# GENERALIZED POST CORRESPONDENCE PROBLEM OF LENGTH 2
## PART II. Cases distinguished by patterns

by

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ABSTRACT

This paper which is the second part of a paper consisting of three parts continues the investigation of the Generalized Post Correspondence Problem of length 2. In this part we demonstrate the decidability of this problem in a number of very "concrete" cases, when one specifies quite precisely the patterns of images of 0 and 1 by the homomorphisms involved.

## INTRODUCTION

In this paper we continue the investigation of the Generalized Post Correspondence Problem of length 2, abbreviated GPCP(2), started in [ER1]. We consider instances I of GPCP(2) of the form $(h,g,a_1,a_2,b_1,b_2)$ where (in the terminology of [ER1]) h,g are marked homomorphisms such that the sequence $(h,g), \mathit{ecol}(h,g), \ldots$ is periodic and moreover one requires that h(0),h(1),g(0) and g(1) have a very specific form. For example one may require that h(0) and g(0) are of the form 0101.... while h(1) and g(1) are of the form 1010.... We distinguish in this way six classes of possible "patterns" for the pair (h,g) and then demonstrate that whenever (h,g) is in one of these classes, it is decidable whether or not I has a solution. These results are very useful in [ER2] when the decidability of GPCP(2) is proved.

## 0. PRELIMINARIES

In addition to the notation and terminology from [ER1] we will use also the following.

For words x and y, *mpref*(x,y) denotes their maximal common prefix. If xPREFy then *dif*(x,y) denotes the unique word z such that either xz = y or yz = x.

In this paper we will consider only marked homomorphisms (and only marked instances of GPCP(2)) from $\{0,1\}^*$ into $\{0,1\}^*$. For a homomorphism f, *Right*(f) = {f(0),f(1)}; if $K \subseteq \{0,1\}^*$ and $f(0), f(1) \in K$ then we say that f is a K-homomorphism. For an instance $I = (h,g,a_1,a_2,b_1,b_2)$ of GPCP(2) *maxr*(I) = max{*maxr*(h), *maxr*(g)}.

We write the composition of functions from right to left, that is, gf means "first apply f and then g".

The following (regular) languages will play an important role in the considerations of this paper.

For $i \in \{0,1\}$,

$A_i = i^+$, $B_i = i(1-i)^*$ and $C_i = i((1-i)i)^*\{\Lambda,(1-i)\}$.

Then $A = A_0 \cup A_1$, $B = B_0 \cup B_1$ and $C = C_0 \cup C_1$.

Based on these languages we define now six classes of marked instances of GPCP(2).

*Definition* 0.1. Let $I = (h,g,a_1,a_2,b_1,b_2)$ be a marked instance of GPCP(2).

(a). For $i \in \{0,1\}$, $I \in CL_{A_i}$ if

$h(0) \in A_0$, $h(1) \in A_1$, $g(i) \in A_0$ and $g(1-i) \in A_1$.

(b). For $i \in \{0,1\}$, $I \in CL_{B_i}$ if

$h(0) \in A_0$, $h(1) \in B_1$, $g(i) \in A_0$ and $g(1-i) \in B_1$.

(c). For $i \in \{0,1\}$, $I \in CL_{C_i}$ if

$h(0) \in C_0$, $h(1) \in C_1$, $g(i) \in C_0$ and $g(1-i) \in C_1$.

(d) Also: $I \in CL_A$ if either $I \in CL_{A_0}$ or $I \in CL_{A_1}$, $I \in CL_B$ if either $I \in CL_{B_0}$ or $I \in CL_{B_1}$ and $I \in CL_C$ if either $I \in CL_{C_0}$ or $I \in CL_{C_1}$. □

The following definition is very basic for this paper.

*Definition* 0.2.

(a). Let (h,g) be an ordered pair of marked homomorphisms. We say that (h.g) is *good* if there exists a pair of marked homomorphisms $(h',g')$ such that *trace* $(h',g')$ is infinite, *thres* $(h',g') = r$ and for some $i \geq r+1, (h,g) = \mathit{ecol}^i(h',g')$.

(b). We say that an instance $I = (h,g,a_1,a_2,b_1,b_2)$ of GPCP(2) is *good* if (h,g) is good.

Indeed, we will investigate here *directly* good instances of GPCP(2) and then as corollaries we will get results about stable instances of GPCP(2) needed in the next part of this paper.

Given an instance $I = (h,g,a_1,a_2,b_1,b_2)$ of GPCP(2) we may:

(i). "switch" the role of *homomorphisms* h and g by considering the instance $I' = (g,h,b_1,b_2,a_1,a_2)$; clearly I has a solution if and only if $I'$ has a solution,

(ii). "switch" the role of 0 and 1 in the *domain* of h and g by considering the instance $I' = (h',g',a_1,a_2,b_1,b_2)$, where $h(0) = h'(1)$, $h(1) = h'(0)$, $g(0) = g'(1)$ and $g(1) = g'(0)$; clearly I has a solution if and only if $I'$ has a solution,

(iii). "switch" the role of 0 and 1 in the *range* of h and g by considering the instance $I' = (h',g',\tilde{a}_1,\tilde{a}_2,\tilde{b}_1,\tilde{b}_2)$ with, for $i \in \{0,1\}$,

$h'(i) = \tilde{\alpha}$ if $h(i) = \alpha$ and $g'(i) = \tilde{\beta}$ if $g(i) = \beta$, where for a word $x$, $\tilde{x}$ denotes the word obtained from $x$ by replacing every occurrence of a 0 in $x$ by 1 and every occurrence of 1 in $x$ by 0; clearly $I$ has a solution if and only if $I'$ has a solution.

