11\|\mathcal{H}\|^4}{2476656} + \frac{\|\mathcal{H}\|^3}{1008}(ck)^3 + \frac{4\|\mathcal{H}\|^2}{2835}(ck)^6 + \frac{\|\mathcal{H}\|}{2177280}(ck)^9\right]|\xi|^{12} + \cdots \Bigg\}, \quad (5.53)
\end{aligned}
$$

where $\|\mathcal{H}\| = \|\mathcal{H}\|_\infty$ and $m = 1, 2, 3$.

First, notice that the coefficients of terms of the form $\|\mathcal{H}\|^n|\xi|^{3n}$ can be identified with the coefficients $|A_{3n}|$ that appear in the ODE series in (5.44) and (5.45). Next, notice that for every $n \ge 3$, the term multiplying $|\xi|^n$ in (5.53) is a polynomial in (ck) of degree $n-3$, where every third term is nonzero. More precisely, we can rewrite (5.53) as

$$
|a_m(x,\tau)| \le \frac{1}{|\xi|}\left[1 + \sum_{n=3}^{\infty}\sum_{p=1}^{\lfloor n/3\rfloor} q_{n,p}(ck)^{n-3p}|\xi|^n\right], \tag{5.54}
$$

where $q_{n,p}$ are nonnegative constants, independent of the mode m. Alternatively, we can change the order of summation and write

$$
|a_m(x,\tau)| \le \frac{1}{|\xi|}\left[1 + \sum_{p=1}^{\infty}\sum_{n=3p}^{\infty} q_{n,p}(ck)^{n-3p}|\xi|^n\right]. \tag{5.55}
$$

The first few constants $q_{n,p}$ are listed in Table 5.2. The index n corresponds to the nth row of the table, while the index p indicates the pth nonzero diagonal of the table. The columns of the table represent powers of (ck).

STEP 5: In this step, we reduce (5.55) to a single infinite sum. Let S_p be the inner sum in (5.55), namely

$$
S_p = \sum_{n=3p}^{\infty} q_{n,p}(ck)^{n-3p}|\xi|^n. \tag{5.56}
$$

n	$(ck)^0$	$(ck)^1$	$(ck)^2$	$(ck)^3$	$(ck)^4$	$(ck)^5$	$(ck)^6$	$(ck)^7$	$(ck)^8$
3	$\frac{\|\mathcal{H}\|}{6}$	0	0	0	0	0	0	0	0
4	0	$\frac{\|\mathcal{H}\|}{6}$	0	0	0	0	0	0	0
5	0	0	$\frac{\|\mathcal{H}\|}{12}$	0	0	0	0	0	0
6	$\frac{\|\mathcal{H}\|^2}{252}$	0	0	$\frac{\|\mathcal{H}\|}{36}$	0	0	0	0	0
7	0	$\frac{\|\mathcal{H}\|^2}{126}$	0	0	$\frac{\|\mathcal{H}\|}{144}$	0	0	0	0
8	0	0	$\frac{\|\mathcal{H}\|^2}{126}$	0	0	$\frac{\|\mathcal{H}\|}{720}$	0	0	0
9	$\frac{\|\mathcal{H}\|^3}{4536}$	0	0	$\frac{\|\mathcal{H}\|^2}{189}$	0	0	$\frac{\|\mathcal{H}\|}{4320}$	0	0
10	0	$\frac{\|\mathcal{H}\|^3}{1512}$	0	0	$\frac{\|\mathcal{H}\|^2}{378}$	0	0	$\frac{\|\mathcal{H}\|}{30240}$	0
11	0	0	$\frac{\|\mathcal{H}\|^3}{1008}$	0	0	$\frac{\|\mathcal{H}\|^2}{945}$	0	0	$\frac{\|\mathcal{H}\|}{241920}$
12	$\frac{11\|\mathcal{H}\|^4}{2476656}$	0	0	$\frac{\|\mathcal{H}\|^3}{1008}$	0	0	$\frac{\|\mathcal{H}\|^2}{2835}$	0	0

Table 5.2: The coefficients $q_{n,p}$ from (5.55). Note that n indicates the nth row of the table, while p indicates the pth nonzero diagonal (that is, the diagonal that begins in the row corresponding to $n = 3p$). For instance, $q_{9,2} = \frac{\|\mathcal{H}\|^2}{189}$.

Then (5.55) becomes

$$|a_m(x,\tau)| \le \frac{1}{|\xi|}\left[1 + \sum_{p=1}^{\infty} S_p\right]. \tag{5.57}$$

We show in Appendix B that

$$q_{n,p} = \frac{p^{n-3p}}{(n-3p)!}\, q_{3p,p}, \quad n \ge 3p, \tag{5.58}$$

where $q_{3p,p}/\|\mathcal{H}\|_\infty^p = |A_{3p}|/|H|^p$, and A_{3p} is the coefficient that appears in the Laurent series solution of the three-wave ODEs in (5.45).

Substituting (5.58) into (5.56) yields

$$\begin{aligned}
S_p &= \sum_{n=3p}^{\infty} \frac{p^{n-3p}}{(n-3p)!}\, q_{3p,p}\, (ck)^{n-3p} |\xi|^n \\
&= \frac{q_{3p,p}}{(ckp)^{3p}} \sum_{n=3p}^{\infty} \frac{(ckp\,|\xi|)^n}{(n-3p)!} \\
&= \frac{q_{3p,p}}{(ckp)^{3p}} \sum_{n=0}^{\infty} \frac{(ckp\,|\xi|)^{n+3p}}{n!} \\
&= q_{3p,p}\,|\xi|^{3p} \sum_{n=0}^{\infty} \frac{(ckp\,|\xi|)^n}{n!} \\
&= q_{3p,p}\,|\xi|^{3p}\, e^{ckp|\xi|}.
\end{aligned}$$

Now (5.57) gives

$$\begin{aligned}
|a_m(x,\tau)| &\le \frac{1}{|\xi|}\left[1+\sum_{p=1}^{\infty} S_p\right] \\
&= \frac{1}{|\xi|}\left[1+\sum_{p=1}^{\infty} q_{3p,p}\,|\xi|^{3p}\,e^{ckp|\xi|}\right] \\
&= \frac{1}{|\xi|}\sum_{p=0}^{\infty} q_{3p,p}\,|\xi|^{3p}\,e^{ckp|\xi|},
\end{aligned} \tag{5.59}$$

where we defined $q_{0,0} = 1$.

Step 6: We now want to use (5.59) in order to determine the radius of convergence of (5.51). From (5.59), we know that the Laurent series for $a_m(x,\tau)$ converges absolutely when

$$\lim_{p\to\infty}\left|\frac{q_{3(p+1),p+1}\,|\xi|^{3(p+1)}\,e^{ck(p+1)|\xi|}}{q_{3p,p}\,|\xi|^{3p}\,e^{ckp|\xi|}}\right| < 1.$$

Simplifying, we have that (5.51) converges if

$$e^{ck|\xi|}|\xi|^3 \lim_{p\to\infty}\left|\frac{q_{3(p+1),p+1}}{q_{3p,p}}\right| \overbrace{=}^{|H|\to\|\mathcal{H}\|_\infty} e^{ck|\xi|}|\xi|^3 \lim_{p\to\infty}\left|\frac{A_{3(p+1)}}{A_{3p}}\right| < 1,$$

where we used the fact that $q_{3p,p} = |A_{3p}|$ if we replace $|H|$ in A_{3p} with $\|\mathcal{H}\|_\infty$. However, recall from Step 1 of this section that

$$\lim_{p\to\infty}\left|\frac{A_{3(p+1)}}{A_{3p}}\right|$$

does not exist. As with the series solution of the three-wave ODEs, we must split (5.59) into its even and odd parts.

Define the following

$$\mathcal{S}_{\text{even}} = \frac{1}{|\xi|}\sum_{p=0}^{\infty} q_{6p,2p}\,|\xi|^{6p}\,e^{2ckp|\xi|} \quad \text{and} \quad \mathcal{S}_{\text{odd}} = \frac{1}{|\xi|}\sum_{p=0}^{\infty} q_{3(2p+1),2p+1}\,|\xi|^{3(2p+1)}\,e^{ck(2p+1)|\xi|}.$$

Then we know from (5.59) that $|a_m(x,\tau)| \le \mathcal{S}_{\text{even}} + \mathcal{S}_{\text{odd}}$. Additionally, the ratio test tells us that $\mathcal{S}_{\text{even}}$ converges when

$$\lim_{p\to\infty}\left|\frac{q_{6(p+1),2(p+1)}\,|\xi|^{6(p+1)}\,e^{2ck(p+1)|\xi|}}{q_{6p,2p}\,|\xi|^{6p}\,e^{2ckp|\xi|}}\right| < 1.$$

Simplifying, we have that $\mathcal{S}_{\text{even}}$ converges under the condition

$$e^{2ck|\xi|}|\xi|^6 \lim_{p\to\infty}\left|\frac{q_{6(p+1),2(p+1)}}{q_{6p,2p}}\right| \overbrace{=}^{|H|\to\|\mathcal{H}\|_\infty} e^{2ck|\xi|}|\xi|^6 \lim_{p\to\infty}\left|\frac{A_{6(p+1)}}{A_{6p}}\right| < 1.$$

Rearranging, we find that the radius of convergence of $\mathcal{S}_{\text{even}}$ is given by

$$e^{\frac{1}{3}ck|\xi|}|\xi| < \left(\lim_{p\to\infty}\left|\frac{A_{6(p+1)}}{A_{6p}}\right|\right)^{-1/6}, \tag{5.60}$$

where $|H|$ is replaced by $\|\mathcal{H}\|_\infty$ in A_{6p}. In this case, we know the value of the right-hand side of (5.60) from (5.50). That is, (5.60) becomes

$$e^{\frac{1}{3}ck|\xi|}|\xi| < \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\,\|\mathcal{H}\|_\infty^{1/3}} \approx \frac{3.05991}{\|\mathcal{H}\|_\infty^{1/3}}. \tag{5.61}$$

Similarly, we find that the radius of convergence of $\mathcal{S}_{\text{odd}}$ is also given by (5.61). Then since $|a_m(x,\tau)| \le \mathcal{S}_{\text{even}} + \mathcal{S}_{\text{odd}}$, it follows that $a_m(x,\tau)$ converges absolutely under the condition (5.61). In other words, the radius of convergence of the formal Laurent series near-general solution of the three-wave PDEs with constant phases when $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$ is given implicitly by

$$|\tau-\tau_0|e^{\frac{1}{3}ck|\tau-\tau_0|} < \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\,\|\mathcal{H}\|_\infty^{1/3}} \approx \frac{3.05991}{\|\mathcal{H}\|_\infty^{1/3}}. \tag{5.62}$$

To summarize, we found that the radius of convergence for the general solution of the three-wave ODEs with $K_2 = K_3 = 0$ is given by

$$|\tau_0-\tau| < R_{\text{ODE}}, \tag{5.63}$$

where

$$R_{\text{ODE}} = \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\,|H|^{1/3}} \approx \frac{3.05991}{|H|^{1/3}}.$$

Meanwhile, we determined that the radius of convergence for the near-general solution of the three-wave PDEs with constant phases and the analogous condition $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$ is given by

$$|\tau_0-\tau|\,e^{\frac{1}{3}ck|\tau-\tau_0|} < R_{\text{ODE}}, \tag{5.64}$$

where we replaced $|H|$ with $\|\mathcal{H}\|_\infty$ in R_{ODE}.

Compare (5.63) to (5.64). The two convergence conditions differ only by the factor of $e^{\frac{1}{3}ck|\tau-\tau_0|}$ in (5.64). This is a known, dimensionless factor that causes the radius of convergence of the PDE solution to be smaller than that of the ODE solution. However, the factor depends only on k, which tells us how quickly the derivatives of $\mathcal{H}(x)$ are growing, and c, which is the largest group velocity (in magnitude) of the three wavetrains. This indicates that for this special case, we are losing very little information in moving from the ODEs to the PDEs. The inclusion of spatial dependence in the three-wave PDEs affects where our near-general solution is valid, but it does not affect the overall structure of the solutions.

5.4 Radius of convergence: Case 2(i)

Consider Case 2(i), in which $\mathcal{H}(x) = 0$ and $\mathcal{K}_2(x) = \mathcal{K}_3(x) \equiv \mathcal{K}(x)/6$ in the solution of the three-wave PDEs, while $H = K_2 = 0$ and $K_3 \equiv K$ in the solution of the three-wave ODEs. In order to find the radius of convergence of the Laurent series solution of the three-wave PDEs in this case, we follow the steps outlined in Section 5.2. Recall that the phases $\psi_m(x)$ in the PDE solution are assumed to be constant.

STEP 1: We begin by finding the series solution of the three-wave ODEs when $H = K_2 = 0$ and $K_3 = K$ using (4.29)-(4.31). The first few terms of the series for $a_m(\tau)$, $m = 1, 2, 3$ are given by

$$a_j(\tau) = \frac{e^{i\psi_j}}{\tau-\tau_0}\Big[1 + \frac{\sigma K}{6}(\tau-\tau_0)^2 + \frac{7K^2}{360}(\tau-\tau_0)^4 + \frac{31\sigma K^3}{15120}(\tau-\tau_0)^6 + \cdots\Big], \tag{5.65a}$$

$$a_3(\tau) = \frac{e^{i\psi_3}}{\tau-\tau_0}\Big[1 - \frac{\sigma K}{3}(\tau-\tau_0)^2 - \frac{K^2}{45}(\tau-\tau_0)^4 - \frac{2\sigma K^3}{945}(\tau-\tau_0)^6 - \cdots\Big], \tag{5.65b}$$

where $j = 1, 2$ in the first line.

It is straightforward to show by induction that when $H = 0$, then (4.29) reduces to

$$a_m(\tau) = \frac{e^{i\psi_m}}{\xi}\sum_{n=0}^{\infty} A_{2n}^m\,\xi^{2n}, \tag{5.66}$$

where $\xi = \tau - \tau_0$, $A_0 = 1$, $A_2^1 = A_2^2 = \sigma K/6$, $A_2^3 = -\sigma K/3$, and (4.31) becomes

$$(2n-1)A_{2n}^m + A_{2n}^{k*} + A_{2n}^{\ell*} = -\sum_{p=1}^{n-1} A_{2p}^{k*}A_{2(n-p)}^{\ell*}, \quad \text{for } n \geq 2.$$

In fact, it is simple to show that when $H = 0$, A_{2n}^m is real for $m = 1, 2, 3$ and for $n \geq 0$. As a result, the recursion relation becomes

$$(2n-1)A_{2n}^m + A_{2n}^k + A_{2n}^\ell = -\sum_{p=1}^{n-1} A_{2p}^k A_{2(n-p)}^\ell, \quad \text{for } n \geq 2. \tag{5.67}$$

Next, we want to determine the radius of convergence of (5.66). There are two options here. First of all, we can apply the ratio test. Then we find that (5.66) converges when

$$\lim_{n\to\infty} \left| \frac{A_{2(n+1)}^m \xi^{2(n+1)}}{A_{2n}^m \xi^{2n}} \right| < 1$$

for $m = 1, 2, 3$. Rearranging, we have that the series converges if

$$|\xi| < \left(\lim_{n\to\infty} \left| \frac{A_{2(n+1)}^m}{A_{2n}^m} \right| \right)^{-1/2}. \tag{5.68}$$

The limit on the right-hand side exists, and is numerically found to equal $\pi/|K|^{1/2}$. That is, we find that (5.66) converges under the condition

$$|\tau - \tau_0| < \frac{\pi}{|K|^{1/2}}. \tag{5.69}$$

In most cases, we can alternatively determine the radius of convergence of the general solution of the three-wave ODEs using the Weierstrass solution in (3.54) and (3.60). However, recall the definition of the discriminant associated with the Weierstrass function in (3.68). That is, we have

$$\Delta = g_2^3 - 27g_3^2,$$

where $g_2 = g_2(K_2, K_3)$ and $g_3 = g_3(K_2, K_3, H)$ are defined in (3.45)-(3.46). In our case, with $H = K_2 = 0$ and $K_3 = K$, we find that

$$\Delta = 0.$$

This is a special degenerate case in which the Weierstrass function is not defined. Instead, the general solution of the three-wave ODEs in this case can be found in terms of $\sec\tau$.

Return to the Hamiltonian system in (3.19). In Chapter 3, we solved the system in order to find the Weierstrass general solution of the three-wave ODEs. However, with $H = K_2 = 0$ and $K_3 = K$, (3.19b) becomes

$$\frac{d\rho}{d\tau} = 2\sqrt{\sigma\rho^2(\rho - K)}. \tag{5.70}$$

The equation is separable, so the solution is found by integrating,

$$\int \frac{d\rho}{2\sqrt{\sigma\rho^2\left(\rho-K\right)}} = \int d\tau.$$

Without loss of generality, assume $\sigma = 1$ and $K > 0$ (the cases for the other signs are analogous). In particular, assume that $\sigma_1 = 1$, which means $\rho(\tau) \geq 0$. Using the change of variables $\rho = K\sec^2\theta$, where $0 \leq \theta < \pi/2$, we obtain

$$\int \frac{d\theta}{\sqrt{K}} = \tau + C,$$

where C is a constant of integration. As a result, we find that $\theta = \sqrt{K}\,(\tau + C)$, so that

$$\rho(\tau) = K\sec^2\left[\sqrt{K}\,(\tau + C)\right]. \tag{5.71}$$

It remains to determine the integration constant C. First, suppose that we are in the explosive regime, and that $\rho \to +\infty$ as $\tau \to \tau_0$, where τ_0 is a real constant. This allows us to determine C, so that our expression for $\rho(\tau)$ becomes

$$\rho(\tau) = K\sec^2\left[\sqrt{K}\,(\tau - \tau_0) + \frac{\pi}{2}\right]. \tag{5.72}$$

In this case, we observe that $\rho(\tau)$ has a double pole at $\tau = \tau_0$, which is also true in the Weierstrass defininition of $\rho(\tau)$ in (3.44).

In the case of the Weierstrass solution, $\rho(\tau)$ is defined in terms of $\wp(\tau - \tau_0)$. Moreover, real values of τ_0 correspond to the explosive regime, while certain complex values of τ_0 correspond to the nonexplosive regime. This suggests that we might be able to let τ_0 be complex in (5.72) in order to find the solution of (5.70) in the nonexplosive case. However, this actually turns out not to be true. The functions $\sec z$ and $\sec^2 z$ for $z \in \mathbb{C}$ are real-valued only along the real axis. There is no complex shift z_0 that can make $\sec^2(z - z_0)$ real along the real axis. Since $\rho(\tau)$ must, by definition, be real along the real axis, it follows that (5.72) with $\tau_0 \in \mathbb{R}$ is the only possible solution when $H = K_2 = 0$. That is, when $H = K_2 = 0$, we are necessarily restricted to the explosive regime.

An alternative way to see this is the following. Return to (5.71). We try to determine the integration constant C without assuming that our solutions lie in the explosive regime. Instead, suppose we know that $\rho(\tau_i) = \rho_i$ for some real τ_i. Then we have

$$K \sec^2\left[\sqrt{K}(\tau_i + C)\right] = \rho_i.$$

This implies that

$$\sec\left[\sqrt{K}(\tau_i + C)\right] = \sqrt{\frac{\rho_i}{K}},$$

where we took the positive square root since we assumed $0 \leq \theta < \pi/2$. Then we have

$$C = \frac{1}{\sqrt{K}} \operatorname{arcsec} \sqrt{\frac{\rho_i}{K}} - \tau_i. \tag{5.73}$$

Recall that arcsecant only takes arguments that are greater than 1, which means we require that $\rho_i \geq K$. By assumption, $\sigma = \sigma_1 = 1$. As a result, the possible regimes are (1) the explosive regime with $\sigma_1 = \sigma_2 = \sigma_3 = 1$, and (2) the nonexplosive regime with $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. Recall from Section 3.2 that if $\sigma_1 = \sigma_2 = \sigma_3 = 1$, then $\rho > \max\{K_2, K_3, 0\}$. In our case, with $K_2 = 0$ and $K_3 = K > 0$, we have that $\rho > K$. In particular, we have that $\rho_i \geq K$, so that the definition of C above makes sense. On the other hand, if $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$, then $\rho(\tau)$ lies in the domain $0 < \rho < \min\{K_2, K_3\}$, where $K_2, K_3 \geq 0$. In our case, with $K_2 = 0$ and $K_3 = K > 0$, it follows that $0 < \rho < 0$, which is impossible. (Recall that $\rho(\tau) = 0$ is not allowed in (3.19), although $|a_m(\tau)|$ is allowed to be zero valued.) As a result, we are restricted to the explosive case.

Using (5.73), we have

$$\rho(\tau) = K \sec^2\left[\sqrt{K}(\tau - \tau_i) + \operatorname{arcsec} \sqrt{\frac{\rho_i}{K}}\,\right]. \tag{5.74}$$

Here we find that a singularity occurs when

$$\tau - \tau_i = \frac{1}{\sqrt{K}}\left[\frac{\pi}{2} - \operatorname{arcsec} \sqrt{\frac{\rho_i}{K}}\,\right].$$

Regardless of whether we formulate the solution to (5.70) as (5.72) or (5.74), the maximum possible distance between two singularities is $\pi/\sqrt{K}$ since $\sec^2 z$ has singularities at $z = \pi/2 \pm n\pi$ for $n \in \mathbb{Z}$. In other words, the radius of convergence of $\rho(\tau)$, and thus $a_m(\tau)$, when $H = K_2 = 0$ is

$$R_{\text{ODE}} = \frac{\pi}{|K|^{1/2}}.$$

This agrees with our findings in (5.69). Again, this is reassuring since the general solution of the three-wave ODEs in terms of $\sec^2\tau$ in (5.72) and (5.74) should be equivalent to the general Laurent series solution.

STEP 2: Next, we find the series solution of the three-wave PDEs, $a_m(x,\tau)$, when the phases are constant, $\mathcal{H}(x)=0$, and $\mathcal{K}_2(x)=\mathcal{K}_3(x)$. For convenience, let

$$\mathcal{K}_2(x)=\mathcal{K}_3(x)\equiv\frac{1}{6}\mathcal{K}(x),$$

where $\mathcal{K}(x)$ is a real function of x, and the factor of $1/6$ is introduced for convenience. Using (5.24)-(5.26), the first few terms of the series for $a_m(x,\tau)$ are

$$\begin{aligned}
a_j(x,\tau)=\frac{e^{i\psi_j}}{\xi}\Bigg\{&1+\frac{\mathcal{K}}{6}\xi^2-\frac{1}{24}\left(3c_j-c_\ell+2c_3\right)\mathcal{K}'\,\xi^3\\
&+\left[\frac{7\mathcal{K}^2}{360}-\frac{1}{240}\left(-12c_j^2+3c_\ell^2+3c_j(c_\ell-3c_3)+c_\ell c_3-6c_3^2\right)\mathcal{K}''\right]\xi^4\\
&+\frac{1}{720}\Big[-\left(5c_j+9c_\ell+14c_3\right)\mathcal{K}\mathcal{K}'+\big(-10c_j^3+2c_\ell^3+2c_j^2(c_\ell-4c_3)\\
&\qquad+c_j\left(2c_\ell^2+c_\ell c_3-6c_3^2\right)+c_\ell^2c_3-4c_3^3\big)\,\mathcal{K}^{(3)}\Big]\xi^5\\
&+\left[\frac{31\mathcal{K}^3}{15120}+(\cdots)(\mathcal{K}')^2+(\cdots)\mathcal{K}\mathcal{K}''+(\cdots)\mathcal{K}^{(4)}\right]\xi^6+\cdots\Bigg\},
\end{aligned}\tag{5.75}$$

$$\begin{aligned}
a_3(x,\tau)=\frac{e^{i\psi_3}}{\xi}\Bigg\{&1-\frac{\mathcal{K}}{3}\xi^2+\frac{1}{24}\left(c_1+c_2+6c_3\right)\mathcal{K}'\,\xi^3\\
&+\left[-\frac{\mathcal{K}^2}{45}-\frac{1}{240}\left(3c_1^2-2c_1c_2+3c_2^2+6c_1c_3+6c_2c_3+24c_3^2\right)\mathcal{K}''\right]\xi^4\\
&+\frac{1}{720}\Big[\left(2(3c_1+3c_2+10c_3)\right)\mathcal{K}\mathcal{K}'+\big(2c_1^3-c_1^2\left(c_2-4c_3\right)\\
&\qquad-c_1\left(c_2^2+2c_2c_3-6c_3^2\right)+2\left(c_2^3+2c_2^2c_3+3c_2c_3^2+10c_3^3\right)\big)\,\mathcal{K}^{(3)}\Big]\xi^5\\
&+\left[-\frac{2\mathcal{K}^3}{945}+(\cdots)(\mathcal{K}')^2+(\cdots)\mathcal{K}\mathcal{K}''+(\cdots)\mathcal{K}^{(4)}\right]\xi^6+\cdots\Bigg\},
\end{aligned}\tag{5.76}$$

where $(j,\ell)=(1,2)$ cyclically. Observe that increasingly higher derivatives of $\mathcal{K}(x)$ appear as the number of terms increases, as well as more nonlinear terms in $\mathcal{K}$ and its derivatives.

As with Case 1, we can see that the series solution of the three-wave PDEs quickly becomes complicated as the number of terms in the series increases. As a result, we restrict our attention to a particular family of functions for $\mathcal{K}(x)$.

Step 3: Following (5.33), we assume that

$$\left\| \frac{d^n}{dx^n}\mathcal{K}(x) \right\|_\infty \leq k^n \|\mathcal{K}\|_\infty, \tag{5.77}$$

where k is a real positive constant.

Step 4: Using (5.77) and the definition of c in (5.31), we bound (5.75)-(5.76) using the triangle inequality. This first few terms in the bounded series of $a_m(x,\tau)$ for $m = 1, 2, 3$ are given by

$$\begin{aligned} |a_j(x,\tau)| \leq \frac{1}{|\xi|}\Bigg\{ & 1 + \frac{\|\mathcal{K}\|}{6}|\xi|^2 + \frac{\|\mathcal{K}\|}{4}ck|\xi|^3 + \left[\frac{7\|\mathcal{K}\|^2}{360} + \frac{17\|\mathcal{K}\|}{120}(ck)^2\right]|\xi|^4 \\ & + \left[\frac{149\|\mathcal{K}\|^2}{3240}(ck) + \frac{\|\mathcal{K}\|}{20}(ck)^3\right]|\xi|^5 \\ & + \left[\frac{31\|\mathcal{K}\|^3}{15120} + \frac{509\|\mathcal{K}\|^2}{11340}(ck)^2 + \frac{11\|\mathcal{K}\|}{840}(ck)^4\right]|\xi|^6 + \cdots \Bigg\}, \end{aligned} \tag{5.78}$$

$$\begin{aligned} |a_3(x,\tau)| \leq \frac{1}{|\xi|}\Bigg\{ & 1 + \frac{\|\mathcal{K}\|}{3}|\xi|^2 + \frac{\|\mathcal{K}\|}{3}ck|\xi|^3 + \left[\frac{\|\mathcal{K}\|^2}{45} + \frac{11\|\mathcal{K}\|}{60}(ck)^2\right]|\xi|^4 \\ & + \left[\frac{\|\mathcal{K}\|^2}{90}(ck) + \frac{\|\mathcal{K}\|}{15}(ck)^3\right]|\xi|^5 \\ & + \left[\frac{2\|\mathcal{K}\|^3}{945} + \frac{331\|\mathcal{K}\|^2}{15120}(ck)^2 + \frac{\|\mathcal{K}\|}{56}(ck)^4\right]|\xi|^6 + \cdots \Bigg\}, \end{aligned} \tag{5.79}$$

where $j = 1, 2$, and $\|\mathcal{K}\| = \|\mathcal{K}\|_\infty$.

Notice that coefficients of the form $\|\mathcal{K}\|^n|\xi|^{2n}$ can be identified with the coefficients $|A^m_{2n}|$ that appear in the ODE series in (5.65) and (5.66). Next, notice that for every $n \geq 2$, the terms multiplying $|\xi|^n$ in (5.78)-(5.79) are polynomials in (ck) of degree $n-2$, where every second term is nonzero. More precisely, we can rewrite (5.78)-(5.79) as

$$|a_m(x,\tau)| \leq \frac{1}{|\xi|}\left[1 + \sum_{n=2}^{\infty}\sum_{p=1}^{\lfloor n/2 \rfloor} q^m_{n,p}(ck)^{n-2p}|\xi|^n\right],$$

where $q^m_{n,p}$ are nonnegative constants. Alternatively, we can change the order of summation and write

$$|a_m(x,\tau)| \leq \frac{1}{|\xi|}\left[1 + \sum_{p=1}^{\infty}\sum_{n=2p}^{\infty} q^m_{n,p}(ck)^{n-2p}|\xi|^n\right], \tag{5.80}$$

which is analagous to (5.55) in Case 1.

The first few coefficients $q^m_{n,p}$ for $m = 1, 2$ are listed in Table 5.3, while the constants $q^3_{n,p}$ are listed in Table 5.4. In both tables, the index n corresponds to the nth row of the table, while the index p indicates the pth nonzero diagonal of the table. The columns of the table represent powers of (ck).

n	$(ck)^0$	$(ck)^1$	$(ck)^2$	$(ck)^3$	$(ck)^4$	$(ck)^5$	$(ck)^6$
2	$\frac{\lVert\mathcal{K}\rVert}{6}$	0	0	0	0	0	0
3	0	$\frac{\lVert\mathcal{K}\rVert}{4}$	0	0	0	0	0
4	$\frac{7\lVert\mathcal{K}\rVert^2}{360}$	0	$\frac{17\lVert\mathcal{K}\rVert}{120}$	0	0	0	0
5	0	$\frac{7\lVert\mathcal{K}\rVert^2}{180}$	0	$\frac{\lVert\mathcal{K}\rVert}{20}$	0	0	0
6	$\frac{31\lVert\mathcal{K}\rVert^3}{15120}$	0	$\frac{7\lVert\mathcal{K}\rVert^2}{180}$	0	$\frac{11\lVert\mathcal{K}\rVert}{840}$	0	0
7	0	$\frac{31\lVert\mathcal{K}\rVert^3}{5040}$	0	$\frac{397\lVert\mathcal{K}\rVert^2}{15120}$	0	$\frac{3\lVert\mathcal{K}\rVert}{1120}$	0
8	$\frac{127\lVert\mathcal{K}\rVert^4}{604800}$	0	$\frac{31\lVert\mathcal{K}\rVert^3}{3360}$	0	$\frac{12293\lVert\mathcal{K}\rVert^2}{907200}$	0	$\frac{83\lVert\mathcal{K}\rVert}{181440}$

Table 5.3: The first few coefficients $q^m_{n,p}$ from (5.80) for $m = 1, 2$. Note that n indicates the nth row of the table, while p indicates the pth nonzero diagonal (that is, the diagonal that begins in the row corresponding to $n = 2p$). For instance, $q^m_{7,2} = \frac{397\|\mathcal{K}\|^2}{15120}$ for $m = 1, 2$.

n	$(ck)^0$	$(ck)^1$	$(ck)^2$	$(ck)^3$	$(ck)^4$	$(ck)^5$	$(ck)^6$
2	$\frac{\lVert\mathcal{K}\rVert}{3}$	0	0	0	0	0	0
3	0	$\frac{\lVert\mathcal{K}\rVert}{3}$	0	0	0	0	0
4	$\frac{\lVert\mathcal{K}\rVert^2}{45}$	0	$\frac{11\lVert\mathcal{K}\rVert}{60}$	0	0	0	0
5	0	$\frac{2\lVert\mathcal{K}\rVert^2}{45}$	0	$\frac{\lVert\mathcal{K}\rVert}{15}$	0	0	0
6	$\frac{2\lVert\mathcal{K}\rVert^3}{945}$	0	$\frac{27\lVert\mathcal{K}\rVert^2}{560}$	0	$\frac{\lVert\mathcal{K}\rVert}{56}$	0	0
7	0	$\frac{2\lVert\mathcal{K}\rVert^3}{315}$	0	$\frac{9\lVert\mathcal{K}\rVert^2}{280}$	0	$\frac{19\lVert\mathcal{K}\rVert}{5040}$	0
8	$\frac{\lVert\mathcal{K}\rVert^4}{4725}$	0	$\frac{\lVert\mathcal{K}\rVert^3}{105}$	0	$\frac{3347\lVert\mathcal{K}\rVert^2}{201600}$	0	$\frac{17\lVert\mathcal{K}\rVert}{25920}$

Table 5.4: The first few coefficients $q^3_{n,p}$ from (5.80). Note that n indicates the nth row of the table, while p indicates the pth nonzero diagonal (that is, the diagonal that begins in the row corresponding to $n = 2p$). For instance, $q^3_{6,2} = \frac{331\|\mathcal{K}\|^2}{15120}$.

Step 5: We now want to reduce (5.80) to a single sum so that we can find where the series converges using the ratio test. However, unlike Case 1 of the previous section, we are unable to find a closed form for the coefficients $q_{n,p}^m$ when $\mathcal{H}(x) = \mathcal{K}_2(x) = 0$. Moreover, recall that the coefficients in (5.55) are independent of the mode m, which is not the case in (5.80). To simplify matters, we proceed to find the radius of convergence of

$$A(x,\tau) = \sum_{m=1}^{3} |a_m(x,\tau)| \tag{5.81}$$

when $\mathcal{H}(x) = \mathcal{K}(x) = 0$, and conclude that $a_m(x,\tau)$ must converge absolutely with an equal or larger radius of convergence for $m = 1, 2, 3$.

From (5.80), we know that

$$A(x,\tau) \leq \frac{1}{|\xi|}\left[3 + \sum_{p=1}^{\infty}\sum_{n=2p}^{\infty}\left(q_{n,p}^1 + q_{n,p}^2 + q_{n,p}^3\right)(ck)^{n-2p}|\xi|^n\right].$$

However, since we do not have a nice formula for $q_{n,p}^m$, this is not necessarily useful in helping us determine the radius of convergence of $A(x,\tau)$. Instead, we return to the series solution of $a_m(x,\tau)$ once more, and try to write $A(x,\tau)$ in a form that is easy to handle.

We know from (5.24)-(5.26) and (5.75)-(5.76) that $a_m(x,\tau)$ is given by

$$a_m(x,\tau) = \frac{e^{i\psi_m}}{\xi}\sum_{n=0}^{\infty}\mathcal{A}_n^m(x)\xi^n$$

for $m = 1, 2, 3$, where $\xi = \tau - \tau_0$ and ψ_m is a real constant. It follows that

$$|a_m(x,\tau)| \leq \frac{1}{|\xi|}\sum_{n=0}^{\infty}\|\mathcal{A}_n^m(x)\|\ |\xi|^n,$$

where $\|\cdot\| = \|\cdot\|_\infty$. Substituting into (5.81) yields

$$A(x,\tau) \leq \frac{1}{|\xi|}\sum_{n=0}^{\infty}\left[\left\|\mathcal{A}_n^1(x)\right\| + \left\|\mathcal{A}_n^2(x)\right\| + \left\|\mathcal{A}_n^3(x)\right\|\right]|\xi|^n.$$

We now seek a bound on $\left\|\mathcal{A}_n^1(x)\right\| + \left\|\mathcal{A}_n^2(x)\right\| + \left\|\mathcal{A}_n^3(x)\right\|$.

