Scientific Report No. 42

STRUCTURE OF THE ELECTROMAGNETIC FIELD OF A VERTICAL ELECTRIC DIPOLE AND OF VERTICAL ANTENNAS IN THE SPACE ABOVE A PLANE EARTH

by

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Translated from the Russian by

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January 1979

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Translator's Preface

This paper, nearly 20 years old, deals with the problem of the numerical evaluation of the integrals of the Sommerfeld type which arise in the analysis of the electromagnetic field of a vertical electric dipole (VED) radiating above a homogeneous, conducting and/or dielectric half-space; e.g., the earth. Although many authors since Sommerfeld have investigated various approximate formulas, Krylov and Makarov seem to be the first to have attempted rigorous series expansions for these integrals. In spite of the fact that convergence of their series is often quite leisurely (as the authors themselves admit), this translation was prepared in hopes of giving their techniques a wider audience and of stimulating further research in this direction.

This translator has corrected a number of misprints from the original Russian article, but assumes full responsibility for any which remain or have been introduced during the translation. The generous assistance of Dr. J.N. Brittingham of Lawrence Livermore Laboratory to the preparation of this manuscript is gratefully acknowledged.

STRUCTURE OF THE ELECTROMAGNETIC FIELD OF A VERTICAL ELECTRIC DIPOLE AND OF VERTICAL ANTENNAS IN THE SPACE ABOVE A PLANE EARTH*

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INTRODUCTION

At the present time, great interest has arisen in the theory of radiowave propagation in determining the structure of the electromagnetic fields of radiators distributed over the surface of the earth. At small distances, the curvature of the earth does not yet influence radiowave propagation and the earth can be taken as planar.** An investigation of this problem with all factors taken into account is an extraordinarily difficult task, therefore we have to limit our considerations only to those factors which exert the strongest influence on the solution of the physical problem. In this paper, the influence of the finite conductivity of the earth on the structure of the electromagnetic field of a vertical electric dipole (VED) and of vertical antennas will be investigated.

The classical statement of the problem of the field of a VED was made by A. Sommerfeld as early as 1909 [2]. Since that time many authors have addressed themselves to this problem, and it is appropriate for us to discuss here the following investigations, the closest in theme to ours of which we are aware: those of K. A. Norton and P. A. Ryazin [3,4].

* Vestnik Leningrad. Univ. (ser. Fiz. Khim.), No. 16, vyp. 3 (1960), pp. 42-66. Translated from the Russian by E. F. Kuester, Dept. of Elec. Eng., University of Colorado, Boulder.

** In the paper of Yu. K. Kalinin and E. A. Feinberg [1] it is shown that this is valid for distances up to 100-150 km.

In Norton's work, the most complete investigation of the electromagnetic field structure of a VED in space was done using an approximate expression for the Hertz vector, borrowed from the work of van der Pol [5], from the very beginning. In his work, this approximate expression is improved so that the first terms of the asymptotic expansions of the exact and approximate Hertz vectors coincided. In later studies, additional approximations were made which were equivalent to assuming i) that the earth constants were such that the problem could be solved by the impedance method [6], ii) that the distance from the dipole to the observation point was large compared to a·wavelength. Norton also considered the electromagnetic field of a vertical antenna, which he obtained by integrating a source function with the approximate expression for the Hertz vector of a vertical dipole; in doing so, only one approximation was used to obtain computational formulas: that the heights of the antenna and of the observation point were small compared to the distance between them. These indicated limitations have a profound effect on the region of applicability of the expansions he obtained and on the accuracy of his computations; thus, they are ill-suited for computations in cases when the source or observation point is elevated, but above all there is no way of checking the accuracy of these computations.

In P. A. Ryazin's paper, a study and systematization of rigorous expansions for the Hertz vector obtained by a number of authors [7-10] was carried out, but these expansions dealt only with one particular case - when the dipole and the observation point are located on the surface of the earth. The expansions obtained allow computations to be carried out only for the vertical component of electric field; for the horizontal components it is then necessary to differentiate with respect to the height, which does not appear in the expansions.

Ryazin relied entirely on Norton's results to investigate the fields in space, and therefore his investigations suffer from the same inadequacies indicated above.

We also note the recently published paper of G. A. Zuikina [11] in which the Hertz vector is studied both in the case of a plane and in the case of a spherical earth. This work is of a purely numerical character and is concerned with a study of the long-known attenuation function of a vertical dipole, and the accuracy of computations using the attenuation function is not studied. The concepts of "relative ground" and "mean phase velocity" introduced by the author, in our opinion, hardly have any meaning. The concept of "mean phase velocity" can have meaning for particular earth parameters and distances between receiver and transmitter, but the variation of this "mean phase velocity" is of the same order as that of the distance or of the properties of the ground. Moreover, the author makes a number of erroneous assertions, in particular having to do with the final values of Δr and $\Delta\phi$ maximum which do not agree with the already long-known limiting phase relationships [12], however, since the author does not indicate how these quantities were computed, we must limit ourselves only to commenting upon and not to discussing details.

The goal of this paper will be to obtain and study exact and asymptotic expansions for the electromagnetic field components of a VED and of vertical antennas located in an arbitrary fashion over the earth's surface, which will allow field computations to be made at an arbitrary point in space. Primary attention was paid to the problems of convergence of the expansions obtained and to the accuracy of the calculations. Field patterns are investigated in the farfield which allows the disturbance of the electromagnetic field structure in space due to the finite conductivity of the earth to be determined, and the influence of other basic factors in the present problem, for instance the ionosphere, to be gauged.

1. Formulation of the Problem

We will assume that a VED is situated at the point (h,o), while the observation point is located at (z', ρ) (in a cylindrical coordinate system). In studying the vertical antenna problem, we will assume that the antenna is located at $\rho = o$ from $z = h_1$ to $z = h_2$ and that current in the antenna is sinusoidally distributed: $j = kj_0 \sin \phi$, where $\phi = k(h + b)$ and b is some phase constant (Figure 1).

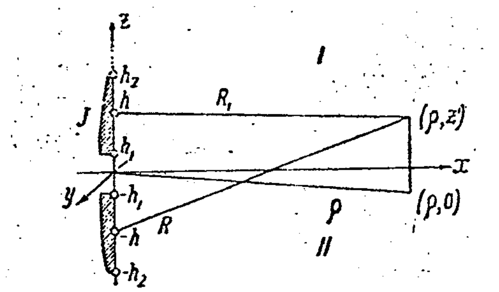

Figure 1 - Positions of Dipoles, Antennas, and Observation Points

As a starting point for our investigations, we take the rigorous solution of the boundary problem for the VED, wherein the Hertz vector is represented in the form [3,14]:

$$\pi_z = \pi_{zo} + \frac{e^{ikR_1}}{R_1} - \frac{e^{ikR}}{R}; \quad \pi_{zo} = \int_{-\infty}^{\infty} \frac{k_3^2 H_0(\lambda\rho)}{k_3^2 n + k^2 m} e^{-nz} \lambda d\lambda, \tag{1}$$

in which $z = z' + h$; $n^2 = \lambda^2 - k^2$; $m^2 = \lambda^2 - k_3^2$; Re $n \geq 0$; Re $m \geq 0$.*

For the following investigations, it is convenient to subject the integrand of the function π_{zo} to some identity transformations:

$$\pi_{zo} = \frac{I_1 - \tau^2 I_2}{1 - \tau^4}, \quad I_1 = \int_{-\infty}^{\infty} \frac{nH_0(\lambda\rho)}{n^2 + k^2 a_1^2} e^{-nz} \lambda d\lambda, \quad I_2 = \int_{-\infty}^{\infty} \frac{mH_0(\lambda\rho)}{m^2 + k_3^2 a_1^2} e^{-nz} \lambda d\lambda, \tag{2}$$

* The numbers k and k_3 refer to the wave numbers in air and in the earth respectively (Translator's note).

where $a_2 = (1 + \tau^2)^{-1/2}$; $\tau = k/k_3$; $a_1 = \tau a_2$; Re $a_2 > 0$; Re $a_1 > 0$, which turns out to be suitable for obtaining rigorous expansions for computing the electromagnetic fields, while for obtaining approximate and asymptotic expansions it is convenient to perform a somewhat different splitting of the initial integral:

$$\pi_{z0} = \frac{I_1^a - \tau I_2^a}{1 - \tau^4}, \quad I_1^a = \int_{-\infty}^{\infty} \frac{H_0(\lambda\rho)e^{-nz}}{n - ika_1} \lambda d\lambda, \quad I_2^a = \int_{-\infty}^{\infty} \frac{H_0(\lambda\rho)e^{-nz}}{m - ik_3a_2} \lambda d\lambda. \tag{3}$$

Having carried out the formulation of the problem, it is appropriate to say a few words on the splitting of the initial form of solution into space wave, surface wave, and earth wave. We must agree completely with the review paper of T. Kahan and C. Eckart that the surface wave is not contained in the radiation of the dipole, and only in view of the arbitrariness with which the cuts are drawn is one allowed to split the initial form of solution into a branch cut integral around $\lambda = k$, a residue, and a branch cut integral around $\lambda = k_3$, which are usually identified with the three indicated waves [15]. However, in this splitting of the integral, the residue comes into the expansion of the space wave with opposite sign, so that their combination has no logarithmic singularity at $\rho = 0$, which will be clear from the rigorous expansions to be obtained herein.