We will refer to these three operations above as the *homomorphisms switch*, the *domain switch* and the *range switch* respectively. Clearly if $I$ is a subject of composition of ( some of ) these switches which yield $I'$ then $I$ has a solution if and only if $I'$ has a solution.

Whenever we refer to a result from Part I of this paper we precede its "identification number" by I; thus, e.g., Theorem I.4.1 refers to Theorem 4.1 in Part I.

## 1. $CL_{A_0}$

In this section we demonstrate that GPCP(2) is decidable for good instances from the class $CL_{A_0}$.

*Theorem* 1.1. It is decidable whether or not an arbitrary good instance I of GPCP(2) such that $I \in CL_{A_0}$ has a solution.

*Proof.*

Let $I = (h,g,a_1,a_2,b_1,b_2)$ be a good instance of GPCP(2) such that $I \in CL_{A_0}$. Hence $h(0) = 0^k, h(1) = 1^{\ell}$, $g(0) = 0^m$ and $g(1) = 1^m$ for some $k,\ell,m,n \geq 1$.

If $k = m$ then, by Theorem I.4.2, it is decidable whether or not I has a solution.

Hence let us assume that $k > m$. (The case of $m > k$ is reduced by the homomorphisms switch to the previous case).

Note that for a word $x \in \{0,1\}^*$ we have

$\#_0 h(x) - \#_0 g(x) = k\#_0 x - m\#_0 x = (k-m)\#_0 x$.

But if x is a solution of I then

$\#_0 h(x) - \#_0 g(x) \leq \#_0 a_1 a_2 b_1 b_2$

and consequently

$$\#_0 x \leq \frac{\#_0 a_1 a_2 b_1 b_2}{k - m}$$

(note that $k - m \neq 0$).

Consequently by Theorem I.1.2 it is decidable whether or not I has a solution. □

## 2. $CL_{A_1}$

In this section we demonstrate that GPCP(2) is decidable for good instances from the class $CL_{A_1}$.

*Theorem* 2.1. It is decidable whether or not an arbitrary good instance I of GPCP(2) such that $I \in CL_{A_1}$ has a solution.

*Proof.*

Let $I = (h,g,a_1,a_2,b_1,b_2)$ be a good instance of GPCP(2) such that $I \in CL_{A_1}$. Hence $h(0) = 0^k$, $h(1) = 1^\ell$, $g(0) = 1^m$ and $g(1) = 0^n$ for some $k,\ell,m,n \geq 1$.

Assume that w is a solution of I. Either *first*(w) = 0 or *first*(w) = 1. Since one case is obtained from the other by the domain switch we will consider only one of them.

Assume that *first*(w) = 0. Hence $w = x_1y_2x_3y_4\ldots$ where $x_1,x_3,\ldots \in 0^+$ and $y_2,y_4,\ldots \in 1^+$ (in the rest of this proof we will use subscripted x to range over strings in $0^+$ and subscripted y to range over strings in $1^+$). We will assume that w is "long enough" (from the rest of the proof it will be clear how long, at least, w should be).

Since $first(h(0)) = 0 \neq 1 = first(g(0))$, it must be that $|a_1| \neq |b_1|$.

Assume then that $|a_1| > |b_1|$ (the case of $|b_1| > |a_1|$ is considered analogously).

Clearly for some $j \geq 2$ we have

$a_1h(x_1) = b_1g(x_1y_2x_3\ldots y_j)$

which implies that, for every $r \geq 1$,

$h(\tau_r) = g(\tau_{r+j-1})$ ............................................(2.1)

where $\tau_r = x_r$ for $r$ odd and $\tau_r = y_r$ for $r$ even.

Let $c_r = |\tau_r|$. We will compute now $c_{r+2j-2}$ as the function of $c_r$. Assume that $r$ is odd.

$|h(\tau_r)| = |h(x_r)| = |h(0)|c_r$

but, by (2.1),

$|h(\tau_r)| = |g(y_{r+j-1})| = |g(1)|c_{r+j-1}$; thus

$|h(0)|c_r = |g(1)|c_{r+j-1}$ ............................................(2.2)

On the other hand

$|h(y_{r+j-1})| = |h(1)|c_{r+j-1}$

but, by (2.1),

$|h(y_{r+j-1})| = |g(x_{r+2j-2})| = |g(0)|c_{r+2j-2}$; thus

$|h(1)|c_{r+j-1} = |g(0)|c_{r+2j-2}$ ............................................(2.3)

From (2.2) and (2.3) we get

$|h(0)||h(1)|c_r c_{r+j-1} = |g(1)||g(0)|c_{r+j-1}\, c_{r+2j-2}$.

Consequently

$c_{r+2j-2} = D\, c_r$ ............................................(2.4)

where $D = \dfrac{|h(0)|\,|h(1)|}{|g(1)|\,|g(0)|}$ .

We get an analogous result if we assume that $r$ is even.

We will consider separately three cases.

<u>$D < 1$.</u>

Then the length of the $x$ and $y$ blocks decreases, and so we run the sequence $x_1y_2x_3\dots$ until we cannot continue it any more (the length of the next block would have to be smaller than 1). Thus, in this case, a solution of I must be shorter than the length of the above sequence, and we can decide whether or not I has a solution.

<u>$D = 1$</u>.

Then clearly the sequence $x_1y_2x_3\ldots$ becomes ultimately periodic and we can effectively construct it until and including the first run of the period. Clearly, if a solution of I exists in this case then a solution of I exists that is no longer than the above sequence.