Recall the recursion relation in (5.26) for $n \geq 4$,

$$(n-1)\mathcal{A}_n^m(x) + \mathcal{A}_n^{k*}(x) + \mathcal{A}_n^{\ell*}(x) = -c_m\left[\mathcal{A}_{n-1}^{m\prime}(x) + i\psi_m'(x)\mathcal{A}_{n-1}^m(x)\right] - \sum_{p=1}^{n-1}\mathcal{A}_p^{k*}(x)\mathcal{A}_{n-p}^{\ell*}(x). \tag{5.82}$$

When $\mathcal{H}(x) = 0$, it follows that all coefficients $\mathcal{A}_n^m(x)$ in the series for $n \geq 0$ are purely real. Consequently, (5.82) becomes

$$(n-1)\mathcal{A}_n^m(x) + \mathcal{A}_n^k(x) + \mathcal{A}_n^\ell(x) = -c_m\left[\mathcal{A}_{n-1}^{m\prime}(x) + i\psi_m'(x)\mathcal{A}_{n-1}^m(x)\right] - \sum_{p=1}^{n-1}\mathcal{A}_p^k(x)\mathcal{A}_{n-p}^\ell(x). \tag{5.83}$$

Inverting the linear system formed by (5.83), it is straightforward to determine that

$$\mathcal{A}_n^m = \frac{1}{(n+1)(n-2)}\left[nb_n^m - b_n^k - b_n^\ell\right], \quad n \geq 3, \tag{5.84}$$

where

$$b_n^m(x) = c_m \mathcal{A}_{n-1}^{m'} - \sum_{p=2}^{n-2} \mathcal{A}_p^k \mathcal{A}_{n-p}^\ell. \tag{5.85}$$

Note that the sum in (5.85) goes from $p = 2$ to $p = n-2$ because $\mathcal{A}_1^m(x) = 0$ for $m = 1, 2, 3$ when the phases ψ_m are constant.

Substituting (5.85) into (5.84), collecting terms, and rearranging yields

$$\begin{aligned}\mathcal{A}_n^m = \frac{1}{(n+1)(n-2)}\Bigg[& -(n+1)c_m \mathcal{A}_{n-1}^{m'} - (n+1)\sum_{p=2}^{n-2} \mathcal{A}_p^k \mathcal{A}_{n-p}^\ell \\ & + \sum_{j=1}^{3} c_j \mathcal{A}_{n-1}^{j'} + \sum_{p=2}^{n-2}\left(\mathcal{A}_p^1\mathcal{A}_{n-p}^2 + \mathcal{A}_p^1\mathcal{A}_{n-p}^3 + \mathcal{A}_p^2\mathcal{A}_{n-p}^3\right)\Bigg].\end{aligned}$$

Notice that the second line is independent of m.

Observe the following

$$\begin{aligned}\|\mathcal{A}_n^m\| \leq \frac{1}{(n+1)(n-2)}\Bigg[& (n+1)c\left\|\mathcal{A}_{n-1}^{m'}\right\| + (n+1)\sum_{p=2}^{n-2}\left\|\mathcal{A}_p^k\right\|\left\|\mathcal{A}_{n-p}^\ell\right\| \\ & + c\sum_{j=1}^{3}\left\|\mathcal{A}_{n-1}^{j'}\right\| + \sum_{p=2}^{n-2}\left(\|\mathcal{A}_p^1\|\,\|\mathcal{A}_{n-p}^2\| + \|\mathcal{A}_p^1\|\,\|\mathcal{A}_{n-p}^3\| + \|\mathcal{A}_p^2\|\,\|\mathcal{A}_{n-p}^3\|\right)\Bigg],\end{aligned}$$

where $c = \max\{|c_1|, |c_2|, |c_3|\}$, as usual. Lastly, we sum over m to obtain

$$\begin{aligned}\sum_{m=1}^{3}\|\mathcal{A}_n^m\| \leq \frac{1}{(n+1)(n-2)}\Bigg[& (n+1)c\sum_{m=1}^{3}\left\|\mathcal{A}_{n-1}^{m'}\right\| \\ & + (n+1)\sum_{p=2}^{n-2}\left(\|\mathcal{A}_p^1\|\,\|\mathcal{A}_{n-p}^2\| + \|\mathcal{A}_p^1\|\,\|\mathcal{A}_{n-p}^3\| + \|\mathcal{A}_p^2\|\,\|\mathcal{A}_{n-p}^3\|\right) \\ & + 3c\sum_{j=1}^{3}\left\|\mathcal{A}_{n-1}^{j'}\right\| + 3\sum_{p=2}^{n-2}\left(\|\mathcal{A}_p^1\|\,\|\mathcal{A}_{n-p}^2\| + \|\mathcal{A}_p^1\|\,\|\mathcal{A}_{n-p}^3\| + \|\mathcal{A}_p^2\|\,\|\mathcal{A}_{n-p}^3\|\right)\Bigg].\end{aligned}$$

Simplifying, we have

$$\sum_{m=1}^{3}\|\mathcal{A}_n^m\| \leq \frac{n+4}{(n+1)(n-2)}\left[c\sum_{m=1}^{3}\left\|\mathcal{A}_{n-1}^{m'}\right\| + \sum_{p=2}^{n-2}\left(\|\mathcal{A}_p^1\|\,\|\mathcal{A}_{n-p}^2\| + \|\mathcal{A}_p^1\|\,\|\mathcal{A}_{n-p}^3\| + \|\mathcal{A}_p^2\|\,\|\mathcal{A}_{n-p}^3\|\right)\right]. \tag{5.86}$$

Finally, observe that

$$\sum_{p=2}^{n-2}\left(\|\mathcal{A}_p^1\|\ \|\mathcal{A}_{n-p}^2\|+\|\mathcal{A}_p^1\|\ \|\mathcal{A}_{n-p}^3\|+\|\mathcal{A}_p^2\|\ \|\mathcal{A}_{n-p}^3\|\right)$$
$$\leq \frac{1}{2}\sum_{p=2}^{n-2}\left(\|\mathcal{A}_p^1\|+\|\mathcal{A}_p^2\|+\|\mathcal{A}_p^3\|\right)\left(\|\mathcal{A}_{n-p}^1\|+\|\mathcal{A}_{n-p}^2\|+\|\mathcal{A}_{n-p}^3\|\right).$$

As a result, (5.86) becomes

$$\sum_{m=1}^{3}\|\mathcal{A}_n^m\| \leq \frac{n+4}{(n+1)(n-2)}\left[c\sum_{m=1}^{3}\left\|\mathcal{A}_{n-1}^{m'}\right\|\right.$$
$$\left. +\frac{1}{2}\sum_{p=2}^{n-2}\left(\|\mathcal{A}_p^1\|+\|\mathcal{A}_p^2\|+\|\mathcal{A}_p^3\|\right)\left(\|\mathcal{A}_{n-p}^1\|+\|\mathcal{A}_{n-p}^2\|+\|\mathcal{A}_{n-p}^3\|\right)\right]. \tag{5.87}$$

Now recall from our work in Step 4 of this section that under the assumption (5.77), we have

$$\|\mathcal{A}_n^m\| \leq q_{n,1}^m(ck)^{n-2}+q_{n,2}^m(ck)^{n-4}+q_{n,3}^m(ck)^{n-6}+\cdots+q_{n,p}^m(ck)^{n-2p},$$
$$\left\|\mathcal{A}_n^{m'}\right\| \leq k\left[q_{n,1}^m(ck)^{n-2}+2q_{n,2}^m(ck)^{n-4}+3q_{n,3}^m(ck)^{n-6}+\cdots+pq_{n,p}^m(ck)^{n-2p}\right],$$

where $p=\lfloor n/2\rfloor$ and $q_{n,p}^m$ are the constants that appear in (5.80). This motivates us to write

$$\|\mathcal{A}_n^1\|+\|\mathcal{A}_n^2\|+\|\mathcal{A}_n^3\| \leq r_{n,1}(ck)^{n-2}+r_{n,2}(ck)^{n-4}+r_{n,3}(ck)^{n-6}+\cdots+r_{n,p}(ck)^{n-2p}, \tag{5.88}$$
$$\left\|\mathcal{A}_n^{1'}\right\|+\left\|\mathcal{A}_n^{2'}\right\|+\left\|\mathcal{A}_n^{3'}\right\| \leq k\left[r_{n,1}(ck)^{n-2}+2r_{n,2}(ck)^{n-4}+3r_{n,3}(ck)^{n-6}+\cdots+pr_{n,p}(ck)^{n-2p}\right], \tag{5.89}$$

where $r_{n,p}$ are constants defined for $n\geq 2p$, independent of m. We seek a closed form for $r_{n,p}$. As a result, we do not define $r_{n,p}=q_{n,p}^1+q_{n,p}^2+q_{n,p}^3$ since a formula for $q_{n,p}^m$ is not known. Rather, we find an expression for $r_{n,p}$ that turns out to satisfy

$$r_{n,p}\geq q_{n,p}^1+q_{n,p}^2+q_{n,p}^3.$$

Substituting (5.88)-(5.89) into (5.87) yields

$$\sum_{m=1}^{3}\|\mathcal{A}_n^m\| \leq \frac{n+4}{(n+1)(n-2)}\left\{ck\left[r_{n-1,1}(ck)^{n-3}+2r_{n-1,2}(ck)^{n-5}+\cdots+pr_{n-1,p}(ck)^{n-1-2p}\right]\right.$$
$$\left.+\frac{1}{2}\sum_{p=2}^{n-2}\left(r_{p,1}(ck)^{p-2}+r_{p,2}(ck)^{p-4}+\cdots\right)\left(r_{n-p,1}(ck)^{n-p-2}+r_{n-p,2}(ck)^{n-p-4}+\cdots\right)\right\}$$
$$=\frac{n+4}{(n+1)(n-2)}\left\{r_{n-1,1}(ck)^{n-2}+2r_{n-1,2}(ck)^{n-4}+\cdots+pr_{n-1,p}(ck)^{n-2p}\right.$$
$$\left.+\frac{1}{2}\sum_{p=2}^{n-2}\left(r_{p,1}(ck)^{p-2}+r_{p,2}(ck)^{p-4}+\cdots\right)\left(r_{n-p,1}(ck)^{n-p-2}+r_{n-p,2}(ck)^{n-p-4}+\cdots\right)\right\}.$$

Finally, assuming n is sufficiently large, collecting powers of (ck) gives

$$\sum_{m=1}^{3}\|\mathcal{A}_n^m\| = \frac{n+4}{(n+1)(n-2)}\Bigg\{r_{n-1,1}(ck)^{n-2} + \left(2r_{n-1,2} + \frac{1}{2}\sum_{p=2}^{n-2} r_{p,1}r_{n-p,1}\right)(ck)^{n-4}$$
$$+ \left(3r_{n-1,3} + \frac{1}{2}\sum_{p=2}^{n-4} r_{p,1}r_{n-p,2} + \frac{1}{2}\sum_{p=4}^{n-2} r_{p,2}r_{n-p,1}\right)(ck)^{n-6}$$
$$+ \left(4r_{n-1,4} + \frac{1}{2}\sum_{p=2}^{n-6} r_{p,1}r_{n-p,3} + \frac{1}{2}\sum_{p=6}^{n-2} r_{p,3}r_{n-p,1} + \frac{1}{2}\sum_{p=4}^{n-4} r_{p,2}r_{n-p,2}\right)(ck)^{n-8}$$
$$+ \cdots\Bigg\}.$$

Simplifying again, we obtain

$$\sum_{m=1}^{3}\|\mathcal{A}_n^m\| \leq \frac{n+4}{(n+1)(n-2)}\Bigg\{r_{n-1,1}(ck)^{n-2} + \left(2r_{n-1,2} + \frac{1}{2}\sum_{j=2}^{n-2} r_{j,1}r_{n-j,1}\right)(ck)^{n-4}$$
$$+ \left(3r_{n-1,3} + \frac{1}{2}\sum_{\ell=1}^{2}\sum_{j=2\ell}^{n-2(3-\ell)} r_{j,\ell}r_{n-j,3-\ell}\right)(ck)^{n-6}$$
$$+ \left(4r_{n-1,4} + \frac{1}{2}\sum_{\ell=1}^{3}\sum_{j=2\ell}^{n-2(4-\ell)} r_{j,\ell}r_{n-j,4-\ell}\right)(ck)^{n-8}$$
$$+ \cdots\Bigg\}.$$

As a result, we have

$$\sum_{m=1}^{3}\|\mathcal{A}_n^m\| \leq \frac{n+4}{(n+1)(n-2)}\sum_{p=1}^{\infty}\left[p\,r_{n-1,p} + \frac{1}{2}\sum_{\ell=1}^{p-1}\sum_{j=2\ell}^{n-2(p-\ell)} r_{j,\ell}\,r_{n-j,p-\ell}\right](ck)^{n-2p}. \tag{5.90}$$

It follows that we can bound $A(x,\tau)$ by

$$A(x,\tau) \leq \frac{1}{|\xi|}\left[3 + \sum_{p=1}^{\infty}\sum_{n=2p}^{\infty} r_{n,p}\|\mathcal{K}\|^p(ck)^{n-2p}|\xi|^n\right], \tag{5.91}$$

where $\|\mathcal{K}\| = \|\mathcal{K}\|_\infty$ and we are motivated by (5.90) to define

$$r_{n,p} = \frac{n+4}{(n+1)(n-2)}\left[p\,r_{n-1,p} + \frac{1}{2}\sum_{\ell=1}^{p-1}\sum_{j=2\ell}^{n-2(p-\ell)} r_{j,\ell}\,r_{n-j,p-\ell}\right], \quad \text{for } n > 2p. \tag{5.92a}$$

$$r_{2p,p} = \frac{1}{|K|^p}\left(|A_{2p}^1| + |A_{2p}^2| + |A_{2p}^3|\right), \tag{5.92b}$$

n	$(ck)^0$	(ck)	$(ck)^2$	$(ck)^3$	$(ck)^4$	$(ck)^5$	$(ck)^6$
2	$\frac{2}{3}$	0	0	0	0	0	0
3	0	$\frac{7}{6}$	0	0	0	0	0
4	$\frac{11}{180}$	0	$\frac{14}{15}$	0	0	0	0
5	0	$\frac{9}{20}$	0	$\frac{7}{15}$	0	0	0
6	$\frac{47}{7560}$	0	$\frac{793}{1008}$	0	$\frac{1}{6}$	0	0
7	0	$\frac{8107}{75600}$	0	$\frac{82423}{100800}$	0	$\frac{11}{240}$	0
8	$\frac{191}{302400}$	0	$\frac{1333}{4200}$	0	$\frac{9161}{15120}$	0	$\frac{11}{1080}$

Table 5.5: The first few coefficients $r_{n,p}$ in (5.91), defined by (5.92). Note that n indicates the nth row of the table, while p indicates the pth nonzero diagonal (that is, the diagonal that begins in the row corresponding to $n = 2p$). For instance, $r_{6,2} = \frac{793}{1008}$. Also, note that the first column contains the entries $r_{2p,p}$, which are defined in terms of the ODE series coefficients in (5.92b).

where (5.92a) is defined for $p \geq 2$ and (5.92b) is defined for $p \geq 1$. Note that in (5.92b), A^m_{2p} are the coefficients from the ODE series solution in (5.66) with $H = K_2 = 0$ and $K_3 = K$.

Finally, note that when $p = 1$, (5.92a) reduces to

$$r_{n,1} = \frac{n+4}{(n+1)(n-2)}\, r_{n-1,1}.$$

It is straightforward to show that this is equivalent to

$$r_{n,1} = \frac{(n+4)(n+3)(n+2)}{120(n-2)!}\, r_{2,1}. \tag{5.93}$$

The first few constants $r_{n,p}$ can be found in Table 5.5.

It still remains to reduce the double sum in (5.91) to a single sum so that we can apply the ratio test. In order to do this, we must use the definition of $r_{n,p}$ in (5.92) and (5.93) to find a nice bound on $r_{n,p}$. To simplify our notation in what follows, we let $r_{2p,p} \equiv r_{2p}$.

It is shown by induction in Appendix C that

$$r_{n,p} \leq \frac{(3p)^{n-2p}}{(n-2p)!} \sum_{\ell=1}^{p-1} r_{2\ell}\, r_{2(p-\ell)} \tag{5.94}$$

for $n \geq 2p$ and for $p \geq 2$. When $p = 1$, (5.93) easily gives

$$r_{n,1} \leq \frac{3^{n-2}}{(n-2)!} r_2, \tag{5.95}$$

which can be thought of as (5.94) for $p = 1$ without the sum.

Next, we show that

$$\sum_{\ell=1}^{p-1} r_{2\ell}\, r_{2(p-\ell)} \le (\delta p + \gamma)\, r_{2p}, \quad p \ge 2, \tag{5.96}$$

where δ and γ are real nonnegative constants.

Let the coefficients in the ODE series solution in (5.66) be decomposed as follows

$$A_{2n}^m = \alpha_{2n}^m K^n, \tag{5.97}$$

where α_{2n}^m is real for $n \ge 0$. The proof of (5.96) requires the following information about the constants α_{2n}^m. Note that these facts are specific to the ODE series solution with $H = K_2 = 0$ and $K_3 = K$. Without loss of generality, assume that $\sigma = 1$.

i. $\alpha_{2n}^1 > 0$, $\alpha_{2n}^2 > 0$, and $\alpha_{2n}^3 < 0$ for all $n \ge 0$.

ii. $\alpha_{2n}^1 = \alpha_{2n}^2$ for all $n \ge 0$.

iii. $\left|\alpha_{2n}^1\right| \le 2\left|\alpha_{2n}^3\right|$ for all $n \ge 0$.

iv. $\left|\alpha_{2n}^3\right| \le 2\left|\alpha_{2n}^1\right|$ for all $n \ge 0$.

Properties (i) and (ii) follow from the ODE series solution in (5.66), with $A_2^1 = A_2^2 = K/6$ and $A_3^1 = -K/3$. Properties (iii)-(iv) require slightly more work to show. We use induction to prove that the results in (iii)-(iv) hold.

We prove (iii)-(iv) together. In particular, we have $\alpha_0^m = 1$ for $m = 1, 2, 3$, as well as $\alpha_2^1 = 1/6$ and $\alpha_2^3 = -1/3$. It follows that (iii)-(iv) hold when $n = 0, 1$. Next, suppose that (iii)-(iv) hold for all n such that $2 \le n < N$. That is, assume that

$$\left|\alpha_{2n}^1\right| \le 2\left|\alpha_{2n}^3\right|, \quad 2 \le n < N, \tag{5.98}$$

$$\left|\alpha_{2n}^3\right| \le 2\left|\alpha_{2n}^1\right|, \quad 2 \le n < N. \tag{5.99}$$

We show that (iii)-(iv) hold for $n = N$.

First, observe that solving the linear system defined by the recursion relation in (5.67) and canceling a factor of $1/K^n$ gives

$$\alpha_{2n}^m = \frac{1}{2(2n+1)(n-1)}\left[2nb_{2n}^m - b_{2n}^k - b_{2n}^\ell\right], \quad n \ge 2, \tag{5.100}$$

where

$$b_{2n}^{m} = -\sum_{j=1}^{n-1} \alpha_{2j}^{k} \alpha_{2(n-j)}^{\ell}$$

and $(k, \ell, m) = (1, 2, 3)$ cyclically. Note that $b_{2n}^1 = b_{2n}^2$ due to (ii). Furthermore, we can use (i)-(ii) to write

$$b_{2n}^1 = b_{2n}^2 = \sum_{j=1}^{n-1} \left|\alpha_{2j}^1 \alpha_{2(n-j)}^3\right| \quad \text{and} \quad b_{2n}^3 = -\sum_{j=1}^{n-1} \left|\alpha_{2j}^1 \alpha_{2(n-j)}^1\right|, \tag{5.101}$$

since $|\alpha_{2n}^3| = -\alpha_{2n}^3$ for $n \geq 0$.

Now we show that (5.98) holds when $n = N$, or that $2\left|\alpha_{2N}^3\right| - \left|\alpha_{2N}^1\right| \geq 0$. Observe the following, which makes use of (i), (ii), and (5.100)-(5.101)

$$\begin{aligned}
2\left|\alpha_{2N}^3\right| - \left|\alpha_{2N}^1\right| &= -2\alpha_{2N}^3 - \alpha_{2N}^1 \\
&= \frac{1}{2(2N+1)(N-1)} \left[-2\left(2Nb_{2N}^3 - 2b_{2N}^1\right) - \left(2Nb_{2N}^1 - b_{2N}^1 - b_{2N}^3\right)\right] \\
&= \frac{1}{2(2N+1)(N-1)} \left[(-4N+1)b_{2N}^3 + (5-2N)b_{2N}^1\right] \\
&= \frac{1}{2(2N+1)(N-1)} \left[(4N-1)\sum_{j=1}^{N-1}\left|\alpha_{2j}^1\alpha_{2(N-j)}^1\right| - (2N-5)\sum_{j=1}^{N-1}\left|\alpha_{2j}^1\alpha_{2(N-j)}^3\right|\right] \\
&= \frac{1}{2(2N+1)(N-1)} \sum_{j=1}^{N-1} \left|\alpha_{2j}^1\right| \left[(4N-1)\left|\alpha_{2(N-j)}^1\right| - (2N-5)\left|\alpha_{2(N-j)}^3\right|\right].
\end{aligned}$$

Since $N > 2$, the terms $(4N-1)|\alpha_{2(N-j)}^1|$ and $(2N-5)|\alpha_{2(N-j)}^3|$ are positive. Moreover, we know from (5.99) that $|\alpha_{2(N-j)}^3| \leq 2|\alpha_{2(N-j)}^1|$ since $N-j$ is at most $N-1$. Thus we have the following

$$\begin{aligned}
2\left|\alpha_{2N}^3\right| - \left|\alpha_{2N}^1\right| &\geq \frac{1}{2(2N+1)(N-1)} \sum_{j=1}^{N-1} \left|\alpha_{2j}^1\right| \left[(4N-1)\left|\alpha_{2(N-j)}^1\right| - 2(2N-5)\left|\alpha_{2(N-j)}^1\right|\right] \\
&= \frac{1}{2(2N+1)(N-1)} \sum_{j=1}^{N-1} \left(4N-1-2(2N-5)\right)\left|\alpha_{2j}^1\alpha_{2(N-j)}^1\right| \\
&= \frac{1}{2(2N+1)(N-1)} \sum_{j=1}^{N-1} 9\left|\alpha_{2j}^1\alpha_{2(N-j)}^1\right| \\
&\geq 0.
\end{aligned}$$

To summarize, we showed that

$$2\left|\alpha_{2N}^3\right| - \left|\alpha_{2N}^1\right| \geq 0 \tag{5.102}$$

as long as both of the inductive hypotheses in (5.98)-(5.99) hold.

It remains to show that (5.99) holds when $n = N$, or that $2\left|\alpha^1_{2N}\right| - \left|\alpha^3_{2N}\right| \geq 0$. Observe the following

$$\begin{aligned}
2\left|\alpha^1_{2N}\right| - \left|\alpha^3_{2N}\right| &= 2\alpha^1_{2N} + \alpha^3_{2N} \\
&= \frac{1}{2(2N+1)(N-1)}\left[2\left(2Nb^1_{2N} - b^1_{2N} - b^3_{2N}\right) + \left(2Nb^3_{2N} - 2b^1_{2N}\right)\right] \\
&= \frac{1}{2(2N+1)(N-1)}\left[(4N-4)b^1_{2N} + (2N-2)b^3_{2N}\right] \\
&= \frac{1}{2(2N+1)(N-1)}\left[(4N-4)\sum_{j=1}^{N-1}\left|\alpha^1_{2j}\alpha^3_{2(N-j)}\right| - (2N-2)\sum_{j=1}^{N-1}\left|\alpha^1_{2j}\alpha^1_{2(N-j)}\right|\right] \\
&= \frac{1}{2(2N+1)(N-1)}\sum_{j=1}^{N-1}\left|\alpha^1_{2j}\right|\left[(4N-4)\left|\alpha^3_{2(N-j)}\right| - (2N-2)\left|\alpha^1_{2(N-j)}\right|\right]
\end{aligned}$$

We know that the terms $(4N-4)|\alpha^3_{2(N-j)}|$ and $(2N-2)|\alpha^1_{2(N-j)}|$ are positive since $N > 2$, and we know from (5.98) that $|\alpha^1_{2(N-j)}| \leq 2|\alpha^3_{2(N-j)}|$. As a result, we have

$$\begin{aligned}
2\left|\alpha^1_{2N}\right| - \left|\alpha^3_{2N}\right| &\geq \frac{1}{2(2N+1)(N-1)}\sum_{j=1}^{N-1}\left|\alpha^1_{2j}\right|\left[(4N-4)\left|\alpha^3_{2(N-j)}\right| - 2(2N-2)\left|\alpha^3_{2(N-j)}\right|\right] \\
&= \frac{1}{2(2N+1)(N-1)}\sum_{j=1}^{N-1}\left(4N-4-2(2N-2)\right)\left|\alpha^1_{2j}\alpha^3_{2(N-j)}\right| \\
&= 0.
\end{aligned}$$

That is, we showed that

$$2\left|\alpha^1_{2N}\right| - \left|\alpha^3_{2N}\right| \geq 0 \tag{5.103}$$

as long as both of the inductive hypotheses in (5.98)-(5.99) hold. The combination of (5.102)-(5.103) with the base cases for $n = 0, 1$ prove by induction that (iii)-(iv) hold for $n \geq 0$.

We are now able to use (i)-(iv) to show that (5.96) holds. That is, we want to prove that

$$\sum_{\ell=1}^{n-1} r_{2\ell}\, r_{2(n-\ell)} \leq (\delta n + \gamma)\, r_{2n}, \quad n \geq 2, \tag{5.104}$$

where δ and γ are real nonnegative constants. We proceed inductively, and determine what values δ and γ can take on in the process. First, from (5.92b), we have that

$$r_{2n} = \left|\alpha^1_{2n}\right| + \left|\alpha^2_{2n}\right| + \left|\alpha^3_{2n}\right|,$$

where we used $A^m_{2n} = \alpha^m_{2n} K^n$. It follows that

$$r_2 = \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \frac{2}{3}.$$

Moreover, for $n = 2$, we have

$$r_4 = \frac{7}{360} + \frac{7}{360} + \frac{1}{45} = \frac{11}{180}.$$

Since $r_2^2 = 4/9$, it follows that (5.104) holds for $n = 2$ as long as

$$\frac{4}{9} \leq (2\delta + \gamma) \frac{11}{180}.$$

In other words, for the base case, $n = 2$, of (5.104) to be satisfied, δ and γ must be chosen such that

$$2\delta + \gamma \geq \frac{80}{11}. \tag{5.105}$$

Henceforth, assume that δ and γ satisfy this condition.

Before we tackle the inductive step of the proof, we first manipulate each side of the inequality in (5.104) into a more useful form. To that end, using (5.100) and (5.101), we have, for $n \geq 1$,

$$\begin{aligned} r_{2n} &= \left|\alpha^1_{2n}\right| + \left|\alpha^2_{2n}\right| + \left|\alpha^3_{2n}\right| \\ &= 2\alpha^1_{2n} - \alpha^3_{2n} \\ &= \frac{1}{2(2n+1)(n-1)} \left[2(2nb^1_{2n} - b^1_{2n} - b^3_{2n}) - (2nb^3_{2n} - 2b^1_{2n})\right] \\ &= \frac{1}{2(2n+1)(n-1)} \left[4nb^1_{2n} - (2n+2)b^3_{2n}\right] \\ &= \frac{1}{2(2n+1)(n-1)} \left[4n \sum_{j=1}^{n-1} \left|\alpha^1_{2j}\alpha^3_{2(n-j)}\right| + (2n+2) \sum_{j=1}^{n-1} \left|\alpha^1_{2j}\alpha^1_{2(n-j)}\right|\right]. \end{aligned} \tag{5.106}$$

We also have

$$\begin{aligned} \sum_{j=1}^{n-1} r_{2j} r_{2(n-j)} &= \sum_{j=1}^{n-1} \left(2\alpha^1_{2j} - \alpha^3_{2j}\right)\left(2\alpha^1_{2(n-j)} - \alpha^3_{2(n-j)}\right) \\ &= \sum_{j=1}^{n-1} \left(4\alpha^1_{2j}\alpha^1_{2(n-j)} - 2\alpha^1_{2j}\alpha^3_{2(n-j)} - 2\alpha^3_{2j}\alpha^1_{2(n-j)} + \alpha^3_{2j}\alpha^3_{2(n-j)}\right) \\ &= \sum_{j=1}^{n-1} \left(4\left|\alpha^1_{2j}\alpha^1_{2(n-j)}\right| + 2\left|\alpha^1_{2j}\alpha^3_{2(n-j)}\right| + 2\left|\alpha^3_{2j}\alpha^1_{2(n-j)}\right| + \left|\alpha^3_{2j}\alpha^3_{2(n-j)}\right|\right). \end{aligned} \tag{5.107}$$

Now we return to the proof of (5.104). Consider the quantity

$$(\delta n+\gamma)r_{2n}-\sum_{j=1}^{n-1} r_{2j}r_{2(n-j)}. \tag{5.108}$$

We know that (5.108) is nonnegative when $n=2$ as long as γ and δ satisfy (5.105). Furthermore, assume that (5.108) is nonnegative for $n \le 3 < N$. Then we show that (5.108) is also nonnegative when $n=N$. This is sufficient to prove (5.104).

Define

$$\kappa(n)=\frac{\delta n+\gamma}{2(2n+1)(n-1)}, \quad n\ge 2. \tag{5.109}$$

Then using (5.106)-(5.107) with $n=N$, we have

$$\begin{aligned}
&(\delta N+\gamma)r_{2N}-\sum_{j=1}^{N-1} r_{2j}r_{2(N-j)}\\
&\quad=\kappa(N)\left[4N\sum_{j=1}^{N-1}\left|\alpha^1_{2j}\alpha^3_{2(N-j)}\right|+(2N+2)\sum_{j=1}^{N-1}\left|\alpha^1_{2j}\alpha^1_{2(N-j)}\right|\right]\\
&\quad\quad-\sum_{j=1}^{N-1}\left(4\left|\alpha^1_{2j}\alpha^1_{2(N-j)}\right|+2\left|\alpha^1_{2j}\alpha^3_{2(N-j)}\right|+2\left|\alpha^3_{2j}\alpha^1_{2(N-j)}\right|+\left|\alpha^3_{2j}\alpha^3_{2(N-j)}\right|\right)\\
&\quad=\sum_{j=1}^{N-1}\Big\{[4N\kappa(N)-2]\left|\alpha^1_{2j}\alpha^3_{2(N-j)}\right|+[(2N+2)\kappa(N)-4]\left|\alpha^1_{2j}\alpha^1_{2(N-j)}\right|\\
&\quad\quad\quad-2\left|\alpha^3_{2j}\alpha^1_{2(N-j)}\right|-\left|\alpha^3_{2j}\alpha^3_{2(N-j)}\right|\Big\}.
\end{aligned}$$

We know from (iv) that $|\alpha^3_{2N}|\le 2|\alpha^1_{2N}|$. It follows that

$$\left|\alpha^3_{2j}\alpha^1_{2(N-j)}\right|\le 2\left|\alpha^1_{2j}\alpha^1_{2(N-j)}\right| \quad\text{and}\quad \left|\alpha^3_{2j}\alpha^3_{2(N-j)}\right|\le 2\left|\alpha^1_{2j}\alpha^3_{2(N-j)}\right|$$

for $j=1,2,\ldots,N-1$. As a result, we have

$$\begin{aligned}
&(\delta N+\gamma)r_{2N}-\sum_{j=1}^{N-1} r_{2j}r_{2(N-j)}\\
&\quad\ge\sum_{j=1}^{N-1}\Big\{[4N\kappa(N)-2]\left|\alpha^1_{2j}\alpha^3_{2(N-j)}\right|+[(2N+2)\kappa(N)-4]\left|\alpha^1_{2j}\alpha^1_{2(N-j)}\right|\\
&\quad\quad\quad-4\left|\alpha^1_{2j}\alpha^1_{2(N-j)}\right|-2\left|\alpha^1_{2j}\alpha^3_{2(N-j)}\right|\Big\}\\
&\quad=\sum_{j=1}^{N-1}\Big\{[4N\kappa(N)-4]\left|\alpha^1_{2j}\alpha^3_{2(N-j)}\right|+[(2N+2)\kappa(N)-8]\left|\alpha^1_{2j}\alpha^1_{2(N-j)}\right|.
\end{aligned}$$

The quantity above is nonnegative if $\kappa(n)$ satisifies

$$4n\kappa(n) - 4 \geq 0 \quad \text{and} \quad (2n+2)\kappa(n) - 8 \geq 0.$$

Consequently, we can satisfy both inequalities by choosing $\kappa(n)$ to satisfy

$$(n+1)\kappa(n) \geq 4,$$

where we note that $\kappa(n)$ is only defined for $n \geq 2$. Using the definiton of $\kappa(n)$ in (5.109), we must pick the real constants δ and γ such that

$$(n+1)\kappa(n) = \frac{(n+1)(\delta n + \gamma)}{2(2n+1)(n-1)} \geq 4.$$

A simple choice is $\delta = 16$ and $\gamma = 0$. Importantly, notice that this choice of δ and γ satisfies (5.105). As a result, with these values, we have shown by induction that (5.104) holds for all $n \geq 2$. That is, we proved

$$\sum_{j=1}^{n-1} r_{2j} r_{2(n-j)} \leq 16n\, r_{2n}. \tag{5.110}$$

Combining (5.94) and (5.110), we have

$$r_{n,p} \leq \frac{(3p)^{n-2p}}{(n-2p)!} 16n\, r_{2p}, \quad n > 2p, \; p \geq 1. \tag{5.111}$$

Note that the result for $p = 1$ is a consequence of (5.95).

Finally, we reduce (5.91) to a single sum. To that end, let S_p be the inner sum in (5.91), namely

$$S_p = \sum_{n=2p}^{\infty} r_{n,p} \|\mathcal{K}\|^p (ck)^{n-2p} |\xi|^n. \tag{5.112}$$

Then (5.91) becomes

$$A(x,\tau) \leq \frac{1}{|\xi|} \left[3 + \sum_{p=1}^{\infty} S_p \right]. \tag{5.113}$$

Using (5.111) in the definition of S_p gives

$$\begin{aligned}
S_p &= \sum_{n=2p}^{\infty} r_{n,p} \|\mathcal{K}\|^p (ck)^{n-2p} |\xi|^n \\
&\le \sum_{n=2p}^{\infty} \frac{(3p)^{n-2p}}{(n-2p)!} \, 16p \, r_{2p} \, \|\mathcal{K}\|^p (ck)^{n-2p} |\xi|^n \\
&= 16p \, \frac{\|\mathcal{K}\|^p \, r_{2p}}{(3ckp)^{2p}} \sum_{n=2p}^{\infty} \frac{(3ckp|\xi|)^n}{(n-2p)!} \\
&= 16p \, \frac{\|\mathcal{K}\|^p \, r_{2p}}{(3ckp)^{2p}} \sum_{n=0}^{\infty} \frac{(3ckp|\xi|)^{n+2p}}{n!} \\
&= 16p \, \|\mathcal{K}\|^p \, r_{2p} \, |\xi|^{2p} \sum_{n=0}^{\infty} \frac{(3ckp|\xi|)^n}{n!} \\
&= 16p \, \|\mathcal{K}\|^p \, r_{2p} \, |\xi|^{2p} \, e^{3ckp|\xi|}.
\end{aligned}$$

Now substituting into (5.113) gives

$$\begin{aligned}
A(x,\tau) &\le \frac{1}{|\xi|} \left[3 + \sum_{p=1}^{\infty} S_p \right] \\
&\le \frac{1}{|\xi|} \left[3 + 16 \sum_{p=1}^{\infty} p \, \|\mathcal{K}\|^p \, r_{2p} \, |\xi|^{2p} \, e^{3ckp|\xi|} \right].
\end{aligned} \tag{5.114}$$

Step 6: Finally, we determine the radius of convergence of $A(x,\tau)$ in (5.81) using (5.114). In particular, we find that the series $A(x,\tau)$ converges when

$$\lim_{p\to\infty} \left| \frac{(p+1) \, \|\mathcal{K}\|^{p+1} \, r_{2(p+1)} \, |\xi|^{2(p+1)} \, e^{3ck(p+1)|\xi|}}{p \, \|\mathcal{K}\|^p \, r_{2p} \, |\xi|^{2p} \, e^{3ckp|\xi|}} \right| < 1.$$

That is, $A(x,\tau)$ converges under the condition

$$\|\mathcal{K}\| \, |\xi|^2 \, e^{3ck|\xi|} \lim_{p\to\infty} \left| \frac{r_{2(p+1)}}{r_{2p}} \right| < 1,$$

where we used the fact that $\lim_{p\to\infty}(p+1)/p = 1$. Finally, rearranging gives that $A(x,\tau)$ converges when

$$e^{\frac{3}{2}ck|\xi|} |\xi| < \left(\|\mathcal{K}\| \lim_{p\to\infty} \left| \frac{r_{2(p+1)}}{r_{2p}} \right| \right)^{-1/2}. \tag{5.115}$$

Recall from (5.68)-(5.69) that

$$\lim_{p\to\infty}\left|\frac{A^m_{2(p+1)}}{A^m_{2p}}\right| = \frac{|K|}{\pi^2}$$

for $m = 1, 2, 3$. Then using $A^m_{2n} = \alpha^m_{2n}K^n$, we have

$$\lim_{p\to\infty}\left|\frac{\alpha^m_{2(p+1)}}{\alpha^m_{2p}}\right| = \frac{1}{\pi^2}$$

for $m = 1, 2, 3$. Moreover, recall that

$$r_{2p} = \left|\alpha^1_{2p}\right| + \left|\alpha^2_{2p}\right| + \left|\alpha^3_{2p}\right| .$$

Then it is straightforward to find that

$$\lim_{p\to\infty}\left|\frac{r_{2(p+1)}}{r_{2p}}\right| = \lim_{p\to\infty}\frac{\left|\alpha^1_{2(p+1)}\right| + \left|\alpha^2_{2(p+1)}\right| + \left|\alpha^3_{2(p+1)}\right|}{\left|\alpha^1_{2p}\right| + \left|\alpha^2_{2p}\right| + \left|\alpha^3_{2p}\right|} = \frac{1}{\pi^2}.$$

Lastly, substituting into (5.115) tells us that the series $A(x,\tau)$ converges when

$$e^{\frac{3}{2}ck|\tau-\tau_0|}|\tau-\tau_0| < \frac{\pi}{\|\mathcal{K}\|^{1/2}_\infty}. \tag{5.116}$$

Since $A(x,\tau) = |a_1(x,\tau)| + |a_2(x,\tau)| + |a_3(x,\tau)|$ converges under the condition (5.116), it follows that $a_m(x,\tau)$ for $m = 1, 2, 3$ must converge absolutely at least under the same condition, or possibly under a less restrictive condition.