It is useful to note that for $z \neq 0$ the values of the integrals I_2 and I_2^a will bring in a large contribution to the final solution with increasing z. Each of these integrals describes a wave progressing along the paths AA' and B'B in medium I and along the path A'B' in medium II (Fig. 2). As will be shown, the influence of medium II on the magnitude of these integrals falls off exponentially with increasing z, leading to an increase in the influence of medium I. Because of this, the values of the integrals I_2 and I_2^a cannot be assumed small as z increases, and these must be computed in what follows.

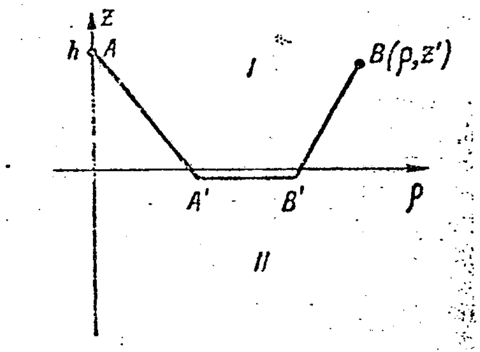

Fig. 2. Influence of the field of the earth wave on the field in space.

2. Transformation of the Initial Solution

Before proceeding with the investigation, we carry out an integral transformation on the integrals introduced in section 1; which leads to a representation of the Hertz vector as a superposition of spherical waves. A similar method has been applied previously to a study of the case z = 0[7,4], but even for z = 0, only one, vertical, electric field component of the dipole can be calculated by known methods.

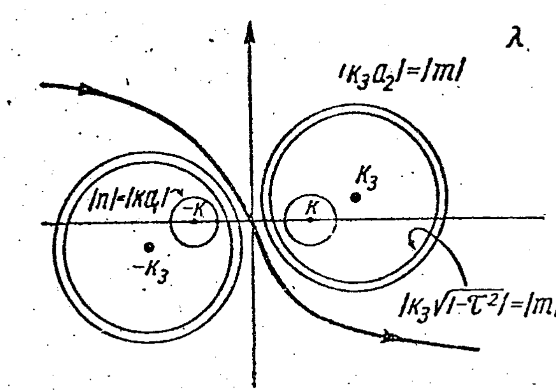

Fig. 3. Deformation of the Original Contour of Integration to Carry out the Integral Transformation

Let us first consider the integral I_1. The denominator of the integrand can be expanded in a series in the neighborhood of the point at infinity in terms of negative powers of n. In order to obtain the final expansions it is necessary to change the order of summation and integration. To justify this change we note that under the condition $n > |ka_1|$ the series can be dominated by a geometric progression, and therefore it is sufficient to show that this condition is fulfilled over the entire contour. The boundary of convergence of the expansion in inverse powers of n is determined by the equation $|n| = |ka_1|$. These are curves of fourth order, which are obtained if a conformal transformation, putting $\lambda = k^2x$ (Fig. 3), is carried out on a circle of radius $|a_1|^2$ centered at $x = 1$ in the x-plane. Based on Cauchy's theorem it can be verified that the original contour can be deformed so that the condition $|n| > |ka_1|$ is satisfied over the entire contour, whence we obtain

$$I_1 = \sum_{\nu=0}^{\infty} (ika_1)^{2\nu} X_{2\nu+1}, \quad X_\nu = \int_{-\infty}^{\infty} H_0(\lambda\rho) e^{-nz} n^{-\nu} \lambda d\lambda$$

If now we use an alternative integral representation for the coefficients X_ν, obtained by Sommerfeld [2, p. 713],

$$X_\nu = \frac{2}{(\nu-2)!} \int_0^{\infty ik^*} \frac{e^{ikR_\lambda}}{R_\lambda} \lambda^{\nu-2} d\lambda, \quad R_\lambda^2 = \rho^2 + (z+\lambda)^2,$$

as well as a representation of the Γ- function as a contour integral [16, p. 279] and interchange the order of the integrations, we obtain a convenient representation for these coefficients

$$X_\nu = \frac{1}{\pi i} \int_c \frac{1}{x^n - 1} \int_0^{\infty ik^*} \frac{e^{ikR_\lambda + \lambda x}}{R_\lambda} d\lambda dx, \quad \nu \geq 2,$$

where the contour c is a circle centered at the origin with radius $(1-\varepsilon)k$ and $\varepsilon > 0$ (Fig. 4). The interchange of the integrations is valid since $|x| < |k|$ and consequently the λ-integrals converge exponentially for any x.

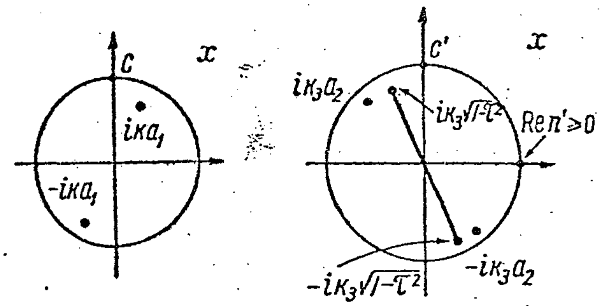

Fig. 4. Contours of Integration for the Inner Integrals After Carrying out the Integral Transformations

If we now substitute the indicated expressions for the X_ν into the summation in (4) and change the order of summation and integration (over x), we arrive at the desired integral representation for I_1.

$$I_1 = \frac{2e^{ikR}}{R} - \frac{1}{\pi i}\int_c \frac{k^2a_1^2}{x^2 + k^2a_1^2}\int_0^{\infty ik^*} \frac{e^{ikR_\lambda + \lambda x}}{R_\lambda}\, d\lambda dx. \qquad (5)$$

Let us now show that the interchange of summation and integration is valid. In expression (5) under the x integral is an analytic function which has two singularities of pole type at $x = \pm ika_1$, whence one can conclude that the expansion in inverse powers of x for this function will converge absolutely over the entire contour under the condition that $|x| > |ka_1|$, which is valid since $|a_1| < 1$. If after this the order of summation and integration (over x) is changed and the new expansion compared with the old one, then we become convinced of their identity. To obtain the final representation it remains to change the order of the integrations in formula (5), which has already been justified once, and we obtain

$$I_1 = \frac{2e^{ikR}}{R} - \frac{1}{\pi i}\int_0^{\infty ik^*} \frac{e^{ikR_\lambda}}{R_\lambda}\int_c \frac{k^2a_1^2e^{\lambda x}}{x^2 + k^2a_1^2}\, dxd\lambda \qquad (6)$$

By completely analogous arguments a transformation of the integral I_1^a could also be performed, and would give

$$I_1^a = \frac{2e^{ikR}}{R} + \frac{1}{\pi}\int_0^{\infty ik^*} \frac{e^{ikR_\lambda}}{R_\lambda} \int_C \frac{ka_1 e^{\lambda x}}{x - ika_1}\, dx d\lambda \tag{7}$$

We now address the problem of obtaining a similar transformation for the integrals I_2 and I_2^a. We note that final representations of the integrals I_1 and I_1^a could have been obtained by following the known methods of [7,4] instead of performing these transformations, but the former lead to a more cumbersome derivation. However, it was not possible to obtain expansions for the integrals I_2 and I_2^a by applying any method known to us up to now, since for $z \neq 0$ there is a multivalued function in the exponent, which cannot be expanded in the neighborhood of the point at infinity. In obtaining the integral representation for the integral I_1, the explicit form of the coefficients in the expansion (4) was of no essential significance, since the series was summed in closed form; this opens up the possibility that transformations for the integrals I_2 and I_2^a can be performed using only the fact that similar expansions do exist.

