A LINEAR-TIME RECOGNITION ALGORITHM FOR INTERVAL DAGS

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1. Introduction

In [PY], Papadimitriou and Yannakakis solve the unit-execution-time scheduling problem [C,pp.51-3] for a family of partial orders called interval orders. Only one other family (the forest orders [H]) is known to have an efficient solution.

The development in [PY] assumes the interval order is given in transitively closed form. This note extends ideas that are implicit in [PY] to remove this assumption. The main tool is a linear-time algorithm to recognize an interval dag, i.e., a dag whose transitive closure is an interval order. This is given in Section 2. Section 3 indicates how the recognition algorithm applies to the scheduling problem.

Before proceeding we establish some notation and give a convenient definition of interval dag. For a dag (directed acyclic graph), n represents the number of vertices and m the number of edges; the vertices are numbered 1, ..., n. For a vertex i, the set of vertices adjacent from i is $A(i) = \{j \mid$ there is an edge from i to $j\}$; the successor set of i is $S(i) = \{j \mid$ there is a path of one or more edges from i to $j\}$.

An interval dag satisfies the following nesting property:

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Definition A dag is an interval dag iff for any two vertices i,j, either $S(i) \subseteq S(j)$ or $S(j) \subseteq S(i)$.

So the distinct successor sets $S(j)$ can be labelled as $\phi = T_0 \subsetneq T_1 \subsetneq \ldots T_{\ell-1} \subsetneq T_\ell$. Figure 1 shows an interval dag. The transitively closed interval dags are exactly the interval orders; this can be seen by examining the proofs of [PY].

2. The Recognition Algorithm

The algorithm processes vertices one-by-one, checking the nesting property is always satisfied. To do this, the vertices are numbered in topological order [K], so if (i,j) is an edge, $i < j$ (Each vertex is identified with its number.). Vertices are processed in decreasing order.

Suppose the nesting property has been verified for all vertices $j > i$. So the distinct successor sets $S(j)$, $j > i$, can be labelled as $T_0 \subsetneq \ldots T_{k-1} \subsetneq T_k$, where for convenience, T_k does not denote a successor set but rather $T_k = \{1, \ldots, n\}$. Let T_q be the smallest set containing $A(i)$. The nesting property implies $T_{q-1} \subsetneq S(i) \subseteq T_q$. So if $T_{q-1} \subsetneq S(i)$ is false, the dag is not interval. Otherwise the dag seen so far is interval. If $S(i) = T_q$, the next vertex i-1 can be processed. If $S(i)$ lies properly between T_{q-1} and T_q, $S(i)$ can be made a new T- set and then vertex i-1 can be processed.

To implement this approach efficiently, the following data structure is used (see Figure 2). A set T_j is represented by a node t; it has a field

COUNT(t) = $|T_j|$.

The nodes T_k, T_{k-1}, ..., T_0 are linked, in this order, to form a linear list L. A vertex i has entries in two arrays:

SUCC(i) points to the node for set $S(i)$;

IN(i) points to the node for the smallest set containing i.

Note the nesting property allows us to use COUNT fields to compare sets. Now we present the algorithm in pseudo-Algol.

```
begin comment  This algorithm halts indicating whether or not the
      given dag is interval.  If it is, a node t in L represents the
      successor set {i|IN(i) is t or is after t in L}, L gives the
      nesting order of successor sets, and vertex i has successor set
      SUCC(i);

initialization:

1.  number the vertices in topological order;

2.  let L be T1, T0, where T1 = {1, ..., n}, T0 = φ; let IN(i) point to T1
      and SUCC(i) point to T0, for 1 ≤ i ≤ n;

processing:

3.  for i ← n to 1 by -1 do

4.    if A(i) ≠ φ then begin

5.      comment find Tq, the smallest successor set containing A(i), and Tp, the
          largest successor set of some j ε A(i);
          let q point to the node that maximizes COUNT(IN(j)) for j ε A(i);
          let p point to the node that maximizes COUNT(SUCC(j)) for j ε A(i);

6.      comment let LO = A(i) ∩ Tq-1 - Tp, HI = A(i) ∩ Tq - Tq-1;
          let LO = {j|j ε A(i), COUNT(p) < COUNT(IN(j)) < COUNT(q)};
          let HI = {j|j ε A(i), IN(j) = q};

7.      comment find Tq-1 and check Tq-1 ⊆ S(i);
        let q' point to the node following q in L;
        if COUNT(q') > COUNT(p) + |LO|  then halt (G is not interval);

8.      comment check if S(i) = Tq;
        if COUNT(q) = COUNT(q') + |HI| then SUCC(i) ← q
        else begin comment create a new successor set for S(i);

9.        let t point to a new node between q and q', with
            COUNT(t) = COUNT(q') + |HI|, SUCC(i) = t, and
            IN(j) = t for j ε HI;

        end end

halt (G is interval)

end.
```