To summarize, we found that the general solution of the three-wave ODEs when $H = K_2 = 0$ and $K_3 = K$ converges when

$$|\tau-\tau_0| < R_{\mathrm{ODE}}, \tag{5.117}$$

where

$$R_{\mathrm{ODE}} = \frac{\pi}{|K|^{1/2}}.$$

Meanwhile, we determined that the analagous near-general solution of the three-wave PDEs with constant phases, $\mathcal{H}(x) = 0$, and $\mathcal{K}_2(x) = \mathcal{K}_3(x) = \mathcal{K}(x)/6$ converges when

$$e^{\frac{3}{2}ck|\tau-\tau_0|}|\tau-\tau_0| < R_{\mathrm{ODE}}, \tag{5.118}$$

where we replace $|K|$ with $\|\mathcal{K}\|_\infty$ in R_{ODE}.

Compare (5.117) with (5.118). As with the convergence criteria (5.63) and (5.64) in Case 1, the two convergence conditions (5.117) and (5.118) differ only by an exponential factor. Indeed, the convergence conditions for the solutions of the ODEs and PDEs differ by the factor of $e^{\frac{3}{2}ck|\tau-\tau_0|}$ in (5.118). This is a known, dimensionless factor that causes the radius of convergence of the PDE solution to be smaller than that of the ODE solution. However, the factor depends only on k, which tells us how quickly the derivatives of $\mathcal{K}(x)$ are growing, and c, which is the largest group velocity (in magnitude) of the three wavetrains. This indicates that we are losing very little information in moving from the ODEs to the PDEs. The inclusion of spatial dependence in the three-wave PDEs affects where our near-general solution is valid, but it does not affect the overall structure of the solutions, at least in the two special cases considered so far.

5.5 Radius of convergence: Case 2(ii)

In this section, we consider Case 2(ii), in which $\mathcal{K}_2(x) = \mathcal{K}(x)/2$ and $\mathcal{H}(x) = \mathcal{K}_3(x) = 0$ in the solution of the three-wave PDEs, while $H = 0$, $K_2 = K$ and $K_3 = 2K$ in the solution of the three-wave ODEs. Additionally, recall that the phases $\psi_m(x)$ in the PDE solution are assumed to be constant. In order to find the radius of convergence of the Laurent series solution of the three-wave PDEs in this case, we follow the steps outlined in Section 5.2. Moreover, the proof of this section follows closely from that in the previous section for Case 2(i).

Step 1: We begin by finding the series solution of the three-wave ODEs when $H = 0$, $K_2 = K$, and $K_3 = 2K$ using (4.29)-(4.31). The first few terms of the series for $a_m(\tau)$, $m = 1, 2, 3$, are

$$a_1(\tau) = \frac{e^{i\psi_1}}{\xi}\left[1 + \frac{K}{2}\xi^2 - \frac{K^2}{40}\xi^4 + \frac{K^3}{80}\xi^6 + \frac{K^4}{9600}\xi^8 + \frac{K^5}{3840}\xi^{10} - \frac{K^6}{1664000}\xi^{12} + \cdots\right], \tag{5.119a}$$

$$a_2(\tau) = \frac{e^{i\psi_2}}{\xi}\left[1 + 0\,\xi^2 + \frac{K^2}{10}\xi^4 + 0\,\xi^6 + \frac{K^4}{600}\xi^8 + 0\,\xi^{10} + \frac{K^6}{26000}\xi^{12} + \cdots\right], \tag{5.119b}$$

$$a_3(\tau) = \frac{e^{i\psi_3}}{\xi}\left[1 - \frac{K}{2}\xi^2 - \frac{K^2}{40}\xi^4 - \frac{K^3}{80}\xi^6 + \frac{K^4}{9600}\xi^8 - \frac{K^5}{3840}\xi^{10} - \frac{K^6}{1664000}\xi^{12} + \cdots\right]. \tag{5.119c}$$

Note that since $H = 0$ in this case, $a_m(\tau)$ has the form (5.66),

$$a_m(\tau) = \frac{e^{i\psi_m}}{\xi} \sum_{n=0}^{\infty} A_{2n}^m \, \xi^{2n}, \tag{5.120}$$

for $m = 1, 2, 3$, where A_{2n}^m is real and defined in (5.67) for $n \geq 2$.

It is straightforward to find the radius of convergence of the series for $a_m(\tau)$ using the Weierstrass general solution of the three-wave ODEs in (3.54). In particular, the elliptic invariants g_2 and g_3 in (3.45)-(3.46) with $K_2 = K$, $K_3 = 2K$, and $H = 0$, are

$$g_2 = 4K^2 \quad \text{and} \quad g_3 = 0.$$

In this case, we know that the lattice associated with the Weierstrass elliptic function in (3.54) is rectangular since the discriminant is $\Delta = g_2^3 - 27g_3^2 > 0$. Moreover, in the special case where $g_3 = 0$, the lattice is lemniscatic, meaning the poles occur on a square lattice [17, §23.5(iii)]. In particular, we have that $\lambda_2 = i\lambda_1$, where λ_1 and λ_2 are the half-periods of the Weierstrass function. In this special case, the relationship between the half-periods and the invariants of $\wp(z; g_2, g_3)$ is known explicitly. In particular, we have

$$g_2 = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^8}{256\pi^2\lambda_1^4}$$

[17, §23.5.4]. Solving for λ_1 yields

$$\lambda_1 = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4 \cdot \pi^{1/2} g_2^{1/4}} = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4 \cdot \pi^{1/2} 4^{1/4} |K|^{1/2}},$$

where we used the fact that $g_2 = 4K^2$. Since the radius of convergence of the Weierstrass general solution of the three-wave ODEs is given by $R = \max\{2|\lambda_1|, 2|\lambda_2|\}$, it follows that the radius of convergence of $a_m(\tau)$ is

$$R_{\text{ODE}} = \frac{2\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4\sqrt{2}\,\pi^{1/2} \cdot |K|^{1/2}} \approx \frac{2.62206}{|K|^{1/2}}. \tag{5.121}$$

Alternatively, we can find the radius of convergence of (5.119a)-(5.119c) by applying the ratio test to (5.120) for $m = 1, 2, 3$. Then we find that the Laurent series for $a_m(\tau)$ converges for $m = 1, 2, 3$ when

$$\lim_{n \to \infty} \left| \frac{A_{2(n+1)}^m \xi^{2(n+1)}}{A_{2n}^m \xi^{2n}} \right| < 1,$$

or

$$|\xi| < \left(\lim_{n\to\infty} \left| \frac{A^m_{2(n+1)}}{A^m_{2n}} \right| \right)^{-1/2}.$$

Unfortunately, the limit on the right-hand side does not exist. As n increases, the value of $|A^m_{2(n+1)}/A^m_{2n}|$ alternates between two numbers. As a result, we follow the procedure outlined for Case 1 in Section 5.3. That is, we split the series (5.120) into its even and odd parts and find the radius of convergence of each.

Define the following

$$S^m_{\text{even}} = \frac{e^{i\psi_m}}{\xi} \sum_{n=0}^{\infty} A^m_{4n}\, \xi^{4n} \quad \text{and} \quad S^m_{\text{odd}} = \frac{e^{i\psi_m}}{\xi} \sum_{n=0}^{\infty} A^m_{2(2n+1)}\, \xi^{2(2n+1)}, \tag{5.122}$$

so that $a_m(\tau) = S^m_{\text{even}} + S^m_{\text{odd}}$. First, we find that S^2_{odd} converges with an infinite radius of convergence because $A^2_{2(2n+1)} = 0$ for all $n \geq 0$. Furthermore, the ratio test tells us that the even sums for $m = 1, 2, 3$ and the odd sums for $m = 1, 3$ converge, respectively, under the conditions

$$\lim_{n\to\infty} \left| \frac{A^m_{4(n+1)} \xi^{4(n+1)}}{A^m_{4n} \xi^{4n}} \right| < 1 \quad \text{and} \quad \lim_{n\to\infty} \left| \frac{A^m_{2(2(n+1)+1)} \xi^{2(2(n+1)+1)}}{A^m_{2(2n+1)} \xi^{2(2n+1)}} \right| < 1.$$

Rearranging, and using $\xi = \tau - \tau_0$ yields the following convergence criteria for the even and odd series in (5.122)

$$S^m_{\text{even}}: \quad |\tau - \tau_0| < \left(\lim_{n\to\infty} \left| \frac{A^m_{4(n+1)}}{A^m_{4n}} \right| \right)^{-1/4}, \quad m = 1, 2, 3 \tag{5.123}$$

$$S^m_{\text{odd}}: \quad |\tau - \tau_0| < \left(\lim_{n\to\infty} \left| \frac{A^m_{2(2(n+1)+1)}}{A^m_{2(2n+1)}} \right| \right)^{-1/4}, \quad m = 1, 3. \tag{5.124}$$

We find numerically that the limits in (5.123) for $m = 1, 3$ yield the same value. In particular, we find

$$\left(\lim_{n\to\infty} \left| \frac{A^m_{4(n+1)}}{A^m_{4n}} \right| \right)^{-1/4} \approx \frac{3.70815}{|K|^{1/2}}, \quad m = 1, 3. \tag{5.125}$$

Furthermore, we find that the limit in (5.123) for $m = 2$ and the limits in (5.124) yield the same value. Specifically, we have

$$\left(\lim_{n\to\infty} \left| \frac{A^2_{4(n+1)}}{A^2_{4n}} \right| \right)^{-1/4} = \left(\lim_{n\to\infty} \left| \frac{A^m_{2(2(n+1)+1)}}{A^m_{2(2n+1)}} \right| \right)^{-1/4} \approx \frac{2.62206}{|K|^{1/2}}, \quad m = 1, 3. \tag{5.126}$$

Since the radius of convergence defined in (5.126) is smaller than the radius in (5.125), it follows that $a_m(\tau) = S^m_{\text{even}} + S^m_{\text{odd}}$ converges for $m = 1, 2, 3$ with a radius of at least $2.62/|K|^{1/2}$. That is, we find that the full series $a_m(\tau)$ in (5.120) converges for $m = 1, 2, 3$ with radius of convergence

$$R_{\text{ODE}} = \frac{2.62}{|K|^{1/2}},$$

which is equivalent to (5.121). In other words, we find the same radius of convergence using both the Weierstrass solution and the Laurent series general solution of the three-wave ODEs.

STEP 2: Now we find the series solution of the three-wave PDEs when $\mathcal{H}(x) = \mathcal{K}_3(x) = 0$ and $\mathcal{K}_2(x) = \mathcal{K}(x)/2$. The first few terms of the series are given by

$$\begin{aligned} a_1(x,\tau) = \frac{e^{i\psi_1}}{\xi}\Bigg\{ 1 &+ \frac{\mathcal{K}}{2}\xi^2 - \left(\frac{3}{8}c_1 + \frac{1}{8}c_3\right)\mathcal{K}'\xi^3 \\ &+ \left[-\frac{1}{40}\mathcal{K}^2 + \left(\frac{3}{20}c_1^2 + \frac{1}{80}c_1c_2 + \frac{1}{16}c_1c_3 - \frac{1}{80}c_2c_3 + \frac{3}{80}c_3^2\right)\mathcal{K}''\right]\xi^4 \\ &+ \left[(\cdots)\,\mathcal{K}\mathcal{K}' + (\cdots)\,\mathcal{K}^{(3)} +\right]\xi^5 \\ &+ \left[(\cdots)\,\mathcal{K}^3 + (\cdots)\,(\mathcal{K}')^2 + (\cdots)\,\mathcal{K}\mathcal{K}'' + (\cdots)\,\mathcal{K}^{(4)}\right]\xi^6 + \cdots \Bigg\}, \end{aligned} \tag{5.127a}$$

$$\begin{aligned} a_2(x,\tau) = \frac{e^{i\psi_2}}{\xi}\Bigg\{ 1 &+ 0\xi^2 + \left(\frac{1}{8}c_1 - \frac{1}{8}c_3\right)\mathcal{K}'\xi^3 \\ &+ \left[\frac{1}{10}\mathcal{K}^2 + \left(-\frac{3}{80}c_1^2 - \frac{1}{20}c_1c_2 + \frac{1}{20}c_2c_3 + \frac{3}{80}c_3^2\right)\mathcal{K}''\right]\xi^4 \\ &+ \left[(\cdots)\,\mathcal{K}\mathcal{K}' + (\cdots)\,\mathcal{K}^{(3)} +\right]\xi^5 \\ &+ \left[(\cdots)\,\mathcal{K}^3 + (\cdots)\,(\mathcal{K}')^2 + (\cdots)\,\mathcal{K}\mathcal{K}'' + (\cdots)\,\mathcal{K}^{(4)}\right]\xi^6 + \cdots \Bigg\}, \end{aligned} \tag{5.127b}$$

$$\begin{aligned} a_3(x,\tau) = \frac{e^{i\psi_3}}{\xi}\Bigg\{ 1 &- \frac{\mathcal{K}}{2}\xi^2 + \left(\frac{1}{8}c_1 + \frac{3}{8}c_3\right)\mathcal{K}'\xi^3 \\ &+ \left[-\frac{1}{40}\mathcal{K}^2 - \left(\frac{3}{80}c_1^2 - \frac{1}{80}c_1c_2 + \frac{1}{16}c_1c_3 + \frac{1}{80}c_2c_3 + \frac{3}{20}c_3^2\right)\mathcal{K}''\right]\xi^4 \\ &+ \left[(\cdots)\,\mathcal{K}\mathcal{K}' + (\cdots)\,\mathcal{K}^{(3)} +\right]\xi^5 \\ &+ \left[(\cdots)\,\mathcal{K}^3 + (\cdots)\,(\mathcal{K}')^2 + (\cdots)\,\mathcal{K}\mathcal{K}'' + (\cdots)\,\mathcal{K}^{(4)}\right]\xi^6 + \cdots \Bigg\}. \end{aligned} \tag{5.127c}$$

As with Case 2(i) in the previous section, notice that increasingly higher derivatives of $\mathcal{K}(x)$ appear as the number of terms in the series increases, as well as more nonlinear terms in $\mathcal{K}$ and its derivatives. The series solution becomes complicated quickly, and thus we consider a particular family of functions for $\mathcal{K}(x)$.

Step 3: Following (5.33), we assume that

$$\left\| \frac{d^n}{dx^n} \mathcal{K}(x) \right\|_\infty \leq k^n \|\mathcal{K}\|_\infty, \tag{5.128}$$

where k is a real positive constant.

Step 4: Using (5.128) and the definition of c in (5.31), we bound the series for $a_m(x,\tau)$ in (5.127) using the triangle inequality. Similar to Case 2(i), we find that $a_m(x,\tau)$ is bounded by a series of the form

$$|a_m(x,\tau)| \leq \frac{1}{|\xi|} \left[1 + \sum_{p=1}^{\infty} \sum_{n=2p}^{\infty} q_{n,p}^m (ck)^{n-2p} |\xi|^n \right], \tag{5.129}$$

where $q_{n,p}^m$ are nonnegative constants for $m = 1, 2, 3$.

We have been unable to find an exact formula for the coefficients $q_{n,p}^m$, so instead we bound the sum

$$A(x,\tau) = \sum_{m=1}^{3} |a_m(x,\tau)|. \tag{5.130}$$

We then find the radius of convergence of $A(x,\tau)$, and infer that $a_m(x,\tau)$ must converge absolutely within at least that radius of convergence for $m = 1, 2, 3$. To that end, we follow the procedure outlined in Step 4 of the previous section to find that

$$A(x,\tau) \leq \frac{1}{|\xi|} \left[3 + \sum_{p=1}^{\infty} \sum_{n=2p}^{\infty} r_{n,p} \|\mathcal{K}\|^p (ck)^{n-2p} |\xi|^n \right], \tag{5.131}$$

where $\|\mathcal{K}\| = \|\mathcal{K}\|_\infty$ and $r_{n,p}$ is defined in (5.92) and (5.93).

It is shown in Appendix C that if $\mathcal{H}(x) = 0$, then $r_{n,p}$ satisifies

$$r_{n,p} \leq \frac{(3p)^{n-2p}}{(n-2p)!} \sum_{\ell=1}^{p-1} r_{2\ell} r_{2(p-\ell)}, \quad n > 2p,\ p \geq 2. \tag{5.132}$$

Moreover, when $p = 1$, (5.93) gives

$$r_{n,1} \leq \frac{3^{n-2}}{(n-2)!}\, r_2. \tag{5.133}$$

Note that we defined $r_{2p} = r_{2p,p}$ for $p \geq 1$. We can also show that when $\mathcal{K}_2(x) = \mathcal{K}(x)/2$ and $\mathcal{H}(x) = \mathcal{K}_3(x) = 0$, we have

$$\sum_{\ell=1}^{p-1} r_{2\ell} r_{2(p-\ell)} \leq 36(2p+1)\, r_{2p}. \tag{5.134}$$

The proof of (5.134) is lengthy and involves a great deal of algebra. As a result, the details of this proof are omitted from the body of this thesis, and instead can be found in Appendix D.

Combining (5.132)-(5.134), we have

$$r_{n,p} \leq \frac{(3p)^{n-2p}}{(n-2p)!}\, 36(2p+1)\, r_{2p}, \quad n > 2p,\ p \geq 1. \tag{5.135}$$

Note that the result for $p = 1$ is a simple consequence of (5.133).

Finally, we reduce (5.131) to a single sum. To that end, let S_p be the inner sum in (5.131), namely

$$S_p = \sum_{n=2p}^{\infty} r_{n,p} \|\mathcal{K}\|^p (ck)^{n-2p} |\xi|^n. \tag{5.136}$$

Then (5.131) becomes

$$A(x,\tau) \leq \frac{1}{|\xi|} \left[3 + \sum_{p=1}^{\infty} S_p \right]. \tag{5.137}$$

Again, following the procedure of the previous section, we find that

$$S_p \leq 36(2p+1)\, \|\mathcal{K}\|^p\, r_{2p}\, |\xi|^{2p}\, e^{3ckp|\xi|},$$

where we used (5.135) in (5.136). Then substituting into (5.137) gives

$$\begin{aligned} A(x,\tau) &\leq \frac{1}{|\xi|} \left[3 + \sum_{p=1}^{\infty} S_p \right] \\ &\leq \frac{1}{|\xi|} \left[3 + \sum_{p=1}^{\infty} 36(2p+1)\, \|\mathcal{K}\|^p\, r_{2p}\, |\xi|^{2p}\, e^{3ckp|\xi|} \right] \\ &= \frac{36}{|\xi|} \sum_{p=0}^{\infty} (2p+1)\, \|\mathcal{K}\|^p\, r_{2p}\, |\xi|^{2p}\, e^{3ckp|\xi|}, \end{aligned} \tag{5.138}$$

where we defined $r_0 = 1/12$.

Step 6: Finally, we determine the radius of convergence of $A(x,\tau)$ in (5.130) using (5.138). In particular, we find that the series $A(x,\tau)$ converges by the ratio test when

$$\lim_{p\to\infty}\left|\frac{(2(p+1)+1)\,\|\mathcal{K}\|^{p+1}\,r_{2(p+1)}\,|\xi|^{2(p+1)}\,e^{3ck(p+1)|\xi|}}{(2p+1)\,\|\mathcal{K}\|^{p}\,r_{2p}\,|\xi|^{2p}\,e^{3ckp|\xi|}}\right|<1.$$

That is, $A(x,\tau)$ converges under the condition

$$\|\mathcal{K}\|\,|\xi|^2\,e^{3ck|\xi|}\lim_{p\to\infty}\left|\frac{r_{2(p+1)}}{r_{2p}}\right|<1,$$

where we used the fact that $\lim_{p\to\infty}(2p+3)/(2p+1)=1$. Finally, rearranging gives that $A(x,\tau)$ converges when

$$e^{\frac{3}{2}ck|\xi|}|\xi|<\left(\|\mathcal{K}\|\lim_{p\to\infty}\left|\frac{r_{2(p+1)}}{r_{2p}}\right|\right)^{-1/2}, \tag{5.139}$$

which is equivalent to (5.115).

It remains to determine $\lim_{p\to\infty} r_{2(p+1)}/r_{2p}$, where we dropped the absolute values since $r_{2p}\geq 0$ by definition for all p. Recall that we can write

$$A_{2n}^m=\alpha_{2n}^m K^n, \tag{5.140}$$

for $n\geq 0$ and for $m=1,2,3$, where A_{2n}^m are the series coefficients in the ODE solution in (5.120). Then by (5.92b), we have that

$$r_{2p}=\left|\alpha_{2p}^1\right|+\left|\alpha_{2p}^2\right|+\left|\alpha_{2p}^3\right|.$$

As a result, we want to determine

$$\lim_{p\to\infty}\frac{\left|\alpha_{2(p+1)}^1\right|+\left|\alpha_{2(p+1)}^2\right|+\left|\alpha_{2(p+1)}^3\right|}{\left|\alpha_{2p}^1\right|+\left|\alpha_{2p}^2\right|+\left|\alpha_{2p}^3\right|}.$$

It turns out that this limit does not exist, which is not surprising since $\lim_{p\to\infty}|\alpha_{2(p+1)}^m/\alpha_{2p}^m|$ does not exist for $m=1,2,3$. As a result, we are forced to split (5.138) into its even and odd parts, and investigate the convergence of each separately. To that end, define

$$\mathcal{S}_{\text{even}}=\sum_{p=0}^{\infty}(4p+1)\,\|\mathcal{K}\|^{2p}\,r_{4p}\,|\xi|^{4p}\,e^{6ckp|\xi|}, \tag{5.141}$$

$$\mathcal{S}_{\text{odd}}=\sum_{p=0}^{\infty}(2(2p+1)+1)\,\|\mathcal{K}\|^{2p+1}\,r_{2(2p+1)}\,|\xi|^{2(2p+1)}\,e^{3ck(2p+1)|\xi|}. \tag{5.142}$$

Then we have that

$$A(x,\tau) \leq \frac{36}{|\xi|}\left(\mathcal{S}_{\text{even}} + \mathcal{S}_{\text{odd}}\right).$$

Using the ratio test and simplifying, we have that $\mathcal{S}_{\text{even}}$ and $\mathcal{S}_{\text{odd}}$ converge when

$$\begin{aligned}
\mathcal{S}_{\text{even}}: &\quad |\xi| e^{\frac{3}{2}ck|\xi|} < \left(\|\mathcal{K}\|^2 \lim_{p\to\infty}\left|\frac{r_{4(p+1)}}{r_{4p}}\right|\right)^{-1/4} \approx \frac{2.62206}{\|\mathcal{K}\|^{1/2}},\\
\mathcal{S}_{\text{odd}}: &\quad |\xi| e^{\frac{3}{2}ck|\xi|} < \left(\|\mathcal{K}\|^2 \lim_{p\to\infty}\left|\frac{r_{2(2(p+1)+1)}}{r_{2(2p+1)}}\right|\right)^{-1/4} \approx \frac{2.62206}{\|\mathcal{K}\|^{1/2}},
\end{aligned}$$

where we used what we know of the ODE series coefficients to determine the limit on the right-hand side. It follows that since $\mathcal{S}_{\text{even}}$ and $\mathcal{S}_{\text{odd}}$ both converge with the same radius of convergence, $A(x,\tau)$ must also converge with this radius. Moreover, since $A(x,\tau) = |a_1(x,\tau)| + |a_2(x,\tau)| + |a_3(x,\tau)|$, we know that $a_m(x,\tau)$ for $m = 1,2,3$ must converge absolutely at least under the same condition, or possibly under a less restrictive condition.

To summarize, we found that the general solution of the three-wave ODEs when $H = 0$, $K_2 = K$ and $K_3 = 2K$ converges when

$$|\tau - \tau_0| < R_{\text{ODE}}, \tag{5.143}$$

where

$$R_{\text{ODE}} = \frac{2\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4\sqrt{2}\,\pi^{1/2}\cdot|K|^{1/2}} \approx \frac{2.62206}{|K|^{1/2}}. \tag{5.144}$$

Meanwhile, we determined that the analagous near-general solution of the three-wave PDEs with constant phases, $\mathcal{K}_2(x) = \mathcal{K}(x)/2$, and $\mathcal{K}_3(x) = \mathcal{H}(x) = 0$, converges when

$$e^{\frac{3}{2}ck|\tau-\tau_0|}|\tau - \tau_0| < R_{\text{ODE}}, \tag{5.145}$$

where we replace $|K|$ with $\|\mathcal{K}\|_\infty$ in R_{ODE}.

As usual, we compare (5.143) with (5.145). Similar to Case 1 and Case 2(i), the two convergence conditions (5.143) and (5.145) for the ODE series solution and the PDE series solution, respectively, differ only by an exponential factor, $e^{\frac{3}{2}ck|\tau-\tau_0|}$. This is a known, dimensionless factor that causes the radius of convergence of the PDE solution to be smaller than that of the ODE solution. It indicates that we are losing very little information in moving from the ODEs to the

PDEs. Moreover, the loss of information can be attributed to the growth rate, k, of the derivatives of $\mathcal{K}(x)$, and the largest group velocity (in magnitude), c, of the three waves.

5.6 Radius of convergence: Case 2(iii)

In this section, we consider Case 2(iii), in which the phases are constant and $\mathcal{H}(x) = 0$, while $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$ are nonzero in the solution of the three-wave PDEs. Similarly, in the solution of the three-wave ODEs, we assume that $H = 0$, and K_2 and K_3 are nonzero. In order to find the radius of convergence of the Laurent series solution of the three-wave PDEs in this case, we must deviate slightly from the set of steps outlined in Section 5.2. The idea of the proof is the same, in that we use the radius of convergence of the ODE series solution in order to determine the radius of convergence of the PDE series solution. However, this case is more complicated than the previous cases, which forces us to take a slightly different approach.

STEP 1: We begin as usual by finding the series solution of the three-wave ODEs when $H = 0$ and $K_2, K_3 \neq 0$ using (4.29)-(4.31). The first few terms of the series for $a_m(\tau)$, $m = 1, 2, 3$ are given by

$$a_1(\tau) = \frac{e^{i\psi_1}}{\xi}\Bigg[1 + \frac{\sigma}{6}(K_2 + K_3)\xi^2 + \frac{1}{360}(7K_2^2 - 22K_2K_3 + 7K_3^2)\xi^4 + \frac{\sigma}{15120}(31K_2^3 - 15K_2^2K_3 - 15K_2K_3^2 + 31K_3^3)\xi^6 + \cdots\Bigg], \tag{5.146a}$$

$$a_2(\tau) = \frac{e^{i\psi_2}}{\xi}\Bigg[1 + \frac{\sigma}{6}(K_3 - 2K_2)\xi^2 + \frac{1}{360}(-8K_2^2 + 8K_2K_3 + 7K_3^2)\xi^4 + \frac{\sigma}{15120}(-32K_2^3 + 48K_2^2K_3 - 78K_2K_3^2 + 31K_3^3)\xi^6 + \cdots\Bigg], \tag{5.146b}$$

$$a_3(\tau) = \frac{e^{i\psi_3}}{\xi}\Bigg[1 + \frac{\sigma}{6}(K_2 - 2K_3)\xi^2 + \frac{1}{360}(7K_2^2 + 8K_2K_3 - 8K_3^2)\xi^4 + \frac{\sigma}{15120}(31K_2^3 - 78K_2^2K_3 + 48K_2K_3^2 - 32K_3^3)\xi^6 + \cdots\Bigg], \tag{5.146c}$$

Note that since $H = 0$ in this case, $a_m(\tau)$ has the form (5.66),

$$a_m(\tau) = \frac{e^{i\psi_m}}{\xi}\sum_{n=0}^{\infty} A_{2n}^m\,\xi^{2n}, \tag{5.147}$$

for $m = 1, 2, 3$, where A_{2n}^m is real and defined in (5.67) for $n \geq 2$.

We can immediately see from (5.146) that this case is more difficult than the previous cases due to the presence of two free constants, K_2 and K_3, in the series coefficients of $a_m(\tau)$. In Case 1, we only had to deal with H in the series coefficients (see (5.44)), and in Case 2(i)-2(ii), a particular relationship existed between K_2 and K_3, which allowed us to reduce the series coefficients to functions of a single variable (see (5.65) and (5.119)). It was then straightforward to find the radius of convergence of the series using a simple application of the ratio test. This is no longer possible in (5.146) since each term in the series is a polynomial in K_2 and K_3. Consequently, we introduce a change of variables in an attempt to simplify our calculations.

We introduce the change of variables

$$K_2 = 2\sigma K(1-\alpha), \quad \text{and} \quad K_3 = 2\sigma K(2+\alpha), \tag{5.148}$$

where K and α are real constants with $\alpha \in [-1,1]$ and $K \neq 0$. The restriction on α amounts to a scaling of the series solution. Note that we still have two free constants to choose from, α and K. Moreover, notice that when $\alpha = 0$, we recover Case 2(ii) in which $K_3 = 2K_2$. Additionally, when $\alpha = 1$, we recover Case 2(i) in which $K_2 = 0$ and $K_3 \neq 0$. Finally, observe that under the change of variables (5.148), we have that $A_2^1 = K$, $A_2^2 = \alpha K$, and $A_2^3 = -(1+\alpha)K$.

Using (5.148), the first few terms of (5.147) become

$$a_1(\tau) = \frac{e^{i\psi_1}}{\xi}\left[1 + K\xi^2 + \frac{K^2}{10}(-1+4\alpha+4\alpha^2)\,\xi^4 + \frac{K^3}{70}(7+12\alpha+12\alpha^2)\,\xi^6 + \cdots\right], \tag{5.149a}$$

$$a_2(\tau) = \frac{e^{i\psi_2}}{\xi}\left[1 + \alpha K\xi^2 + \frac{K^2}{10}(4+4\alpha-\alpha^2)\,\xi^4 + \frac{K^3}{70}(12\alpha+12\alpha^2+7\alpha^3)\,\xi^6 + \cdots\right], \tag{5.149b}$$

$$a_3(\tau) = \frac{e^{i\psi_3}}{\xi}\left[1 - K(1+\alpha)\,\xi^2 - \frac{K^2}{10}(1+6\alpha+\alpha^2)\,\xi^4 - \frac{K^3}{70}(7+9\alpha+9\alpha^2+7\alpha^3)\,\xi^6 + \cdots\right]. \tag{5.149c}$$

Observe that $a_m(\tau)$ can be written

$$a_m(\tau) = \frac{e^{i\psi_m}}{\xi}\sum_{n=0}^{\infty}\sum_{p=0}^{n} d_{2n,p}^{m}\alpha^p K^n \xi^{2n}, \tag{5.150}$$

where $d_{n,p}^m$ are real constants that could be zero. If we try to apply the ratio test to one of the series in (5.149), we find that the series converges when

$$\lim_{n\to\infty}\left|\frac{\sum_{p=0}^{n+1} d^m_{2(n+1),p}\alpha^p K^{n+1}\xi^{2(n+1)}}{\sum_{p=0}^{n} d^m_{2n,p}\alpha^p K^n\xi^{2n}}\right| < 1.$$

Rearranging, the condition for convergence becomes

$$|\xi| < \frac{1}{|K|^{1/2}}\left(\lim_{n\to\infty}\left|\frac{\sum_{p=0}^{n+1} d^m_{2(n+1),p}\alpha^p}{\sum_{p=0}^{n} d^m_{2n,p}\alpha^p}\right|\right)^{-1/2}.$$

Assuming the limit on the right-hand side exists, we see that the radius of convergence should go like $1/|K|^{1/2}$. This is the same form as the radius of convergence of the ODE series solution in Case 2(i) and 2(ii), given in (5.69) and (5.144), respectively. However, computing the limit is less straightforward since it involves a ratio of polynomials. Moreover, if we try to use the Weierstrass general solution of the three-wave ODEs in order to determine the radius of convergence of (5.146) or (5.149), we find that there is not a nice analytic formula for the half-periods, λ_1 and λ_2, of the Weierstrass function when K_2 and K_3 are arbitrary. Case 1 and Case 2(i) were special in that one of the elliptic invariants, g_2 and g_3, was zero in each case, which makes the relationship between the invariants and the half-periods simple. When g_2 and g_3 are both nonzero, there is not an explicit formula for $\lambda_j = \lambda_j(g_2, g_3)$, $j = 1, 2$.