<u>$D > 1$</u>.

Then we run the sequence until the length of each of current $(2j-1)$ consecutive blocks will be longer than $|a_2b_2|$ . Clearly, in this case it suffices to check whether I has a solution not longer than the length of the above sequence. $\square$

## 3. $CL_{B_0}$

In this section we demonstrate that GPCP(2) is decidable for good instances from the class $CL_{B_0}$.

*Theorem* 3.1. It is decidable whether or not an arbitrary good instance I of GPCP(2) such that $I \in CL_{B_0}$ has a solution.

*Proof*.

Let $I = (h,g,a_1,a_2,b_1,b_2)$ be a good instance of GPCP(2) such that $I \in CL_{B_0}$.

Hence $h(0) = 0^k$, $h(1) = 10^\ell$, $g(0) = 0^m$ and $g(1) = 10^n$ for some $k,m \geq 1$ and $\ell,n \geq 0$.

We consider separately two cases ((a) and (b)).

(a). Either $h(10^t) = g(10^t)$ for some $t \geq 0$............................(3.1)

or $k = m$.................................................(3.2)

Then we proceed as follows.

(a.1). If (3.1) holds then let $t_0$ be the smallest t for which (3.1) holds. Note that then $\overline{h}(1) = 10^{t_0}$, $\overline{g}(1) = 10^{t_0}$ and so, by Theorem I.4.2, we can decide whether or not ECOL(I) contains an instance which has a solution and consequently by Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

(a.2). If (3.2) holds then, by Theorem I.4.2 we can decide whether or not I has a solution.

(b). Neither of the conditions (3.1), (3.2) holds.

The following construction is very basic for the considerations of this case.

A *base* is a sequence of words $\tau = \tau_0, \tau_1, \ldots$ satisfying the following conditions:

(0). $\tau_0 = \Lambda$,

(1). if $\tau_{i+1}$ is defined then $|\tau_{i+1}| = |\tau_i| + 1$ and $\tau_i \, \mathit{pref} \, \tau_{i+1}$,

(2). for each $i \geq 0$, $a_1 h(\tau_i)$ PREF $b_1 g(\tau_i)$,

(3). for each $i \geq 0$ and each $0 < j < i$, $a_1 h(\tau_j) \neq b_1 g(\tau_j)$, and

(4). if $\tau$ is finite, $\tau = \tau_0, \tau_1, \ldots, \tau_s$ and $c \in \{0,1\}$, then either $a_1 h(\tau_s) = b_1 g(\tau_s)$ or it is not true that $a_1 h(\tau_s c)$ PREF $b_1 g(\tau_s c)$.

The following result follows easily from the definition of a base.

*Claim* 3.1. Let $\tau$ be a base, $\tau = \tau_0, \tau_1, \ldots$ and let $i \geq 0$ be such that $\tau_{i+1}$ is defined. If $\mathit{first}\,(\mathit{dif}(a_1 h(\tau_i), b_1 g(\tau_i))) = c$ then $\tau_{i+1} = \tau_i c$. □

As straightforward corollaries of this claim we get the following two results.

*Claim* 3.2. Let $a_1 = b_1$. Then there exist precisely two bases, denoted $\tau^{(0)}$ and $\tau^{(1)}$. Both are infinite and

(1). $\tau^{(0)}_{r+1} = 0^r$ for every $r \geq 0$,

(2). $\tau^{(1)}_{r+1} = 10^r$ for every $r \geq 0$. □

Note that $k \neq m$ guarantees that $\tau^{(0)}$ is infinite and the negation of (3.1) guarantees that $\tau^{(1)}$ is infinite.

*Claim* 3.3. Let $a_1 \neq b_1$. Then there exists precisely one base, denoted $\tau$, where $\tau_1 = \mathit{first}\,(\mathit{dif}(a_1, b_1))$. □

We will consider separately cases $a_1 = b_1$ and $a_1 \neq b_1$.

(b.1). $a_1 = b_1$.

Assume that x is a solution of I. Then

$$abs\,(|a_1h(x)| - |b_1g(x)|) \leq |a_2b_2| \qquad (3.3)$$

If $first\,(x) = 0$ then Claim 3.2 implies that

$abs\,(|a_1h(x)| - |b_1g(x)|) =$

$abs\,(|x|\;|h(0)| + |a_1| - |b_1| - |x||g(0)|) = |x|\,abs(|h(0)| - |g(0)|)$.

Thus, by (3.3),

$$|x| \leq \frac{|a_2b_2|}{abs(k-m)}$$

(note that since $k \neq m$, $k - m \neq 0$).

Hence, it is decidable whether or not I has a solution.

If $first\,(x) = 1$ then Claim 3.2 implies that

$abs\,(|a_1h(x)| - |b_1g(x)|) =$

$abs\,(|a_1| + |h(1)| + (|x| - 1)|h(0)| - |b_1| - |g(1)| - (|x|-1)|g(0)|)$

and so, by (3.3),

$$|x| \leq \frac{|a_2b_2|}{abs(k-m)} + |a_1b_1h(1)g(1)| + 1;$$

thus it is decidable whether or not I has a solution.

(b.2). $a_1 \neq b_1$.

By Claim 3.3 we know that there exists precisely one base $\tau$.

(b.2.1). We assume first that $dif\,(a_1,b_1) \in 0^+$.

Then we have two cases to consider.

(i). Assume that $a_1h(u) = b_1g(u)$ for some $u \in \{0\}^*$.