Theorem: An interval dag can be recognized in time O(n+m).

Proof: Correctness of the algorithm follows from the comments and the preceding discussion. The linear time bound is obvious. □

In an actual implementation, the algorithm would be modified for greater speed. The topological numbering could be eliminated by exploring the graph depth-first, processing successors before predecessors. In line 6 it is more economical to compute only |LO| and |HI|; only in line 9 might it be necessary to reexamine A(i) to find HI.

3. Application to Scheduling

To implement the scheduling algorithm of Papadimitriou and Yannakakis, it is necessary to form a "priority list" of the vertices i in decreasing order of |S(i)|. This is done by traversing L, listing all vertices i with SUCC(i) = t when node t is visited. It is easy to implement this in time O(n+m).

The complete scheduling algorithm runs in almost-linear time, $O(n\,\alpha(n)+m)$.* Here $\alpha(n)$ is the inverse of Ackermann's function and is ≤ 3 for practical applications [T]. The factor $\alpha(n)$ results from the time to convert the priority list to a schedule [S]. There seems to be no simple way to take advantage of the special structure of interval dags to do the conversion faster.

*This is slightly worse than the linear bound claimed in [PY].

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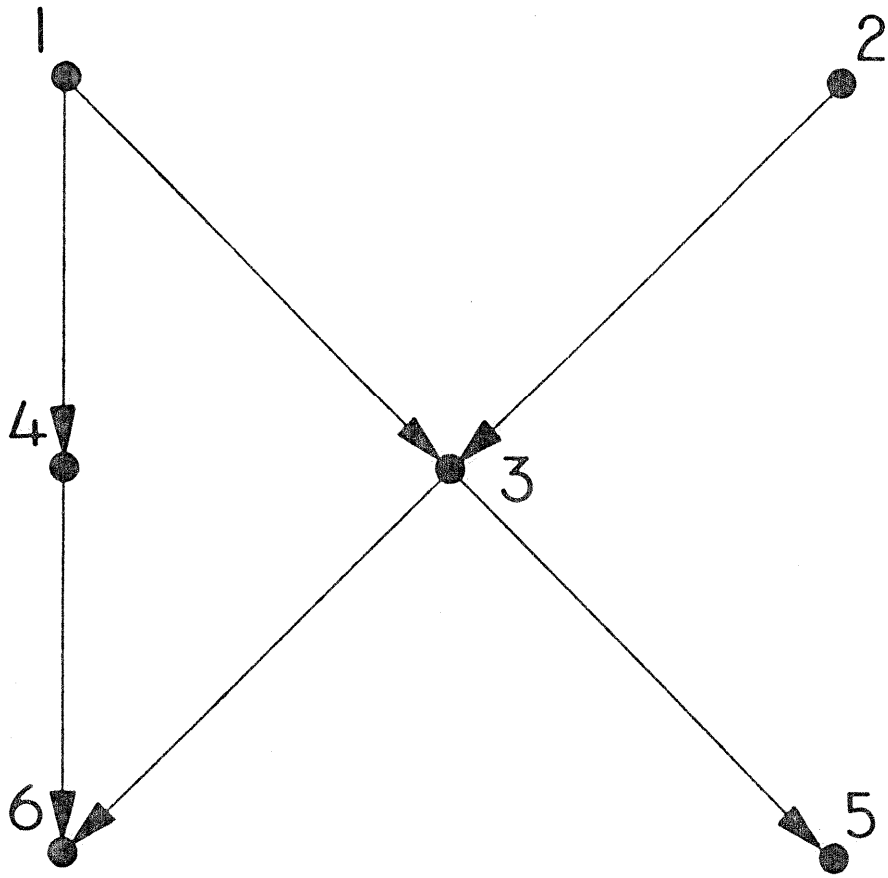
Figure 1. Example graph