The best we can do for now is to bound (5.149) using the triangle inequality as follows,

$$|a_1(\tau)| = \frac{1}{|\xi|}\left[1 + |K||\xi|^2 + \frac{9}{10}|K|^2|\xi|^4 + \frac{31}{70}|K|^3|\xi|^6 + \cdots\right], \tag{5.151a}$$

$$|a_2(\tau)| = \frac{1}{|\xi|}\left[1 + |K||\xi|^2 + \frac{9}{10}|K|^2|\xi|^4 + \frac{31}{70}|K|^3|\xi|^6 + \cdots\right], \tag{5.151b}$$

$$|a_3(\tau)| = \frac{1}{|\xi|}\left[1 + 2|K||\xi|^2 + \frac{4}{5}|K|^2|\xi|^4 + \frac{16}{35}|K|^3|\xi|^6 + \cdots\right]. \tag{5.151c}$$

A straightforward numerical application of the ratio test tells us that each of the functions converges absolutely approximately when

$$|\xi| < \frac{1.2826}{|K|^{1/2}}. \tag{5.152}$$

Next, observe the following. Rearranging the relations in (5.148), we have that

$$\frac{1}{|K|} = \frac{2|1-\alpha|}{|K_2|} \le \frac{4}{|K_2|}, \quad \text{and} \quad \frac{1}{|K|} = \frac{2|2+\alpha|}{|K_3|} \le \frac{6}{|K_3|}.$$

This implies that

$$\frac{1}{|K|} \leq \min\left\{\frac{4}{|K_2|}, \frac{6}{|K_3|}\right\}.$$

Consequently, the convergence condition in (5.152) becomes

$$\begin{aligned} |\xi| &< 1.2826 \min\left\{\sqrt{\frac{4}{|K_2|}}, \sqrt{\frac{6}{|K_3|}}\right\} \\ &\approx \min\left\{\frac{2.57}{|K_2|^{1/2}}, \frac{\pi}{|K_3|^{1/2}}\right\}. \end{aligned} \tag{5.153}$$

Consider the results of Table 5.1. For instance, consider Case 2(ii), in which $K_3 = 2K_2$. It follows that $4/|K_2| > 6/|K_3|$. As a result, the convergence criteria in (5.153) becomes

$$|\xi| < 1.2826\sqrt{\frac{6}{|K_3|}} \approx \frac{\pi}{\sqrt{2|K_2|}} \approx \frac{2.22}{|K_2|^{1/2}}.$$

This is a slightly smaller radius (by about 15%) than the one we found in Section 5.5, which was $R_{\mathrm{ODE}} \approx 2.62/|K_2|^{1/2}$. It is not surprising, however, that we lose some information by computing the radius of convergence of the ODE solution using (5.151). The bound in (5.151) is not tight due to our use of the triangle inequality. Nevertheless, we have still managed to find a good estimate for where the general solution of the three-wave ODEs converges in the case where $H = 0$ and K_2 and K_3 are nonzero.

Finally, notice that as $K_2 \to 0$, we have that $4/|K_2| > 6/|K_3|$. Thus for small values of K_2, we have

$$|\xi| < \frac{\pi}{|K_3|^{1/2}},$$

which is the same as the radius of convergence found in Table 5.1 for Case 2(i), in which $K_2 = 0$.

Later, it turns out to be useful to formulate (5.146) and (5.149) in yet another way, which uses the known Laurent series expansion of the Weierstrass function. We return to this in Step 5.

Step 2: Following the usual procedure, now we construct the formal series solution of the three-wave PDEs when $\mathcal{H}(x) = 0$ and $\mathcal{K}_2(x), \mathcal{K}_3(x) \neq 0$ using (5.24)-(5.26). Moreover, recall that when

$\mathcal{H}(x) = 0$, the recursion relation in (5.26) reduces to (5.83). The first few terms of the series are

$$\begin{aligned} a_1(x,\tau) = \frac{e^{i\psi_1}}{\xi}\Bigg\{ & 1 + \mathcal{K}_2\xi^2 + \frac{1}{4}\Big[-(3c_1+c_3)\mathcal{K}_2' + (c_2-c_3)\mathcal{K}_3'\Big]\xi^3 \\ & + \frac{1}{40}\Big[\left(12c_1^2 + c_1c_2 + 5c_1c_3 - c_2c_3 + 3c_3^2\right)\mathcal{K}_2'' \\ & + (c_3-c_2)(4c_1+3(c_2+c_3))\mathcal{K}_3'' + 16\mathcal{K}_2\mathcal{K}_3 - 4\mathcal{K}_2^2 + 16\mathcal{K}_3^2\Big]\xi^4 + \cdots\Bigg\}, \end{aligned} \tag{5.154a}$$

$$\begin{aligned} a_2(x,\tau) = \frac{e^{i\psi_2}}{\xi}\Bigg\{ & 1 + \mathcal{K}_3\xi^2 + \frac{1}{4}\Big[(c_1-c_3)\mathcal{K}_2' - (3c_2+c_3)\mathcal{K}_3'\Big]\xi^3 \\ & + \frac{1}{40}\Big[(c_3-c_1)(3c_1+4c_2+3c_3)\mathcal{K}_2'' \\ & + \left(c_1c_2 - c_1c_3 + 12c_2^2 + 5c_2c_3 + 3c_3^2\right)\mathcal{K}_3''(x) + 16\mathcal{K}_2\mathcal{K}_3 + 16\mathcal{K}_2^2 - 4\mathcal{K}_3^2\Big]\xi^4 + \cdots\Bigg\}, \end{aligned} \tag{5.154b}$$

$$\begin{aligned} a_3(x,\tau) = \frac{e^{i\psi_3}}{\xi}\Bigg\{ & 1 - (\mathcal{K}_2+\mathcal{K}_3)\xi^2 + \frac{1}{4}\Big[(c_1+3c_3)\mathcal{K}_2' + (c_2+3c_3)\mathcal{K}_3'\Big]\xi^3 \\ & + \frac{1}{40}\Big[-\left(3c_1^2 - c_1(c_2-5c_3) + c_3(c_2+12c_3)\right)\mathcal{K}_2'' \\ & - \left(c_1(c_3-c_2) + 3c_2^2 + 5c_2c_3 + 12c_3^2\right)\mathcal{K}_3'' - 4\left(6\mathcal{K}_2\mathcal{K}_3 + \mathcal{K}_2^2 + \mathcal{K}_3^2\right)\Big]\xi^4 + \cdots\Bigg\}. \end{aligned} \tag{5.154c}$$

Note that even low order terms are complicated due to the presence of two free functions. As more terms are computed, higher derivatives of $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$ appear, as well as nonlinear terms in $\mathcal{K}_2(x)$, $\mathcal{K}_3(x)$, and their derivatives. In order to gain traction in our convergence proof, we restrict our attention to certain families of functions for $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$.

STEP 3: Following (5.34), we assume that

$$\left\|\frac{d^n}{dx^n}\mathcal{K}_j(x)\right\|_\infty \le k_j^n\|\mathcal{K}_j\|_\infty, \quad j = 2,3. \tag{5.155}$$

where k_j is a real positive constant for $j = 2,3$. Moreover, define

$$k_M = \max\{k_2, k_3\} \quad \text{and} \quad \mathcal{K}_M = \max\{\|\mathcal{K}_2\|_\infty, \|\mathcal{K}_3\|_\infty\}.$$

It follows that

$$\left\|\frac{d^n}{dx^n}\mathcal{K}_j(x)\right\|_\infty \le k_M^n\mathcal{K}_M, \quad j = 2,3. \tag{5.156}$$

Step 4: Next, we use (5.156) and the definition of c in (5.31) in order to bound the series in (5.154). We obtain, as usual, a series of the form

$$|a_m(x,\tau)| \leq \frac{1}{|\xi|}\left[1+\sum_{p=1}^{\infty}\sum_{n=2p}^{\infty} q_{n,p}^m (ck_M)^{n-2p}|\xi|^n\right], \tag{5.157}$$

where $q_{n,p}^m$ are nonnegative constants for $m = 1, 2, 3$.

As with Case 2(i) and 2(ii), it is difficult to find an exact formula for the coefficients $q_{n,p}^m$. Instead we bound the sum

$$A(x,\tau) = \sum_{m=1}^{3} |a_m(x,\tau)|\,. \tag{5.158}$$

We then find the radius of convergence of $A(x,\tau)$, and infer that $a_m(x,\tau)$ must converge absolutely within at least that radius of convergence for $m = 1, 2, 3$.

Step 5: We follow the procedure outlined in Step 4 of Section 5.4 to find that

$$A(x,\tau) \leq \frac{1}{|\xi|}\left[3+\sum_{p=1}^{\infty}\sum_{n=2p}^{\infty} r_{n,p}\,\mathcal{K}_M^p (ck_M)^{n-2p}|\xi|^n\right], \tag{5.159}$$

where $r_{n,p}$ is defined in (5.92a) for $p > 1$ and for $n > 2p$. In the previous cases, $r_{n,p}$ for $n = 2p$ is defined in (5.92b) as

$$r_{2p,p} = \frac{1}{|K|^p}\left(\left|A_{2p}^1\right| + \left|A_{2p}^2\right| + \left|A_{2p}^3\right|\right).$$

However, in our case, A_{2p}^m is a polynomial in α. Indeed, we know from (5.150) that

$$A_{2p}^m = \sum_{s=0}^{p} d_{2p,s}^m \alpha^s.$$

It follows that

$$\left|A_{2p}^m\right| \leq \sum_{s=0}^{p}\left|d_{2p,s}^m\right|,$$

since $\alpha \in [-1, 1]$. As a result, in the case where $\mathcal{H}(x) = 0$, and $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$ are arbitrary (except for (5.156)), we define

$$r_{2p,p} = r_{2p} = \sum_{m=1}^{3}\sum_{s=0}^{p}\left|d_{2p,s}^m\right|. \tag{5.160}$$

It is shown in Appendix C that if $\mathcal{H}(x) = 0$, then $r_{n,p}$ satisifies

$$r_{n,p} \leq \frac{(3p)^{n-2p}}{(n-2p)!} \sum_{\ell=1}^{p-1} r_{2\ell} r_{2(p-\ell)}, \quad n > 2p,\ p \geq 2. \tag{5.161}$$

Moreover, when $p = 1$, (5.93) gives

$$r_{n,1} \leq \frac{3^{n-2}}{(n-2)!} r_2. \tag{5.162}$$

In Case 2(i) and 2(ii), we also have the bound

$$\sum_{\ell=1}^{p-1} r_{2\ell} r_{2(p-\ell)} \leq (\delta p + \gamma)\, r_{2p}, \quad p \geq 2, \tag{5.163}$$

where δ and γ are nonnegative constants. The combination of (5.161) and (5.163) allowed us to reduce the double sums in (5.80) and (5.129) to a single sum. We then applied the ratio test to the single sum in order to determine where $A(x,\tau)$ converges in those cases. In the present case, however, it is difficult to prove (5.163). In fact, proving the bound is prohibitively complicated. Instead, we seek an alternative bound on

$$\sum_{\ell=1}^{p-1} r_{2\ell} r_{2(p-\ell)}, \quad p \geq 2. \tag{5.164}$$

To be precise, we seek to bound (5.164), not in terms of r_{2p}, which involves the coefficients of the series for $a_m(\tau)$, but rather in terms of the coefficients of the series for $|a_m(\tau)|^2$.

To that end, recall the Weierstrass general solution of the three-wave ODEs in Chapter 3. In particular, recall (3.54), which gives $|a_m(\tau)|$ in terms of the Weierstrass elliptic function for $m = 1, 2, 3$ as

$$|a_m(\tau)|^2 = \sigma_m \left[\sigma \wp(\tau - \tau_0; g_2, g_3) + C_m\right], \tag{5.165}$$

where $C_m = 2A_2^m$. That is, we have

$$C_1 = \frac{K_2 + K_3}{3}, \tag{5.166}$$

$$C_2 = \frac{K_3 - 2K_2}{3}, \tag{5.167}$$

$$C_3 = \frac{K_2 - 2K_3}{3}. \tag{5.168}$$

Moreover, recall the Laurent series expansion of $\wp(\xi; g_2, g_3)$ in (4.33),

$$\wp(\xi; g_2, g_3) = \frac{1}{\xi^2} + \sum_{n=2}^{\infty} c_{2n}\xi^{2n-2}, \tag{5.169}$$

where $\xi = \tau - \tau_0, c_4 = g_2/20$, $c_6 = g_3/28$, and c_{2n} is determined by

$$c_{2n} = \frac{3}{(2n+1)(n-3)} \sum_{j=2}^{n-2} c_{2j}c_{2(n-j)}, \quad n \geq 4. \tag{5.170}$$

Without loss of generality, assume we are in the explosive regime with $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$. It follows from (5.165) and (5.169) that $|a_m(\tau)|^2$ can be expanded as

$$|a_m(\tau)|^2 = \frac{1}{\xi^2}\left[1 + \sum_{n=1}^{\infty} c_{2n}^m \xi^{2n}\right], \tag{5.171}$$

where $c_2^m = C_m$ and $c_{2n}^1 = c_{2n}^2 = c_{2n}^3 \equiv c_{2n}$ for $n \geq 2$, with c_{2n} defined in (5.170). We want to bound r_{2n} in (5.164) using the coefficients c_{2n} in (5.171).

Recall that in Step 1 of this section, we made the change of variables (5.148). Under this change of variables, $A_2^1 = K$, $A_2^2 = \alpha K$, and $A_2^3 = -(1+\alpha)K$. As a result, using $c_2^m = C_m = 2A_2^m$, we have

$$c_2^1 = 2K, \quad c_2^2 = 2\alpha K, \quad \text{and} \quad c_2^3 = -2(1+\alpha)K. \tag{5.172}$$

Furthermore, under the same change of variables, we have that the elliptic invariants g_2 and g_3 become

$$g_2 = 16K^2(1 + \alpha + \alpha^2) \quad \text{and} \quad g_3 = 32\sigma K^3(\alpha + \alpha^2).$$

Consequently, we have

$$c_4 = \frac{g_2}{20} = \frac{4}{5}K^2(1 + \alpha + \alpha^2) \quad \text{and} \quad c_6 = \frac{g_3}{28} = \frac{8}{7}K^3(\alpha + \alpha^2), \tag{5.173}$$

where we set $\sigma = 1$ in the formula for g_3. In particular, we notice from (5.170) and (5.172)-(5.173) that c_{2n}^m is always an nth degree (or less) polynomial in α, and we write $c_{2n}^m = c_{2n}^m(\alpha)$.

We introduce the notation

$$c_{2n}^m(\alpha) = K^n\left[h_{2n,0}^m + h_{2n,1}^m\alpha + h_{2n,2}^m\alpha^2 + \cdots + h_{2n,n}^m\alpha^n\right] = \sum_{s=0}^{n} K^n h_{2n,s}^m \alpha^s. \tag{5.174}$$

Then define

$$f_{2n}^m(\alpha) = \sum_{s=0}^{n} h_{2n,s}^m \alpha^s, \quad n \geq 1, \tag{5.175}$$

so that

$$c_{2n}^m(\alpha) = K^n f_{2n}^m(\alpha).$$

As with c_{2n}^m, when $n \geq 2$, we have that $f_{2n}^1 = f_{2n}^2 = f_{2n}^3 \equiv f_{2n}$. It follows from (5.170) that $f_{2n}(\alpha)$ can be computed via

$$f_{2n}(\alpha) = \frac{3}{(2n+1)(n-3)} \sum_{j=2}^{n-2} f_{2j}(\alpha) f_{2(n-j)}(\alpha), \quad n \geq 4, \tag{5.176}$$

using $f_4 = g_2/20$ and $f_6 = g_3/28$.

When $n = 1$, (5.172) and (5.174) imply that

$$h_{2,0}^1 = h_{2,1}^2 = -h_{2,0}^3 = -h_{2,1}^3 = 2, \quad \text{and} \quad h_{2,1}^1 = h_{2,0}^2 = 0. \tag{5.177}$$

Additionally, for $n \geq 2$, it is clear from (5.170) and (5.173) that $h_{2n,s}^1 = h_{2n,s}^2 = h_{2n,s}^3 \equiv h_{2n,s} \geq 0$. Furthermore, observe that since $h_{2n,s} \geq 0$ for $n \geq 2$, we have

$$|f_{2n}(\alpha)| \leq f_{2n}(1) = \sum_{s=0}^{n} h_{2n,s}, \quad n \geq 2.$$

Later, we need the notation

$$t_{2n} = \sum_{m=1}^{3} \sum_{s=0}^{n} \left| h_{2n,s}^m \right|, \quad n \geq 1. \tag{5.178}$$

It follows from (5.177) that $t_2 = 8$. Moreover, since $h_{2n,s} \geq 0$ for $n \geq 2$, (5.175) gives

$$t_{2n} = 3 f_{2n}(1). \tag{5.179}$$

We now want to bound (5.164) in terms of t_{2p}. To begin, we can see from Figure 5.1 that

$$r_{2n} \leq t_{2n}, \tag{5.180}$$

for $n \geq 1$, where r_{2n} is defined in (5.160). In fact, it is clear from Figure 5.2 that both r_{2n} and t_{2n} behave like $Ce^{-n/2}$ for sufficiently large n, where C is a real positive constant. As a result, as $n \to \infty$, we conjecture that r_{2n} and t_{2n} approach zero exponentially fast, with $r_{2n} \leq t_{2n}$.

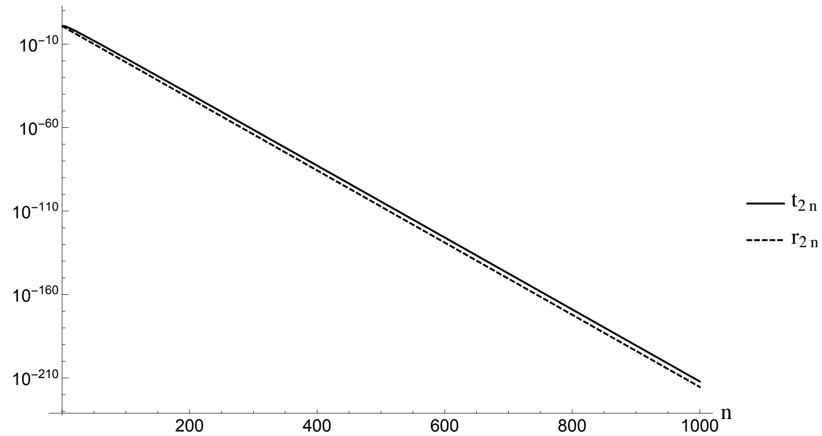


Figure 5.1: A log plot of r_{2n} and t_{2n} up to $n = 1000$, with r_{2n} defined in (5.160) and t_{2n} defined in (5.178). Note that $r_{2n} \leq t_{2n}$ for $n \geq 1$.

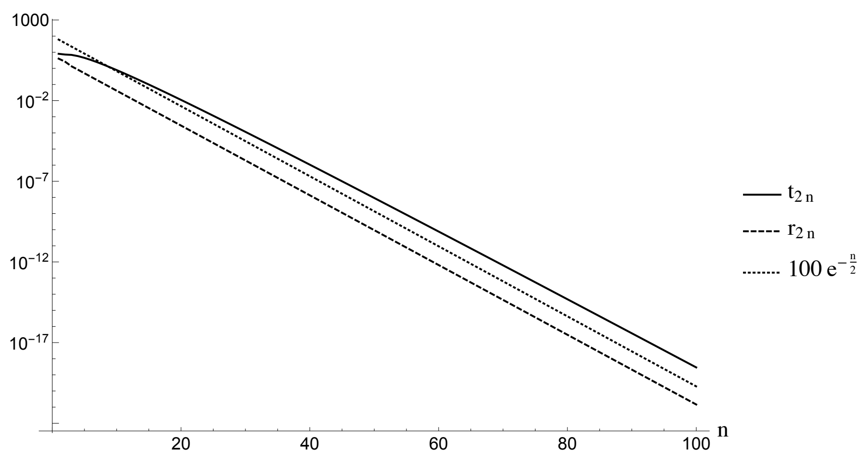


Figure 5.2: A log plot of r_{2n}, t_{2n}, and $100e^{-n/2}$ up to $n = 100$, with r_{2n} defined in (5.160) and t_{2n} defined in (5.178). Note that for sufficiently large n, $r_{2n} \leq 100e^{-n/2} \leq t_{2n}$.

Assume that $n \geq 4$. Observe the following, making use of (5.180),

$$
\begin{aligned}
\sum_{j=1}^{n-1} r_{2j} r_{2(n-j)} &\leq \sum_{j=1}^{n-1} t_{2j} t_{2(n-j)} \\
&= 2t_2 t_{2(n-1)} + \sum_{j=2}^{n-2} t_{2j} t_{2(n-j)} \\
&= 16 t_{2(n-1)} + 9 \sum_{j=2}^{n-2} f_{2j}(1) f_{2(n-j)}(1) \\
&= 48 f_{2(n-1)}(1) + 9 \sum_{j=2}^{n-2} f_{2j}(1) f_{2(n-j)}(1) \\
&= 48 f_{2(n-1)}(1) + 3(2n+1)(n-3) f_{2n}(1) \\
&\leq \Big[48 + 3(2n+1)(n-3)\Big] f_{2(n-1)}(1) \\
&= \Big[6n^2 - 15n + 39\Big] f_{2(n-1)}(1), \qquad (5.181)
\end{aligned}
$$

where we used the definitions of f_{2n} and t_{2n} in (5.176) and (5.179), respectively, and the last inequality used the fact that

$$
f_{2n}(1) \leq f_{2(n-1)}(1), \quad n \geq 2. \qquad (5.182)
$$

Note that this fact can be easily proved using the technique of Section 5.4, Step 5 (in particular, see the proofs of (iii)-(iv) on page 115).

Next, consider the quantity

$$
\left(\beta n^2 + \gamma n + \delta\right) t_{2(n-1)} - \Big[6n^2 - 15n + 39\Big] f_{2(n-1)}(1), \quad n \geq 4,
$$

for some real constants β, γ, and δ. Observe the following

$$
\begin{aligned}
&\left(\beta n^2 + \gamma n + \delta\right) t_{2(n-1)} - \Big[6n^2 - 15n + 39\Big] f_{2(n-1)}(1) \\
&\qquad = \Big[3\left(\beta n^2 + \gamma n + \delta\right) - 6n^2 + 15n - 39\Big] f_{2(n-1)}(1) \\
&\qquad = \Big[(3\beta - 6)n^2 + (3\gamma + 15)n + (3\delta - 39)\Big] f_{2(n-1)}(1),
\end{aligned}
$$

where we used the fact that $t_{2n} = 3f_{2n}(1)$ for $n \geq 2$. This quantity is clearly nonnegative if we choose $\beta = 2$, $\gamma = 0$, and $\delta = 13$. As a result, we know that

$$
(2n^2 + 13) t_{2(n-1)} \geq (6n^2 - 15n + 39) f_{2(n-1)}(1), \qquad (5.183)
$$

for $n \geq 4$. Consequently, combining (5.183) with (5.181), we find that

$$\sum_{j=1}^{n-1} r_{2j} r_{2(n-j)} \leq (2n^2+13) t_{2(n-1)}, \quad n \geq 4.$$

The inequality (5.6) can also be verified for $n = 2$ and $n = 3$ directly.

We now have a bound for the sum in (5.164). Combined with (5.161), we find that

$$r_{n,p} \leq \frac{(3p)^{n-2p}}{(n-2p)!} (2p^2+13) t_{2(p-1)}, \quad n \geq 2p,\ p \geq 1. \tag{5.184}$$

Note that the result for $p = 1$ follows from (5.162) if we define $t_0 = r_2$.

Returning to the original problem, we were investigating the convergence of the series solution of the three-wave PDEs. In particular, we seek the radius of convergence of $A(x,\tau)$ in (5.159). Applying (5.184) to (5.159) and following the procedure of the previous sections yields

$$\begin{aligned}
A(x,\tau) &\leq \frac{1}{|\xi|}\left[3 + \sum_{p=1}^{\infty}\sum_{n=2p}^{\infty} r_{n,p}\, \mathcal{K}_M^p (ck_M)^{n-2p} |\xi|^n\right] \\
&\leq \frac{1}{|\xi|}\left[3 + \sum_{p=1}^{\infty}\sum_{n=2p}^{\infty} \frac{(3p)^{n-2p}}{(n-2p)!}\,(2p^2+13)\, t_{2(p-1)} \mathcal{K}_M^p (ck_M)^{n-2p}\, |\xi|^n\right] \\
&\leq \frac{1}{|\xi|}\left[3 + \sum_{p=1}^{\infty} (2p^2+13)\, t_{2(p-1)}\, \mathcal{K}_M^p\, |\xi|^{2p} e^{3ck_M p|\xi|}\right].
\end{aligned}$$

Step 6: The series for $A(x,\tau)$ converges by the ratio test if

$$\lim_{p\to\infty}\left|\frac{(2(p+1)^2+13)\, t_{2p}\, \mathcal{K}_M^{p+1}\, |\xi|^{2(p+1)} e^{3ck_M(p+1)|\xi|}}{(2p^2+13)\, t_{2(p-1)}\, \mathcal{K}_M^p\, |\xi|^{2p} e^{3ck_M p|\xi|}}\right| < 1.$$

Simplifying, the convergence condition becomes

$$\mathcal{K}_M\, |\xi|^2 e^{3ck_M|\xi|} \lim_{p\to\infty}\left|\frac{t_{2p}}{t_{2(p-1)}}\right| < 1,$$

where we used the fact that $\lim_{p\to\infty}(2(p+1)^2+13)/(2p^2+13) = 1$. Simplifying further, we obtain the condition

$$|\xi| e^{\frac{3}{2}ck_M|\xi|} < \left(\mathcal{K}_M \lim_{p\to\infty}\left|\frac{t_{2p}}{t_{2(p-1)}}\right|\right)^{-1/2}.$$

Numerically, we can see that

$$\lim_{p\to\infty}\left|\frac{t_{2p}}{t_{2(p-1)}}\right| \approx 0.608,$$

which gives

$$\left(\lim_{p\to\infty}\left|\frac{t_{2p}}{t_{2(p-1)}}\right|\right)^{-1/2} \approx 1.2826.$$

As a result, we know that the series (5.159) converges approximately when

$$|\tau_0-\tau|\, e^{\frac{3}{2}ck_M|\tau_0-\tau|} < \frac{1.28}{\mathcal{K}_M^{1/2}} = \frac{1.28}{\left[\max\left\{\|\mathcal{K}_2\|, \|\mathcal{K}_3\|\right\}\right]^{1/2}}. \tag{5.185}$$

Consider the results summarized in Table 5.1. For instance, consider Case 2(i), in which $\mathcal{K}_2(x) = \mathcal{K}_3(x) = \mathcal{K}(x)/6$. Substituting into (5.185), we find that the convergence condition for the PDE series solution becomes

$$|\tau_0-\tau|\, e^{\frac{3}{2}ck|\tau_0-\tau|} < \frac{1.28\sqrt{6}}{\|\mathcal{K}\|^{1/2}} \approx \frac{\pi}{\|\mathcal{K}\|^{1/2}}.$$

This is exactly the condition we determined in Case 2(i). Next, consider Case 2(ii), in which $\mathcal{K}_2(x) = \mathcal{K}(x)/2$ and $\mathcal{K}_3(x) = 0$. In this case, (5.185) becomes

$$|\tau_0-\tau|\, e^{\frac{3}{2}ck|\tau_0-\tau|} < \frac{1.28\sqrt{2}}{\|\mathcal{K}\|^{1/2}} \approx \frac{1.81}{\|\mathcal{K}\|^{1/2}}.$$

The radius of convergence we determined in Section 5.5 for this case was approximately $2.62/\|\mathcal{K}\|^{1/2}$. As a result, we did lose some information by using (5.185) here, but this is not surprising. We had to make several bounds on the series for $a_m(x,\tau)$ in order to derive (5.185), and most of those bounds were not tight due to our use of the triangle inequality. As a result, we can expect some loss of information in the more general convergence criteria, (5.185), which applies to any case in which $\mathcal{H}(x) = 0$, and $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$ are arbitrary (as long as (5.156) is satisfied).

To summarize, we found that the general solution of the three-wave ODEs when $H = 0$ and $K_2, K_3 \neq 0$ converges when

$$|\tau-\tau_0| < R_{\mathrm{ODE}}, \tag{5.186}$$

where, following (5.153),

$$
\begin{aligned}
R_{\mathrm{ODE}} &= \frac{1.283}{|K|^{1/2}} \\
&= \min\left\{\frac{2.565}{|K_2|^{1/2}}, \frac{\pi}{|K_3|^{1/2}}\right\},
\end{aligned}
\tag{5.187}
$$

with K defined in (5.148). Meanwhile, we determined that the analagous near-general solution of the three-wave PDEs with constant phases, $\mathcal{H}(x) = 0$, and $\mathcal{K}_2(x), \mathcal{K}_3(x) \neq 0$, converges when

$$
e^{\frac{3}{2}ck_M|\tau-\tau_0|}|\tau - \tau_0| < R_{\mathrm{PDE}}, \tag{5.188}
$$

where

$$
R_{\mathrm{PDE}} = \frac{1.283}{\max\left\{\|\mathcal{K}_2\|^{1/2}, \|\mathcal{K}_3\|^{1/2}\right\}}, \tag{5.189}
$$

and $\|\cdot\| = \|\cdot\|_\infty$.

The two convergence conditions (5.143) and (5.145) for the ODE series solution and the PDE series solution, respectively, differ only by an exponential factor, $e^{\frac{3}{2}ck_M|\tau-\tau_0|}$. This is a known, dimensionless factor that causes the radius of convergence of the PDE solution to be smaller than that of the ODE solution. It indicates that we are losing very little information in moving from the ODEs to the PDEs. Moreover, the loss of information can be attributed to the maximum growth rate, k_M, of the derivatives of $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$, and the largest group velocity (in magnitude), c, of the three waves. Finally, if we try to apply the convergence criteria in (5.186)-(5.189) to previously studied cases like Case 2(ii), we find another small loss of information. Again, this is not surprising since we had to make several bounds on the series for $a_m(\tau)$ and $a_m(x,\tau)$ in order to derive (5.186)-(5.189), and most of these bounds were not tight.

5.7 The nonexplosive case

Our work in this chapter has focused so far on the explosive regime in which $\sigma_1 = \sigma_2 = \sigma_3$. In this section, we discuss the nonexplosive case of the three-wave PDEs. This work of this section is less rigorous than the rest of the chapter, but it gives some insight into the bounded solutions of the three-wave PDEs. This section follows closely from the end of Section 4.4, which gives a possible solution of the three-wave ODEs in the nonexplosive regime.

Motivated by (4.67), we seek a solution of the three-wave PDEs in the nonexplosive regime of the form

$$a_m(x,\tau) = e^{i\psi_m} T_m(x,\tau), \quad \text{where} \quad T_m(x,\tau) = \frac{\alpha_m}{\xi} \sum_{n=0}^{\infty} \mathcal{A}_n^m(x) \xi^n, \tag{5.190}$$

for $m = 1, 2, 3$, and $\xi = \tau - \tau_0$. We assume that $T_m(x,\tau)$ is real-valued for real x and τ, and that $\mathrm{Im}\,(\tau_0) \neq 0$. Additionally, $\alpha_m \in \mathbb{C}$ is a constant for $m = 1, 2, 3$. Note that we could generalize and assume that $\psi_m = \psi_m(x)$, but for convenience, we assume that ψ_m is constant for $m = 1, 2, 3$.

Suppose that $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. Then under the assumption that $T_m(x,\tau)$ is real for real x and τ, if we substitute (5.190) into the three-wave PDEs, we find that for $m = 1, 2, 3$, $\mathcal{A}_0^m(x) = 1$ and $\mathcal{A}_1^m(x) = 0$. Moreover, we have

$$\alpha_1 = 1, \quad \text{and} \quad \alpha_2 = \alpha_3 = -i,$$

as well as $\psi_1 + \psi_2 + \psi_3 = 2n\pi$, $n \in \mathbb{Z}$. Finally, we have that

$$\mathcal{A}_2^1(x) \equiv \mathcal{K}_2(x), \quad \mathcal{A}_2^2(x) \equiv \mathcal{K}_3(x), \quad \text{and} \quad \mathcal{A}_2^3(x) = -\left(\mathcal{K}_2(x) + \mathcal{K}_3(x)\right),$$

where $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$ are free functions of x. The remaining coefficients in the series are determined by (5.83).

As with the ODE series solution in the nonexplosive case, if we write α_m in complex exponential form for $m = 1, 2, 3$, then we obtain

$$a_1(x,\tau) = \frac{e^{i\psi_1}}{\xi} \sum_{n=0}^{\infty} \mathcal{A}_n^1(x) \xi^n, \quad \text{and} \quad a_j(x,\tau) = \frac{e^{i(\psi_j - \pi/2)}}{\xi} \sum_{n=0}^{\infty} \mathcal{A}_n^j(x) \xi^n, \tag{5.191}$$

for $j = 2, 3$. It follows that the sum of the phases becomes $\psi_1 + \psi_2 + \psi_3 - \pi = (2n-1)\pi$, which recovers (5.6) since $\sigma = 1$ when $\sigma_1 = -\sigma_2 = -\sigma_3 = 1$. As a result, the solution (5.191) is equivalent to the formal Laurent series solution of the three-wave PDEs described in (5.24)-(5.26) when $\mathcal{H}(x) = 0$ and the phases are constant. The first few terms of the series are given in (5.154). The only difference is that τ_0 is now assumed to be complex.

The investigation of the radius of convergence for the nonexplosive series solution is identical to that of Sections 5.4-5.6. Note, however, that our derivation of the PDE series solution in the nonexplosive regime is not rigorous, since we have not proven that the solution is indeed real along the real τ-axis. Moreover, in the PDE solution (5.191), it is not clear what values $\mathrm{Im}\,(\tau_0)$ is allowed to take on. In the ODE series solution in the nonexplosive regime, we were able to determine $\mathrm{Im}\,(\tau_0)$ using the Weierstrass general solution. For the PDEs, we do not have this advantage.

Chapter 6

Numerical Verification

The purpose of this chapter is to provide some numerical insight into the Laurent series solution of the three-wave PDEs. We present our PDE series solution in a variety of cases in the explosive regime. In particular, we turn our focus to the numerical verification of the Laurent series, showing that it does indeed converge to the exact PDE solution. We begin by demonstrating that using a relatively small number of terms in the partial sum approximation of the Laurent series is sufficient in practice. Next, we analyze the simplest case of equal group velocities, in which the three-wave PDEs can be transformed back into the three-wave ODEs, whose solutions are exact. Finally, we turn to more complicated regimes, where known solutions do not exist. In order to verify the PDE solution in this case, we consider a numerical solution of the equations using a simple finite difference approach, and verify that known convergence results hold. It should be noted that we are not comparing our solution technique to the performance of other numerical methods. Rather, we are using numerical methods to verify the partial sum approximations of our solution.

6.1 Convergence of partial sums

In Section 5.1, we derived a Laurent series solution of the three-wave PDEs in the explosive regime. The solution involves five free functions of x and one free constant, and is given by (5.24)-(5.26). Although the solution is defined in terms of an infinite sum, here we show that only a small number of terms is required in order to achieve machine precision. This is beneficial, because from

Sections 5.3-5.5, we know that even for the simple cases in which the phases are constant and one or more of the remaining free functions is set to zero, the coefficients in the series still become prohibitively complicated as more terms are computed (see equations (5.44), (5.65), (5.119), and (5.146)).

Consider the PDEs series solution of Section 5.3, in which the phases are constant and $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$. We know from Table 5.1 on page 91, or from equation (5.63), that the radius of convergence of the solution in this case is given by

$$|\tau - \tau_0|\, e^{\frac{1}{3}ck|\tau-\tau_0|} < \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi \|H\|^{1/3}} \approx \frac{3.06}{\|\mathcal{H}\|^{1/3}}, \tag{6.1}$$

where $\|\cdot\| = \|\cdot\|_\infty$.

Suppose that $\mathcal{H}(x)$ is given by

$$\mathcal{H}(x) = 2 + \sin x, \tag{6.2}$$

and that $c_1 = c_2 = c_3 = 1$. Then (6.1) implies that the radius of convergence of the series solution under these conditions is defined by

$$|\tau - \tau_0|\, e^{\frac{1}{3}ck|\tau-\tau_0|} < \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\, 3^{1/3}} \approx 2.12, \tag{6.3}$$

since $\|\mathcal{H}\|_\infty = 3$. Next, we solve the inequality in (6.3) for $|\tau - \tau_0|$. This yields the condition,

$$|\tau - \tau_0| \lesssim 1.35, \tag{6.4}$$

where $\lesssim$ is used to denote the fact that the right-hand side of the inequality is an approximation.