We consider first the integral I_2, which it is convenient to write in the following form:

$$I_2 = \int_{-\infty}^{\infty} H_0(\lambda\rho)e^{-zm}F(\lambda)\lambda d\lambda,\ F(\lambda) = \frac{me^{z(m-n)}}{m^2 + k_3^2 a_2^2} .$$

If we realize that the contour of integration is so positioned that as $\lambda \to \infty$ the arguments of both square roots in the function $F(\lambda)$ become identical, then it can be shown that the maximum of $|m-n|$ on the upper sheet of the Riemann surface is reached when λ coincides with one of the branch points, and its magnitude is $|k_3\sqrt{1 - \tau^2}|$. Thus we conclude that the exponential factor can be expanded into a series which is absolutely convergent over the entire contour, and upon doing so we obtain a series of integrals

$$I_2 = \sum_{\nu=0}^{\infty} \frac{z^\nu}{\nu!} Y_\nu, \quad Y_\nu = \int_{-\infty}^{\infty} \frac{H_0(\lambda\rho)(m-n)^\nu}{m^2 + k_3^2 a_2^2} m\lambda d\lambda.$$

Let us concentrate on calculating the coefficients Y_ν. First, the denominator of the integrand of Y can be expanded as a series in negative powers of m which will be absolutely convergent over the entire contour if the condition $|a_2| < 1$ is satisfied, as in the case of the integral I_1. Secondly, the factor $(m-n)^\nu$ can be expanded into a series in negative powers of m which will be absolutely convergent over the entire contour if the condition $|1-\tau^2| < 1$ is satisfied. Having carried out the indicated operations and using the representation for the coefficients X_ν (with $k=k_3$), a new representation for the coefficients Y_ν can be obtained analogously to the case of the integral I_1. If we then carry out the summation over ν, we obtain a final representation for the integral I_2 (Fig. 4),

$$I_2 = \frac{2e^{ik_3R}}{R} + \frac{1}{\pi i} \int_0^{\infty i k_3^*} \frac{e^{ik_3R_\lambda}}{R_\lambda} \int_{c'} \left[\frac{x^2 e^{z(x-n')}}{x^2 + k_3^2 a_2^2} - 1 \right] e^{\lambda x} dx d\lambda, \tag{8}$$

where $n' = \sqrt{x^2 + k^2 - k_3^2}$; $c' = c/|\tau|$.

Analogous arguments could also be made for the integral I_2^a, and we have finally:

$$I_2^a = \frac{2e^{ik_3R}}{R} + \frac{1}{\pi i} \int_0^{\infty i k^*_3} \frac{e^{ik_3R_\lambda}}{R_\lambda} \int_{c'} \left[\frac{x e^{z(x-n')}}{x - ik_3 a_2} - 1 \right] e^{\lambda x} dx d\lambda, \tag{9}$$

In obtaining the integral representations the conditions $|a_1| < 1$, $|a_2| < 1$ and $|1-\tau^2| < 1$ were used, which always hold if k is real and k_3 is complex and $\mathrm{Re}\, k_3 \geq \mathrm{Re}\, k$. However, inasmuch as the integrals are analytic functions of τ, the identities obtained will be valid for those values of τ where the integrals make sense.

3. Rigorous Calculation of the Integral I_1

We turn now to obtaining rigorous expansions for computing the Hertz vector, beginning with the integral I_1. On the basis of the residue theorem the x-integration in expression (6) is equal to sin $(ka_1\lambda)$, so that after a change of variables I_1 can be represented in the form

$$I_1 = \frac{2e^{ikR}}{R} - 2ka_1\int_z^{\infty ik^*} \frac{e^{ikR'}}{R'} \sin[ka_1(\lambda - z)]d\lambda, \quad R' = \sqrt{\rho^2 + \lambda^2}. \tag{10}$$

We now show that if the condition $|a_1| < 1$ is satisfied, sin $(ka_1\lambda)$ and cos $(ka_1\lambda)$, which are obtained upon expansion of sin $[ka_1(\lambda-z)]$, can be expanded into series and the order of summation and integration with respect to λ with infinite limits can be interchanged. To do this it suffices to justify the interchange for integration between 0 and ∞ik^*, since it is always justified for the integration from 0 to z. For the justification we consider only the integral with $\sin(ka_1\lambda)$; clearly we have:

$$\int_0^{Qik^*} \frac{e^{ikR'}}{R'} \sin(ka_1\lambda)d\lambda = \sum_{\nu=0}^{\infty} V_\nu, \quad V_\nu = \frac{(ika_1)^{2\nu + 1}}{i(2\nu + 1)!} \int_0^{Qik^*} \frac{e^{ikR'}}{R'} \lambda^{2\nu + 1} d\lambda,$$

where the integration from 0 to Qik^* is valid. Estimating each of the integrals of this sum,

$$|V_\nu| \lesssim \frac{A|a_1|^{2\nu + 1}}{(2\nu + 1)!} \int_0^\infty e^{-\lambda}\lambda^{2\nu + 1} d\lambda = A|a_1|^{2\nu + 1}$$

where A is independent of Q and ν; consequently an N can be chosen such that the remainder of the series for $\nu > N$ is smaller than any $\varepsilon > 0$, independent of Q, if $|a_1| < 1$. In view of the uniform covergence of the series the limiting transition $Q \to \infty$ can be performed under the summation sign, so that all integrals make sense, which demonstrates the validity of the change of order of integration and summation.

Carrying out the indicated expansions, we obtain an expression for the integral I_1 in the form of a series

$$I_1 = 2\left[\frac{e^{ikR}}{R} + \sum_{\nu=0}^{\infty} A_\nu \gamma_\nu\right], A_\nu = \frac{(ika_1)^{\nu+1}}{\nu!}\int_z^{\infty ik^*} \frac{e^{ikR'}}{R'} \lambda^\nu d\lambda, \tag{11}$$

where $\gamma_{2\nu} = -i\,\sin(ka_1 z)$ and $\gamma_{2\nu+1} = \cos(ka_1 z)$.

Convenient recurrence relations can be obtained for computing the coefficients A by integrating twice by parts, twice putting $du = ikde^{ikR'}$ and $u = e^{ikR'}$

$$A_{\nu+2} = \frac{\nu a_1^2}{\nu+2} A_\nu - \frac{(k\rho a_1^2)^2}{\nu(\nu+2)} A_{\nu-2} + M_\nu[ikR(\nu+1) + (kz)^2], \tag{12}$$

$$M_\nu = ika_1^4(ika_1 z)^{\nu-1} e^{ikR}/(\nu+2)!$$

Thus, to finally compute the series, the first four coefficients of the series must be known. The odd coefficients can be obtained in closed form

$$A_1 = -\,ika_1^2 e^{ikR},\ A_3 = ka_1^4[i(kz)^2 - 2i - 2kR]e^{ikR}/6.$$

The even coefficients can be evaluated in terms of certain integrals which will be studied in one of the succeeding sections:

$$A_0 = ika_1\alpha(kz;\,k\rho),\ A_2 = ka_1^3\beta(kz;\,k\rho)/2i,$$

where

$$\alpha(x;y) = \int_x^{i\infty} \frac{e^{i\sqrt{y^2+\lambda^2}}}{\sqrt{y^2+\lambda^2}} d\lambda;\ \infty(x;y) = \int_x^{i\infty} \frac{e^{i\sqrt{y^2+\lambda^2}}}{\sqrt{y^2+\lambda^2}} \lambda^2 d\lambda. \tag{13}$$

To obtain the horizontal field components it is necessary to differentiate the Hertz vector with respect to ρ; it is therefore convenient to obtain an expansion to calculate $U_1 = D_\rho I_1$, where $D_\rho = (1/k)(\partial/\partial\rho)$. Denoting $D_\rho A_\nu = C_\nu$, then an expansion for U_1 can be obtained as follows:

$$U_1 = \left[\frac{e^{ikR}}{R}\left(i - \frac{1}{kR}\right)\frac{\rho}{R} + \sum_{\nu=0}^{\infty} C_\nu \gamma_\nu\right], \tag{14}$$

where

$$C_{\nu+2} = \frac{\nu a_1^2}{\nu + 2} C_\nu - \frac{(k\rho a_2^2)^2 C_{\nu-2} + 2k\rho a_1^4 A_{\nu-2}}{\nu(\nu + 2)} +$$

$$M_\nu[(\nu + 1)(i - kR) + i(kz)^2] \frac{\rho}{R} . \tag{15}$$

The starting values for this recurrence relation are:

$$C_0 = ia_1[\beta(kz;\ k\rho) + kze^{ikR}(1/kR - i)]/\rho;\ C_1 = i\rho A_1/R;$$

$$C_2 = ika_1^3 k\rho[ze^{ikR}/R + \alpha(kz;\ k\rho)]/2;\ C_3 = i\rho(A_3 - a_1^2 A_1/3)/R.$$

A direct analysis of the behavior of the coefficients A_ν and C_ν on the basis of recurrence relations (12) and (15) shows that for $|k\rho a_1^2 > \nu$ and $|kza_1| > \nu$ they increase; in this case the terms furthest right in the recurrence relations dominate. When the opposite inequality holds the coefficients A_ν and C_ν begin to decrease, and starting with some ν the relations $|A_{\nu+2}| \lesssim |A_\nu a_1^2|$ and $|C_{\nu+2}| \lesssim |C_\nu a_1^2|$ will hold, i.e., the series is dominated by a geometric progression with ratio $|a_1^2|$. The indicated expansion is suitable for computations when the magnitude $|k\rho a_1^2|$ is not too large, so that no loss of significant figures will arise. In the opposite situation one can pass over to computing via asymptotic expansions.