Figure 2. Data structure after vertex 3 is processed.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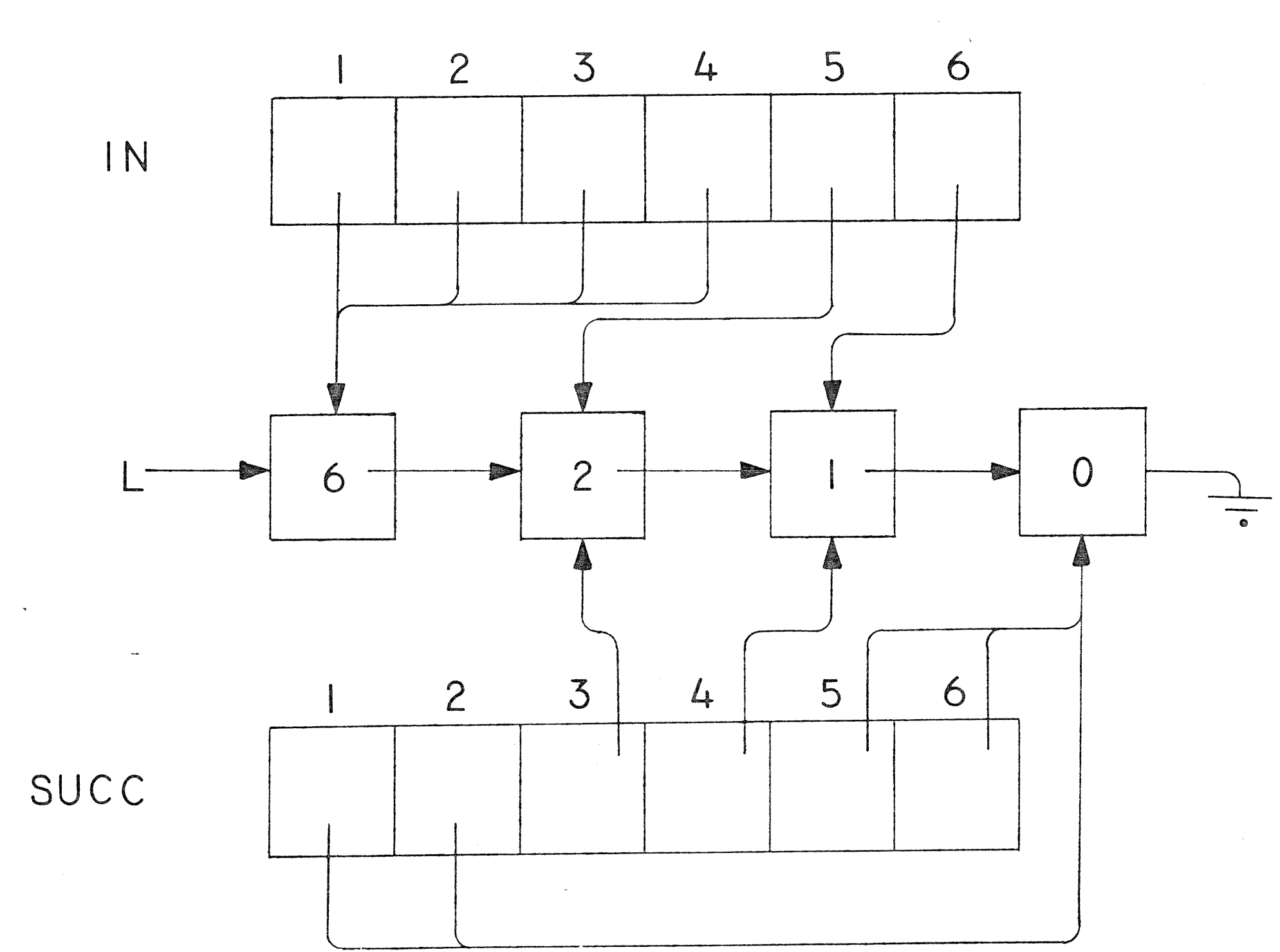
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