Next, let $\mathcal{S}_N^m(x,\tau)$ denote the Nth partial sum of the series solution for $a_m(x,\tau)$ in (5.24)-(5.26). That is,

$$\mathcal{S}_N^m(x,\tau) = \frac{e^{i\psi_m}}{\xi} \sum_{n=0}^{N} \mathcal{A}_n^m(x)\xi^n, \quad m = 1, 2, 3,$$

where $\xi = \tau - \tau_0$ and $N \geq 1$. With $\mathcal{H}(x)$ defined in (6.2), and $c_1 = c_2 = c_3 = 1$, Figure 6.1 depicts a contour plot of $|\mathcal{S}_{100}^1(x,\tau)|$ when $\psi_1 = 0$ and $\tau_0 = 1$. We used the domain $x \in [0, 4\pi]$ and $\tau \in [-0.35, 2.35]$, so that τ lies within the radius of convergence defined in (6.4). The figure shows the fixed pole at $\tau_0 = 1$, and the small variations in x away from $\tau = \tau_0$. Note that plots of the partial sums of $|a_2(x,\tau)|$ and $|a_3(x,\tau)|$ would appear the same since the coefficients in the PDE series when $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$ and $c_1 = c_2 = c_3$ are equal (see (5.51)).

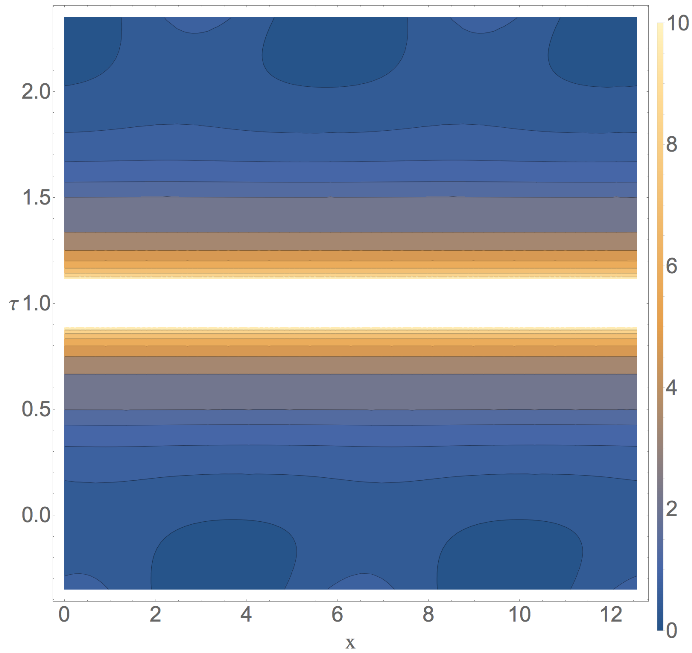


Figure 6.1: Contour plot of $|\mathcal{S}^1_{100}(x,\tau)|$ in the explosive regime with $\tau_0 = 1$, $c_1 = c_2 = c_3 = 1$. The phases, ψ_m are constant for $m = 1, 2, 3$, and $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$. $\mathcal{H}(x)$ is defined in (6.2).

If N is sufficiently large, then $a_m(x,\tau) \approx \mathcal{S}^m_N(x,\tau)$. This motivates us to consider the following error term

$$\mathcal{E}^m_j(x,\tau) = \left| \frac{\mathcal{S}^m_N(x,\tau) - \mathcal{S}^m_j(x,\tau)}{\mathcal{S}^m_N(x,\tau)} \right|, \quad j = 0, 1, 2, \ldots, N-1, \tag{6.5}$$

for $m = 1, 2, 3$. For large N, $\mathcal{E}^m_j(x,\tau)$ is a good estimate of the relative error between the exact PDE series solution, $a_m(x,\tau)$ in (5.24), and its partial sum approximation, $S^m_j(x,\tau)$, for $m = 1, 2, 3$, $j = 0, 1, \ldots, N-1$, and for all x and τ in the domain.

Computationally, we do the following. We discretize the (x,τ) domain, so that

$$\mathbf{x} = (x_1, x_2, \ldots, x_h), \quad \text{and} \quad \boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_k), \tag{6.6}$$

where $x_{p+1} = x_1 + p\Delta x$, $p = 1, 2, \ldots, h-1$, with $\Delta x = (x_h - x_1)/(h-1)$. Similarly, we have $\tau_{q+1} = \tau_1 + q\Delta\tau$, $q = 1, 2, \ldots, k-1$, with $\Delta\tau = (\tau_k - \tau_1)/(k-1)$. We compute the approximate

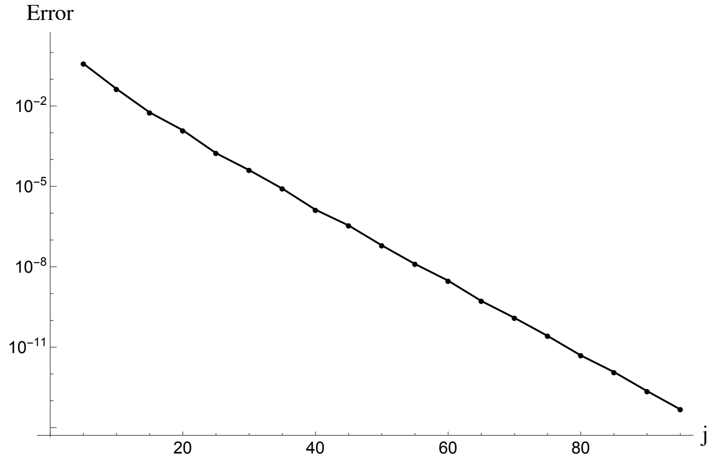


Figure 6.2: $\|\mathcal{S}_{100}^m - \mathcal{S}_j^m\|_{L_R^2}$. The figure is generated using $x \in [0, 4\pi]$ and $\tau \in [-0.35, 2.35]$, so that τ extends all the way out to the edge of the solution's region of convergence.

L^2-norm of the relative error (henceforth referred to as the "L^2-relative error" for convenience), which we denote,

$$\left\|\mathcal{S}_N^m - \mathcal{S}_j^m\right\|_{L_R^2} = \sqrt{\sum_{p=1}^{h}\sum_{q=1}^{k}\left[\mathcal{E}_j^m(x_p, \tau_q)\right]^2 \Delta x\, \Delta t}. \tag{6.7}$$

Suppose that $N = 100$. With $x \in [0, 4\pi]$, $\tau \in [-0.35, 2.35]$, $\Delta x = 0.1$ and $\Delta\tau = 0.051$, we compute $\|\mathcal{S}_N^m - \mathcal{S}_j^m\|_{L_R^2}$ for $j = 1, 2, \ldots, 99$. The results are depicted in Figure 6.2 on a log scale for $m = 1$ (the results for $m = 2$ and $m = 3$ are the same). The L^2-relative error between S_j^1 and S_{100}^1 is monotonically decreasing. Indeed, on the log scale in Figure 6.2, the error decreases linearly with j, meaning it decreases exponentially fast as $j \to N$. In fact, we can make the same type of plot for the error between $\mathcal{S}_j^m$ and $\mathcal{S}_N^m$ for $N > 100$ and $j = 1, 2, \ldots, N-1$. The qualitative results are unchanged. As a result, if we let $\hat{S}^m$ denote the exact solution of the three-wave PDEs (meaning $\hat{\mathcal{S}}^m = a_m(x, \tau)$ with infinite terms in the series (5.24)), then

$$\left\|\hat{\mathcal{S}}^m - \mathcal{S}_j^m\right\|_{L_R^2} = \lim_{N\to\infty}\left\|\mathcal{S}_N^m - \mathcal{S}_j^m\right\|_{L_R^2} \sim \mathcal{O}\left(D\, C^{-j}\right), \tag{6.8}$$

where C and D are real positive constants, and $\|\cdot\|_{L^2}$ is used to denote the L^2-relative error. In Figure 6.2, $C \approx 1.435$ and $D \approx 0.592$.

For many purposes, we should not need to go far beyond the partial sum with a few dozen

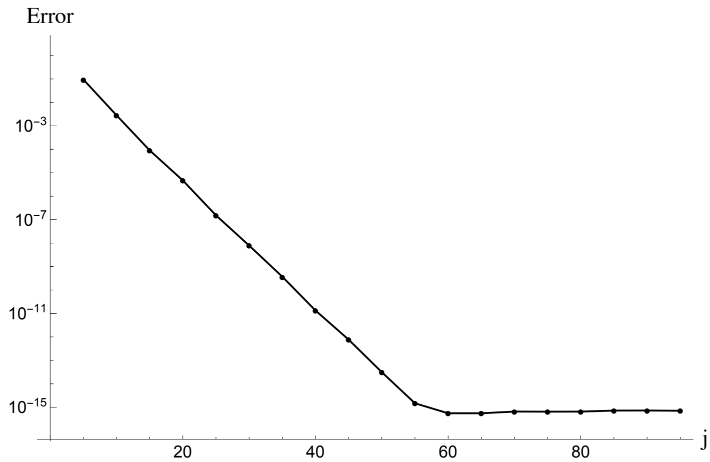


Figure 6.3: $\|\mathcal{S}^m_{100} - \mathcal{S}^m_j\|_{L^2_R}$. The figure is generated using $x \in [0, 4\pi]$ and $\tau \in [0, 2]$, so that τ remains well within the solution's region of convergence.

terms, since beyond this, the difference between the true solution and the partial sum approximation is small. This is especially true when we restrict our attention to the interior of the region of convergence, away from the boundaries. To see this, suppose we set $N = 100$ and compute $\|\mathcal{S}^m_N - \mathcal{S}^m_j\|_{L^2_R}$ once more for $j = 1, 2, \ldots, 99$; this time, however, we restrict the τ-domain to $\tau \in [0, 2]$, while keeping the x-domain the same as before, with $x \in [0, 4\pi]$. The important point is that we have now restricted τ to lie well within its radius of convergence, which extends below $\tau_0 = 1$ to approximately $\tau = -0.35$, and above $\tau_0 = 1$ to approximately $\tau = 2.35$. Figure 6.3 depicts the results in this case for $m = 1$. We see that $\|\mathcal{S}^m_{100} - \mathcal{S}^m_j\|_{L^2_R}$ approaches a constant after $j \approx 60$; the constant is $\mathcal{O}(10^{-15})$, which is close to machine epsilon. That is, after 60 terms, there appears to be no advantage to increasing the number of terms in the sum. In combination with the results of Figure 6.2, this implies that fewer terms are needed to capture the behavior of the solution accurately near the pole at $\tau_0 = 1$, while more terms are needed in order to describe the behavior of the solution near the boundary of its region of convergence. This is further seen in Figure 6.4, which depicts $\mathcal{E}^m_{25}(x, \tau)$ when $N = 100$ and $m = 1$, where $\mathcal{E}^m_j(x, \tau)$ is given in (6.5). It is evident from the plot that $\|\mathcal{S}^m_{100} - \mathcal{S}^m_{25}\|_{L^2_R}$ increases as τ approaches the edge of the region of convergence. In the interior of the domain, the error remains low.

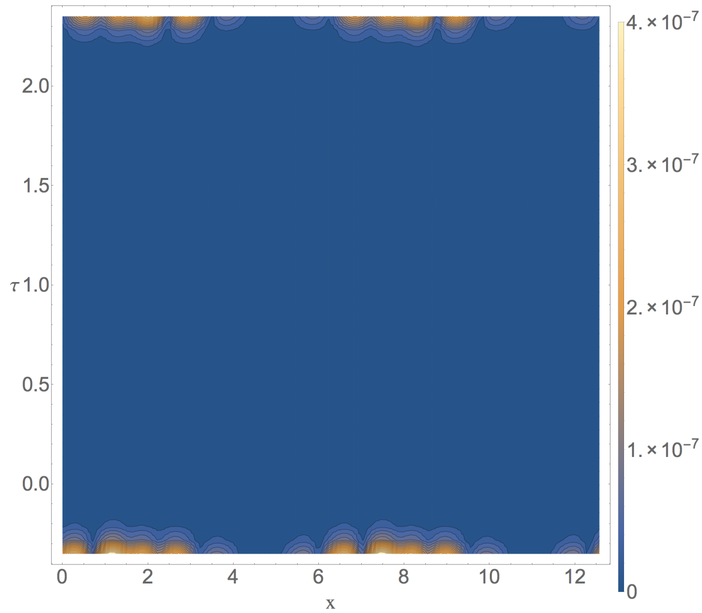


Figure 6.4: Contour plot of the relative error, $\mathcal{E}_{25}^{1}(x,\tau)$, between the 100th partial sum approximation of $a_1(x,\tau)$ and the 25th partial sum when $\tau_0 = 1$, $c_1 = c_2 = c_3 = 1$, $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$, and $\mathcal{H}(x)$ is defined in (6.2). Note that $\mathcal{E}_j^m(x,\tau)$ is defined in (6.5), and we set $N = 100$.

6.2 The special case of equal group velocity

The three-wave PDEs when $c_1 = c_2 = c_3$ is a special case, because under a change of variables, the PDEs can be reduced to the three-wave ODEs. Since the solution of the three-wave ODEs is known analytically in the entire complex τ-plane in terms of Weierstrass elliptic functions, we can use the ODE solution to verify the accuracy of our PDE series solution in this case.

Let $c_1 = c_2 = c_3 \equiv c$ and define the following change of variables

$$\zeta = x - c\tau, \quad \text{and} \quad \eta = \tau. \tag{6.9}$$

Under this change of variables, the three-wave equations (5.1) become

$$\frac{\partial a_m}{\partial \eta} = \sigma_m a_k^* a_\ell^*, \tag{6.10}$$

where $(k,\ell,m) = (1,2,3)$ cyclically, $a_m = a_m(\zeta,\eta)$, and (6.10) holds along lines of constant ζ. Equation (2.20) is equivalent to the three-wave ODEs for constant ζ, and its general solution in terms of elliptic functions is given by (3.54) and (3.60).

In order to compare the PDE series solution with the exact analytic solution of the ODEs, we restrict our attention to $|a_m(x,\tau)|^2$. Moreover, we assume that the phases are constant and $\mathcal{K}_2(x) = \mathcal{K}_2(x) = 0$. This choice is for simplicity, and the computation can be repeated for other cases with similar results.

First, we construct the partial sum approximation of the solution, $|a_m(x,\tau)|^2 \approx |\mathcal{S}_N^m(x,\tau)|^2$, where $N \geq 1$ and we restrict τ to be within the radius of convergence of the solution. To compute the corresponding ODE solution using (3.54), we need the values of the constants H, K_2, and K_3 along each line $\zeta = x - ct$ (since H, K_2, and K_3 determine the elliptic invariants, g_2 and g_3). Since $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$ in the PDE case, it follows that $K_2 = K_3 = 0$ in the corresponding ODEs. One can easily verify this using the Manley-Rowe relations in (2.30)-(2.31), since $|a_1(x,\tau)|^2 = |a_2(x,\tau)|^2 = |a_3(x,\tau)|^2$ for all x, τ when $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$ and $c_1 = c_2 = c_3$. Next, H is determined from our choice of $\mathcal{H}(x)$ in the PDE series solution. In particular, consider the line $\tau = \tau_0$. Suppose we fix a particular value of x along the line $\tau = \tau_0$ and call it $\tilde{x}$. Then $\tilde{x}$ has a corresponding line of constant ζ defined by $\zeta = \tilde{x} - c\tau_0$. Along each of these constant ζ lines, we set $H = \mathcal{H}(\tilde{x})$.

Now that K_2, K_3, and H are known for each ζ line, we simply evaluate the Weierstrass solution in (3.54) along this line, using the fact that $\tau = \eta$. This provides an exact solution of the three-wave PDEs when $c_1 = c_2 = c_3$ in the entire (x,τ)-plane. We denote the exact solution of the three-wave PDEs by $\hat{\mathcal{S}}^m(x,\tau)$ for $m = 1, 2, 3$.

Consider the case of the PDE series solution outlined in the previous section, in which the phases are constant, $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$, $c = 1$, $\tau_0 = 1$, and $\mathcal{H}(x)$ is given by

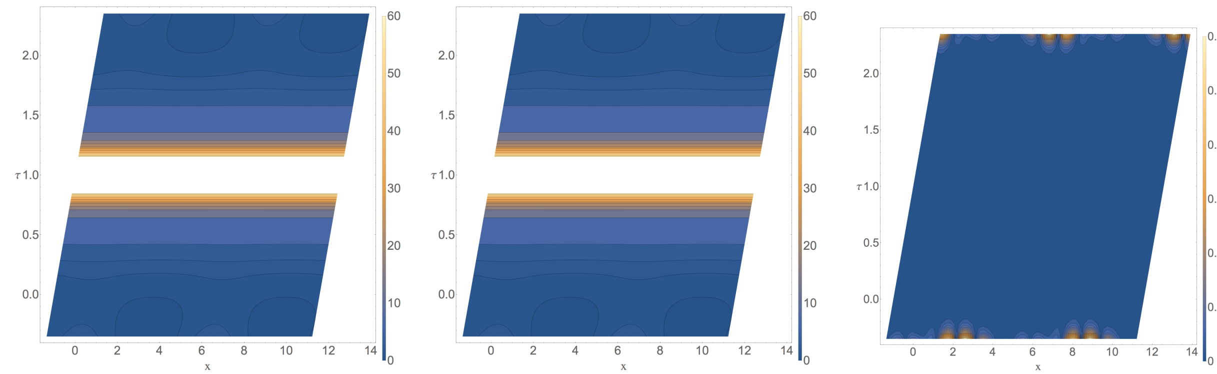
$$\mathcal{H}(x) = 2 + \sin x. \tag{6.11}$$

We know from (6.4) that the radius of convergence in this case is

$$|\tau - \tau_0| \lesssim 1.35,$$

where $\lesssim$ is used to denote the fact that the right-hand side of the inequality is an approximation.

Figure 6.5a depicts $|\mathcal{S}_{25}^1(x,\tau)|^2$ within the radius of convergence of the solution. Figure 6.5b, on the other hand, depicts the exact solution of the three-wave PDEs in the same domain, where the three-wave ODEs in (6.10) were used to compute the solution. Finally, Figure 6.5c shows the pointwise relative error between Figures 6.5a and 6.5b, which is larger for values of τ that are near the radius of convergence.



(a) $|\mathcal{S}_{25}^1|^2$ (b) $|\hat{\mathcal{S}}^1|^2$ (c) $(|\hat{\mathcal{S}}^1|^2 - |\mathcal{S}_{25}^1|^2)/|\hat{\mathcal{S}}^1|^2$

Figure 6.5: A comparison of the formal Laurent series solution of the three-wave PDEs with 25 terms when $\tau_0 = c_1 = c_2 = c_3 = 1$, and the corresponding solution of the three-wave ODEs along lines of constant ζ under the change of variables in (6.9). We use $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$, with $\mathcal{H}(x)$ defined in (6.11), and we consider τ within its region of convergence, $\tau \in [-0.35, 2.35]$. Note that $\mathcal{S}_{25}^m(x,\tau)$ denotes the 25th partial sum approximation of $a_m(x,\tau)$ and $\hat{\mathcal{S}}^m(x,\tau)$ denotes the true solution, $a_m(x,\tau)$.

Following the notation of Section 6.1, define the approximate L^2-norm of the relative error between $\hat{\mathcal{S}}^m(x,\tau)$ and $\mathcal{S}_N^m(x,\tau)$ as

$$\left\|\hat{\mathcal{S}}^m - \mathcal{S}_N^m\right\|_{L_R^2} = \sqrt{\sum_{p=1}^{h}\sum_{q=1}^{k}\left[\mathcal{E}_N^m(x_p,\tau_q)\right]^2 \, \Delta x \, \Delta t}, \tag{6.12}$$

where in this case $\mathcal{E}_N^m(x,\tau)$ is defined to be

$$\mathcal{E}_N^m(x,\tau) = \left|\frac{\hat{\mathcal{S}}^m(x,\tau) - \mathcal{S}_N^m(x,\tau)}{\hat{\mathcal{S}}^m(x,\tau)}\right|. \tag{6.13}$$

In Figure 6.6, we depict the L^2-relative error between the the exact solution and the series solution with N terms up to $N = 100$. We see that increasing the number of terms in the series solution

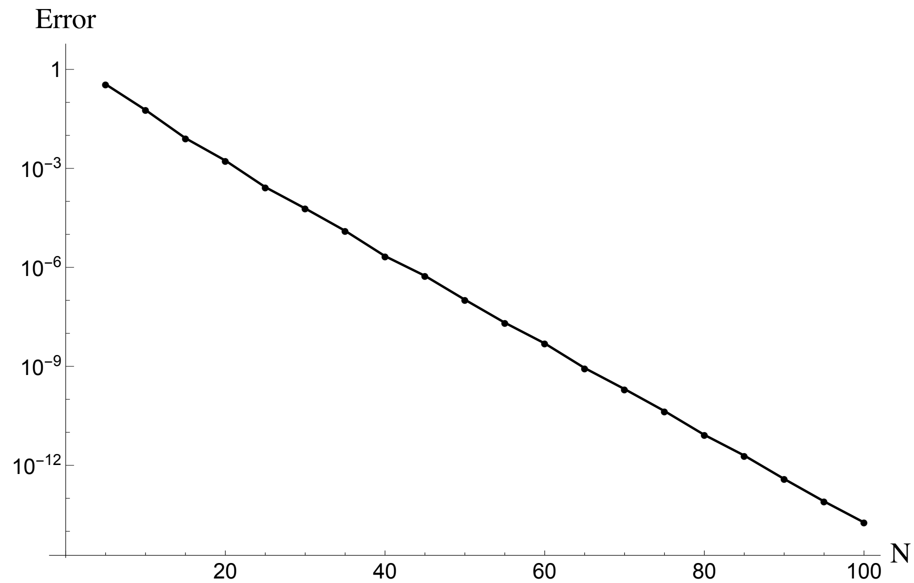


Figure 6.6: $\|\hat{\mathcal{S}}^1 - \mathcal{S}_N^1\|_{L^2}$, the relative L^2-error between the partial sum approximation $\mathcal{S}_N^m$ of $a_m(x,\tau)$ for $m = 1$ and the exact solution $\hat{S}$ of the three-wave PDEs, generated using the ODEs in (6.10). The figure is generated using $x \in [0, 4\pi]$ and $\tau \in [-0.35, 2.35]$, so that τ extends all the way out to its radius of convergence.

exponentially increases the accuracy of the solution. Indeed, we find that

$$\left\|\hat{\mathcal{S}}^m - \mathcal{S}_N^m\right\|_{L_R^2} \sim \mathcal{O}(D\,C^{-N}), \tag{6.14}$$

where C and D are positive constants. In the case of Figure 6.6, we have $C \approx 1.379$ and $D \approx 0.827$. Note that (6.14) is consistent with (6.8).

Figure 6.6 gives us confidence that the PDE series solution we generated in Chapter 5 accurately describes the behavior of solutions of the three-wave equations, even when we truncate the series after a relatively small number of terms. Moreover, it is clear from Figure 6.5c that the loss of information in truncating the series solution is primarily restricted to the edges of the region of convergence.

Since the PDE series solution when $c_1 = c_2 = c_3$ can be computed exactly using the three-wave ODEs, it is useful to observe the behavior of the solution outside of the radius of convergence of the typical series solution. This allows us to see what information we are missing by restricting ourselves to τ values that lie within the radius of convergence. To that end, in Figure 6.7, we depict the exact solution of the three-wave PDEs when $c_1 = c_2 = c_3 = 1$, $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$, and $\mathcal{H}(x)$ is given in (6.11). In this case, we allow τ to go far outside the radius of convergence of the series solution, so that $\tau \in [-2.75, 4.75]$.

It is clear from Figure 6.7 that there are other poles away from the pole we put in at $\tau_0 = 1$. Moreover, the locations of the other poles depend on x. The exact solution in the figure is found by using the Weierstrass solution along lines of constant ζ. Since each line has a different value of H, it follows that the Weiertrass function along each line has its own arrangement of poles. In particular, recall from Section 5.3, equation (5.47), that the radius of convergence of the three-wave ODEs when $K_2 = K_3 = 0$ is given by

$$R_{\mathrm{ODE}} = \frac{2\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{4\pi\left|H\right|^{1/3}} \approx \frac{3.06}{\left|H\right|^{1/3}}.$$

As a result, along constant ζ lines where H is small, the distance between poles will be large; conversely, along ζ lines where H is large, the distance between poles will be small. This accounts for the fact that the location of the pole away from $\tau_0 = 1$ is not constant in x.

We would like to know how much of the behavior of the poles away from $\tau_0 = 1$ can be captured by our PDE series solution, without the use of the exact Weierstrass solution. To that end, Figure 6.8a depicts $|\mathcal{S}^1_{25}(x,\tau)|$ for $\tau \in [-0.95, 2.95]$. This allows τ to extend beyond the region of convergence of the solution, outside of where we know the solution is valid. Figure 6.8b depicts the exact solution for the same range of τ for comparison purposes. It is clear from the figures that the PDE series solution does begin to capture the behavior of the poles away from $\tau = \tau_0$. However, the series solution also introduces spurious artifacts near the poles that do not exist in the exact solution. We continue to work on the problem of accurately extending our PDE series solution outside of its radius of convergence.

6.3 Finite difference validation

In this section, we provide an alternative check on the Laurent series solution of the three-wave PDEs, given in (5.24)-(5.26). In particular, we use a finite difference scheme in order to solve

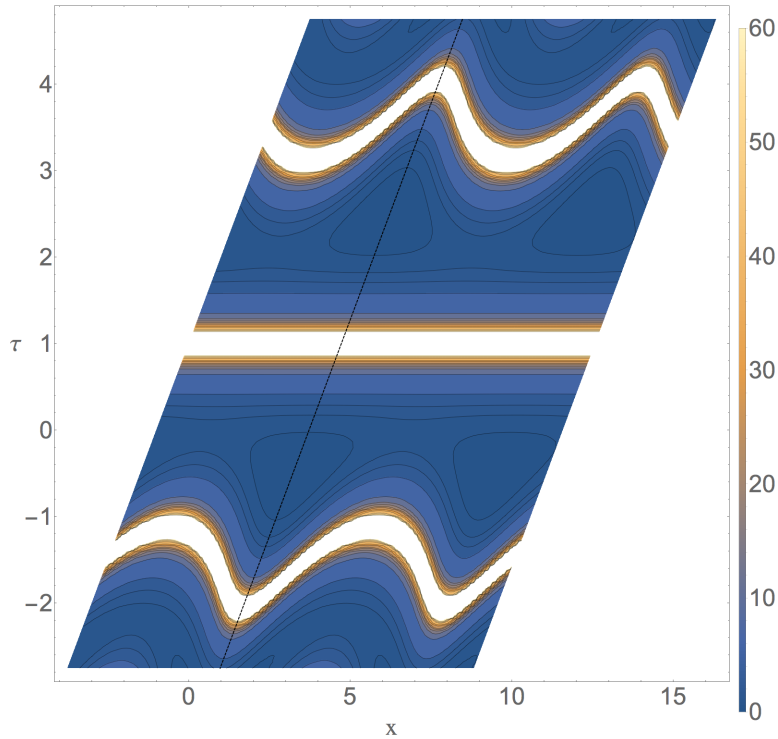


Figure 6.7: Contour plot of the exact solution of the three-wave PDEs, $|a_1(x,\tau)|^2$, when $\tau_0 = c_1 = c_2 = c_3 = 1$, the phases are constant, $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$, and $\mathcal{H}(x)$ is defined in (6.11). The domain is $\tau \in [-2.75, 4.75]$, which is far beyond the radius of convergence of the PDE series solution. Note that there is a fixed pole at $\tau_0 = 1$, and poles with spatial dependence away from there. The dashed black line represents a line of constant ζ, where $\zeta = x - t$.

the three-wave PDEs, which we compare to the partial sum approximation $\mathcal{S}_N^m(x,\tau)$ of our series solution. The finite difference comparison is advantageous because it allows us to consider the case where the group velocities, c_m, $m = 1, 2, 3$, are not all equal. This was not possible in the previous section.

We choose one of the simplest possible finite difference methods in order to solve the three-wave PDEs in (5.1). This allows us to validate the solution via standard convergence analysis. Specifically, we use a first-order upwind method. That is, we use a forward Euler scheme in τ, and a backward difference scheme in x. To that end, we discretize the (x,τ)-domain using (6.6), and define the following

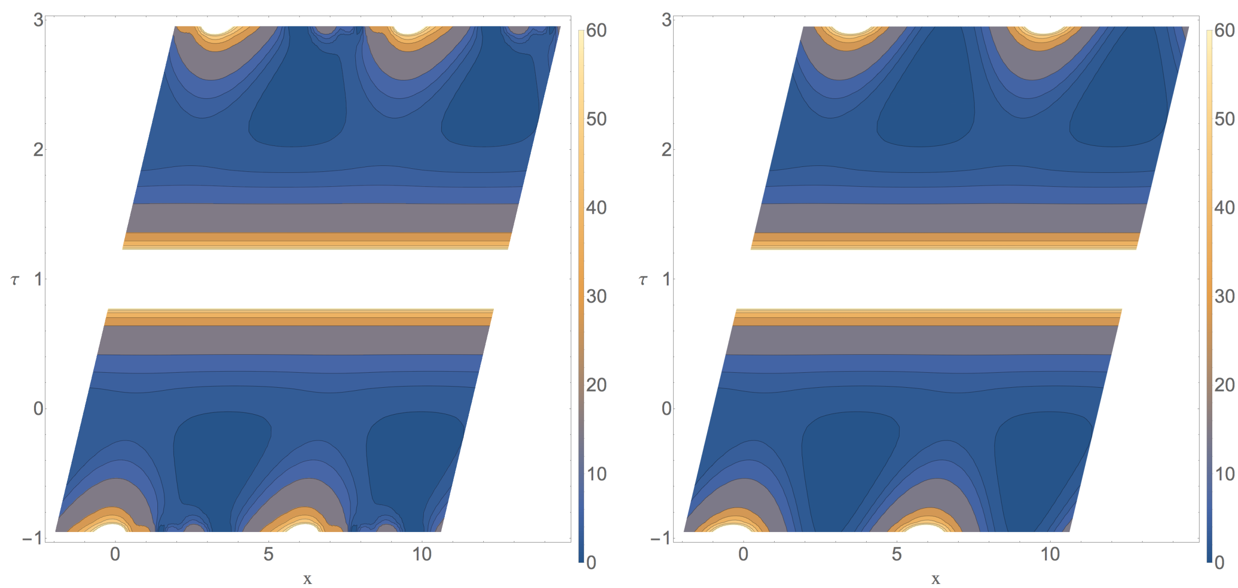
$$[a_m]_p^q = a_m(x_p, \tau_q).$$

Then the explicit finite difference method for solving the three-wave PDEs becomes

$$\frac{1}{\Delta t}\left([a_m]_p^{q+1} - [a_m]_p^q\right) + \frac{c_m}{\Delta x}\left([a_m]_p^q - [a_m]_{p-1}^1\right) = \sigma_m\, [a_k^*]_p^q\, [a_\ell^*]_p^q, \tag{6.15}$$

where $(k, \ell, m) = (1, 2, 3)$ cyclically. Given an initial condition and a boundary condition on the left of the domain, (6.15) allows us to step forward in time and space in order to find the solution of the three-wave PDEs in a given (x, τ)-domain. Note that specifying a boundary condition on the left means we specify $[a_m]_1^q$ for $q = 1, 2, \ldots, k$, and specifying an in initial condition means specifying $[a_m]_p^1$ for $p = 1, 2, \ldots, h$.

It is well known that the accuracy of the first-order upwind scheme is $\mathcal{O}(\Delta\tau) + \mathcal{O}(\Delta x)$. In



(a) $|\mathcal{S}_{25}^1|^2$ (b) $|\hat{\mathcal{S}}^1|^2$

Figure 6.8: A comparison of the formal Laurent series solution of the three-wave PDEs with 25 terms when $c_1 = c_2 = c_3 = 1$, and the corresponding solution of the three-wave ODEs along lines of constant ζ under the change of variables in (6.9). We use $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$, with $\mathcal{H}(x)$ defined in (6.11), and we consider τ outside its region of convergence, $\tau \in [-0.95, 2.95]$. Note that $\mathcal{S}_{25}^1(x, \tau)$ denotes the 25th partial sum approximation of $a_m(x, \tau)$ and $\hat{S}^m(x, \tau)$ denotes the true solution, $a_m(x, \tau)$.

order to see this, we first use Taylor expansions to find

$$a_m(x,\tau+\Delta\tau) = a_m(x,\tau) + \Delta\tau\frac{\partial}{\partial\tau}a_m(x,\tau) + \frac{(\Delta t)^2}{2}\frac{\partial^2}{\partial\tau^2}a_m(x,\tau) + \mathcal{O}(\Delta\tau^3),$$
$$a_m(x-\Delta x,\tau) = a_m(x,\tau) - \Delta x\frac{\partial}{\partial x}a_m(x,\tau) + \frac{(\Delta x)^2}{2}\frac{\partial^2}{\partial x^2}a_m(x,\tau) + \mathcal{O}(\Delta\tau^3).$$

Rearranging, we obtain

$$\frac{a_m(x,\tau+\Delta\tau) - a_m(x,\tau)}{\Delta t} = \frac{\partial}{\partial\tau}a_m(x,\tau) + \mathcal{O}(\Delta t),$$
$$\frac{a_m(x,\tau) - a_m(x-\Delta x,\tau)}{\Delta x} = \frac{\partial}{\partial x}a_m(x,\tau) + \mathcal{O}(\Delta x).$$

If we set $x = x_p$ and $\tau = \tau_q$, then $x + \Delta x = x_{p+1}$ and $\tau + \Delta\tau = \tau_{q+1}$. Thus, we have

$$\frac{1}{\Delta t}\left([a_m]_p^{q+1} - [a_m]_p^q\right) = \frac{\partial}{\partial\tau}a_m(x_p,\tau_q) + \mathcal{O}(\Delta t),$$
$$\frac{1}{\Delta x}\left([a_m]_p^q - [a_m]_{p-1}^1\right) = \frac{\partial}{\partial x}a_m(x_p,\tau_q) + \mathcal{O}(\Delta x).$$

This is sufficient to show that the error term for the first-order upwind method is $\mathcal{O}(\Delta\tau)+\mathcal{O}(\Delta x)$.

The first-order upwind scheme is stable as long as the Courant-Friedrichs-Lewy (CFL) condition is satisfied. In particular, $\Delta\tau$ and Δx must be chosen such that

$$c\,\Delta\tau \leq \Delta x, \tag{6.16}$$

where $c = \max\{|c_1|, |c_2|, |c_3|\}$.

In practice, in order to ensure that (6.16) is satisfied, we fix Δx and set $\Delta\tau = \Delta x/c$. Consider the usual example of the PDE series solution in which the phases are constant, $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$, $\mathcal{H}(x)$ is given in (6.11), and $\tau_0 = 1$. This time, however, we assume that the group velocities are not equal, and that in particular $c_1 = 1, c_2 = 0.5$, and $c_3 = 0.75$. In order to compare the partial sum approximation of the solution with the finite difference approximation, we restrict our attention to a small domain in x and τ. Since the finite difference solution will not do well near the pole, we restrict τ to lie in the domain $[-0.35, 0.75]$. This means τ extends to the lower bound of its region of convergence, but it does not get too close to the pole at $\tau_0 = 1$. For simplicity, we consider a square domain, so that $x \in [-0.35, 0.75]$.