Clearly it is decidable whether or not (i) is satisfied and if it is one can find a u, say $u_0$, satisfying (i).

If I has a solution then

either (i.1). I has a solution not longer than $u_0$,

or (i.2). I has a solution of the form $u_0y$ for some $y \in \{0,1\}^*$.

If we assume that (i.1) holds then clearly we can effectively find out whether or not I has a solution.

If we assume that (i.2) holds then let us construct the instance $I_{u_0}$ of GPCP(2) defined by $I_{u_0} = (h,g,a_1h(u_0),a_2,b_1g(u_0),b_2)$. Obviously $I_{u_0}$ has a solution if and only I has a solution of the form $u_0y$ for some $y \in \{0,1\}^*$. Observe that $I_{u_0}$ belongs to the category (b.1) and so we can decide whether or not $I_{u_0}$ has a solution.

(ii). Assume that (i) does not hold.

Clearly for every word $x \in \{0,1\}^*$ we have

$abs\,(|a_1h(x)a_2| - |b_1g(x)b_2|) \geq abs\,(|h(x)|-|g(x)|) - |a_1a_2b_1b_2|$ ..........(3.4)

Since $dif\,(a_1,b_1) \in 0^+$ and (i) does not hold, we have

$abs\,(|h(x)|-|g(x)|) - |a_1a_2b_1b_2| = |x|\,abs\,(k-m) - |a_1a_2b_1b_2|$ .............(3.5)

If x is a solution of I, then $|a_1h(x)a_2| - |b_1g(x)b_2| = 0$

and so, by (3.4) and (3.5), we have

$$|x| \leq \frac{|a_1a_2b_1b_2|}{abs(k-m)}$$

(note that $k \neq m$ and so $abs\,(k-m) \neq 0$).

Thus, by Theorem I.1.2, it is decidable whether or not I has a solution.

(b.2.2). Assume now that $dif\,(a_1,b_1) \notin 0^+$. Clearly we can assume that

$|a_1| > |b_1|$.

Let $p = \#_1\, \mathit{dif}(a_1, b_1)$. The reader can easily prove (by induction on i) the following result.

*Claim* 3.4. For every $i \geq 0$, $\#_1\, \mathit{dif}(a_1 h(\tau_i), b_1 g(\tau_i)) = p$. □

Note that Claim 3.4 does not imply that $\tau$ is infinite in this case. Given $\tau_i = X_{1,i} \dots X_{q,i}$, $i \geq 1$, $X_{1,i}, \dots, X_{q,i} \in \{0,1\}$ we say that the sequence $X_{j,i} \dots X_{j+r,i}$ of(occurrences of) letters from $\tau_i$ is a *block* if $X_{j,i}$ is (an occurrence of) 1, $X_{j+r+1,i}$ is (an occurrence of) 1 and $X_{j+1,i}, \dots, X_{j+r,i}$ are (occurrences of) zeros.

It is easily seen, by induction on j, that Claim 3.4 implies the following result.

*Claim* 3.5. Let $i \geq 1$ and let $\tau_i = \alpha W_1 W_2 \dots W_{s_i} \beta$ where $W_1, W_2, \dots, W_{s_i}$ are all the blocks of $\tau_i$. Then for every $j \geq 1$ such that $j + p \leq s_i$ we have $a_1 h(\alpha W_1 \dots W_j) = b_1 g(\alpha W_1 \dots W_{j+p})$ and $h(W_j) = g(W_{j+p})$. □

Let $z_j = \#_0 W_j$ for $1 \leq j < j + p \leq s_i$.

We will compute now $z_{j+p}$ as the function of $z_j$.

Clearly

$\#_0 h(W_j) = \ell + k z_j$ and $\#_0 g(W_{j+p}) = n + m z_{j+p}$.

Hence, by Claim 3.5, we have

$\ell + k z_j = n + m z_{j+p}$

and so

$z_{j+p} = \frac{k}{m} z_j + \frac{(\ell - n)}{m}$

(note that since $k \neq m$, $\frac{k}{m} \neq 1$).

Thus if we set $D = \frac{k}{m}$ and $F = \frac{\ell - n}{m}$ we get the following linear equation

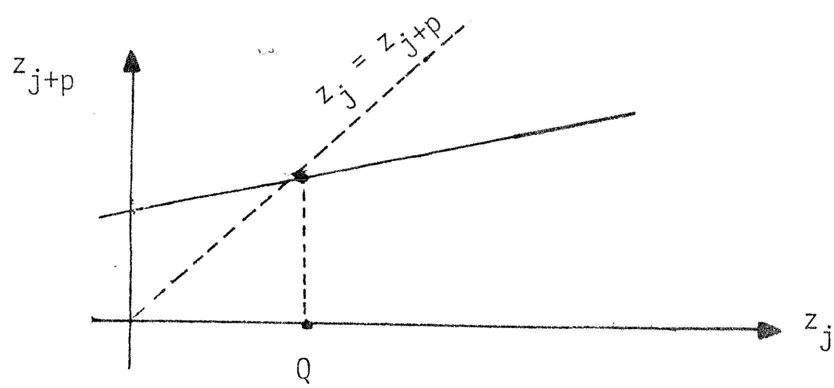
$z_{j+p} = D z_j + F$ ............................................... (3.7)

Note that if $D > 1$ and $\ell \geq n$, then by Theorem I.1.1 we can decide whether or not I has a solution. Thus we can assume that if $D > 1$ then $F < 0$. Similarly we can assume that if $D < 1$ then $F > 0$.