4. Rigorous Calculation of the Integral I_2

We now address ourselves to the exact computation of the integral I_2, for which we must evaluate the inner integral in expression (8) as a prelude. Besides the two residues, a branch cut integral also appears in the evaluation of this integral, whence (Fig. 4)

$$\frac{i}{2\pi}\int_{C'} \frac{x^2 e^{(x-n')z+\lambda x}}{x^2 + k_3^2 a_2^2}\,dx = \frac{\theta}{\pi i}\int_{-i}^{i} \frac{x^2 e^{\theta x(z+\lambda)}}{x^2 + b^2}\,\mathrm{sh}\,(\theta z\sqrt{1 + x^2})dx$$

$$+ k_3 a_2 \sin[k_3 a_2(\lambda + z - \tau^2 z)], \tag{16}$$

where $\theta = k_3\sqrt{1-\tau^2}$; $\mathrm{Re}\sqrt{1-\tau^2} \geq 0$; $b = k_3 a_2/\theta$.

If the solution is written in two parts corresponding to the latter terms of expression (16), then having first performed a change of variable, we obtain

$$I_2 = I_2' + I_2'', \; I_2' = \frac{2e^{ik_3R}}{R} - 2k_3a_2\int_z^{\infty ik_3^*} \frac{e^{ik_3R'}}{R'} \sin[k_3a_2(\lambda - \tau^2 z)]d\lambda. \tag{17}$$

Comparing expression (17) with the initial expression for the expansion of the integral I_1 from (10), it is seen that they can be obtained from one another by exchanging k and k_3, and λ-z and $\lambda-\tau^2 z$. This circumstance allows us to assert that all the expansions for I_2' can be obtained if, in the expansions for I_1 we put $k = k_3$, but leave the values of γ_ν unchanged. In the following, we will denote the coefficients A_ν and C_ν in which k and k_3 have been so exchanged by A_ν' and C_ν'. The expansions for the integral I_2' will clearly be valid under the condition $|a_2| < 1$.

We turn now to evaluating the second term (I_2''), whose appearance is intimately connected with the introduction of two branch points into the original integral I_2 and which vanishes if we put z = 0. Performing the change of variable $z + \lambda \to \lambda'$, then expanding the factor $e^{\theta x\lambda'}$ into a series under the integral sign and interchanging summation and integration, we obtain:

$$I_2'' = 2\sum_{\nu=0}^{\infty} A_{2\nu}'B_\nu, \; B_\nu = \frac{(-1)^\nu}{\pi b^{2\nu+1}}\int_{-i}^{i} \frac{x^{2\nu+2}\mathrm{sh}(\theta z\sqrt{1+x^2})}{x^2+b^2}\,dx. \tag{1}$$

Interchanging the order of summation and integration over x turns out to be justified because the limits of the x-integral are finite. If we argue analogously to that for the evaluation of the integral I_1, then it can be shown that interchanging the summation with the λ-integration with infinite limits is valid if the condition $|1-\tau^2| < 1$ is satisfied. Since the $A'_{2\nu}$ have already been considered, it remains only to evaluate the B_ν.

If the equation $x^{2\nu+2}/(x^2+b^2) = x^{2\nu} - b^2x^{2\nu}/(x^2+b^2)$ is used, a recurrence relation for the coefficients B_ν can be obtained:

$$B_\nu = \frac{i(2\nu - 1)!!}{(\theta z)^\nu b^{2\nu + 1}} I_{\nu+1}(z\theta) + B_{\nu-1}. \tag{19}$$

The free term of this recurrence relation is calculated by integration of the expansion for $sh(\theta z\sqrt{1+x^2})$, and I_ν is the Bessel function of imaginary argument.

To evaluate the initial term of this recurrence relation (B_{-1}), $sh(\theta z\sqrt{1+x^2})$ must be expanded into a series, as a result of which a final expression of the form

$$B_{-1} = - ib\sum_{\nu=1}^{\infty} \frac{(\theta z)^{2\nu + 1}}{(2\nu + 1)!} M'_\nu + \Phi_0; \; M'_{\nu+1} = (1 - b^2)M'_\nu + \frac{(2\nu + 1)!!}{(2\nu + 2)!!},$$

is obtained, where

$$\Phi_0 = \gamma_0(1 - \tau^{-2}); \; M'_0 = 0.$$

Differentiation of the integral I_2 with respect to ρ is obviously accomplished by a simple change from the expansion coefficients A'_ν into C'_ν and e^{ik_3R}/R into $D_\rho e^{ik_3R}/R$, which gives the final answer to the question of how to compute I_2 and $U_2 = D_\rho I_2$.

The convergence of the expansion for I'_2 can be studied in precisely the same way as for I_1, since their structures are identical. If we turn to the coefficients B_ν, we see that they fall off as $(b^{2\nu}\nu)^{-1}$ as ν increases, from which it can be concluded that the coefficients in the expansion of I''_2 can be dominated by a progression with ratio $|1-\tau^2|$, while the factor ν^{-1}

serves only to accelerate the decrease of the terms.

5. Evaluation of the integrals $\alpha(x;y)$ and $\beta(x;y)$

For the final evaluation of the Hertz vector, it only remains to obtain expansions for computing $\alpha(x;y)$ and $\beta(x;y)$ in terms of which the first coefficients of the series were obtained. To do this, it is convenient to make the change of variable in expression (13)

$$u = i(y - \sqrt{y^2 + \lambda^2}); \quad \lambda = (1 + i)\sqrt{y}\left(u + \frac{iu^2}{2y}\right)^{1/2}; \quad \frac{-i\lambda d\lambda}{\sqrt{y^2 + \lambda^2}} = du.$$

We will present the derivation only for $\alpha(x;y)$, after which we will quote the final result for $\beta(x;y)$. As a result of the change of variable, we obtain

$$\alpha(x;\, y) = \frac{e^{iy}}{(1 - i)\, y} \int_{-ib'}^{\infty} \left(u + \frac{iu^2}{2y}\right)^{-1/2} e^{-u} du, \tag{20}$$

where $\quad b' = \sqrt{x^2 + y^2} - y.$

First of all we note that $\alpha(x;y)$ and $\beta(x;y)$ are closely connected with Hankel functions. If we put $z = 0$ in the expressions for I_1 and I_2, we obtain that $\alpha(0;y) = i\pi H_0(y)/2$ and $\beta(0;y) = \pi y H_1(y)/2i$ [17, p. 169]. Using this fact, the contour of integration for $\alpha(x;y)$ can be split into two parts: from ∞ to 0 and from $-ib'$ to 0; the integral over the first contour can be expressed in terms of the Hankel function, while the integral from $-ib'$ to 0 is expanded by developing $(1 + iu/2y)^{-1/2}$ in a binomial series in positive powers of u. The latter will converge absolutely over the entire contour if the condition $|b'| < |2y|$ is satisfied or, returning to the original geometry, for $z < 2\sqrt{2}\,\rho$, i.e., for practically all cases of interest; the order of integration and summation can then be interchanged. We note that for $z > 2\sqrt{2}\,\rho$, other expansions can be better applied for computing the Hertz vector which will be obtained a little later. After carrying out the indicated operations, we obtain an expression for the coefficient $\alpha(x;y)$ in series form

$$\alpha(x;\ y) = \frac{i\pi}{2} H_0(y) - \frac{e^{iy}}{y} \sum_{\nu=0}^{\infty} D_\nu,$$

$$D_\nu = \frac{\sqrt{iy}(2\nu - 1)!!}{2\nu!!(2iy)^\nu\sqrt{2}} \int_0^{-ib'} e^{-u} u^{\nu - 1/2}\, du \tag{21}$$

A recurrence relation can be obtained for evaluating the D_ν connecting successive coefficients by integrating by parts:

$$D_\nu = \frac{(2\nu - 1)^2}{8i\nu y} D_{\nu-1} - \frac{i(-b')^\nu(2\nu - 1)!!}{(2y)^\nu 2\nu!!} e^{ib'} \sqrt{\frac{y}{2b'}}.$$

For a final representation it remains to express the initial coefficient D_0 of the series, which after the change of variables $x = -u^2$ can be represented in the form

$$D_0 = \frac{\sqrt{\pi y}}{2} (1 + i) - \sqrt{y}(1 - i)e^{ib'}\Phi(\eta),\quad \Phi(x) = e^{-x^2}\int_x^{\infty} e^{x^2} dx;\ \eta = \sqrt{ib'}.$$

There are tables in the literature for computing $\Phi(x)$ [18,19], so that the problem of computing $\alpha(x;y)$ is completely solved.