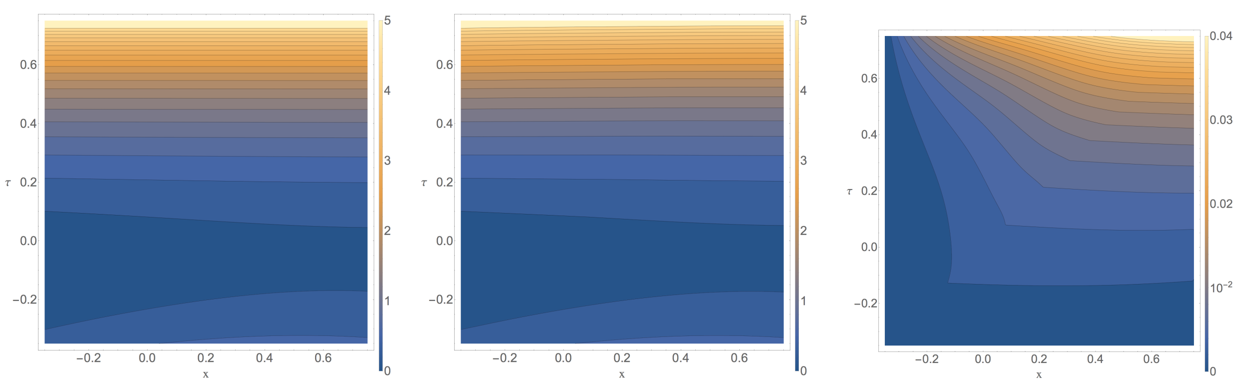
Since $c = 1$, (6.16) is satisfied as long as we pick $\Delta\tau = \Delta x$. The finite difference solution is computed by using a boundary condition on the left and an initial condition taken from the partial sum approximation of $a_m(x, \tau)$. That is, we set

$$[a_m]_p^q = \mathcal{S}_N^m(x_p, \tau_1) = \mathcal{S}_N^m(x_p, -0.35), \quad \text{and} \quad [a_m]_1^q = \mathcal{S}_N^m(x_1, \tau_q) = \mathcal{S}_N^m(-0.35, \tau_q).$$

Figure 6.9a depicts the partial sum approximation of $|a_1(x, \tau)|$ with 25 terms for the conditions described above. Similarly, Figure 6.9b depicts the finite difference approximation of $|a_1(x, \tau)|$ on a 256×256 grid. This corresponds to a value of $\Delta x = \Delta t \approx 0.0043$. Figure 6.9c shows the magnitude of the relative pointwise error between the two approximations,

$$\mathcal{E}_N(x, \tau) = \left| \frac{\mathcal{S}_N^m(x, \tau) - \mathcal{S}_{FD}^m(x, \tau)}{\mathcal{S}_N^m(x, \tau)} \right|,$$

where $\mathcal{S}_{\mathrm{FD}}^m$ denotes the finite difference approximation of $\hat{\mathcal{S}}^m$. Note that the error increases towards the top, right corner of the plot. This is due to the fact that numerical dispersion is introduced into the finite difference approximation as the solution propagates away from the initial and boundary data.



(a) $|\mathcal{S}_{25}^1|$ (b) $|\mathcal{S}_{\mathrm{FD}}^1|$ (c) $\mathcal{E}_{25}(x, \tau)$

Figure 6.9: A comparison of the partial sum approximation of the solution of the three-wave PDEs with 25 terms when $\tau_0 = 1$, $c_1 = 1, c_2 = 0.5$, and $c_3 = 0.75$, and the corresponding finite difference approximation with $\Delta t = \Delta x \approx 0.0043$. The domain is $[-0.35, 0.75] \times [-0.35, 0.75]$. Note that $\mathcal{S}_{25}^m(x, \tau)$ denotes the 25th partial sum approximation of $a_m(x, \tau)$, $\mathcal{S}_{\mathrm{FD}}^m(x, \tau)$ denotes the finite difference approximation of $a_m(x, \tau)$, and $\mathcal{E}_{25}(x, \tau)$ denotes the relative error between them.

Returning to the problem at hand, recall that we want to use the finite difference scheme in (6.15) in order to validate our PDE series solution in (5.24)-(5.26). The difficulty is that computationally, we can only compute the partial sum approximation of our PDE series solution. Consequently, we are using one approximation (the finite difference approximation) in order to validate another approximation (the partial sum approximation).

In order to verify our results, we need to know a priori how the error between the finite difference and partial sum approximations of the exact solution should behave. To that end, let $\hat{S}^m$ denote the exact solution of the three-wave PDEs as usual. Observe the following

$$\begin{aligned}\left\|\mathcal{S}_N^m - \mathcal{S}_{\mathrm{FD}}^m\right\|_{L_R^2} &= \left\|\hat{\mathcal{S}}^m + (\mathcal{S}_N^m - \hat{\mathcal{S}}^m) - \mathcal{S}_{\mathrm{FD}}^m\right\|_{L_R^2} \\ &\leq \left\|\hat{\mathcal{S}}^m - \mathcal{S}_{\mathrm{FD}}^m\right\|_{L_R^2} + \left\|\mathcal{S}_N^m - \hat{\mathcal{S}}^m\right\|_{L_R^2} \\ &\sim \mathcal{O}(\Delta\tau) + \mathcal{O}(\Delta x) + \mathcal{O}(D\,C^{-N}),\end{aligned} \tag{6.17}$$

where as usual, $\|\cdot\|_{L_R^2}$ denotes the approximate L^2-norm of the relative error. The last line used the fact that we know the error of our finite difference scheme is $\left\|\hat{\mathcal{S}}^m - \mathcal{S}_{\mathrm{FD}}^m\right\|_{L_R^2} \sim \mathcal{O}(\Delta\tau) + \mathcal{O}(\Delta x)$, and we know from (6.8) and (6.14) that $\|\mathcal{S}_N^m - \hat{\mathcal{S}}^m\|_{L_R^2} \sim \mathcal{O}(D\,C^{-N})$. If N is sufficiently large, then $D\,C^{-N}$ is small, and the dominant behavior of the error between the two approximations becomes $\mathcal{O}(\Delta\tau) + \mathcal{O}(\Delta x)$. Since we fixed $\Delta\tau = \Delta x/c$, it follows that the error in this case behaves like $\mathcal{O}(\Delta x)$.

We compute the error on the left-hand side of (6.17) for increasingly small values of Δx in the finite difference approximation, while holding N constant in the partial sum approximation. Figure 6.10 depicts the results of this computation on a log-log scale for $N = 8, 14, 20, 32$, and 50. Notice that for small values of N, the curves do not linearly decrease as $\mathcal{O}(\Delta x)$, but instead approach a constant, as predicted by (6.17). On the other hand, the curves for larger N are linear, with slopes of approximately one. This means the error is $\mathcal{O}(\Delta x)$ as predicted, which indicates that when N is large, the partial sum is a good approximation of the exact solution. For the larger values of N, we hypothesize that if Δx was taken small enough, the curves would still approach a constant predicted by (6.17). Due to computational limitations this hypothesis could not be tested.

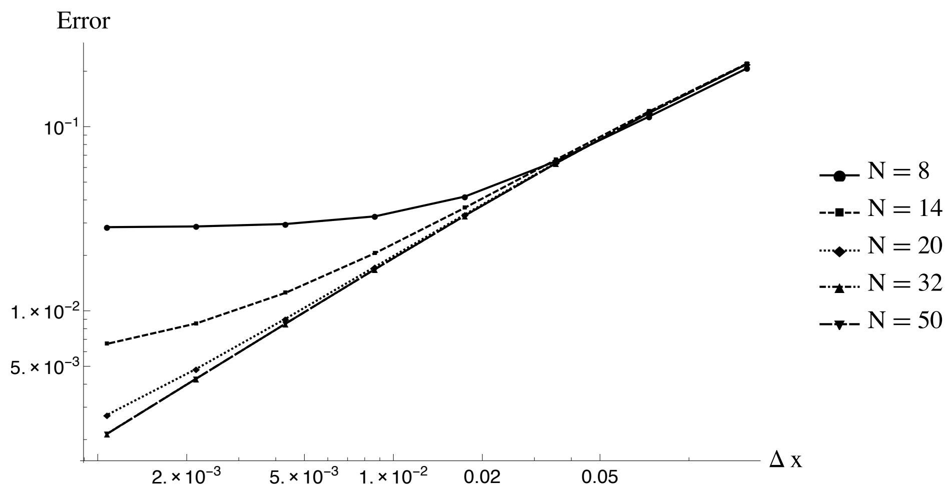


Figure 6.10: $\|\mathcal{S}^1_N - \mathcal{S}^1_{\text{FD}}\|_{L^2_R}$ when $c_1 = 1, c_2 = 0.5$, $c_3 = 0.75$, and $\tau_0 = 1$. We assumed $x, \tau \in [-0.35, 0.75]$, so that τ reaches the lower boundary of the region of convergence, but does not get close to the pole. We assumed constant phases and that $\mathcal{K}_2(x) = \mathcal{K}_3(x) = 0$, with $\mathcal{H}(x)$ given in (6.11).

Chapter 7

Discussion

We derived a solution of the three-wave PDEs in terms of a formal Laurent series in time, the coefficients of which have spatial dependence. The series solution involves five real free functions of x and one real free constant, namely the position of the pole at $\tau = \tau_0$. A fully general solution of the three-wave PDEs would involve six free functions, so our work stops short of a general solution. In spite of this, we are only one function short of having a general solution of the three-wave PDEs, a set of completely integrable coupled nonlinear complex-valued PDEs.

We focused our attention on the case in which the phase functions are constant. This allowed us to derive the radius of convergence of the PDE series solution in several cases, specifically when one or more of the remaining functions is set to zero. We determined that if the nonzero free functions satisfy some differentiability constraints, then the radius of convergence of the solution of the three-wave PDEs is related to the radius of convergence of a corresponding solution of the three-wave ODEs. In particular, the radius of convergence of the PDE series solution is smaller than the radius of convergence of the corresponding three-wave ODE solution, but it is smaller by a known dimensionless factor. That factor depends on the maximum group velocity of the three interacting waves, and how quickly the derivatives of the free functions are growing.

After making the formal Laurent series solution of the three-wave PDEs more rigorous by determining where the solution converges, we used numerical methods in order to verify our solution within its region of convergence. More specifically, we demonstrated that in practice, we can truncate the formal Laurent series solution of the three-wave PDEs after a few dozen terms without much loss of accuracy. More terms are required for τ-values that are near the radius of convergence, whereas fewer terms are needed to obtain the same level of accuracy for τ-values in the interior of the τ-domain.

Our series solution provides an alternative to using inverse scattering when finding a solution of the three-wave PDEs. Moreover, our method imposes no boundary conditions in x, meaning our solution should be consistent with a wide variety of boundary conditions, including periodic boundary conditions. This is useful since, to our knowledge, the three-wave PDEs have never been solved with periodic boundary conditions. Our solution also makes no restriction on the number of spatial dimensions allowed.

There is more work to be done in the nonexplosive regime of the three-wave PDEs, particularly in determining what values τ_0 is allowed to take on in the complex plane. Additionally, we have not rigorously considered what happens when all three of the functions $\mathcal{K}_2(x)$, $\mathcal{K}_3(x)$, and $\mathcal{H}(x)$ are nonzero, although we expect this to be a natural extension of the cases outlined in Sections 5.3-5.6. Furthermore, we have not addressed the question of what happens when we allow the phase functions to be nonconstant. This problem is more difficult because the phases do not appear in the coefficients of the series solution of the three-wave ODEs; they appear only in the exponential term in front of the series. As a result, it is not clear whether we can find the radius of convergence of the PDE series solution in terms of the radius of convergence of a corresponding ODE solution. In spite of this, if we impose some differentiability conditions on $\psi_m(x)$, then we expect the general process of bounding the PDE solution in terms of polynomials in (ck) to be the same, although we do not know whether or not the resulting series converges.

Finally, in order to develop a fully general solution of the three-wave PDEs, spatial dependence must be incorporated into τ_0. Acquiring a general solution of a PDE is a rare occurrence, so it will be significant if this is accomplished. The question of how to specify initial data in the general (or near-general) solution of the three-wave PDEs must also be addressed, since this would make the PDE series solution more physically applicable.

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Appendix A

Coefficient γ_1

$$
\begin{aligned}
&\gamma_1 = \\
&\Bigg\{ \frac{1}{2}\left(gk_1+\omega_1^2\right)\left(\rho_1\cosh\left(h_1k_3\right)+\rho_2\sinh\left(h_1k_3\right)\coth\left(h_2k_3\right)\right) \\
&-ik_1\omega_1\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\left(\rho_1\cosh\left(h_1k_3\right)+\rho_2\sinh\left(h_1k_3\right)\coth\left(h_2k_3\right)\right) \\
&+2k_1k_2\rho_1e^{h_1k_1+h_1k_2}\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)-2k_1k_2\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\left(\rho_1\cosh\left(h_1k_3\right)\right. \\
&\left.+\rho_2\sinh\left(h_1k_3\right)\coth\left(h_2k_3\right)\right) \\
&+ik_2\omega_2\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\left(\rho_1\cosh\left(h_1k_3\right)+\rho_2\sinh\left(h_1k_3\right)\coth\left(h_2k_3\right)\right) \\
&+\frac{1}{2}\left(gk_2+\omega_2^2\right)\left(\rho_1\cosh\left(h_1k_3\right)+\rho_2\sinh\left(h_1k_3\right)\coth\left(h_2k_3\right)\right) \\
&-\frac{ik_2\rho_1\omega_2e^{h_1k_2-h_1k_1}\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{2\omega_1^2} \\
&+\frac{\rho_1e^{-h_1k_1-h_1k_2}\left(gk_2+\omega_2^2\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{4\omega_1^2} \\
&+\frac{\rho_1e^{-h_1k_1-h_1k_2}\left(gk_1+\omega_1^2\right)\left(gk_2+\omega_2^2\right)}{2\omega_1\omega_2} \\
&-\frac{\left(gk_1+\omega_1^2\right)\left(gk_2+\omega_2^2\right)\left(\rho_1\cosh\left(h_1k_3\right)+\rho_2\sinh\left(h_1k_3\right)\coth\left(h_2k_3\right)\right)}{2\omega_1\omega_2} \\
&-\frac{\rho_2e^{-h_1k_1-h_1k_2}\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{4\omega_1^2\left(e^{2h_1k_2+2(-h_1-h_2)k_2}-1\right)} \\
&+\frac{\rho_2\exp\left(-h_1k_1+h_1k_2+2\left(-h_1-h_2\right)k_2\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{4\omega_1^2\left(e^{2h_1k_2+2(-h_1-h_2)k_2}-1\right)} \\
&\qquad\qquad\qquad\qquad\cdot\frac{\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{4\omega_1^2\left(e^{2h_1k_2+2(-h_1-h_2)k_2}-1\right)} \\
&+\frac{ik_1\rho_1\omega_1e^{h_1k_1-h_1k_2}\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{2\omega_2^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{e^{h_1k_1+2(-h_1-h_2)k_1+h_1k_2+2(-h_1-h_2)k_2}\rho_2\left(e^{2h_1k_1}\omega_1^2+\omega_1^2-e^{2h_1k_1}gk_1+gk_1\right)\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)}{2\left(-1+e^{2h_1k_1+2(-h_1-h_2)k_1}\right)\left(-1+e^{2h_1k_2+2(-h_1-h_2)k_2}\right)\omega_1\omega_2}\\
&-\frac{e^{-h_1k_1-h_1k_2}\rho_2\left(e^{2h_1k_1}\omega_1^2+\omega_1^2-e^{2h_1k_1}gk_1+gk_1\right)\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)}{2\left(-1+e^{2h_1k_1+2(-h_1-h_2)k_1}\right)\left(-1+e^{2h_1k_2+2(-h_1-h_2)k_2}\right)\omega_1\omega_2}\\
&-\frac{e^{-h_1k_1-h_1k_2}\rho_1\left(\omega_1^2+gk_1\right)\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)}{4\omega_2^2}\\
&-\frac{e^{-h_1k_1-h_1k_2}\rho_2\left(e^{2h_1k_1}\omega_1^2+\omega_1^2-e^{2h_1k_1}gk_1+gk_1\right)\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)}{4\left(-1+e^{2h_1k_1+2(-h_1-h_2)k_1}\right)\omega_2^2}\\
&+\frac{e^{h_1k_1+2(-h_1-h_2)k_1-h_1k_2}\rho_2\left(e^{2h_1k_1}\omega_1^2+\omega_1^2-e^{2h_1k_1}gk_1+gk_1\right)\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)}{4\left(-1+e^{2h_1k_1+2(-h_1-h_2)k_1}\right)\omega_2^2}\\
&+\frac{igk_1k_3\left(\rho_2-\rho_1\right)\omega_1\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\sinh\left(h_1k_3\right)}{\omega_3^2}-\frac{gk_3\left(\rho_2-\rho_1\right)\left(gk_1+\omega_1^2\right)\sinh\left(h_1k_3\right)}{2\omega_3^2}\\
&+\frac{2gk_1k_2k_3\left(\rho_2-\rho_1\right)\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\sinh\left(h_1k_3\right)}{\omega_3^2}-\frac{igk_2k_3\left(\rho_2-\rho_1\right)\omega_2\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\sinh\left(h_1k_3\right)}{\omega_3^2}\\
&-\frac{gk_3\left(\rho_2-\rho_1\right)\left(gk_2+\omega_2^2\right)\sinh\left(h_1k_3\right)}{2\omega_3^2}+\frac{gk_3\left(\rho_2-\rho_1\right)\left(gk_1+\omega_1^2\right)\left(gk_2+\omega_2^2\right)\sinh\left(h_1k_3\right)}{2\omega_1\omega_2\omega_3^2}\\
&+\frac{igk_1^2\left(\rho_1-\rho_2\right)\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\cosh\left(h_1k_3\right)}{\omega_3}-\frac{igk_1k_2\left(\rho_1-\rho_2\right)\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\cosh\left(h_1k_3\right)}{\omega_3}\\
&+\frac{igk_2^2\left(\rho_1-\rho_2\right)\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\cosh\left(h_1k_3\right)}{\omega_3}-\frac{igk_1k_2\left(\rho_1-\rho_2\right)\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\cosh\left(h_1k_3\right)}{\omega_3}\\
&-\frac{gk_2\left(\rho_1-\rho_2\right)\left(gk_1+\omega_1^2\right)\cosh\left(h_1k_3\right)}{2\omega_1\omega_3}+\frac{gk_1\left(\rho_1-\rho_2\right)\left(gk_1+\omega_1^2\right)\cosh\left(h_1k_3\right)}{2\omega_1\omega_3}\\
&+\frac{igk_1k_2\left(\rho_1-\rho_2\right)e^{h_1k_2-h_1k_1}\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{2\omega_1^2\omega_3}\\
&-\frac{igk_2^2\left(\rho_1-\rho_2\right)e^{h_1k_2-h_1k_1}\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{2\omega_1^2\omega_3}\\
&-\frac{gk_1\left(\rho_1-\rho_2\right)e^{-h_1k_1-h_1k_2}\left(gk_2+\omega_2^2\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{4\omega_1^2\omega_2\omega_3}\\
&+\frac{gk_2\left(\rho_1-\rho_2\right)e^{-h_1k_1-h_1k_2}\left(gk_2+\omega_2^2\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{4\omega_1^2\omega_2\omega_3}\\
&+\frac{gk_1\left(\rho_1-\rho_2\right)\left(gk_2+\omega_2^2\right)\cosh\left(h_1k_3\right)}{2\omega_2\omega_3}-\frac{gk_2\left(\rho_1-\rho_2\right)\left(gk_2+\omega_2^2\right)\cosh\left(h_1k_3\right)}{2\omega_2\omega_3}\\
&+\frac{igk_1k_2\left(\rho_1-\rho_2\right)e^{h_1k_1-h_1k_2}\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{2\omega_2^2\omega_3}\\
&-\frac{igk_1^2\left(\rho_1-\rho_2\right)e^{h_1k_1-h_1k_2}\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{2\omega_2^2\omega_3}\\
&+\frac{gk_2\left(\rho_1-\rho_2\right)e^{-h_1k_1-h_1k_2}\left(gk_1+\omega_1^2\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{4\omega_1\omega_2^2\omega_3}\\
&-\frac{gk_1\left(\rho_1-\rho_2\right)e^{-h_1k_1-h_1k_2}\left(gk_1+\omega_1^2\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{4\omega_1\omega_2^2\omega_3}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{ik_2k_1\rho_1\omega_3\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\sinh\left(h_1k_3\right)}{k_3}+\frac{ik_1^2\rho_1\omega_3\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\sinh\left(h_1k_3\right)}{k_3}\\
&+\frac{ik_1^2\rho_2\omega_3\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\cosh\left(h_1k_3\right)\coth\left(h_2k_3\right)}{k_3}-\frac{ik_2k_1\rho_2\omega_3\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\cosh\left(h_1k_3\right)\coth\left(h_2k_3\right)}{k_3}\\
&+\frac{k_1\rho_1\omega_3\left(gk_1+\omega_1^2\right)\sinh\left(h_1k_3\right)}{2k_3\omega_1}-\frac{k_2\rho_1\omega_3\left(gk_1+\omega_1^2\right)\sinh\left(h_1k_3\right)}{2k_3\omega_1}\\
&+\frac{k_1\rho_2\omega_3\left(gk_1+\omega_1^2\right)\cosh\left(h_1k_3\right)\coth\left(h_2k_3\right)}{2k_3\omega_1}-\frac{k_2\rho_2\omega_3\left(gk_1+\omega_1^2\right)\cosh\left(h_1k_3\right)\coth\left(h_2k_3\right)}{2k_3\omega_1}\\
&+\frac{ik_2^2\rho_1\omega_3\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\sinh\left(h_1k_3\right)}{k_3}-\frac{ik_1k_2\rho_1\omega_3\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\sinh\left(h_1k_3\right)}{k_3}\\
&+\frac{ik_2^2\rho_2\omega_3\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\cosh\left(h_1k_3\right)\coth\left(h_2k_3\right)}{k_3}-\frac{ik_1k_2\rho_2\omega_3\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\cosh\left(h_1k_3\right)\coth\left(h_2k_3\right)}{k_3}\\
&+\frac{ik_1k_2\rho_2\omega_3e^{h_1k_2-h_1k_1}\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\coth\left(h_2k_3\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{2k_3\omega_1^2}\\
&-\frac{ik_2^2\rho_2\omega_3e^{h_1k_2-h_1k_1}\left(\frac{ig}{2\omega_2}-\frac{i\omega_2}{2k_2}\right)\coth\left(h_2k_3\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{2k_3\omega_1^2}\\
&+\frac{k_1\rho_1\omega_3\left(gk_2+\omega_2^2\right)\sinh\left(h_1k_3\right)}{2k_3\omega_2}-\frac{k_2\rho_1\omega_3\left(gk_2+\omega_2^2\right)\sinh\left(h_1k_3\right)}{2k_3\omega_2}\\
&+\frac{k_2\rho_2\omega_3e^{-h_1k_1-h_1k_2}\left(gk_2+\omega_2^2\right)\coth\left(h_2k_3\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{4k_3\omega_1^2\omega_2}\\
&-\frac{k_1\rho_2\omega_3e^{-h_1k_1-h_1k_2}\left(gk_2+\omega_2^2\right)\coth\left(h_2k_3\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{4k_3\omega_1^2\omega_2}\\
&+\frac{k_1\rho_2\omega_3\left(gk_2+\omega_2^2\right)\cosh\left(h_1k_3\right)\coth\left(h_2k_3\right)}{2k_3\omega_2}-\frac{k_2\rho_2\omega_3\left(gk_2+\omega_2^2\right)\cosh\left(h_1k_3\right)\coth\left(h_2k_3\right)}{2k_3\omega_2}\\
&-\frac{ik_1^2\rho_2\omega_3e^{h_1k_1-h_1k_2}\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\coth\left(h_2k_3\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{2k_3\omega_2^2}\\
&+\frac{ik_2k_1\rho_2\omega_3e^{h_1k_1-h_1k_2}\left(\frac{i\omega_1}{2k_1}-\frac{ig}{2\omega_1}\right)\coth\left(h_2k_3\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{2k_3\omega_2^2}\\
&-\frac{k_1\rho_2\omega_3e^{-h_1k_1-h_1k_2}\left(gk_1+\omega_1^2\right)\coth\left(h_2k_3\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{4k_3\omega_1\omega_2^2}\\
&+\frac{k_2\rho_2\omega_3e^{-h_1k_1-h_1k_2}\left(gk_1+\omega_1^2\right)\coth\left(h_2k_3\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{4k_3\omega_1\omega_2^2}\\
&-\frac{k_1\rho_2\omega_3e^{-h_1k_1-h_1k_2}\coth\left(h_2k_3\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{4k_3\omega_1\omega_2^2\left(e^{2h_1k_1+2(-h_1-h_2)k_1}-1\right)}\\
&-\frac{k_1\rho_2\omega_3e^{h_1k_1+2(-h_1-h_2)k_1-h_1k_2}\coth\left(h_2k_3\right)\left(-gk_1e^{2h_1k_1}+gk_1+\omega_1^2e^{2h_1k_1}+\omega_1^2\right)}{4k_3\omega_1\omega_2^2\left(e^{2h_1k_1+2(-h_1-h_2)k_1}-1\right)}\\
&\qquad\cdot\frac{\left(-gk_2e^{2h_1k_2}+gk_2+\omega_2^2e^{2h_1k_2}+\omega_2^2\right)}{4k_3\omega_1\omega_2^2\left(e^{2h_1k_1+2(-h_1-h_2)k_1}-1\right)}\\
&+\frac{e^{-h_1k_1-h_1k_2}\coth\left(h_2k_3\right)k_2\rho_2\left(e^{2h_1k_1}\omega_1^2+\omega_1^2-e^{2h_1k_1}gk_1+gk_1\right)\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)\omega_3}{4\left(-1+e^{2h_1k_1+2(-h_1-h_2)k_1}\right)k_3\omega_1\omega_2^2}\\
&+\frac{e^{-h_1k_1+h_1k_2+2(-h_1-h_2)k_2}\coth\left(h_2k_3\right)k_2\rho_2\left(e^{2h_1k_1}\omega_1^2+\omega_1^2-e^{2h_1k_1}gk_1+gk_1\right)}{4\left(-1+e^{2h_1k_2+2(-h_1-h_2)k_2}\right)k_3\omega_1^2\omega_2}\\
&\qquad\cdot\frac{\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)\omega_3}{4\left(-1+e^{2h_1k_2+2(-h_1-h_2)k_2}\right)k_3\omega_1^2\omega_2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{e^{-h_1k_1-h_1k_2}\coth\left(h_2k_3\right)k_2\rho_2\left(e^{2h_1k_1}\omega_1^2+\omega_1^2-e^{2h_1k_1}gk_1+gk_1\right)\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)\omega_3}{4\left(-1+e^{2h_1k_2+2(-h_1-h_2)k_2}\right)k_3\omega_1^2\omega_2}\\
&+\frac{e^{h_1k_1+2(-h_1-h_2)k_1-h_1k_2}\coth\left(h_2k_3\right)k_2\rho_2\left(e^{2h_1k_1}\omega_1^2+\omega_1^2-e^{2h_1k_1}gk_1+gk_1\right)}{4\left(-1+e^{2h_1k_1+2(-h_1-h_2)k_1}\right)k_3\omega_1\omega_2^2}\\
&\qquad\qquad\cdot\frac{\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)\omega_3}{4\left(-1+e^{2h_1k_1+2(-h_1-h_2)k_1}\right)k_3\omega_1\omega_2^2}\\
&-\frac{e^{-h_1k_1+h_1k_2+2(-h_1-h_2)k_2}\coth\left(h_2k_3\right)k_1\rho_2\left(e^{2h_1k_1}\omega_1^2+\omega_1^2-e^{2h_1k_1}gk_1+gk_1\right)}{4\left(-1+e^{2h_1k_2+2(-h_1-h_2)k_2}\right)k_3\omega_1^2\omega_2}\\
&\qquad\qquad\cdot\frac{\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)\omega_3}{4\left(-1+e^{2h_1k_2+2(-h_1-h_2)k_2}\right)k_3\omega_1^2\omega_2}\\
&-\left.\frac{e^{-h_1k_1-h_1k_2}\coth\left(h_2k_3\right)k_1\rho_2\left(e^{2h_1k_1}\omega_1^2+\omega_1^2-e^{2h_1k_1}gk_1+gk_1\right)\left(e^{2h_1k_2}\omega_2^2+\omega_2^2-e^{2h_1k_2}gk_2+gk_2\right)\omega_3}{4\left(-1+e^{2h_1k_2+2(-h_1-h_2)k_2}\right)k_3\omega_1^2\omega_2}\right\}\\
&\Bigg/\left\{-\frac{ie^{2h_1k_3}\rho_1\omega_3^3}{k_3\left(e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3\right)}+\frac{2e^{h_1k_3}i\sinh\left(h_1k_3\right)\rho_1\omega_3^3}{k_3\left(e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3\right)}\right.\\
&+\frac{2e^{h_1k_3}i\cosh\left(h_1k_3\right)\coth\left(h_2k_3\right)\rho_2\omega_3^3}{k_3\left(e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3\right)}+\frac{e^{h_1k_3}i\left(\cosh\left(h_1k_3\right)\rho_1+\coth\left(h_2k_3\right)\sinh\left(h_1k_3\right)\rho_2\right)\omega_3^3}{k_3\left(e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3\right)}\\
&+\frac{e^{2h_1k_3+2(-h_1-h_2)k_3}i\rho_2\omega_3}{\left(-1+e^{2h_1k_3+2(-h_1-h_2)k_3}\right)k_3}+\frac{i\rho_2\omega_3}{\left(-1+e^{2h_1k_3+2(-h_1-h_2)k_3}\right)k_3}+\frac{e^{2h_1k_3}gi\rho_1\omega_3}{e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3}\\
&+\frac{2e^{h_1k_3}gi\cosh\left(h_1k_3\right)\left(\rho_1-\rho_2\right)\omega_3}{e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3}-\frac{ie^{h_1k_3}g\sinh\left(h_1k_3\right)\left(\rho_2-\rho_1\right)\omega_3}{e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3}\\
&-\frac{ie^{h_1k_3}g\left(\cosh\left(h_1k_3\right)\rho_1+\coth\left(h_2k_3\right)\sinh\left(h_1k_3\right)\rho_2\right)\omega_3}{e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3}+\frac{i\rho_1\left(\omega_3^2+gk_3\right)\omega_3}{k_3\left(e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3\right)}\\
&-\frac{ie^{h_1k_3}\left(\cosh\left(h_1k_3\right)\rho_1+\coth\left(h_2k_3\right)\sinh\left(h_1k_3\right)\rho_2\right)\left(\omega_3^2+gk_3\right)\omega_3}{k_3\left(e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3\right)}\\
&+\left.\frac{e^{h_1k_3}g^2i\sinh\left(h_1k_3\right)k_3\left(\rho_2-\rho_1\right)}{\omega_3\left(e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3\right)}+\frac{e^{h_1k_3}gi\sinh\left(h_1k_3\right)\left(\rho_2-\rho_1\right)\left(\omega_3^2+gk_3\right)}{\omega_3\left(e^{2h_1k_3}\omega_3^2+\omega_3^2-e^{2h_1k_3}gk_3+gk_3\right)}-\frac{ig\left(\rho_1-\rho_2\right)}{\omega_3}\right\}
\end{aligned}
$$

Appendix B

Proof of (5.58)

In this section, we outline how to prove the result (5.58). That is, we want to show that

$$q_{n,p} = \frac{p^{n-3p}}{(n-3p)!} q_{3p,p}, \quad n \geq 3p. \tag{B.1}$$

Note that the equation can be verified by direct computation for $n = 3p, 3p+1, 3p+2$, $p = 1, 2, 3$. However, we want to show that (B.1) is true in general.

First, recall that the solutions of the three-wave ODEs when $K_2 = K_3 = 0$ are described by (5.45),

$$a_m(\tau) = \frac{e^{i\psi_m}}{\xi} \sum_{n=0}^{\infty} A_{3n}\, \xi^{3n},$$

where the coefficients A_{3n}^m are determined via

$$(3n-1)A_{3n}^m + A_{3n}^{k^*} + A_{3n}^{\ell^*} = -\sum_{p=3}^{3n-3} A_p^{k^*} A_{3n-p}^{\ell^*}, \quad n \geq 2.$$

When $K_2 = K_3 = 0$, we have that $A_n^1 = A_n^2 = A_n^3$, so we can drop the superscripts and simplify to obtain

$$(3n-1)A_{3n} + 2A_{3n}^* = -\sum_{p=1}^{n-1} A_{3p}^* A_{3(n-p)}^*, \quad n \geq 2 \tag{B.2}$$

By induction, we can show that since $A_0 = 1$, $A_3 = i\sigma H/6$, then (B.2) gives that A_{3n} is real when n is even and pure imaginary when n is odd.

(1) Suppose n is odd. It follows that A_{3n} is pure imaginary, so that $A_{3n}^* = -A_{3n}$. Then (B.2) yields

$$3(n-1)A_{3n} = -\sum_{p=1}^{n-1} A_{3p}^* A_{3(n-p)}^*. \tag{B.3}$$

Notice that when p is odd, $3p$ is also odd, so that A_{3p} must be pure imaginary. Since n is odd, the quantity $n-p$ is even when p is odd, so that $A_{3(n-p)}$ is real. That is, when p is odd, we have that

$$A^*_{3p}A^*_{3(n-p)} = (-A_{3p})\,A_{3(n-p)} = -A_{3p}A_{3(n-p)}.$$

If we write $A_{3n} = i\alpha_{3n}$ for n odd and $A_{3n} = \alpha_{3n}$ for n even, then we have

$$A^*_{3p}A^*_{3(n-p)} = -i\alpha_{3p}\,\alpha_{3(n-p)}.$$

If p is even, then A_{3p} is real. Since n is odd, the quantity $n-p$ is odd when p is even, so it follows that $A_{3(n-p)}$ is pure imaginary. Thus, when p is even, we have

$$A^*_{3p}A^*_{3(n-p)} = A_{3p}\left(-A_{3(n-p)}\right) = -i\alpha_{3p}\,\alpha_{3(n-p)}.$$

In summary, if n is odd, we have that

$$3(n-1)\alpha_{3n} = \sum_{p=1}^{n-1} \alpha_{3p}\,\alpha_{3(n-p)},$$

or alternatively,

$$3(n-1)\alpha_{3n} = 2\sum_{p=1}^{\frac{n-1}{2}} \alpha_{3p}\,\alpha_{3(n-p)}, \tag{B.4}$$

where the two negative signs on the right hand side cancelled.

(2) Suppose n is even. It follows that A_{3n} is real, so that $A^*_{3n} = A_{3n}$. Then (B.2) yields

$$(3n+1)A_{3n} = -\sum_{p=1}^{n-1} A^*_{3p}A^*_{3(n-p)}. \tag{B.5}$$

Notice that when p is odd, A_{3p} must be pure imaginary. Since n is even, the quantity $n-p$ is odd when p is odd, so that $A_{3(n-p)}$ is also pure imaginary. That is, when p is odd, we have that

$$A^*_{3p}A^*_{3(n-p)} = (-A_{3p})\left(-A_{3(n-p)}\right) = A_{3p}A_{3(n-p)}.$$

As before, if we write $A_{3p} = i\alpha_{3p}$ for p odd, then we have

$$A^*_{3p}A^*_{3(n-p)} = -\alpha_{3p}\alpha_{3(n-p)},$$

where the minus sign is due to the i^2 term.

If p is even, then A_{3p} is real. Since n is even, the quantity $n-p$ is even when p is even, so it follows that $A_{3(n-p)}$ is also real. Thus, when p is even, we have

$$A_{3p}^{*}A_{3(n-p)}^{*} = A_{3p}A_{3(n-p)} = \alpha_{3p}\,\alpha_{3(n-p)}.$$

In summary, if n is even, we have that

$$\begin{aligned}(3n+1)\alpha_{3n} &= -\sum_{p=1}^{n-1}(-1)^{p}\alpha_{3p}\,\alpha_{3(n-p)}\\ &= \sum_{p=1}^{n-1}(-1)^{p+1}\alpha_{3p}\,\alpha_{3(n-p)},\end{aligned}$$

or alternatively,

$$(3n+1)\alpha_{3n} = (-1)^{\frac{n}{2}+1}\alpha_{3n/2}^{2} + 2\sum_{p=1}^{\frac{n}{2}-1}(-1)^{p+1}\alpha_{3p}\,\alpha_{3(n-p)}. \tag{B.6}$$

We use the relations (B.4) and (B.6) later in the proof.