We will analyse equation (3.7) for each of the above two cases separately.

<u>Assume that $D < 1$ and $F > 0$.</u>

Then we have the following situation.

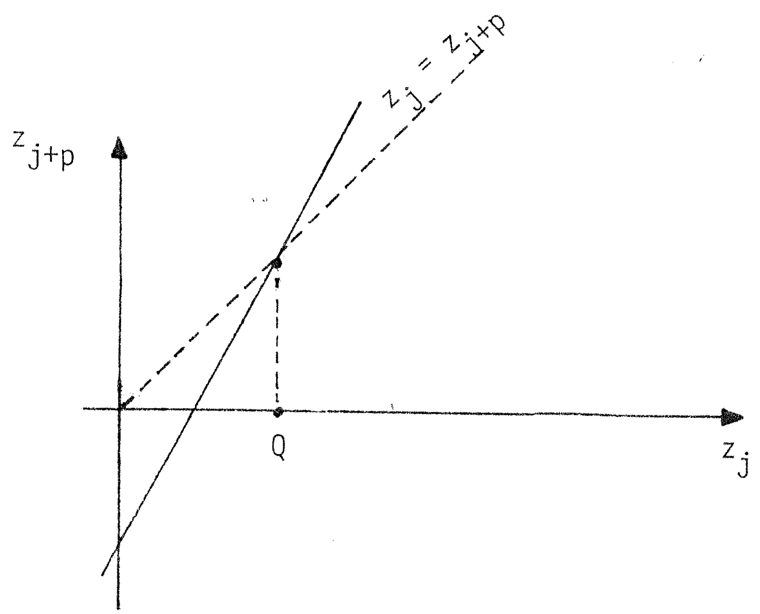



Thus eventually the length of every block becomes $Q + 1$. Let then $\tau_{i_0}$ be the first element of $\tau$ which has $p + 1$ consecutive blocks of length $Q + 1$. Clearly if I has a solution then it has a solution not longer than $|\tau_{i_0}|$. Thus we can effectively find out whether or not I has a solution.

Thus the length of consecutive blocks eventually grows. Now let $\tau_{i_0}$ be the first element of $\tau$ which contains $p + 1$ constructive blocks such that the number of occurrences of 0 in each of them is bigger than $|a_2 b_2|$. Clearly if I has a solution then it has a solution smaller than $|\tau_{i_0}|$ and so we can effectively find out whether or not I has a solution.

<u>Assume that $D > 1$ and $F < 0$</u>.

Then we have the following situation.




Now let $\tau_{i_0}$ be the first element of $\tau$ containing p blocks.

We consider separately three cases.

If one of the p blocks in $\tau_{i_0}$ is shorter than $Q + 1$ then, clearly, $\tau$ is finite and we can effectively decide whether or not I has a solution.

If all p blocks of $\tau_{i_0}$ are not shorter than Q + 1 and at least one of them, say $W_{i_0,q}$ (where $W_{i_0,1}, W_{i_0,2},\ldots, W_{i_0,p}$ are all blocks of $\tau_{i_0}$), is longer than Q + 1 then we proceed as follows. Let $\tau_{i_1}$, $i_1 > i_0$, be the first $\tau_i$ such that for some $t \geq 0$ the length of the block $W_{i_1,q+tp}$ is longer than $|a_2b_2|\mathit{maxr}(I)$. Clearly if a solution of I exists then it is no longer than $|\tau_{i_1}|$ and so we can decide whether or not I has a solution.

If all p blocks in $\tau_{i_0}$ are of length Q + 1 then, clearly, if I has a solution then it has also a solution not longer than $|\tau_{i_0}|$. Hence we can decide whether or not I has a solution.

This concludes the proof of the theorem. $\square$

## 4. $CL_{B_1}$

In this section we demonstrate that GPCP(2) is decidable for good instances from the class $CL_{B_1}$.

*Theorem* 4.1. It is decidable whether or not an arbitrary stable instance I of GPCP(2) such that $I \in CL_{B_1}$ has a solution.

*Proof.*

Let $I = (h,g,a_1,a_2,b_1,b_2)$ be a good instance of GPCP(2) such that $I \in CL_{B_1}$. Hence $h(0) = 0^k$, $h(1) = 10^\ell$, $g(0) = 10^m$ and $g(1) = 0^n$ for some $k$, $n \geq 1$ and $\ell$, $m \geq 0$.

Construct *ecol* $(h,g) = (\overline{h},\overline{g})$. It is easily seen that $\overline{h}(0) = 0^{\overline{k}}$, $\overline{h}(1) = 10^{\overline{\ell}}$, $\overline{g}(0) = 1^{\overline{n}}$ and $\overline{g}(1) = 01^{\overline{m}}$ where $\overline{k}$, $\overline{n} \geq 1$ and $\overline{\ell}$, $\overline{m} \geq 0$.

Note that it cannot be that both $\overline{\ell} \neq 0$ and $\overline{m} \neq 0$ because this contradicts Lemma I.3.7.

Assume then that $\overline{\ell} = 0$ (the case of $\overline{m} = 0$ reduces to this one by applying the homomorphisms switch and the range switch). Hence we have $\overline{h}(0) = 0^{\overline{k}}$, $\overline{h}(1) = 1$, $\overline{g}(0) = 1^{\overline{n}}$ and $\overline{g}(1) = 01^{\overline{m}}$.