Completely analogous arguments can be applied to evaluating the coefficients of $\beta(x;y)$, as a result of which we obtain

$$\beta(x;y) = \frac{\pi y}{2i} H_1(y) - 8ye^{iy}\sum_{\nu=0}^{\infty} \frac{(\nu + 1)}{4\nu^2 - 1} D_{\nu+1} \tag{22}$$

The evaluation of the coefficients $\alpha(x;y)$ and $\beta(x;y)$ using the rigorous expansions requires a knowledge of the values of Hankel functions. If the computation of the Hankel functions must be done by asymptotic expansion, expansions. To obtain asymptotic representations for the Hankel functions from integral representations analogous to (20), $(1 + iu/2y)_{+\,1/2}$ must be expanded as a binomial series in positive powers of u and integrated over the infinite limits, which gives a series of an asymptotic nature. In our case the lower limit 0 must be replaced by -ib', and as a result we obtain

$$\alpha(x;\, y) \sim \frac{e^{iy}}{y} \sum_{\nu=0}^{\infty} D^{a}_{\nu}, \quad \beta(x;\, y) \sim 8ye^{iy} \sum_{\nu=0}^{\infty} \frac{\nu + 1}{4\nu^{2} - 1}\; D^{a}_{\nu+1}, \tag{23}$$

where the coefficients D^{a}_{ν} are related by the same recurrence formula as D_{ν} except for the sign in front of the free term, and

$$D^{a}_{0} = (1 - i)\sqrt{ye^{ib'}}\; \Phi(\eta)$$

Application of the asymptotic expansions to calculate $\alpha(x;y)$ and $\beta(x;y)$ significantly simplifies the computations, and turns out to be possible for the majority of practical cases. We note that, in spite of the fact that the coefficients $\alpha(x;y)$ and $\beta(x;y)$ are the initial coefficients in a recurrence relation for computing the Hertz vector, they can be computed to the same accuracy (not greater) as the Hertz vector itself. This can be shown from first principles, since it can be verified that an approximate calculation of $\alpha(x;y)$ and $\beta(x;y)$ for exact values of x and y is an exact calculation for some different values, the difference being of the same order as the accuracy of the calculation.

6. Evaluation of the electric field components

In order to finally obtain the electromagnetic field components, it remains to find expressions for them in terms of the Hertz vector. In the general case the electric and magnetic field components can be obtained by differentiations of the Hertz vector:

$$\vec{E} = \omega\mu(1 + k^{2}\nabla\nabla)\pi_{z}\vec{e}_{z}, \quad \vec{H} = -i[\nabla, \pi_{z}\vec{e}_{z}].$$

Before proceeding any further it is appropriate to show that the operations D_z, D_ρ, and integration over h with a source function $j = kj_0 \sin \phi$ can be performed on expressions (2) and (3) under the integral sign. Using Jordan's lemma and Cauchy's theorem, it can be shown that the original contour of integration in λ for (2) and (3) can be deformed so that both endpoints will approach $+ i\infty$, but on different sheets of the Riemann surface of the λ-plane. If an estimate is then made of the integrand of either of these integrals, for sufficiently large λ, they can be dominated by the quantity $Ae^{-\lambda\rho}$, where A is independent of h. From this it follows that both integrals converge exponentially for $\rho > 0$ for any h, which allows the indicated operations to be applied under the integral sign any finite number of times. In what follows, we shall only evaluate the electric field vector, which has only the two components

$$E_z = \omega\mu(1 + D_z^2)\pi_z, \; E_\rho = \omega\mu D_{\rho z}^2 \pi_z,$$

where $D_z = (1/k)(\partial/\partial z')$.

Operating with $(1 + D_z^2)$ under the integral sign of I_1 from (2) and performing an identity transformation, we obtain

$$(1 + D_z^2)I_1 = a_2^2 I_1 + 2D_z^2 e^{ikR}/R.$$

An analogous relation can be obtained for the integral I_2', by operating with $(1 + D_z^2)$ on expression (17), whence

$$(1 + D_z^2)I_2' = a_2^2 I_2' + 2D_z^2 e^{ik_3R}/R + \psi_1(h),$$

where $\psi_1(h) = 2a_2[a_1(\tau^2 - \cos\theta' z) + D_z \sin\theta' z] e^{ik_3R}/\tau R$;
$\theta' = a_2\theta\sqrt{1 - \tau^2}$.

This allows a final representation for the vertical component of electric field of the dipole to be obtained:

$$E_z = \omega\mu\left\{\pi_z - a_1^2\pi_{z0} + D_z^2\left(\frac{e^{ikR_1}}{R_1} + \frac{1+\tau^4}{1-\tau^4}\,\frac{e^{ikR}}{R}\right) - \frac{\tau^2}{1-\tau^4}\times\right.$$

$$\left.\left[(D_z^2 + a_1^2)\; I''_2 + D_z^2\,\frac{e^{ik_3R}}{R} + \psi_1(h)\right]\right\}. \tag{24}$$

All the expansions necessary to calculate E_z have been obtained in previous sections except for the term $D_z^2\ I''_2$. To obtain an expansion for this term, we need only apply the operation D_z^2 to the series (18) termwise, which is valid since for $|1 - \tau^2| < 1$ the series can be dominated by a geometric progression; it must be noted that both coefficients $A'_{2\nu}$ and B_ν depend on z and that

$$D_z^2 B_\nu = [(1 - \tau^2)B_\nu - a_2^2 B_{\nu+1}]/\tau^2;\; D_z A'_\nu = a_2(ik_3a_2z)^\nu e^{ik_3R}/i\tau R\nu!$$

Thus, to calculate $D_z^2 I''_2$ it is only necessary to have the values of $D_z\ B_\nu$, for which the operation D_z can be applied to the recurrence relation (19), in which only the free term changes, and to the initial term B_{-1}, in which only θz and Φ_0 depend on z.

If we put z = 0 in (24), then we obtain the known expression for the vertical electric field component of the dipole in terms of the Hertz vector [4, p. 88]. For $z \neq o$ the field E_z is expressed in terms of the Hertz vector differing from the previous case only in the operations D_z^2.

We now turn to obtaining the horizontal electric field component of the dipole, and obtain at once:

$$E_\rho = \omega\mu \left\{ D^2_{z\rho} \left[\frac{e^{ikR_1}}{R_1} - \frac{e^{ikR}}{R} \right] + \frac{D_z(U_1 - \tau^2 U_2)}{1 - \tau^4} \right\} . \tag{25}$$

In applying the operation D_z to U_1 and U_2, it must be remembered that γ_ν, C'_ν, C_ν and B_ν depend on z and that $D_z C_\nu$ and $D_z C'_\nu$ are expressible in elementary functions, and also that $D_z B_\nu$ was already evaluated in obtaining the vertical component.

The rigorous expansions obtained here for the horizontal field component appear to be new, even for z = 0. Only rigorous expansions for π_z at z = 0 exist in the literature, and in view of the absence of the parameter z, the passage from the Hertz vector to E_ρ cannot be made using the operation D_z[4].

Turning now to obtaining the field components of a vertical antenna, we find it necessary to integrate the Hertz vector for a vertical dipole with the source function $j = kj_0 \sin\phi$ in order to obtain the Hertz vector for such an antenna. We consider first how to obtain the vertical electric field component E^a_z of the antenna. First of all we obtain those terms of E^a_z which are obtained from the terms of expression (1) which are outside the integral; we get

$$k(1 + D^2_z) \int_{h_1}^{h_2} \sin\phi [e^{ikR_1}/R_1 - e^{ikR}/R] dh = - \psi_2(h) \Big|_{h_1}^{h_2}$$

where

$$\psi_2(h) = \sin\phi D_z (e^{ikR_1}/R_1 + e^{ikR}/R) + \cos\phi (e^{ikR_1}/R_1 - e^{ikR}/R).$$

We are left with finding the other terms of the expression for E_z^a. Now only the exponential factor in the integrals I_1 and I_2 of (2) depends on h; its integration with the source function between h_1 and h_2 gives

$$\int_{h_1}^{h_2} \sin\phi e^{-nh} dh = - e^{-nh}(k\cos\phi + n\sin\phi)/\lambda^2 \Big|_{h_1}^{h_2} . \tag{2}$$

To find E_z^a the operator $(1 + D_z^2)$ must be applied under the integral sign, and an expression for the vertical field component of the antenna can be obtained as

$$E_z^a = \omega\mu j_o \left\{\psi_2(h) + \cos\phi\pi_{zo} - \sin\phi D_z\pi_{zo}\right\}\Big|_{h_2}^{h_1} . \tag{27}$$

The problem of computing π_{zo} and $D_z\pi_{zo}$ has already been treated in previous sections. The rigorous expressions obtained here for computing E_z^a allow field values to be found at any point of space, and are free from all the limitations of those applied for similar purposes by K. A. Norton [3].

It turned out to be possible to obtain such a convenient formula for computing E_z^a because λ^2 in the numerator and denominator of I_1 and I_2 cancelled. This does not occur for the horizontal field component of the antenna; thus, a more appropriate method for obtaining this component using the present expansions is to numerically integrate the corresponding field components of the dipole with the source function.