Next, consider the formal Laurent series solution of the three-wave PDEs. For the series (5.24), the series coefficients can be found by solving the linear system in (5.22). This yields

$$\begin{aligned}\mathcal{A}_n^m(x) = \frac{1}{f(n)}\Big[\left(n^3-3n^2+2\right)b_n^m + (n-1)\,b_n^k + (n-1)\,b_n^\ell\\ -2b_n^{m*} + \left(-n^2+2n+1\right)b_n^{k*} + \left(-n^2+2n+1\right)b_n^{\ell*}\Big],\end{aligned} \tag{B.7}$$

where

$$b_n^m(x) = -\sum_{p=1}^{n-1}\mathcal{A}_p^{k*}\mathcal{A}_{n-p}^{\ell*} \tag{B.8}$$

and

$$f(n) = n(n+1)(n-2)(n-3).$$

We split b_n^m into its real and imaginary parts by writing $b_n^m = b_{n_{\mathrm{Re}}}^m + ib_{n_{\mathrm{Im}}}^m$. Then (B.7) becomes

$$\begin{aligned}\mathcal{A}_n^m = \frac{1}{f(n)}\Big[&\left(n^3-3n^2+2\right)\left(b_{n_{\mathrm{Re}}}^m + ib_{n_{\mathrm{Im}}}^m\right) + (n-1)\left(b_{n_{\mathrm{Re}}}^k + ib_{n_{\mathrm{Im}}}^k\right) + (n-1)\left(b_{n_{\mathrm{Re}}}^\ell + ib_{n_{\mathrm{Im}}}^\ell\right)\\ &-2\left(b_{n_{\mathrm{Re}}}^m - ib_{n_{\mathrm{Im}}}^m\right) + \left(-n^2+2n+1\right)\left(b_{n_{\mathrm{Re}}}^k - ib_{n_{\mathrm{Im}}}^k\right) + \left(-n^2+2n+1\right)\left(b_{n_{\mathrm{Re}}}^\ell + ib_{n_{\mathrm{Im}}}^\ell\right)\Big].\end{aligned}$$

Let $c_m = \max\{|c_1|, |c_2|, |c_3|\} \equiv c$ for $m = 1, 2, 3$, and note that the more general case is analagous. Then $\mathcal{A}_n^1(x) = \mathcal{A}_n^2(x) = \mathcal{A}_n^3(x)$, so that $b_n^1 = b_n^2 = b_n^3$. Thus, we have

$$\begin{aligned}
\mathcal{A}_n^m &= \frac{1}{f(n)}\Big\{ \left[n^3 - 3n^2 + 2 + 2(n-1) - 2 + 2(-n^2 + 2n + 1)\right] b_{n_{\mathrm{Re}}} \\
&\qquad\qquad + \left[n^3 - 3n^2 + 2 + 2(n-1) + 2 - 2(-n^2 + 2n + 1)\right] i b_{n_{\mathrm{Im}}} \Big\} \\
&= \frac{1}{f(n)}\Big\{ \left(n^3 - 5n^2 + 6n\right) b_{n_{\mathrm{Re}}} + \left(n^3 - n^2 - 2n\right) i b_{n_{\mathrm{Im}}} \Big\} \\
&= \frac{1}{n(n+1)(n-2)(n-3)} \Big\{ n(n-2)(n-3) b_{n_{\mathrm{Re}}} + n(n-2)(n+1) i b_{n_{\mathrm{Im}}} \Big\} \\
&= \frac{b_{n_{\mathrm{Re}}}}{n+1} + \frac{i b_{n_{\mathrm{Im}}}}{n-3}.
\end{aligned} \tag{B.9}$$

When $c_1 = c_2 = c_3 = c$ and $\mathcal{H}(x)$ satisfies (5.32), we have from (5.54)

$$|\mathcal{A}_n(x)| \leq \sum_{p=1}^{\lfloor n/3 \rfloor} q_{n,p} (ck)^{n-3p},$$

where $q_{n,p}$ comes from the imaginary part of $\mathcal{A}_n(x)$ if p is odd, and $q_{n,p}$ comes from the real part of $\mathcal{A}_n(x)$ if p is even. Then from (B.8) and (B.9), we know that if p is odd,

$$q_{n,p} = \frac{1}{n-3} \left[p\, q_{n-1,p} + 2 \sum_{\ell=1}^{\frac{p-1}{2}} \sum_{j=3\ell}^{n-3(p-\ell)} q_{j,\ell} q_{n-j,p-\ell} \right], \tag{B.10}$$

and if p is even,

$$q_{n,p} = \frac{1}{n+1} \left[p\, q_{n-1,p} + (-1)^{\frac{p}{2}+1} \sum_{j=\frac{p}{2}}^{n-\frac{3p}{2}} q_{j,\frac{p}{2}} q_{n-j,\frac{p}{2}} + 2 \sum_{\ell=1}^{\frac{p}{2}-1} \sum_{j=3\ell}^{n-3(p-\ell)} (-1)^{\ell+1} q_{j,\ell} q_{n-j,p-\ell} \right]. \tag{B.11}$$

We now use induction to prove that (B.10)-(B.11) are equivalent to (B.1). To start, it is trivial to show that $q_{n,1} = 1/(n-3)!$, which gives us a base case. Now suppose that (B.1) holds for $p = 1, 2, 3, \ldots, P-1$ (for some $P > 2$), and for all $n > 3p$.

First, suppose that P is odd. Then from (B.10) we have

$$q_{n,P} = \frac{1}{n-3} \left[P q_{n-1,P} + 2 \sum_{\ell=1}^{\frac{P-1}{2}} \sum_{j=3\ell}^{n-3(P-\ell)} \frac{\ell^{j-3\ell}}{(j-3\ell)!} \frac{(P-\ell)^{n-j-3(P-\ell)}}{(n-j-3(P-\ell))!} q_{3\ell,\ell} q_{3(p-\ell)p-\ell} \right], \tag{B.12}$$

where we applied (B.1) for $p < P$. Let R_ℓ be the inner sum above, defined by

$$R_\ell = \sum_{j=3\ell}^{n-3(P-\ell)} \frac{\ell^{j-3\ell}}{(j-3\ell)!} \frac{(P-\ell)^{n-j-3(P-\ell)}}{(n-j-3(P-\ell))!}. \tag{B.13}$$

Observe the following

$$
\begin{aligned}
R_\ell &= \sum_{k=0}^{n-3P} \frac{\ell^k}{k!} \frac{(P-\ell)^{n-k-3P}}{(n-k-3P)!} \\
&= \sum_{k=0}^{m} \frac{\ell^k}{k!} \frac{(P-\ell)^{m-k}}{(m-k)!},
\end{aligned}
$$

where we defined $m = n - 3P$. It follows that

$$
\begin{aligned}
m!R_\ell &= \sum_{k=0}^{m} m! \frac{\ell^k}{k!} \frac{(P-\ell)^{m-k}}{(m-k)!} \\
&= \sum_{k=0}^{m} \binom{m}{k} \ell^k (P-\ell)^{m-k} \\
&= (P-\ell)^m \sum_{k=0}^{m} \binom{m}{k} \left(\frac{\ell}{P-\ell}\right)^k \\
&= (P-\ell)^m \left(1 + \frac{\ell}{P-\ell}\right)^m \\
&= (P-\ell)^m \left(\frac{P}{P-\ell}\right)^m \\
&= P^m,
\end{aligned}
$$

where the fourth equality above is due to [17, §26.3.4]. Finally, using $m = n - 3P$, we have

$$
R_\ell = \frac{P^{n-3P}}{(n-3P)!}. \tag{B.14}
$$

Consequently, (B.12) becomes

$$
q_{n,P} = \frac{1}{n-3} \left[P q_{n-1,P} + 2 \frac{P^{n-3P}}{(n-3P)!} \sum_{\ell=1}^{\frac{P-1}{2}} q_{3\ell,\ell} q_{3(p-\ell)p-\ell} \right].
$$

Now, substitute (B.1) for $p = P$ into the equation above. We obtain

$$
\begin{aligned}
\frac{P^{n-3P}}{(n-3P)!} q_{3P,P} &= \frac{1}{n-3} \left[P \frac{P^{n-1-3P}}{(n-1-3P)!} q_{3P,P} + 2 \frac{P^{n-3P}}{(n-3P)!} \sum_{\ell=1}^{\frac{P-1}{2}} q_{3\ell,\ell} q_{3(p-\ell)p-\ell} \right] \\
&= \frac{1}{n-3} \left[\frac{(n-3P)P^{n-3P}}{(n-3P)!} q_{3P,P} + 2 \frac{P^{n-3P}}{(n-3P)!} \sum_{\ell=1}^{\frac{P-1}{2}} q_{3\ell,\ell} q_{3(p-\ell)p-\ell} \right].
\end{aligned}
$$

Simplifying, we have

$$
\Big[(n-3) - (n-3P)\Big] q_{3P,P} = 2 \sum_{\ell=1}^{\frac{P-1}{2}} q_{3\ell,\ell} q_{3(p-\ell)p-\ell}.
$$

That is,

$$3(P-1)q_{3P,P} = 2\sum_{\ell=1}^{\frac{P-1}{2}} q_{3\ell,\ell}q_{3(p-\ell)p-\ell}.$$

However, this is equivalent to (B.4) with $q_{3P,P} = \alpha_{3P}$. As a result, we know that (B.1) satisfies (B.10) exactly. This is now true for any $n > 3p$ and for $p \geq 1$. Equation (B.10) gives a unique sequence because the sequence is defined explicitly starting at the known value of $q_{3p,p}$. Consequently, if the sequence defined by (B.1) (again, starting at $q_{3p,p}$) also satisfies (B.10), then it must be the unique sequence produced by (B.10).

Next, suppose that P is even. Then from (B.11) we have

$$\begin{aligned} q_{n,P} = \frac{1}{n+1}\Bigg[&Pq_{n-1,P} + (-1)^{\frac{P}{2}+1}\sum_{j=\frac{3P}{2}}^{n-\frac{3P}{2}} \frac{\left(\frac{P}{2}\right)^{j-\frac{3P}{2}}}{(j-\frac{3P}{2})!}\frac{(\frac{P}{2})^{n-j-\frac{3P}{2}}}{(n-j-\frac{3P}{2})!}q^2_{\frac{3P}{2},\frac{P}{2}} \\ &+2\sum_{\ell=1}^{\frac{P}{2}-1}\sum_{j=3\ell}^{n-3(P-\ell)}(-1)^{\ell+1}\frac{\ell^{j-3\ell}}{(j-3\ell)!}\frac{(P-\ell)^{n-j-3(P-\ell)}}{(n-j-3(P-\ell))!}q_{3\ell,\ell}q_{3(p-\ell)p-\ell}\Bigg], \end{aligned} \tag{B.15}$$

where we applied (B.1) for $p < P$.

Using (B.13)-(B.14), we obtain

$$\begin{aligned} q_{n,P} &= \frac{1}{n+1}\left[Pq_{n-1,P} + (-1)^{\frac{P}{2}+1}\frac{P^{n-3P}}{(n-3P)!}q^2_{\frac{3P}{2},\frac{P}{2}} + 2\frac{P^{n-3P}}{(n-3P)!}\sum_{\ell=1}^{\frac{P}{2}-1}(-1)^{\ell+1}q_{3\ell,\ell}q_{3(p-\ell)p-\ell}\right], \\ &= \frac{1}{n+1}\left\{Pq_{n-1,P} + \frac{P^{n-3P}}{(n-3P)!}\left[(-1)^{\frac{P}{2}+1}q^2_{\frac{3P}{2},\frac{P}{2}} + 2\sum_{\ell=1}^{\frac{P}{2}-1}(-1)^{\ell+1}q_{3\ell,\ell}q_{3(p-\ell)p-\ell}\right]\right\}. \end{aligned}$$

Now, substitute (B.1) for $p = P$ into the equation above. We obtain

$$\begin{aligned} \frac{P^{n-3P}}{(n-3P)!}q_{3P,P} &= \frac{1}{n+1}\Bigg\{P\frac{P^{n-1-3P}}{(n-1-3P)!}q_{3P,P} \\ &\qquad + \frac{P^{n-3P}}{(n-3P)!}\left[(-1)^{\frac{P}{2}+1}q^2_{\frac{3P}{2},\frac{P}{2}} + 2\sum_{\ell=1}^{\frac{P}{2}-1}(-1)^{\ell+1}q_{3\ell,\ell}q_{3(p-\ell)p-\ell}\right]\Bigg\} \\ &= \frac{1}{n+1}\Bigg\{\frac{(n-3P)P^{n-3P}}{(n-3P)!}q_{3P,P} \\ &\qquad + \frac{P^{n-3P}}{(n-3P)!}\left[(-1)^{\frac{P}{2}+1}q^2_{\frac{3P}{2},\frac{P}{2}} + 2\sum_{\ell=1}^{\frac{P}{2}-1}(-1)^{\ell+1}q_{3\ell,\ell}q_{3(p-\ell)p-\ell}\right]\Bigg\}. \end{aligned}$$

Simplifying, we have

$$\Big[(n+1)-(n-3P)\Big]q_{3P,P} = (-1)^{\frac{P}{2}+1}q^2_{\frac{3P}{2},\frac{P}{2}} + 2\sum_{\ell=1}^{\frac{P}{2}-1}(-1)^{\ell+1}q_{3\ell,\ell}q_{3(p-\ell)p-\ell}.$$

That is,

$$(3P+1)q_{3P,P} = (-1)^{\frac{P}{2}+1}q^2_{\frac{3P}{2},\frac{P}{2}} + 2\sum_{\ell=1}^{\frac{P}{2}-1}(-1)^{\ell+1}q_{3\ell,\ell}q_{3(p-\ell)p-\ell}.$$

However, this is equivalent to (B.6) with $q_{3P,P} = \alpha_{3P}$. As a result, we know that (B.1) satisfies (B.11) exactly. This is now true for any $n > 3p$ and for $p \geq 1$. Equation (B.11) gives a unique sequence because the sequence is defined explicitly starting at the known value of $q_{3p,p}$. Consequently, if the sequence defined by (B.1) (again, starting at $q_{3p,p}$) also satisfies (B.11), then it must be the unique sequence produced by (B.11).

Appendix C

Proof of (5.94)

In this section, we outline how to prove the result (5.94). That is, we want to show that

$$r_{n,p} \le \frac{(\gamma\, p)^{n-2p}}{(n-2p)!} \sum_{\ell=1}^{p-1} r_{2\ell,\ell}\, r_{2(p-\ell),p-\ell}, \tag{C.1}$$

for $n \ge 2p$ and for $p \ge 2$, where $r_{n,p}$ is defined in (5.92) and γ is a real, nonnegative constant. In order to prove (C.1), we follow the procedure below.

STEP 1: It is straightforward to show from the recursion relation for $\mathcal{A}_n^m(x)$ in (5.26) with constant phases that

$$\sum_{m=1}^{3} |\mathcal{A}_n^m| \le \frac{n+4}{(n+1)(n-2)} \left[c \sum_{m=1}^{3} \left|\mathcal{A}_{n-1}^{m'}\right| + \sum_{p=2}^{n-2} \left(|\mathcal{A}_p^1|\; |\mathcal{A}_{n-p}^2| + |\mathcal{A}_p^1|\; |\mathcal{A}_{n-p}^3| + |\mathcal{A}_p^2|\; |\mathcal{A}_{n-p}^3| \right) \right]. \tag{C.2}$$

Moreover, we know that

$$\left|\mathcal{A}_n^1\right| + \left|\mathcal{A}_n^2\right| + \left|\mathcal{A}_n^3\right| \le r_{n,1}(ck)^{n-2} + r_{n,2}(ck)^{n-4} + r_{n,3}(ck)^{n-6} + \cdots + r_{n,p}(ck)^{n-2p},$$

$$\left|\mathcal{A}_n^{1'}\right| + \left|\mathcal{A}_n^{2'}\right| + \left|\mathcal{A}_n^{3'}\right| \le k \left[r_{n,1}(ck)^{n-2} + 2r_{n,2}(ck)^{n-4} + 3r_{n,3}(ck)^{n-6} + \cdots + p r_{n,p}(ck)^{n-2p} \right].$$

STEP 2: Recall the definition of $r_{n,p}$ in (C.3),

$$r_{n,p} = \frac{n+4}{(n+1)(n-2)} \left[p\, r_{n-1,p} + \frac{1}{2} \sum_{\ell=1}^{p-1} \sum_{j=2\ell}^{n-2(p-\ell)} r_{j,\ell}\, r_{n-j,p-\ell} \right] \quad \text{for} \quad n > 2p, \tag{C.3}$$

for $p \ge 2$. Assume that

$$r_{n,p} \le \frac{(\gamma\, p)^{n-2p}}{(n-2p)!} \sum_{\ell=1}^{p-1} r_{2\ell,\ell}\, r_{2(p-\ell),p-\ell}$$

holds for $p < P$. Substitute into (C.3) to obtain

$$r_{n,P} \leq \frac{n+4}{(n+1)(n-2)}\Bigg[P\, r_{n-1,P} + \frac{1}{2}\sum_{\ell=1}^{P-1}\sum_{j=2\ell}^{n-2(P-\ell)} \frac{(\gamma\ell)^{j-2\ell}}{(j-2\ell)!}\,\frac{(\gamma(P-\ell))^{n-j-2(P-\ell)}}{(n-j-2(P-\ell))!}\, r_{2\ell,\ell}\, r_{2(P-\ell),P-\ell}\Bigg].$$

Rearrange the inner sum to find that

$$r_{n,P} \leq \frac{n+4}{(n+1)(n-2)}\left[P r_{n-1,P} + \frac{(\gamma P)^{n-2P}}{2(n-2P)!}\sum_{\ell=1}^{P-1} r_{2\ell,\ell}\, r_{2(P-\ell),P-\ell}\right]. \tag{C.4}$$

Define

$$X_{n-1,P} = \frac{(\gamma P)^{n-2P}}{2(n-2P)!}\sum_{\ell=1}^{P-1} r_{2\ell,\ell}\, r_{2(P-\ell),P-\ell}, \tag{C.5}$$

so that (C.4) becomes

$$r_{n,P} \leq \frac{n+4}{(n+1)(n-2)}\Big[P r_{n-1,P} + X_{n-1,P}\Big]. \tag{C.6}$$

Step 4: Substitute (C.6) into itself recursively, to obtain

$$r_{n,P} \leq \frac{P^{n-2P}(n+4)(n+3)(n+2)\cdots(6+2P)(5+2P)}{(n+1)n(n-1)(n-2)^2(n-3)^2(n-4)^2\cdots(2+2P)^2(1+2P)(2P)(2P-1)}\, r_{2P,P} + \sum_{k=1}^{n-2P} S_k, \tag{C.7}$$

where

$$S_1 = \frac{n+4}{(n+1)(n-2)} X_{n-1,P} \quad \text{and} \quad S_k = \frac{(n+5-k)(n+2-k-2p)}{3(n+2-k)(n-1-k)}\, S_{k-1},$$

for $k = 2, 3, 4, \ldots, n-2P$. Observe that

$$S_k \leq \frac{n+5-k}{3(n-1-k)}\, S_{k-1}. \tag{C.8}$$

Notice that $n+5-k \leq 3(n-1-k)$ whenever $k \leq n-4$. Moreover, notice that $n-2P \leq n-4$ when $P \geq 2$. Since S_k is only defined when $k \leq n-2P$, it follows that $k \leq n-2P \leq n-4$, so that $n+5-k \leq 3(n-1-k)$ for all allowed values of k. As a result, (C.8) tells us that

$$S_k \leq S_{k-1} \quad \text{for } k = 2, 3, 4, \ldots, n-2P.$$

Finally, substituting the bound on S_k into (C.7) yields

$$r_{n,P} \le \frac{P^{n-2P}(n+4)(n+3)(n+2)\cdots(6+2P)(5+2P)}{(n+1)n(n-1)(n-2)^2(n-3)^2(n-4)^2\cdots(2+2P)^2(1+2P)(2P)(2P-1)}\, r_{2P,P} \\ + \frac{(n-2P)(n+4)}{(n+1)(n-2)}\,\frac{(\gamma P)^{n-2P}}{2(n-2P)!}\sum_{\ell=1}^{P-1} r_{2\ell,\ell}\, r_{2(P-\ell),P-\ell}. \tag{C.9}$$

STEP 5: It is straightforward to show from the three-wave ODE problem that

$$r_{2n,n} \le \frac{n+2}{2(2n+1)(n-1)}\sum_{p=1}^{n-1} r_{2p,p} r_{2(n-p),n-p}.$$

Substituting into (C.9) and simplifying yields

$$\begin{aligned} r_{n,P} &\le \Bigg[\frac{P^{n-2P}(n+4)!(2P-2)!(1+2P)!}{(4+2P)!(n+1)!(n-2)!}\,\frac{P+2}{(2P+1)(P-1)} \\ &\qquad + \frac{(n-2P)(n+4)}{(n+1)(n-2)}\,\frac{(\gamma P)^{n-2P}}{(n-2P)!}\Bigg]\frac{1}{2}\sum_{\ell=1}^{P-1} r_{2\ell,\ell}\, r_{2(P-\ell),P-\ell} \\ &= \frac{(4+n)P^{n-2P}}{4}\Bigg[\frac{2\,\gamma^{n-2P}}{(n+1)(n-2)\,\Gamma(n-2P)} \\ &\qquad + \frac{(n+3)(n+2)\,\Gamma(2P-2)}{(2P+3)(2P+1)(P+1)\,\Gamma(n-1)}\Bigg]\sum_{\ell=1}^{P-1} r_{2\ell,\ell}\, r_{2(P-\ell),P-\ell}. \end{aligned}$$

It can be shown that

$$\frac{(4+n)P^{n-2P}}{4}\left[\frac{2\,\gamma^{n-2P}}{(n+1)(n-2)\,\Gamma(n-2P)}+\frac{(n+3)(n+2)\,\Gamma(2P-2)}{(2P+3)(2P+1)(P+1)\,\Gamma(n-1)}\right] \le \frac{\gamma^{n-2P}}{(n-2P)!}$$

whenever $n \ge 2p$. Thus, we have

$$r_{n,P} \le \frac{\gamma^{n-2P}}{(n-2P)!}\sum_{\ell=1}^{P-1} r_{2\ell,\ell}\, r_{2(P-\ell),P-\ell}.$$

A base case analysis tells us that $\gamma = 3$ is sufficient.

Appendix D

Proof of (5.134)

In this section, we outline how to prove the result (5.134). That is, we want to show that

$$\sum_{j=1}^{n-1} r_{2j} \cdot r_{2(n-j)} \leq \kappa(n)\, r_{2n} \tag{D.1}$$

for some function $\kappa(n)$ to be determined.

Recall from (5.66) that the Laurent series solution of the three-wave ODEs with $H = 0$ has the expansion

$$a_m(\tau) = \frac{e^{i\psi_m}}{\xi} \sum_{n=0}^{\infty} A_{2n}^m \xi^{2n},$$

where

$$A_{2n}^m = \frac{1}{2(2n+1)(n-1)} \left[2n\, b_{2n}^m - b_{2n}^k - b_{2n}^\ell \right], \tag{D.2}$$

and

$$b_{2n}^m = -\sum_{j=1}^{n-1} A_{2j}^k A_{2(n-j)}^\ell, \tag{D.3}$$

for $n \geq 2$. Additionally, recall from (5.140) that $A_{2n}^m = \alpha_{2n}^m K^n$. Canceling a common factor of K^n, we find that (D.2)-(D.3) become

$$\alpha_{2n}^m = \frac{1}{2(2n+1)(n-1)} \left[2n\, b_{2n}^m - b_{2n}^k - b_{2n}^\ell \right], \tag{D.4}$$

where now

$$b_{2n}^m = -\sum_{j=1}^{n-1} \alpha_{2j}^k \alpha_{2(n-j)}^\ell. \tag{D.5}$$

Finally, recall that

$$r_{2n} = \left|\alpha_{2n}^1\right| + \left|\alpha_{2n}^2\right| + \left|\alpha_{2n}^3\right|. \tag{D.6}$$

In the special case where $K_3 = 2K_2$, we have that

$$\alpha_2^1 = \frac{1}{2}, \quad \alpha_2^2 = 0, \quad \text{and} \quad \alpha_2^3 = -\frac{1}{2}.$$

Additionally, from (D.4)-(D.5), we find the coefficients α_{2n}^m for $n = 2, 3, 4$ are given by

$$\begin{aligned}
\alpha_4^1 &= -\frac{1}{40}, & \alpha_4^2 &= \frac{1}{10}, & \text{and} \quad \alpha_4^3 &= -\frac{1}{40}, \\
\alpha_6^1 &= \frac{1}{80}, & \alpha_6^2 &= 0, & \text{and} \quad \alpha_6^3 &= -\frac{1}{80}, \\
\alpha_8^1 &= \frac{1}{9600}, & \alpha_8^2 &= \frac{1}{600} & \text{and} \quad \alpha_8^3 &= \frac{1}{9600}.
\end{aligned}$$

The values of α_{2n}^m for $n = 2, 3, 4$ indicate that there are three distinct cases to consider. In particular, we consider the following cases

Case A: n is odd, $n = 2\mu + 1$ $(\mu = 0, 1, 2, 3, \ldots)$,

$$\begin{aligned}
&\alpha_{2n}^1 > 0, \quad \alpha_{2n}^2 = 0, \quad \text{and} \quad \alpha_{2n}^3 = -\alpha_{2n}^1, \\
&b_{2n}^1 > 0, \quad b_{2n}^2 = 0, \quad \text{and} \quad b_{2n}^3 = -b_{2n}^1.
\end{aligned} \tag{D.7}$$

Case B: n is even and not a multiple of four, $n = 4\mu + 2$ $(\mu = 0, 1, 2, 3, \ldots)$,

$$\begin{aligned}
&\alpha_{2n}^1 < 0, \quad \alpha_{2n}^2 > 0, \quad \text{and} \quad \alpha_{2n}^3 = \alpha_{2n}^1. \\
&b_{2n}^1 > 0, \quad b_{2n}^2 > 0, \quad \text{and} \quad b_{2n}^3 = b_{2n}^1.
\end{aligned} \tag{D.8}$$

Case C: n is even and a multiple of four, $n = 4\mu$ $(\mu = 1, 2, 3, \ldots)$,

$$\begin{aligned}
&\alpha_{2n}^1 > 0, \quad \alpha_{2n}^2 > 0, \quad \text{and} \quad \alpha_{2n}^3 = \alpha_{2n}^1. \\
&b_{2n}^1 > 0, \quad b_{2n}^2 > 0, \quad \text{and} \quad b_{2n}^3 = b_{2n}^1.
\end{aligned} \tag{D.9}$$

Note that (D.7)-(D.9) can be proven inductively using the definitions in (D.4)-(D.5).

First, consider Case A, where n is odd. Using (D.7) in the definition of r_{2n} in (D.6), we have that for n odd,

$$r_{2n} = \left|\alpha_{2n}^1\right| + \left|\alpha_{2n}^2\right| + \left|\alpha_{2n}^3\right| = 2\alpha_{2n}^1. \tag{D.10}$$

Then from (D.4) and (D.7), it follows that

$$\begin{aligned}
r_{2n} &= 2\alpha_{2n}^1 \\
&= \frac{1}{(2n+1)(n-1)}\left[2n\, b_{2n}^1 - b_{2n}^2 - b_{2n}^3\right] \\
&= \frac{1}{(2n+1)(n-1)} \cdot (2n+1) b_{2n}^1 \\
&= \frac{1}{n-1} \cdot b_{2n}^1 \\
&= -\frac{1}{n-1} \sum_{j=1}^{n-1} \alpha_{2j}^2\, \alpha_{2(n-j)}^3 \\
&= -\frac{1}{n-1}\left[\alpha_2^2 \alpha_{2(n-1)}^3 + \alpha_4^2 \alpha_{2(n-2)}^3 + \alpha_6^2 \alpha_{2(n-3)}^3 + \cdots + \alpha_{2(n-3)}^2 \alpha_6^3 + \alpha_{2(n-2)}^2 \alpha_4^3 + \alpha_{2(n-1)}^2 \alpha_2^3\right] \\
&= -\frac{1}{n-1}\left[\alpha_4^2 \alpha_{2(n-2)}^3 + \alpha_8^2 \alpha_{2(n-4)}^3 + \alpha_{12}^2 \alpha_{2(n-6)}^3 + \cdots + \alpha_{2(n-5)}^2 \alpha_{10}^3 + \alpha_{2(n-3)}^2 \alpha_6^3 + \alpha_{2(n-1)}^2 \alpha_2^3\right] \\
&= \frac{1}{n-1}\left[\alpha_4^2 \alpha_{2(n-2)}^1 + \alpha_8^2 \alpha_{2(n-4)}^1 + \alpha_{12}^2 \alpha_{2(n-6)}^1 + \cdots + \alpha_{2(n-5)}^2 \alpha_{10}^1 + \alpha_{2(n-3)}^2 \alpha_6^1 + \alpha_{2(n-1)}^2 \alpha_2^1\right],
\end{aligned} \tag{D.11}$$

where we used the fact that when j is odd, then $\alpha_{2n}^2 = 0$. Additionally, when j is even, then $n-j$ is odd, so that $\alpha_{2(n-j)}^3 = -\alpha_{2(n-j)}^1$.

Next, consider

$$\sum_{j=1}^{n-1} r_{2j}\, r_{2(n-j)}$$

We know from (D.10) that when j is odd, then $r_{2j} = 2\alpha_{2j}^1$. From (D.8)-(D.9), we can see that when j is even,

$$r_{2j} = \left|\alpha_{2j}^1\right| + \left|\alpha_{2j}^2\right| + \left|\alpha_{2j}^3\right| = 2\left|\alpha_{2j}^1\right| + \left|\alpha_{2j}^2\right|.$$

It follows that

$$\begin{aligned}
&\sum_{j=1}^{n-1} r_{2j}\, r_{2(n-j)} \\
&= r_2 r_{2(n-1)} + r_4 r_{2(n-2)} + r_6 r_{2(n-3)} + \cdots + r_{2(n-3)} r_6 + r_{2(n-2)} r_4 + r_{2(n-1)} r_2 \\
&= 2\alpha_2^1 \cdot \left(2\left|\alpha_{2(n-1)}^1\right| + \left|\alpha_{2(n-1)}^2\right|\right) + \left(2\left|\alpha_4^1\right| + \left|\alpha_4^2\right|\right) \cdot 2\alpha_{2(n-2)}^1 + 2\alpha_6^1 \cdot \left(2\left|\alpha_{2(n-3)}^1\right| + \left|\alpha_{2(n-3)}^2\right|\right) \\
&\qquad + \cdots + \left(2\left|\alpha_{2(n-1)}^1\right| + \left|\alpha_{2(n-1)}^2\right|\right) \cdot 2\alpha_2^1
\end{aligned}$$

$$= 4\Big[\alpha_2^1\left|\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\right|\alpha_{2(n-2)}^1 + \alpha_6^1\left|\alpha_{2(n-3)}^1\right| + \cdots + \alpha_{2(n-2)}^1\left|\alpha_4^1\right| + \left|\alpha_{2(n-1)}^1\right|\alpha_2^1\Big]$$
$$+ 2\Big[\alpha_2^1\left|\alpha_{2(n-1)}^2\right| + \left|\alpha_4^2\right|\alpha_{2(n-2)}^1 + \alpha_6^1\left|\alpha_{2(n-3)}^2\right| + \cdots + \alpha_{2(n-2)}^1\left|\alpha_4^2\right| + \left|\alpha_{2(n-1)}^2\right|\alpha_2^1\Big].$$

Since α_{2n}^1 is always positive when n is odd, we can rewrite the above as

$$\begin{aligned}
\sum_{j=1}^{n-1} r_{2j}\, r_{2(n-j)} &= 4\Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots + \left|\alpha_{2(n-2)}^1\alpha_4^1\right| + \left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Big] \\
&\quad + 2\Big[\left|\alpha_2^1\alpha_{2(n-1)}^2\right| + \left|\alpha_4^2\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^2\right| + \cdots + \left|\alpha_{2(n-2)}^1\alpha_4^2\right| + \left|\alpha_{2(n-1)}^2\alpha_2^1\right|\Big] \\
&= 8\Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots + \left|\alpha_{2(\frac{n-1}{2})}^1\alpha_{2(\frac{n+1}{2})}^1\right|\Big] \\
&\quad + 4\Big[\left|\alpha_2^1\alpha_{2(n-1)}^2\right| + \left|\alpha_4^2\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^2\right| + \cdots + \left|\alpha_{2(\frac{n\pm1}{2})}^1\alpha_{2(\frac{n\mp1}{2})}^2\right|\Big], \qquad \text{(D.12)}
\end{aligned}$$

where the choice of $\pm$ in the last line will depend on whether $\frac{n+1}{2}$ is even or odd. Furthermore, since α_{2n}^2 is always positive when n is even, we can write (D.11) as

$$r_{2n} = \frac{1}{n-1}\Big[\left|\alpha_4^2\alpha_{2(n-2)}^1\right| + \left|\alpha_8^2\alpha_{2(n-4)}^1\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^1\right| + \cdots + \left|\alpha_{2(n-3)}^2\alpha_6^1\right| + \left|\alpha_{2(n-1)}^2\alpha_2^1\right|\Big] \qquad \text{(D.13)}$$

Observe the following

$$\begin{aligned}
&\kappa(n)r_{2n} - \sum_{j=1}^{n-1} r_{2j}r_{2(n-j)} \\
&= \frac{\kappa(n)}{n-1}\Big[\left|\alpha_4^2\alpha_{2(n-2)}^1\right| + \left|\alpha_8^2\alpha_{2(n-4)}^1\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^1\right| + \cdots + \left|\alpha_{2(n-3)}^2\alpha_6^1\right| + \left|\alpha_{2(n-1)}^2\alpha_2^1\right|\Big] \quad \leftarrow \tfrac{(n-1)}{2} \text{ terms} \\
&\quad - 8\Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots + \left|\alpha_{2(\frac{n-1}{2})}^1\alpha_{2(\frac{n+1}{2})}^1\right|\Big] \quad \leftarrow \tfrac{(n-1)}{2} \text{ terms} \\
&\quad - 4\Big[\left|\alpha_2^1\alpha_{2(n-1)}^2\right| + \left|\alpha_4^2\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^2\right| + \cdots + \left|\alpha_{2(\frac{n\pm1}{2})}^1\alpha_{2(\frac{n\mp1}{2})}^2\right|\Big] \quad \leftarrow \tfrac{(n-1)}{2} \text{ terms} \\
&= \frac{\kappa(n)-4(n-1)}{n-1}\Big[\left|\alpha_2^1\alpha_{2(n-1)}^2\right| + \left|\alpha_4^2\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^2\right| + \cdots + \left|\alpha_{2(\frac{n\pm1}{2})}^1\alpha_{2(\frac{n\mp1}{2})}^2\right|\Big] \\
&\qquad - 8\Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots + \left|\alpha_{2(\frac{n-1}{2})}^1\alpha_{2(\frac{n+1}{2})}^1\right|\Big].
\end{aligned}$$

Define

$$\beta(n) = \frac{\kappa(n)-4(n-1)}{n-1}.$$

Now we have

$$
\begin{aligned}
&\kappa(n) r_{2n} - \sum_{j=1}^{n-1} r_{2j} r_{2(n-j)} \\
&= \left|\alpha_2^1\right| \left[\beta(n) \left|\alpha_{2(n-1)}^2\right| - 8 \left|\alpha_{2(n-1)}^1\right|\right] + \left|\alpha_6^1\right| \left[\beta(n) \left|\alpha_{2(n-3)}^2\right| - 8 \left|\alpha_{2(n-3)}^1\right|\right] \\
&+ \left|\alpha_{10}^1\right| \left[\beta(n) \left|\alpha_{2(n-5)}^2\right| - 8 \left|\alpha_{2(n-5)}^1\right|\right] + \cdots + \left|\alpha_{2\left(\frac{n\pm 1}{2}\right)}^1\right| \left[\beta(n) \left|\alpha_{2\left(\frac{n\mp 1}{2}\right)}^2\right| - 8 \left|\alpha_{2\left(\frac{n\mp 1}{2}\right)}^1\right|\right].
\end{aligned}
$$

Using induction, we can show that when n is even

$$
\left|\alpha_{2n}^1\right| \leq \left|\alpha_{2n}^2\right|. \tag{D.14}
$$

As a result, we can see that $\kappa(n) r_{2n} - \sum_{j=1}^{n-1} r_{2j} r_{2(n-j)}$ is certainly positive if $\beta(n) \geq 8$. That is, if

$$
\beta(n) = \frac{\kappa(n) - 4(n-1)}{n-1} \geq 8.
$$

It is sufficient to choose

$$
\kappa(n) = 12(n-1).
$$

Thus, using (D.7)-(D.9) and (D.14), we have shown that for n odd

$$
12(n-1) r_{2n} - \sum_{j=1}^{n-1} r_{2j} r_{2(n-j)} \geq 0,
$$

or

$$
\sum_{j=1}^{n-1} r_{2j} r_{2(n-j)} \leq 12(n-1)\, r_{2n}.
$$

In other words, we have proven (D.1) when n is odd with $\kappa(n) = 12(n-1)$ for Case A.