If $\overline{k} = 1$ then $|\overline{h}(0)| \leq |\overline{g}(0)|$ and $|\overline{h}(1)| \leq |\overline{g}(1)|$ and so by Theorem I.1.1 we can decide whether or not ECOL(I) contains an instance which has a solution and consequently, by Theorem I.3.1 and Theorem I.4.1, we can decide whether or not I has a solution.

Assume then that $\overline{k} > 1$.

Then, by Lemma I.3.6, it must be that $\overline{m} = 0$. Hence $\overline{h}(0) = 0^{\overline{k}}$, $\overline{h}(1) = 1$, $\overline{g}(0) = 1^{\overline{n}}$ and $\overline{g}(1) = 0$, and so if $\overline{k} \leq \overline{n}$ then $|\overline{h}(0)| \leq |\overline{g}(0)|$ and $|\overline{h}(1)| \leq |\overline{g}(1)|$ and if $\overline{n} \geq \overline{k}$ then $|\overline{g}(0)| \leq |\overline{h}(0)|$ and $|\overline{g}(1)| \leq |\overline{h}(1)|$. Hence by Theorem I.1.1 we can decide whether or not ECOL(I) contains an instance which has a solution and consequently, by Theorem I.3.1 and Theorem I.4.1, we can decide whether or not I has a solution. $\square$

## 5. REDUCTION THEOREM FOR $CL_C$.

In this section we begin the investigation of the class $CL_C$. The main result of this section, Theorem 5.6, allows us to consider only those instances $I = (h,g,a_1,a_2,b_1,b_2)$ from $CL_C$ for which if $i \in \{0,1\}$ then *first* (h(i)) = *last* (h(i)) and *first* (g(i)) = *last* (g(i)).

We start by introducing a classification of C-homomorphisms.

*Definition* 5.1. Let f be a C-homomorphism. Then

(a). $f \in$ SAME if and only if *first* (f(i)) = *last* (f(i)) for $i \in \{0,1\}$,

(b). $f \in$ FLIP if and only if *first* (f(i)) $\neq$ *last* (f(i)) for $i \in \{0,1\}$,

(c). $f \in LAST_0$ if and only if *last* (f(0)) = *last* (f(1)) = 0.

(d). $f \in LAST_1$ if and only if *last* (f(0)) = *last* (f(1)) = 1. □

Clearly the above classification exhausts all possibilities for a C-homomorphism.

Let (h,g) be a good pair of C-homomorphisms. Then the following four results hold.

*Lemma* 5.1. Both h $\overline{h}$ and g $\overline{g}$ are C-homomorphisms.

*Proof.*

Obvious. □

*Lemma* 5.2.

(a). If $h \in$ FLIP, then $\overline{h}$ is an A-homomorphism.

(b). If $g \in$ FLIP, then $\overline{g}$ is an A-homomorphism.

*Proof.*

(a). It follows from Lemma 5.1 and from the simple observation that if $h \in$ FLIP then neither h(0)h(1) nor h(1)h(0) are in C.

(b). The proof is analogous to the proof of (a). □

*Lemma* 5.3. Let $j \in \{0,1\}$.

(a). If $h \in LAST_j$, then there exists an $i \in \{0,1\}$ such that either $\overline{h}(0) \in A_i$ and $\overline{h}(1) \in B_{1-i}$ or $\overline{h}(0) \in B_{1-i}$ and $\overline{h}(1) \in A_i$.

(b). If $g \in LAST_j$ then there exists an $i \in \{0,1\}$ such that either $\overline{g}(0) \in A_i$ and $\overline{g}(1) \in B_{1-i}$ or $\overline{g}(0) \in B_{1-i}$ and $\overline{g}(1) \in A_i$.

*Proof.*

(a). Let k be such that $\mathit{first}(h(k)) = \mathit{last}(h(k)) = j$. Then, obviously, $h(k)h(k) \notin C$ and $h(1-k)h(k) \notin C$. Hence, by Lemma 5.1, neither $\overline{h}(0)$ nor $\overline{h}(1)$ can have $k\,k$ or $(1-k)k$ as a subword. Thus if we set $i=1-k$, (a) follo

(b). The proof is analogous to the proof of (a). □

*Lemma* 5.4.

(a). If $h \in SAME$, then $\overline{h}$ is a C-homomorphism.

(b). If $g \in SAME$, then $\overline{g}$ is a C-homomorphism.

*Proof.*

(a). If $h \in SAME$ then, for $i \in \{0,1\}$, $h(i)h(i) \notin C$ and so the result follows from Lemma 5.1.

(b). Is proved in the same way. □

*Definition* 5.2. Let $I = (h,g,a_1,a_2,b_1,b_2)$ be a good instance of GPCP(2) and let $X, Y \in \{SAME, FLIP, LAST_0, LAST_1\}$. Then we say that I is a $(X,Y)$ *instance* if $h \in X$ and $g \in Y$. □

*Theorem* 5.1. It is decidable whether or not an arbitrary (FLIP,FLIP) instance I of GPCP(2) has a solution.

*Proof.*

Assume that $ECOL(I) \neq \emptyset$ and let $J \in ECOL(I)$. Then, by Lemma 5.2, (perhaps with the use of the domain switch) $J \in CL_A$. Thus the theorem follows from Theorem 1.1, Theorem 1.2, Theorem I.3.1 and Theorem 1.4.1.

*Theorem* 5.2. It is decidable whether or not an arbitrary $(LAST_0, LAST_0)$ instance I of GPCP(2) has a solution.