7. Rigorous Evaluation of the Integral I_1^a

A study of the convergence of integrals I_2' and I_2'' shows that the series for computing them converge as progressions with ratio $|a_2^2|$ or $|1 - \tau^2|$, and thus for small $|\tau|$ a large number of terms of the series must be taken. On the other hand, a study of the terms A'_ν and C'_ν has established that they increase as long as the inequality $2\nu < |k_3\rho a_2^2|$ is satisfied. This limits the possible application of the expansions obtained in Section 6 when large distances, $|k_3\rho| >> 1$, or good conductivities, $|\tau| << 1$, are involved. Presently, we will obtain other expansions for the Hertz vector based on the alternative splitting (3) of the original formula instead of (2), whose computational accuracy increases with increasing $|k_3\rho|$ or decreasing $|\tau|$. To obtain rigorous expansions, we will first of all consider the exact, and then the asymptotic computation of the integral I_1^a. Only an asymptotic expansion will be obtained for the second integral I_2^a in the general case.

First of all, we show that for small values of kz the computation of the integral I_1^a can be performed by adding a certain correction to the value of this integral at z = o. To do this, the x-integral in expression (7) must be evaluated, and after an identity transformation, a part of the solution must be again written as an integral

$$I_1^a = 2\left[\frac{e^{ikR}}{R} - \gamma\frac{e^{ik\rho}}{\rho} - L\right] + \gamma I_1^a\Big|_{z=o}, L = ika_1\gamma\int_0^z \frac{e^{ik(R'+a_1\lambda)}}{R'}\, d\lambda, \tag{28}$$

where $\gamma = e^{-ika_1 z}$.

This representation of I_1^a for z≠o turns out to be convenient if its value at z=o is known and if kz is not very large, so that an approximate quadrature formula can be profitably applied to the computation of L. If the calculation of L is performed by expansions, which will be completely analogous to those obtained for the computation of I_1^a, then it can be seen that the numerical difficulties in this case will be twice as great as for the computation of the original integral I_1^a. The computation of $U_1^a = D_\rho I_1^a$ can be done immediately by differentiating the expression for I_1^a.

Let us address ourselves to obtaining rigorous expansions of this integral at once from a knowledge of the initial parameters. In this case we must expand the factor $e^{ika_1\lambda}$, which is obtained after evaluating the x-integral in (7), as a series and change the order of summation and the integration with infinite limits; expression (7), as in the case of I_1, turns out to be valid if $|a_1| < 1$. If the coefficients A_ν and C_ν are used, which have already been considered previously, then we obtain:

$$I_1^a = 2\left[\frac{e^{ikR}}{R} + \gamma\sum_{\nu=0}^{\infty} A_\nu\right], U_1^a = 2\left[D_\rho \frac{e^{ikR}}{R} + \gamma\sum_{\nu=0}^{\infty} C_\nu\right]. \tag{29}$$

The given representations for the integrals I_1^a and U_1^a turn out to be convenient if the initial coefficients of the series (29) can be computed sufficiently simply, but we note that if $z \simeq 2\rho$, the error in calculating $\alpha(x;y)$ and $\beta(x;y)$ by the asymptotic formulas is rather large, and most importantly it increases with growing z, and application of the same exact expansions for the computation of these coefficients as in (21) and (22) is somewhat cumbersome.

Based upon a different principle, we will now obtain a rigorous expansion for the integral I_1^a whose convergence improves as z grows, and which then allows an asymptotic expansion for I_1^a to be found. To obtain this expansion we make a change of variable after evaluating the inner x-integral in (7):

$$u = k(R' + a_1\lambda), \sqrt{u^2 - k^2a_2^2\rho^2}\,d\lambda = R'du,$$

and then I_1^a, can be obtained in the form

$$I_1^a = \frac{2e^{ikR}}{R} + 2ika_1\gamma\int_{y_1}^{i\infty}\frac{e^{iu}du}{\sqrt{u^2 - k^2a_2^2\rho^2}},\ y_1 = k(R + a_1z). \tag{30}$$

In view of the fact that the condition $|u| > |k\rho a_2|$ is satisfied over the entire contour, the radical under the integral sign can be expanded into a series, and the order of summation and integration reversed. To do this, it is convenient to use the formula

$$\int_{y_1}^{i\infty}\frac{e^{iu}du}{u^{2\nu+1}} = \frac{(-1)^\nu}{2\nu!}\left[E_i(y_1) + \sum_{j=o}^{2\nu-1}\frac{e^{iy_1}j!}{(iy_1)^{j+1}}\right],\quad E_i(x) = \int_x^{i\infty}\frac{e^{iu}}{u}\,du,$$

which is obtained by integrating by parts, and then interchanging the summations, as a result of which we obtain a final representation for the integral I_1^a

$$I_1^a = \frac{2e^{ikR}}{R} + 2ika_1\gamma\left[J_0(ka_2\rho)E_i(y_1) + e^{iy_1}\sum_{\nu=0}^{\infty}\left(\frac{1}{2\nu+1} + \frac{1}{iy_1}\right)\Phi_\nu\right] \tag{31}$$

where the Φ_ν are related by the recurrence relation

$$\Phi_{\nu+1} = \frac{(2\nu+3)(2\nu+2)}{(iy_1)^2}\left[\Phi_\nu + \frac{(2\nu+1)!!(a_2k\rho)^{2\nu+2}}{(2\nu+2)!!iy^{2\nu+1}}\right];\ \Phi_0 = \frac{J_0(ka_2\rho) - 1}{iy_1},$$

and J_o is the Bessel function.

A direct estimate of the convergence of the expansion obtained here shows that the series (31) can be dominated by a geometrical progression with ratio $|ka_2\rho/y_1|$, which can be used to determine the number of terms necessary for a given computational accuracy.

8. Asymptotic Computation of the Integral I_1^a

Carrying out calculations based on the expansions obtained in the previous sections requires an increasing number of terms for larger and larger values of the parameters z or ρ, thus it is appropriate to study the problem of how to obtain an asymptotic expansion for the integral I_1^a. To do this, we expand the exponential integral function as an asymptotic series, and then group together terms of identical order in y_1, so that we obtain an expression for I_1^a in a form convenient for computation:

$$I_1^a \sim 2e^{ikR}y_3 \left(\frac{z}{R} - a_1 \sum_{\nu=0}^{\infty} \widetilde{A}_\nu\right),$$

$$\widetilde{A}_\nu = \frac{\nu!}{(iy_1)^{\nu+1}} F\left[\frac{\nu+1}{2}; \frac{\nu+2}{2}; 1; \left(\frac{k\rho a_2}{y_1}\right)^2\right], \tag{32}$$

where $y_2 = k(a_1R + z)$; $y_3 = y_2^{-1}$.

To compute the coefficients $\widetilde{A}_\nu$ one can use the relations between hypergeometric functions [13, § 9.137] after which, noting that the coefficients A_ν can be expressed linearly in terms of Legendre polynomials [20, p. 12], a recurrence relation for the A_ν suitable for computation can be obtained:

$$\widetilde{A}_\nu = y_3^2[iy_1(1 - 2\nu)\widetilde{A}_{\nu-1} + (\nu - 1)^2\widetilde{A}_{\nu-2}],\ \widetilde{A}_1 = -iy_1y_3^2,\ \widetilde{A}_2 = 3\widetilde{A}_1^2 + y_3^2.$$

To obtain an asymptotic expansion for U_1^a, it is clearly necessary to apply the operator D_ρ to the asymptotic expansion of the integral I_1^a, however, this same result can be obtained by applying this operator to the rigorous expansion (31) for I_1^a and subsequently obtaining an asymptotic expansion; thence

$$U_1^a \sim \frac{\rho}{R}\left[(i - a_1y_3)I_1^a - \frac{2zy_3}{kR^2}e^{ikR} - 2a_1y_3e^{ikR}\sum_{\nu=0}^{\infty} \tilde{C}_\nu\right],$$

$$\tilde{C}_\nu = y_3^2[iy_1(1-2\nu)\tilde{C}_{\nu-1} + (\nu-1)^2\tilde{C}_{\nu-2} - i(2\nu-1)\tilde{A}_{\nu-1}] - 2a_1y_3\tilde{A}_\nu, \tag{33}$$

$$\tilde{C}_1 = iy_3^2(2a_1y_1y_3 - 1),\ \tilde{C}_2 = 6\tilde{A}_1\tilde{C}_1 - 2a_1y_3^3.$$

If we put z=o in the asymptotic expansion for I_1^a and multiply by $(1-\tau^4)^{-1}$, then we obtain the asymptotic expansion for $\Theta^0 + P^0$ quoted in the paper of P. A. Ryazin [4]. The asymptotic expansion obtained by Ryazin turned out to be possible thanks to a transformation of the original integral for z=o due to van der Pol, for which the equality z=o is crucial [9]. In our case a more general asymptotic expansion was obtained by going a different route.

The asymptotic expansion for I_1^a could have been obtained by starting not from the rigorous expansion (31), but directly from (30), if an integration by parts were carried out an infinite number of times, always taking $d\nu = e^{iu}du$ and $\nu = -ie^{iu}$. Naturally, both results are the same. The method presented here serves to emphasize that the asymptotic nature of the expansion is completely determined by that of the exponential integral function.