Next, consider Case B, in which n is even, but not a multiple of four ($n = 4\mu + 2$ for $\mu = 0, 1, 2, \ldots$). This time, we have

$$\begin{aligned}
r_{2n} &= \left|\alpha^1_{2n}\right| + \left|\alpha^2_{2n}\right| + \left|\alpha^3_{2n}\right| \\
&= 2\left|\alpha^1_{2n}\right| + \left|\alpha^2_{2n}\right| \\
&= -2\alpha^1_{2n} + \alpha^2_{2n} \\
&= -\frac{2}{2(2n+1)(n-1)}\left[2n\, b^1_{2n} - b^2_{2n} - b^3_{2n}\right] + \frac{1}{2(2n+1)(n-1)}\left[2n\, b^2_{2n} - b^1_{2n} - b^3_{2n}\right] \\
&= \frac{1}{2(2n+1)(n-1)}\left[-(4n+1)\, b^1_{2n} + (2+2n)\, b^2_{2n} + b^3_{2n}\right] \\
&= \frac{1}{2(2n+1)(n-1)}\left[-4n\, b^1_{2n} + 2(n+1) b^2_{2n}\right] \\
&= \frac{1}{(2n+1)(n-1)}\left[-2n\, b^1_{2n} + (n+1) b^2_{2n}\right], \\
&= \frac{1}{(2n+1)(n-1)}\left[-2n\, b^3_{2n} + (n+1) b^2_{2n}\right],
\end{aligned}$$

where we used the fact that $b^3_{2n} = b^1_{2n}$ from (D.8). Substituting from (D.5) yields

$$\begin{aligned}
r_{2n} &= \frac{1}{(2n+1)(n-1)}\left[2n\sum_{j=1}^{n-1}\alpha^2_{2j}\alpha^3_{2(n-j)} - (n+1)\sum_{j=1}^{n-1}\alpha^1_{2j}\alpha^3_{2(n-j)}\right] \\
&= \frac{1}{(2n+1)(n-1)}\left\{2n\left[\alpha^2_2\alpha^3_{2(n-1)} + \alpha^2_4\alpha^3_{2(n-2)} + \alpha^2_6\alpha^3_{2(n-3)} + \cdots + \alpha^2_{2(n-2)}\alpha^3_4 + \alpha^2_{2(n-1)}\alpha^3_2\right]\right. \\
&\qquad\left. - (n+1)\left[\alpha^1_2\alpha^3_{2(n-1)} + \alpha^1_4\alpha^3_{2(n-2)} + \alpha^1_6\alpha^3_{2(n-3)} + \cdots + \alpha^1_{2(n-2)}\alpha^3_4 + \alpha^1_{2(n-1)}\alpha^3_2\right]\right\} \\
&= \frac{1}{(2n+1)(n-1)}\left\{2n\left[\alpha^2_4\alpha^1_{2(n-2)} + \alpha^2_8\alpha^1_{2(n-4)} + \alpha^2_{12}\alpha^1_{2(n-6)} + \cdots + \alpha^2_{2(n-4)}\alpha^1_8 + \alpha^2_{2(n-2)}\alpha^1_4\right]\right. \\
&\qquad\left. - (n+1)\left[-\alpha^1_2\alpha^1_{2(n-1)} + \alpha^1_4\alpha^1_{2(n-2)} - \alpha^1_6\alpha^1_{2(n-3)} + \cdots + \alpha^1_{2(n-2)}\alpha^1_4 - \alpha^1_{2(n-1)}\alpha^1_2\right]\right\},
\end{aligned} \tag{D.15}$$

where we used the fact when n is even, then j and $n-j$ are either both even or both odd. We also used properties from (D.8).

Next, observe the following, noting that if $n = 4m+2$, then $n/2$ is odd.

$$\sum_{j=1}^{n-1} r_{2j}\, r_{2(n-j)} = r_2 r_{2(n-1)} + r_4 r_{2(n-2)} + r_6 r_{2(n-3)} + \cdots + r^2_{2\left(\frac{n}{2}\right)} + \cdots + r_{2(n-3)} r_6 + r_{2(n-2)} r_4 + r_{2(n-1)} r_2$$

$$\begin{aligned}
&= 2\alpha_2^1 \cdot 2\alpha_{2(n-1)}^1 + \left(2\left|\alpha_4^1\right| + \left|\alpha_4^2\right|\right)\left(2\left|\alpha_{2(n-2)}^1\right| + \left|\alpha_{2(n-2)}^2\right|\right) + 2\alpha_6^1 \cdot 2\alpha_{2(n-3)}^1 \\
&+ \cdots + \left(2\alpha^1_{2\left(\frac{n}{2}\right)}\right)^2 + \cdots + \left(2\left|\alpha_{2(n-2)}^1\right| + \left|\alpha_{2(n-2)}^2\right|\right)\left(2\left|\alpha_4^1\right| + \left|\alpha_4^2\right|\right) + 2\alpha_2^1 \cdot 2\alpha_{2(n-1)}^1 \\
&= 4\Bigg[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots + \left|\alpha^1_{2\left(\frac{n}{2}\right)}\alpha^1_{2\left(\frac{n}{2}\right)}\right| + \cdots \\
&\qquad\qquad + \left|\alpha_{2(n-2)}^1\alpha_4^1\right| + \left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Bigg] \\
&+ 2\Bigg[\left|\alpha_4^1\alpha_{2(n-2)}^2\right| + \left|\alpha_8^1\alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^1\alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^1\alpha_8^2\right| + \left|\alpha_{2(n-2)}^1\alpha_4^2\right|\Bigg] \\
&+ 2\Bigg[\left|\alpha_4^2\alpha_{2(n-2)}^1\right| + \left|\alpha_8^2\alpha_{2(n-4)}^1\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^1\right| + \cdots + \left|\alpha_{2(n-4)}^2\alpha_8^1\right| + \left|\alpha_{2(n-2)}^2\alpha_4^1\right|\Bigg] \\
&+ \Bigg[\left|\alpha_4^2\alpha_{2(n-2)}^2\right| + \left|\alpha_8^2\alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^2\alpha_8^2\right| + \left|\alpha_{2(n-2)}^2\alpha_4^2\right|\Bigg] \\
&= 4\Bigg[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots + \left|\alpha^1_{2\left(\frac{n}{2}\right)}\alpha^1_{2\left(\frac{n}{2}\right)}\right| + \cdots \\
&\qquad\qquad + \left|\alpha_{2(n-2)}^1\alpha_4^1\right| + \left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Bigg] \\
&+ 4\Bigg[\left|\alpha_4^1\alpha_{2(n-2)}^2\right| + \left|\alpha_8^1\alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^1\alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^1\alpha_8^2\right| + \left|\alpha_{2(n-2)}^1\alpha_4^2\right|\Bigg] \\
&+ \Bigg[\left|\alpha_4^2\alpha_{2(n-2)}^2\right| + \left|\alpha_8^2\alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^2\alpha_8^2\right| + \left|\alpha_{2(n-2)}^2\alpha_4^2\right|\Bigg].
\end{aligned}$$

Finally, consider $\tilde{\kappa}(n) r_{2n} - \sum_{j=1}^{n-1} r_{2j}\, r_{2(n-j)}$, where $\tilde{\kappa}(n)$ is a function to be determined.

$$\begin{aligned}
&\tilde{\kappa}(n) r_{2n} - \sum_{j=1}^{n-1} r_{2j}\, r_{2(n-j)} \\
&= \frac{\tilde{\kappa}(n)}{(2n+1)(n-1)}\Bigg\{2n\Big[\alpha_4^2\alpha_{2(n-2)}^1 + \alpha_8^2\alpha_{2(n-4)}^1 + \alpha_{12}^2\alpha_{2(n-6)}^1 + \cdots + \alpha_{2(n-4)}^2\alpha_8^1 + \alpha_{2(n-2)}^2\alpha_4^1\Big] \\
&\qquad - (n+1)\Big[-\alpha_2^1\alpha_{2(n-1)}^1 + \alpha_4^1\alpha_{2(n-2)}^1 - \alpha_6^1\alpha_{2(n-3)}^1 + \cdots + \alpha_{2(n-2)}^1\alpha_4^1 - \alpha_{2(n-1)}^1\alpha_2^1\Big]\Bigg\} \\
&- 4\Bigg[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots + \left|\alpha^1_{2\left(\frac{n}{2}\right)}\alpha^1_{2\left(\frac{n}{2}\right)}\right| + \cdots \\
&\qquad\qquad + \left|\alpha_{2(n-2)}^1\alpha_4^1\right| + \left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&-4\Big[\left|\alpha_4^1\alpha_{2(n-2)}^2\right|+\left|\alpha_8^1\alpha_{2(n-4)}^2\right|+\left|\alpha_{12}^1\alpha_{2(n-6)}^2\right|+\cdots+\left|\alpha_{2(n-4)}^1\alpha_8^2\right|+\left|\alpha_{2(n-2)}^1\alpha_4^2\right|\Big]\\
&-\quad\Big[\left|\alpha_4^2\alpha_{2(n-2)}^2\right|+\left|\alpha_8^2\alpha_{2(n-4)}^2\right|+\left|\alpha_{12}^2\alpha_{2(n-6)}^2\right|+\cdots+\left|\alpha_{2(n-4)}^2\alpha_8^2\right|+\left|\alpha_{2(n-2)}^2\alpha_4^2\right|\Big]\\
&=\frac{\tilde{\kappa}(n)}{(2n+1)(n-1)}\Bigg\{2n\Big[\left|\alpha_4^2\alpha_{2(n-2)}^1\right|-\left|\alpha_8^2\alpha_{2(n-4)}^1\right|+\left|\alpha_{12}^2\alpha_{2(n-6)}^1\right|-\cdots\\
&\qquad\qquad+\left|\alpha_{2(n-4)}^2\alpha_8^1\right|-\left|\alpha_{2(n-2)}^2\alpha_4^1\right|\Big]\\
&\qquad+(n+1)\Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right|+\left|\alpha_4^1\alpha_{2(n-2)}^1\right|+\left|\alpha_6^1\alpha_{2(n-3)}^1\right|+\cdots+\left|\alpha_{2(n-2)}^1\alpha_4^1\right|+\left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Big]\Bigg\}\\
&-4\Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right|+\left|\alpha_4^1\alpha_{2(n-2)}^1\right|+\left|\alpha_6^1\alpha_{2(n-3)}^1\right|+\cdots+\left|\alpha_{2(\frac{n}{2})}^1\alpha_{2(\frac{n}{2})}^1\right|+\cdots\\
&\qquad+\left|\alpha_{2(n-2)}^1\alpha_4^1\right|+\left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Big]\\
&-4\Big[\left|\alpha_4^1\alpha_{2(n-2)}^2\right|+\left|\alpha_8^1\alpha_{2(n-4)}^2\right|+\left|\alpha_{12}^1\alpha_{2(n-6)}^2\right|+\cdots+\left|\alpha_{2(n-4)}^1\alpha_8^2\right|+\left|\alpha_{2(n-2)}^1\alpha_4^2\right|\Big]\\
&-\quad\Big[\left|\alpha_4^2\alpha_{2(n-2)}^2\right|+\left|\alpha_8^2\alpha_{2(n-4)}^2\right|+\left|\alpha_{12}^2\alpha_{2(n-6)}^2\right|+\cdots+\left|\alpha_{2(n-4)}^2\alpha_8^2\right|+\left|\alpha_{2(n-2)}^2\alpha_4^2\right|\Big].
\end{aligned}
$$

For Case B, define

$$\tilde{\beta}(n)=\frac{\tilde{\kappa}(n)}{(2n+1)(n-1)}. \tag{D.16}$$

Now we have

$$
\begin{aligned}
&\tilde{\kappa}(n)r_{2n}-\sum_{j=1}^{n-1}r_{2j}\,r_{2(n-j)}\\
&=\Big(2n\tilde{\beta}(n)-4\Big)\Big[\left|\alpha_4^2\alpha_{2(n-2)}^1\right|+\left|\alpha_{12}^2\alpha_{2(n-6)}^1\right|+\left|\alpha_{20}^2\alpha_{2(n-10)}^1\right|+\cdots+\left|\alpha_{2(n-8)}^2\alpha_{16}^1\right|+\left|\alpha_{2(n-4)}^2\alpha_8^1\right|\Big]\\
&-\Big(2n\tilde{\beta}(n)+4\Big)\Big[\left|\alpha_8^2\alpha_{2(n-4)}^1\right|+\left|\alpha_{16}^2\alpha_{2(n-8)}^1\right|+\left|\alpha_{24}^2\alpha_{2(n-12)}^1\right|+\cdots+\left|\alpha_{2(n-6)}^2\alpha_{12}^1\right|+\left|\alpha_{2(n-2)}^2\alpha_4^1\right|\Big]\\
&+\Big((n+1)\tilde{\beta}(n)-4\Big)\Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right|+\left|\alpha_4^1\alpha_{2(n-2)}^1\right|+\left|\alpha_6^1\alpha_{2(n-3)}^1\right|+\cdots+\left|\alpha_{2(n-2)}^1\alpha_4^1\right|+\left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Big]\\
&-\quad\Big[\left|\alpha_4^2\alpha_{2(n-2)}^2\right|+\left|\alpha_8^2\alpha_{2(n-4)}^2\right|+\left|\alpha_{12}^2\alpha_{2(n-6)}^2\right|+\cdots+\left|\alpha_{2(n-4)}^2\alpha_8^2\right|+\left|\alpha_{2(n-2)}^2\alpha_4^2\right|\Big].
\end{aligned}\tag{D.17}
$$

We can prove the following facts using induction. We omit the proofs here for brevity, but they follow exactly the procedure of the proofs of (5.98)-(5.99) in Section 5.4.

$$\left|\alpha_{2j}^{1}\right| \leq \frac{1}{2}\left|\alpha_{2(j+1)}^{1}\right|, \quad j = 4, 6, 8, \ldots, \tag{D.18}$$

$$\left|\alpha_{2j}^{1}\right| \leq \frac{1}{2}\left|\alpha_{2j}^{2}\right|, \quad j = 2, 4, 6, \ldots, \tag{D.19}$$

$$\left|\alpha_{2j}^{2}\right| \leq \frac{1}{2}\left|\alpha_{2(j-1)}^{1}\right|, \quad j = 2, 4, 6, \ldots, \tag{D.20}$$

$$\left|\alpha_{2j}^{2}\right| \leq 16\left|\alpha_{2(j+1)}^{1}\right|, \quad j = 2, 4, 6, \ldots, \tag{D.21}$$

$$\left|\alpha_{2j}^{2}\right| \geq \frac{1}{10}\left|\alpha_{2(j-1)}^{1}\right|, \quad j = 2, 4, 6, \ldots, \tag{D.22}$$

$$\left|\alpha_{2j}^{2}\right| \geq 3\left|\alpha_{2(j+1)}^{1}\right|, \quad j = 2, 4, 6, \ldots. \tag{D.23}$$

Note that (D.19)-(D.20) imply that $\left|\alpha_{2j}^{1}\right| \leq \frac{1}{4}\left|\alpha_{2(j-1)}^{1}\right|$ for $j = 2, 4, 6, \ldots$.

For $j = 4, 8, 12, \ldots, n-6$, we have the following bound

$$\begin{aligned}
\left(2n\tilde{\beta}(n) + 4\right)\left|\alpha_{2j}^{2}\alpha_{2(n-j)}^{1}\right| + 2\left|\alpha_{2j}^{2}\alpha_{2(n-j)}^{2}\right| &\leq \left(2n\tilde{\beta}(n) + 4\right) \cdot \frac{1}{2}\left|\alpha_{2(j-1)}^{1}\right| \cdot \frac{1}{2}\left|\alpha_{2(n-j+1)}^{1}\right| \\
&\quad + 2 \cdot \frac{1}{2}\left|\alpha_{2(j-1)}^{1}\right| \cdot 16\left|\alpha_{2(n-j+1)}^{1}\right| \\
&= \left(\frac{n\tilde{\beta}(n)}{2} + 17\right)\left|\alpha_{2(j-1)}^{1}\alpha_{2(n-j+1)}^{1}\right|.
\end{aligned} \tag{D.24}$$

In order to keep the quantity in (D.17) nonnegative, we compare the quantity on the right-hand side of (D.24) with

$$\left((n+1)\tilde{\beta}(n) - 4\right)\left|\alpha_{2(j-1)}^{1}\alpha_{2(n-j+1)}^{1}\right|$$

for $j = 4, 8, 12, \ldots$. In particular, for (D.17) to be nonnegative, we require

$$(n+1)\tilde{\beta}(n) - 4 \geq \frac{n\tilde{\beta}(n)}{2} + 17.$$

Rearranging, we have

$$\left(\frac{n}{2} + 1\right)\tilde{\beta}(n) \geq 21.$$

Substituting the definition of $\tilde{\beta}(n)$ from (D.16) yields

$$\frac{\left(\frac{n}{2} + 1\right)\tilde{\kappa}(n)}{(2n+1)(n-1)} \geq 21.$$

A simple choice for $\tilde{\kappa}(n)$ that satisfies the inequality above is

$$\tilde{\kappa}(n) = 21\,(4n-1)\,. \tag{D.25}$$

Finally, consider the remaining negative terms in the quantity in (D.17). We are left with

$$\left|\alpha^2_{2j}\alpha^2_{2(n-j)}\right|, \quad \text{for } j = 2, 6, 10, \ldots.$$

Again, we can use (D.20)-(D.21) to show that

$$\left|\alpha^2_{2j}\alpha^2_{2(n-j)}\right| \leq 5\left|\alpha^1_{2(j-1)}\alpha^1_{2(n-j+1)}\right|.$$

With the choice of $\tilde{\kappa}(n)$ in (D.25), it is straightforward to show that

$$\Big((n+1)\tilde{\beta}(n) - 4\Big)\left|\alpha^1_{2(j-1)}\alpha^1_{2(n-j+1)}\right| \geq 5\left|\alpha^1_{2(j-1)}\alpha^1_{2(n-j+1)}\right|$$

for $j = 2, 6, 10, \ldots$. As a result, we have that (D.17) is nonnegative at least when $\tilde{\kappa}(n)$ is given in (D.25).

This completes the proof for Case B; we showed that when $n = 4\mu + 2$, $\mu = 0, 1, 2, \ldots$,

$$21\,(4n-1)\,r_{2n} - \sum_{j=1}^{n-1} r_{2j}r_{2(n-j)} \geq 0,$$

or

$$\sum_{j=1}^{n-1} r_{2j}r_{2(n-j)} \leq 21\,(4n-1)\,r_{2n}.$$

Finally, consider Case C, in which n is even and a multiple of four ($n = 4\mu$ for $\mu = 1, 2, \ldots$). We have

$$\begin{aligned}
r_{2n} &= \left|\alpha^1_{2n}\right| + \left|\alpha^2_{2n}\right| + \left|\alpha^3_{2n}\right| \\
&= 2\left|\alpha^1_{2n}\right| + \left|\alpha^2_{2n}\right| \\
&= 2\alpha^1_{2n} + \alpha^2_{2n} \\
&= \frac{2}{2(2n+1)(n-1)}\left[2n\,b^1_{2n} - b^2_{2n} - b^3_{2n}\right] + \frac{1}{2(2n+1)(n-1)}\left[2n\,b^2_{2n} - b^1_{2n} - b^3_{2n}\right] \\
&= \frac{1}{2(2n+1)(n-1)}\left[(4n-1)\,b^1_{2n} + (-2+2n)\,b^2_{2n} - 3b^3_{2n}\right] \\
&= \frac{1}{2(2n+1)(n-1)}\left[4(n-1)\,b^1_{2n} + 2(n-1)b^2_{2n}\right] \\
&= \frac{1}{(2n+1)}\left[2\,b^1_{2n} + b^2_{2n}\right],
\end{aligned}$$

where we used the fact that $b_{2n}^3 = b_{2n}^1$ from (D.9). Substituting from (D.5) yields

$$\begin{aligned}
r_{2n} &= \frac{1}{2n+1}\left[-2\sum_{j=1}^{n-1}\alpha_{2j}^2\alpha_{2(n-j)}^3 - \sum_{j=1}^{n-1}\alpha_{2j}^1\alpha_{2(n-j)}^3\right] \\
&= \frac{1}{2n+1}\Bigg\{-2\Big[\alpha_2^2\alpha_{2(n-1)}^3 + \alpha_4^2\alpha_{2(n-2)}^3 + \alpha_6^2\alpha_{2(n-3)}^3 + \cdots + \alpha_{2(n-2)}^2\alpha_4^3 + \alpha_{2(n-1)}^2\alpha_2^3\Big] \\
&\qquad - \Big[\alpha_2^1\alpha_{2(n-1)}^3 + \alpha_4^1\alpha_{2(n-2)}^3 + \alpha_6^1\alpha_{2(n-3)}^3 + \cdots + \alpha_{2(n-2)}^1\alpha_4^3 + \alpha_{2(n-1)}^1\alpha_2^3\Big]\Bigg\} \\
&= \frac{1}{2n+1}\Bigg\{-2\Big[\alpha_4^2\alpha_{2(n-2)}^1 + \alpha_8^2\alpha_{2(n-4)}^1 + \alpha_{12}^2\alpha_{2(n-6)}^1 + \cdots + \alpha_{2(n-4)}^2\alpha_8^1 + \alpha_{2(n-2)}^2\alpha_4^1\Big] \\
&\qquad - \Big[-\alpha_2^1\alpha_{2(n-1)}^1 + \alpha_4^1\alpha_{2(n-2)}^1 - \alpha_6^1\alpha_{2(n-3)}^1 + \cdots + \alpha_{2(n-2)}^1\alpha_4^1 - \alpha_{2(n-1)}^1\alpha_2^1\Big]\Bigg\},
\end{aligned} \tag{D.26}$$

where we used the fact when n is even, then j and $n-j$ are either both even or both odd. We also used properties from (D.9).

Next, observe the following, noting that if $n = 4\mu$, then $n/2$ is even.

$$\begin{aligned}
\sum_{j=1}^{n-1} r_{2j}\, r_{2(n-j)} &= r_2r_{2(n-1)} + r_4r_{2(n-2)} + r_6r_{2(n-3)} + \cdots + r_{2\left(\frac{n}{2}\right)}^2 + \cdots \\
&\qquad + r_{2(n-3)}r_6 + r_{2(n-2)}r_4 + r_{2(n-1)}r_2 \\
&= 2\alpha_2^1\cdot 2\alpha_{2(n-1)}^1 + \left(2\left|\alpha_4^1\right| + \left|\alpha_4^2\right|\right)\left(2\left|\alpha_{2(n-2)}^1\right| + \left|\alpha_{2(n-2)}^2\right|\right) + 2\alpha_6^1\cdot 2\alpha_{2(n-3)}^1 \\
&\quad + \cdots + \left(2\left|\alpha_{2\left(\frac{n}{2}\right)}^1\right| + \left|\alpha_{2\left(\frac{n}{2}\right)}^2\right|\right)^2 + \cdots + \left(2\left|\alpha_{2(n-2)}^1\right| + \left|\alpha_{2(n-2)}^2\right|\right)\left(2\left|\alpha_4^1\right| + \left|\alpha_4^2\right|\right) \\
&\qquad + 2\alpha_2^1\cdot 2\alpha_{2(n-1)}^1 \\
&= 4\Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots + \left|\alpha_{2\left(\frac{n}{2}\right)}^1\alpha_{2\left(\frac{n}{2}\right)}^1\right| + \cdots \\
&\qquad + \left|\alpha_{2(n-2)}^1\alpha_4^1\right| + \left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Big] \\
&\quad + 2\Big[\left|\alpha_4^1\alpha_{2(n-2)}^2\right| + \left|\alpha_8^1\alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^1\alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^1\alpha_8^2\right| + \left|\alpha_{2(n-2)}^1\alpha_4^2\right|\Big] \\
&\quad + 2\Big[\left|\alpha_4^2\alpha_{2(n-2)}^1\right| + \left|\alpha_8^2\alpha_{2(n-4)}^1\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^1\right| + \cdots + \left|\alpha_{2(n-4)}^2\alpha_8^1\right| + \left|\alpha_{2(n-2)}^2\alpha_4^1\right|\Big] \\
&\quad + \Big[\left|\alpha_4^2\alpha_{2(n-2)}^2\right| + \left|\alpha_8^2\alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^2\alpha_8^2\right| + \left|\alpha_{2(n-2)}^2\alpha_4^2\right|\Big].
\end{aligned}$$

Finally, consider $\hat{\kappa}(n) r_{2n} - \sum_{j=1}^{n-1} r_{2j}\, r_{2(n-j)}$, where $\hat{\kappa}(n)$ is a function to be determined.

$$
\begin{aligned}
&\hat{\kappa}(n) r_{2n} - \sum_{j=1}^{n-1} r_{2j}\, r_{2(n-j)} \\
&= \frac{\hat{\kappa}(n)}{2n+1}\Bigg\{ -2\Big[\alpha_4^2\alpha_{2(n-2)}^1 + \alpha_8^2\alpha_{2(n-4)}^1 + \alpha_{12}^2\alpha_{2(n-6)}^1 + \cdots + \alpha_{2(n-4)}^2\alpha_8^1 + \alpha_{2(n-2)}^2\alpha_4^1\Big] \\
&\qquad\qquad - \Big[-\alpha_2^1\alpha_{2(n-1)}^1 + \alpha_4^1\alpha_{2(n-2)}^1 - \alpha_6^1\alpha_{2(n-3)}^1 + \cdots + \alpha_{2(n-2)}^1\alpha_4^1 - \alpha_{2(n-1)}^1\alpha_2^1\Big]\Bigg\} \\
&- 4\Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots + \left|\alpha_{2\left(\frac{n}{2}\right)}^1\alpha_{2\left(\frac{n}{2}\right)}^1\right| + \cdots \\
&\qquad\qquad + \left|\alpha_{2(n-2)}^1\alpha_4^1\right| + \left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Big] \\
&- 2\Big[\left|\alpha_4^1\alpha_{2(n-2)}^2\right| + \left|\alpha_8^1\alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^1\alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^1\alpha_8^2\right| + \left|\alpha_{2(n-2)}^1\alpha_4^2\right|\Big] \\
&- 2\Big[\left|\alpha_4^2\alpha_{2(n-2)}^1\right| + \left|\alpha_8^2\alpha_{2(n-4)}^1\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^1\right| + \cdots + \left|\alpha_{2(n-4)}^2\alpha_8^1\right| + \left|\alpha_{2(n-2)}^2\alpha_4^1\right|\Big] \\
&- \Big[\left|\alpha_4^2\alpha_{2(n-2)}^2\right| + \left|\alpha_8^2\alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^2\alpha_8^2\right| + \left|\alpha_{2(n-2)}^2\alpha_4^2\right|\Big] \\
&= \frac{\hat{\kappa}(n)}{2n+1}\Bigg\{ 2\Big[\left|\alpha_4^2\alpha_{2(n-2)}^1\right| - \left|\alpha_8^2\alpha_{2(n-4)}^1\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^1\right| - \cdots - \left|\alpha_{2(n-4)}^2\alpha_8^1\right| + \left|\alpha_{2(n-2)}^2\alpha_4^1\right|\Big] \\
&\qquad\qquad + \Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| - \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots - \left|\alpha_{2(n-2)}^1\alpha_4^1\right| + \left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Big]\Bigg\} \\
&- 4\Big[\left|\alpha_2^1\alpha_{2(n-1)}^1\right| + \left|\alpha_4^1\alpha_{2(n-2)}^1\right| + \left|\alpha_6^1\alpha_{2(n-3)}^1\right| + \cdots + \left|\alpha_{2\left(\frac{n}{2}\right)}^1\alpha_{2\left(\frac{n}{2}\right)}^1\right| + \cdots \\
&\qquad\qquad + \left|\alpha_{2(n-2)}^1\alpha_4^1\right| + \left|\alpha_{2(n-1)}^1\alpha_2^1\right|\Big] \\
&- 2\Big[\left|\alpha_4^1\alpha_{2(n-2)}^2\right| + \left|\alpha_8^1\alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^1\alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^1\alpha_8^2\right| + \left|\alpha_{2(n-2)}^1\alpha_4^2\right|\Big] \\
&- 2\Big[\left|\alpha_4^2\alpha_{2(n-2)}^1\right| + \left|\alpha_8^2\alpha_{2(n-4)}^1\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^1\right| + \cdots + \left|\alpha_{2(n-4)}^2\alpha_8^1\right| + \left|\alpha_{2(n-2)}^2\alpha_4^1\right|\Big] \\
&- \Big[\left|\alpha_4^2\alpha_{2(n-2)}^2\right| + \left|\alpha_8^2\alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^2\alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^2\alpha_8^2\right| + \left|\alpha_{2(n-2)}^2\alpha_4^2\right|\Big]
\end{aligned}
$$

For Case C, define

$$\hat{\beta}(n) = \frac{\hat{\kappa}(n)}{2n+1}.$$

Now we have

$$
\begin{aligned}
&\hat{\kappa}(n) r_{2n} - \sum_{j=1}^{n-1} r_{2j}\, r_{2(n-j)} \\
&= \left(2\hat{\beta}(n) - 4\right) \left[\left|\alpha_4^2 \alpha_{2(n-2)}^1\right| + \left|\alpha_{12}^2 \alpha_{2(n-6)}^1\right| + \left|\alpha_{20}^2 \alpha_{2(n-10)}^1\right| + \cdots + \left|\alpha_{2(n-8)}^2 \alpha_{16}^1\right| + \left|\alpha_{2(n-4)}^2 \alpha_8^1\right| \right] \\
&+ \left(\hat{\beta}(n) - 4\right) \left[\left|\alpha_2^1 \alpha_{2(n-1)}^1\right| + \left|\alpha_6^1 \alpha_{2(n-3)}^1\right| + \left|\alpha_{10}^1 \alpha_{2(n-5)}^1\right| + \cdots + \left|\alpha_{2(n-3)}^1 \alpha_6^1\right| + \left|\alpha_{2(n-1)}^1 \alpha_2^1\right| \right] \\
&- \left(2\hat{\beta}(n) + 4\right) \left[\left|\alpha_8^2 \alpha_{2(n-4)}^1\right| + \left|\alpha_{16}^2 \alpha_{2(n-8)}^1\right| + \left|\alpha_{24}^2 \alpha_{2(n-12)}^1\right| + \cdots + \left|\alpha_{2(n-8)}^2 \alpha_{16}^1\right| + \left|\alpha_{2(n-4)}^2 \alpha_8^1\right| \right] \\
&- \left(\hat{\beta}(n) + 4\right) \left[\left|\alpha_4^1 \alpha_{2(n-2)}^1\right| + \left|\alpha_8^1 \alpha_{2(n-4)}^1\right| + \left|\alpha_{12}^1 \alpha_{2(n-6)}^1\right| + \cdots + \left|\alpha_{2(n-4)}^1 \alpha_8^1\right| + \left|\alpha_{2(n-2)}^1 \alpha_4^1\right| \right] \\
&- \quad \left[\left|\alpha_4^2 \alpha_{2(n-2)}^2\right| + \left|\alpha_8^2 \alpha_{2(n-4)}^2\right| + \left|\alpha_{12}^2 \alpha_{2(n-6)}^2\right| + \cdots + \left|\alpha_{2(n-4)}^2 \alpha_8^2\right| + \left|\alpha_{2(n-2)}^2 \alpha_4^2\right| \right].
\end{aligned} \tag{D.27}
$$

Observe that when $j = 2, 6, 10, \ldots$, we have

$$
\left(\hat{\beta}(n) + 4\right) \left|\alpha_{2j}^1 \alpha_{2(n-j)}^1\right| \leq \frac{1}{2} \left(\hat{\beta}(n) + 4\right) \left|\alpha_{2j}^2 \alpha_{2(n-j)}^1\right|,
$$

where we used (D.19). The first condition for the nonnegativity of the quantity in (D.27) is

$$
\frac{1}{2} \left(\hat{\beta}(n) + 4\right) \leq 2\hat{\beta}(n) - 4,
$$

or equivalently

$$
\hat{\beta}(n) \geq 4. \tag{D.28}
$$

If $j = 4, 8, 12, \ldots$, then we have

$$
\begin{aligned}
\left(2\hat{\beta}(n) + 4\right) &\left|\alpha_{2j}^2 \alpha_{2(n-j)}^1\right| + \left(\hat{\beta}(n) + 4\right) \left|\alpha_{2j}^1 \alpha_{2(n-j)}^1\right| \\
&\leq \left[\frac{1}{2} \cdot \frac{1}{2} \left(2\hat{\beta}(n) + 4\right) + \frac{1}{4} \cdot \frac{1}{2} \left(\hat{\beta}(n) + 4\right) \right] \left|\alpha_{2(j-1)}^1 \alpha_{2(n-j+1)}^1\right| \\
&= \left(\frac{5}{8} \hat{\beta}(n) + \frac{3}{2} \right) \left|\alpha_{2(j-1)}^1 \alpha_{2(n-j+1)}^1\right|.
\end{aligned} \tag{D.29}
$$

Additionally, notice that for j even,

$$
\left|\alpha_{2j}^2 \alpha_{2(n-j)}^2\right| \leq \frac{1}{2} \cdot 16 \left|\alpha_{2(j-1)}^1 \alpha_{2(n-j+1)}^1\right| = 8 \left|\alpha_{2(j-1)}^1 \alpha_{2(n-j+1)}^1\right|. \tag{D.30}
$$

The negative terms in (D.27) of the form (D.29) and (D.30) must be balanced by the positive term in the second line of (D.27) in order for (D.27) to be nonnegative. Thus we require

$$
\left(\frac{5}{8} \hat{\beta}(n) + \frac{3}{2} \right) \leq (1 - s) \left(\hat{\beta}(n) - 4\right) \quad \text{and} \quad 8 \leq s \left(\hat{\beta}(n) - 4\right),
$$

for $0 \leq s \leq 1$. Equivalently, this yields the conditions

$$\hat{\beta}(n) \geq \frac{44 - 32s}{3 - 8s} \quad \text{and} \quad \hat{\beta}(n) \geq \frac{8 + 4s}{s}.$$

For instance, we can pick $s = \frac{1}{4}$. This yields the condition

$$\hat{\beta}(n) \geq 36.$$

That is, we must pick $\hat{\kappa}(n)$ so that

$$\frac{\hat{\kappa}(n)}{2n+1} \geq 36.$$

For simplicity, we choose

$$\hat{\kappa}(n) = 36(2n+1).$$

In summary, we have now shown that when $n = 4\mu$ for $\mu = 1, 2, 3, \ldots$, then

$$36\,(2n+1)\,r_{2n} - \sum_{j=1}^{n-1} r_{2j} r_{2(n-j)} \geq 0,$$

or

$$\sum_{j=1}^{n-1} r_{2j} r_{2(n-j)} \leq 36\,(2n+1)\,r_{2n}.$$

This completes the proof of (D.1) for Case C.

Comparing Cases A-C, we see that for any n,

$$\sum_{j=1}^{n-1} r_{2j,j} r_{2(n-j),n-j} \leq 36\,(2n+1)\,r_{2n,n}.$$