*Proof.*

Assume that $ECOL(I) \neq \emptyset$ and let $J \in ECOL(I)$. Then, by Lemma 5.3, (perhaps with the use of the domain switch) $J \in CL_B$. Thus the theorem follows from Theorem 2.1, Theorem 2.2, Theorem I.3.1 and Theorem I.4.1. □

*Theorem* 5.3. It is decidable whether or not an arbitrary (SAME,FLIP) instance I of GPCP(2) has a solution.

*Proof.*

From Lemma 5.2 and Lemma 5.4 it follows that $Right(\overline{h}) = \{\alpha,\beta\}$ and $Right(\overline{g}) = \{0^n,1^m\}$ where $\alpha \in C_0$, $\beta \in C_1$, $|\alpha| = k+1$, $|\beta| = \ell+1$, $k, \ell \geq 0$ and $m, n \geq 1$.

We consider separately two cases.

<u>$n \geq 2$</u>.

It is easily seen that, by Lemma I.3.6, either $k = 0$ and $\ell = 0$ or $k = 0$ and $\ell = 1$.

If $k = 0$ and $\ell = 0$ then by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

Assume then that $k = 0$ and $\ell = 1$.

It is easily seen that, by Lemma I.3.6, $m = 1$ and so we have $Right(\overline{h}) = \{0,10\}$ and $Right(\overline{g}) = \{0^n,1\}$.

If for some $i \in \{0,1\}$, $\overline{h}(i) = 0$ and $\overline{g}(i) = 1$ then by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

Otherwise, if ECOL(I) $\neq \emptyset$ and $J \in$ ECOL(I), then (perhaps with the use of the domaine switch) $J \in CL_B$ and so by Theorem 3.1, Theorem 3.2, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

$\underline{n = 1}$.

If $m = 1$ then by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

If $m \geq 2$ then by the range switch we get the previous case of $n \geq 2$. □

*Theorem* 5.4. It is decidable whether or not an arbitrary (SAME,$LAST_0$) instance I of GPCP(2) has a solution.

*Proof*.

By Lemma 5.3 and Lemma 5.4 we have that $Right(\overline{h}) = \{\alpha,\beta\}$ with $\alpha \in C_0$, $\beta \in C_1$, $|\alpha| = k + 1$, $|\beta| = \ell + 1$, $k, \ell \geq 0$ and either $Right(\overline{g}) = \{0^n,10^m\}$ or $Right(\overline{g}) = \{1^n,01^m\}$ for $n \geq 1$, $m \geq 0$. It suffices to consider $Right(\overline{g}) = \{0^n,10^m\}$ since the other case reduces by the range switch to this one.

We consider separately the cases of $n \geq 2$ and $n = 1$.

$\underline{n \geq 2}$.

By Lemma I.3.6, either $Right(\overline{h}) = \{0,1\}$ or $Right(\overline{h}) = \{0,10\}$.

If $Right(\overline{h}) = \{0,1\}$ then by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

We consider two cases. If $m \geq 1$ then by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution. If $m = 0$ then by Theorem 3.1, Theorem 4.1, Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

If $Right(\overline{h}) = \{0,10\}$ then by Theorem 5.2, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

$\underline{n = 1}$.

If $m \geq 2$ then, by Lemma I.3.6, $k = 0$. Then by Theorem I.3.1, Theorem I.4.1 and Theorem I.4.2 we can decide whether or not I has a solution.

If $m = 0$ then $|\overline{g}(0)| \le |\overline{h}(0)|$ and $|\overline{g}(1)| \le |\overline{h}(1)|$ and so by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 it is decidable whether or not I has a solution.

Let us then assume that $m = 1$. Then let $j \in \{0,1\}$ be such that $\overline{g}(j) = 10$. If $|\overline{h}(j)| \ge 2$, then $|\overline{g}(0)| \le |\overline{h}(0)|$ and $|\overline{g}(1)| \le |\overline{h}(1)|$; consequently by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

Thus it suffices to consider two cases: $k = 0$ and $\ell = 0$ (with $k \ne 0$).

<u>$k = 0$</u>.

Then *Right*$(\overline{h}) = \{0,\beta\}$ and *Right*$(\overline{g}) = \{0,10\}$. Thus by Theorem I.4.2, Theorem I.3.1 and Theorem I.4.1 one can decide whether or not I has a solution.

<u>$\ell = 0$ and $k \ne 0$</u>.

Then *Right*$(\overline{h}) = \{\alpha,1\}$ and *Right*$(\overline{g}) = \{0,10\}$.

If, for an $i \in \{0,1\}$, $\overline{h}(i) = 1$ and $\overline{g}(i) = 0$ then by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

Thus we assume that, for some $i \in \{0,1\}$, $\overline{h}(i) = \alpha$, $\overline{h}(1-i) = 1$, $\overline{g}(i) = 0$ and $\overline{g}(1-i) = 10$; note that by Lemma I.3.7 $k$ must be even. So we can assume that $\overline{h}(0) = \alpha$, $\overline{h}(1) = 1$, $\overline{g}(0) = 0$ and $\overline{g}(1) = 10$; the other case reduces to this one after the domain switch.

Construct *ecol* $(\overline{h},\overline{g}) = (\overline{\overline{h}},\overline{\overline{g}})$.

It is easily seen that $\overline{\overline{h}}(0) = 0$, $\overline{\overline{h}}(1) = 10$, $\overline{\overline{g}}(0) = 01^{\frac{k}{2}}$ and $\overline{\overline{g}}(1) = 1^{\frac{k}{2}+1}$.

Consequently by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution. □

*Theorem* 5.5. It is decidable whether or not an arbitrary $(\mathrm{FLIP},\mathrm{LAST}_0)$ instance I of GPCP(2) has a solution.