The asymptotic expansions of I_1^a and U_1^a obtained here essentially complement the rigorous expansions of these integrals and allow their computation by efficient methods over all space.

9. A New Representation of the Electric Field Components

Applying the new splitting of the original Hertz vector in the form (3) compels us once again to consider the question of how to compute the electric fiel components using the Hertz vector. As already noted, I_1^a forms the basic component of the value of the field, and at the same time I_2^a is usually small and, as a rule, is neglected. Thus it appears to be convenient to consider this question from the start, and then to estimate the magnitude of the earth wave and compute it approximately. If the operation D_z is applied inside the integral I_1^a, and then an identity transform done and the expression for the incident excitation used, then we obtain

$$D_z I_1^a = - ia_1 I_1^a + 2D_z e^{ikR}/R, \tag{34}$$

as a result of which, the final expression for the vertical field component of the dipole can be represented in the form

$$E_z = \omega\mu \left\{ (1 + D_z^2) \frac{e^{ikR_1}}{R_1} + \frac{(1 + \tau^4)D_z^2 - 2ia_1 D_z - 1 + \tau^4}{1 - \tau^4} \times \frac{e^{ikR}}{R} + \frac{a_2^2 I_1^a}{1 - \tau^4} + W_1 \right\} \tag{35}$$

where $W_1 = - \tau^2(1 + D_z^2)I_2^a/(1 - \tau^4)$.

If now the operation D_ρ is applied to (34), then an analogous form for the horizontal field component of the dipole can be obtained

$$E_\rho = \omega\mu \left\{ D_{\rho z}^2 \left[\frac{e^{ikR_1}}{R_1} + \frac{1 + \tau^4}{1 - \tau^4} \frac{e^{ikR}}{R} \right] - \frac{ia_1 U_1^a}{1 - \tau^4} + W_2 \right\}, \tag{36}$$

where

$$W_2 = - \tau^2 D_{z\rho}^2 I_2^a/(1 - \tau^4).$$

We now address ourselves to obtaining the horizontal and vertical field components of an antenna. To obtain the vertical field component, expression (27) can be used, along with relation (34), after which we obtain

$$E_z^a = \omega\mu j_o \left\{ \psi_2(h) + \frac{\cos\phi + ia_1\sin\phi}{1-\tau^4} I_1^a - \frac{2\sin\phi D_z}{1-\tau^4} \frac{e^{ikR}}{R} - W_3 \right\} \Bigg|_{h_2}^{h_1} , \qquad (37)$$

where

$$W_3 = \tau^2(\cos\phi - \sin\phi \ D_z)I_2^a/(1-\tau^4).$$

Now it remains to obtain the horizontal field component. First of all we obtain that part of E_ρ^a which corresponds to the terms outside of the integral in expression (1), whence we have

$$kD_{z\rho}\int_{h_1}^{h_2} \sin\phi e^{ikR_1} dh/R_1 = \psi_3(h)\Bigg|_{h_1}^{h_2} , \ \psi_3(h) = D_z \frac{e^{ikR_1}}{\rho}\left[\frac{(z'-h)}{R_1} \sin\phi + i\cos\phi\right].$$

To obtain the analogous terms with R in place of R_1 it suffices to replace R_1 by R and z' by - z' and denote the right side of the equation by $\psi_4(h)$.

Now we deal with the term corresponding to the integral I_1^a:

$$kD_{\rho z}^2\int_{h_1}^{h_2} \sin\phi I_1^a dh = \psi_5(h) + (ia_1\cos\phi - a_1^2\sin\phi)U_1^a/a_2^2\Bigg|_{h_1}^{h_2}, \qquad (38)$$

only postulating for the time being that such a representation might exist.

To obtain this relation, all integral and differential operations are carried out under the integral sign. If the denominator of the integral thus obtained is considered, then as a function of the variable n it has zeroes at $\pm$ ik and at ika_1, which permits it to be expanded into simple partial fractions. Relation (38) can be proven as a result, and $\psi_5(h)$ turns out to be equal to

$$\psi_5(h) = kD_\rho[\cos\phi + ia_1\sin\phi - (\sin\phi - ia_1\cos\phi)D_z]I_3/a_2^2,$$

$$I_3 = \int_{-\infty}^{\infty} H_o(\lambda_\rho)e^{-nz}\frac{d\lambda}{\lambda}.$$

To compute the integral I_3 we use equation (30), which we equate to I_1^a in the form of (3). If in the resulting identity we let first τ tend to ∞, a_1 to 1 and a_2 to 0, and then $\tau \to -\infty$, $a_1 \to -1$ and $a_2 \to 0$, and finally subtract the first result from the second and apply D_ρ, we obtain

$$kD_\rho I_3 = 2ze^{ikR}/\rho R, \tag{39}$$

from which a final expression for the horizontal field component of the antenna can be written down

$$E_\rho^a = \omega\mu j_o \Big\{\psi_3(h) - \psi_4(h) + [\psi_5(h) + ia_1U_1^a(\cos\phi + ia_1\sin\phi)]/(1 - \tau^4) + W_4\Big\}\Big|_{h_1}^{h_2}, \tag{40}$$

where

$$W_4 = \frac{k^2\tau^2D_{z\rho}^2(\cos\phi - \sin\phi \;\; D_z)}{1 - \tau^4}\int_{-\infty}^{\infty}\frac{H_o(\lambda\rho)e^{-nz}}{\lambda(m - ik_3a_2)}\,d\lambda.$$

To compute the fields of a dipole or an antenna it remains to evaluate the integrals W_i or to estimate them, which we shall now proceed to do.

10. Evaluation of the Integral I_2^a

Before obtaining the asymptotic expansion for I_2^a, one special case can be indicated where this integral can be represented as a rapidly convergent series - the case $z = o$. For $z = o$, the expansions for I_1^a and I_2^a are obtained from one another simply by interchanging k and k_3, and expansion (31) will be more convenient from a computational viewpoint, with $k = k_3$, since for $z = o$ a similar expansion can be dominated by a progression with ratio $q = |a_1|$, and almost always $q << 1$. The indicated expansion converges rapidly for all cases of practical importance, while the expansions for this case of which we are aware, obtained, it is true, from other splittings of the original Hertz vector, converge as progressions with ratio $|a_2|$, and $|a_2| \simeq 1$.

Consequently, for small $|\tau|$, they are practically useless for computation [4, p.68]. An asymptotic expansion could be obtained at once for I_2^a if $z = o$ and $k = k_3$ in (32), however, we will presently obtain a more general asymptotic expansion for I_2^a for $z \neq 0$, and the agreement of the result for $z = o$ will clearly be a good check on the intervening arguments. Because of the rather cumbersome intermediate computations, we will outline here only the course of the arguments and quote at once the final result.

First of all we evaluate the inner x-integral in expression (9), as the residue at the point ik_3a_2 and a branch cut integral. (As for the exact evaluation of I_2, it is convenient to justify this just as in (16), by a change of variable). We denote the term corresponding to the residue by I_{22}^a, and the multiple integral by I_{23}^a. The second step is to change the order of integrations in I_{23}^a, which turns out to be valid for $|1 - \tau^2| < 1$, when the λ-integral converges exponentially for all x. After this, we perform the change of variable $u = k_3(R' + a_2\lambda)$ in the integral I_{22}^a, and then integrate by parts on both u-integrals an infinite number of times, always putting $e^{iu}du = dv$ and $v = - ie^{iu}$, as was done to obtain the asymptotic expansion for I_1^a. Having integrated by parts, we have to evaluate a series of contour integrals in the expression for I_{23}^a. Their integrands have pole-type singularities for $x = \alpha = k_3z/\theta R$, for $x = b$, as well as at the point at infinity. Here it turns out that the term outside the integral in (9) cancels with the residue of the first term of I_{23}^a at the point at infinity (the other terms of I_{23}^a have no residue at the point at infinity). Secondly, the terms outside the integral of I_{22}^a cancel with those of I_{23}^a, if in calculating the x-integral only the residue at $x = b$ is taken, after which the integral I_2^a can be represented in the form:

$$I_2^a \sim \sum_{\nu=o}^{\infty} \frac{(2\nu - 1)!!}{(ik_3R)^{\nu+1}} e^{ik_3R} \sum_{j=o}^{[\nu/2]} \frac{\nu!(-1)^j(2\nu + 1 - 2j)!!}{2j!!(2\nu + 1)!!(\nu - 2j)!} I^{\nu-2j}_{2\nu+1-2j}, \tag{41}$$

where

$$I^{\nu-2j}_{2\nu+1-2j} = 2ik_3 \left(\frac{d}{dx}\right)^{2\nu-2j} e^{\theta z\left(ix-\sqrt{1-x^2}\right)} x$$