*Proof.*

By Lemma 5.2 and Lemma 5.3, $Right(\overline{h}) = \{0^k, 1^\ell\}$ and either $Right(\overline{g}) = \{0^m, 10^n\}$ or $Right(\overline{g}) = \{1^m, 01^n\}$ where $k, \ell, m \geq 1$ and $n \geq 0$; it suffices to consider the case of $Right(\overline{g}) = \{0^m, 10^n\}$ because the other case can be obtained by the range switch.

We consider separately two cases.

$\underline{\ell \geq 2}$.

Lemma I.3.6 implies that $n = 0$ and so (perhaps using the domain switch) by Theorem 1.1, Theorem 2.1, Theorem I.3.1 and Theorem I.4.1 we can decide whether or not I has a solution.

$\underline{\ell = 1}$.

Then (perhaps using the domain switch) by Theorem 3.1, Theorem 4.1, Theorem 1.3.1 and Theorem 1.4.1 one can decide whether or not I has a solution. □

*Theorem* 5.6. It is decidable whether or not an arbitrary good instance I of GPCP(2), such that $I \in CL_C$ but I is not a (SAME,SAME) instance, has a solution.

*Proof.*

Clearly, for some X, Y $\in$ {SAME,FLIP,$LAST_0$,$LAST_1$}, I is a (X,Y) instance.

First of all we notice that by Lemma I.3.7 neither I is a ($LAST_1$,$LAST_0$) instance, nor I is a ($LAST_0$,$LAST_1$) instance.

Then we notice that

(i). by the homomorphisms switch (FLIP,SAME), ($LAST_0$,SAME), ($LAST_1$,SAME), ($LAST_0$,FLIP) and ($LAST_1$,FLIP) cases reduce to (SAME,FLIP), (SAME,$LAST_0$), (SAME,$LAST_1$), (FLIP,$LAST_0$) and (FLIP,$LAST_1$) cases respectively;

(ii). by the range switch (SAME,$LAST_1$), (FLIP,$LAST_1$) and ($LAST_1$,$LAST_1$) cases reduce to (SAME,$LAST_0$), (FLIP,$LAST_0$) and ($LAST_0$,$LAST_0$) cases respectively.

Consequently the theorem follows by Theorems 5.1 through 5.5. □

## 6. $CL_C$

In this section we demonstrate that GPCP(2) is decidable for good instances from the class $CL_C$.

*Theorem* 6.1. It is decidable whether or not an arbitrary good instance I of GPCP(2) such that $I \in CL_C$ has a solution.

*Proof.*

Let $I = (h,g,a_1,a_2,b_1,b_2)$ be a good instance of GPCP(2) such that $I \in CL_C$. By Theorem 5.6 we can assume that $I \in$ (SAME,SAME). Thus by Lemma 5.4 both $\overline{h}$ and $\overline{g}$ are C-homomorphisms. Consequently:

for each $i \in \{0,1\}$ there exist $k_i$, $\ell_i \geq 0$ and $u_i, t_i \in \{0,1,\Lambda\}$ such that
$\overline{h}(i) \in \{01,10\}^{k_i} u_i$ and $\overline{g}(i) \in \{01,10\}^{\ell_i} t_i$ ......................(6.1)

Since for each $i \in \{0,1\}$ we have $h\overline{h}(i) = g\overline{g}(i)$, (6.1) implies that

$k_i \mid h(0)h(1) \mid + \mid h(u_i) \mid = \ell_i \mid g(0)g(1) \mid + \mid g(t_i) \mid$..........(6.2).

We consider separately three cases.

(a). Assume that $|h(0)h(1)| = |g(0)g(1)|$ .

If there exist $i,j \in \{0,1\}$ such that $h(i) = g(j)$ then by Theorem I.4.2 it is decidable whether or not I has a solution.

Otherwise, $|\overline{h}(0)| = |\overline{h}(1)| = |\overline{g}(0)| = |\overline{g}(1)| = 2$ and so by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 it is decidable whether or not I has a solution.

(b). Assume that $|h(0)h(1)| > |g(0)g(1)|$ .

Let $i \in \{0,1\}$.

If $k_i = \ell_i$, then (because $|h(0)h(1)| > |g(0)g(1)|$) it must be that $t_i \neq \Lambda$ and so $|\overline{h}(i)| \leq |\overline{g}(i)|$.

If $k_i < \ell_i$, then, by (6.1), $|\overline{h}(i)| < |\overline{g}(i)|$.

If $k_i > \ell_i$, then

$$\ell_i|g(0)g(1)| + |g(t_i)| < \ell_i|g(0)g(1)| + |g(0)g(1)| = (\ell_i+1)\ |g(0)g(1)| <$$
$$< k_i|h(0)h(1)| + |h(u_i)|$$

which contradicts (6.2); thus $k_i > \ell_i$ cannot hold.

Consequently, for each $i \in \{0,1\}$, $|\overline{h}(i)| \leq |\overline{g}(i)|$ and so by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 it is decidable whether or not I has a solution.

(c). The case of $|h(0)h(1)| < |g(0)g(1)|$ reduces to the previous one by the homomorphisms switch. □

## 7. THE MAIN THEOREM

As a straightforward corollary of Theorems 1.1, 2.1, 3.1, 4.1 and 6.1 we get the main result of this paper.

*Theorem* 7.1. It is decidable whether or not an arbitrary stable instance I of GPCP(2) such that either $I \in CL_A$ or $I \in CL_B$ or $I \in CL_C$ has a solution.

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