$$(1 + z^2x/R^2\alpha^2)^{\nu-2j}/(2\nu - 2j)!(1 - \tau^2)^{\nu-j}(x - b).$$

Direct analysis of the asymptotic expansion shows that it can be useful at least in three cases: first, to obtain the limiting characteristics as $\rho \to \infty$ for any finite z; second, to obtain an asymptotic expansion for I_2^a at z=o; third, an asymptotic expansion for $D_zI_2^a$ at z=o, which is needed to calculate the horizontal field component and which cannot be obtained by any known method since, as already noted, the integrand of $D_zI_2^a$, even for z=o, has two sets of branch points. For the special case z=o the computations simplify considerably, and we have

$$I^{\nu-2j}_{2\nu-2j+1} = \frac{2k_3}{ia_2^{2\nu-2j}}, \quad D_zI^{\nu-2j}_{2\nu-2j+1} =$$

$$\frac{2k_3}{i\tau}\left[\sum_{\sigma=o}^{\nu-j-1} \frac{(2\sigma-3)!!(1-\tau^2)^{\frac{1}{2}-\sigma}}{2\sigma!2\sigma!!a_2^{2\nu-2j-2\sigma}} + \frac{i}{a_2^{2\nu-2j-1}}\right], \tag{42}$$

where we have denoted (-1)!! = 1 and (-3)!! = -1 for brevity. If the first expression from (42) is substituted into (41) and compared with the already known asymptotic expansion of I_2^a for z=o, then it can easily be seen that they are completely equivalent [4].

After obtaining the asymptotic expansion for I_2^a and considering how it may be applied, it is clear that the expansion thus obtained can be used only for sufficiently small $|k_3 z|$. We thus proceed next to obtain asymptotic expansions valid for arbitrary z which will allow the integrals W_i to be evaluated.

11. Asymptotic Evaluation of the Integrals Along the Cut $\lambda = k$

Upon investigation of the integrands of these integrals it can be seen at once that on the upper sheet of the Riemann surface, where the original contour of integration is taken, there are only two (sets of) branch points among all the singularities. If the branch cuts are taken so that Re n = const and Re m = const, then each of these integrals can be split into two: W_i', which are obtained by integrating around the cut $\lambda = k$, and W_i'', by integrating around the cut $\lambda = k_3$, and $W_i = W_i' + W_i''$. We deform the contour of integration for these integrals as shown in Figure 5.

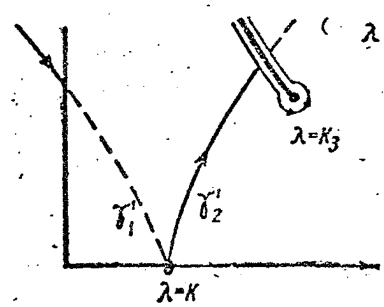

Figure 5 - Deformation of the Integration Contour to Obtain the Asymptotic Expansion

First of all we must obtain the asymptotic expansion for the integral W_i', for which it is useful to study some properties of the integrands of these integrals. The function $H_0(\lambda\rho)$ can already for $k_\rho > 2$ be replaced by its asymptotic expansion at least to an accuracy of ten percent; this contains a rapidly oscillating exponential part. Hence, it is clear that in the vicinity of $\lambda = k$ the integrands of these integrals can be represented as the product of a rapidly varying function of the order of $e^{i\rho\lambda \pm nz}$, where the plus sign corresponds to the part of the contour γ_1' and the minus sign to γ_2' for Re $n \geq 0$, and a slowly varying function. Clearly one can always take the original contours of integration γ_1' and γ_2' that Im $(i\rho\lambda \pm nz) = \rho$; then the exponential factor will decrease most rapidly on the given contour, and the integral can profitably be evaluated using the method of steepest descent [16,21]. The integral in the vicinity of the point where the real part of the exponent is maximum is conveniently reduced for this purpose to an expression for an incident excitation and its derivatives. We note that the asymptotic expansion for $H_0(\lambda\rho)$ is necessary only to establish the possibility of applying the steepest descent technique.

In view of the specific singularities of the integrands of these integrals, it is considerably more convenient to expand the slowly varying function not about the point =k, but = ka_2, and almost always $a_2 \simeq 1$. Clearly, we have:

$$\frac{\tau^2}{(m - ik_3a_2)(\tau^4 - 1)} \sum_{\nu=0}^{\infty} C_\nu \frac{(n^2 + k^2a_1^2)^\nu}{k^{2\nu+1}} ; \; C_\nu = \frac{\tau^{2\nu+2}a_1(2\nu - 1)!!}{i(1 - \tau^2)(2\nu + 2)!!a_2^{2\nu}} , \tag{43}$$

whence it follows immediately that W_1' can be represented in the form

$$W_1' \sim \sum_{\nu=0}^{\infty} C_\nu \, (S_{\nu+1}^{(1)} + a_2^2 S_\nu^{(1)}), \; S_\nu^{(1)} = \int_{\nu=0}^{\infty} H_0(\lambda\rho)e^{-nz}(n^2 + k^2a_1^2)^{\nu} d\lambda/k^{2\nu+1}.$$

The integrals W_2' and W_3' can likewise be expressed in terms of the coefficients $S_\nu^{(1)}$ and their derivatives, and are equal to

$$W_2' \sum_{\nu=0}^{\infty} C_\nu S_\nu^{(3)}, \; W_3' \sim \sum_{\nu=0}^{\infty} C_\nu(\sin\phi S_\nu^{(2)} - \cos\phi S_\nu^{(1)}),$$

where $S_\nu^{(2)} = D_z S_\nu^{(1)}$; $S_\nu^{(3)} = D_\rho S_\nu^{(2)}$.

The computation of W_4' is somewhat more complicated. For this calculation we must use the identity $(n^2 + k^2 a_1^2)/\lambda^2 = 1 - a_2^2 k^2/\lambda^2$, after which, using the expression already obtained for the first terms of the asymptotic expansion in the form of (39), we obtain

$$W_4' \sim \sum_{\nu=0}^{\infty} C_\nu (\sin\phi\, N_\nu - \cos\phi\, L_\nu), \text{ where}$$

$$L_{\nu+1} = S_\nu^{(3)} - a_2^2 L_\nu;\; L_0 = -kD_z{}^2_\rho I_3;\; S_\nu^{(4)} = D_z S_\nu^{(3)};\; N_{\nu+1} = S_\nu^{(4)} - a_2^2 N_\nu;\; N_0 = D_z L_0.$$

Thus, to obtain the final asymptotic expansions for these integrals it remains to compute the $S_\nu^{(1)}$, which are expressed as derivatives of an incident excitation:

$$S_\nu^{(1)} = -\sum_{\sigma=0}^{\nu} \frac{\nu!\, a_1^{2\nu-2\sigma}}{\sigma!(\nu-\sigma)!} \phi_{2\sigma+1}^{(1)},\; \phi_j^{(1)} = 2D_z^j e^{ikR}/R,\; \phi_j^{(2)} = D_\rho \phi_j^{(1)}. \tag{44}$$

The value of $S_\nu^{(2)}$ is obtained from $S_\nu^{(1)}$ if $\phi_{2\sigma+1}^{(1)}$ is replaced by $\phi_{2\sigma+2}^{(1)}$ in (44), and $S_\nu^{(3)}$ and $S_\nu^{(4)}$ by replacing it with $\phi_{2\sigma+2}^{(2)}$ and $\phi_{2\sigma+3}$ respectively. Therefore it remains to give an expansion to compute $\phi_\sigma^{(1)}$ and $\phi_\sigma^{(2)}$.

By induction it can be shown that $\phi_\sigma^{(1)}$ and $\phi_\sigma^{(2)}$ can be given in the form

$$\phi_\sigma^{(1)} = \frac{P_\sigma^{(1)} + kRQ_\sigma^{(1)}}{(kR)^{2\sigma}} \frac{2e^{ikR}}{R},\quad \phi_\sigma^{(2)} = \frac{P_{\sigma+1}^{(2)} + kRQ^{(2)}{}_{\sigma+1}}{(kR)^{2\sigma+2}} \frac{2e^{ikR}}{R}, \tag{45}$$

where $P_\sigma^{(i)}$ and $Q_\sigma^{(i)}$ are some polynomials connected by recurrence relations.

The asymptotic expansions obtained in this section allow a rather simple estimate to be made of the values of these integrals and calculations to be made with great accuracy in a number of cases of interest.

12. Asymptotic Evaluation of the Integrals along the Cut $\lambda = k_3$

Now let us proceed to study the last problem in obtaining rigorous expansions for the Hertz vector: the computation of the integrals W_i''. These integrals can also be computed using the method of steepest descent, but again not in the usual sense. This is similar to the case of W_i', where $\sqrt{\lambda^2 - k^2}$ was not expanded into a series at the branch point, and consequently the usual path was avoided.

If numerator and denominator in W_ν are multiplied by $m + ik_3a_2$, and remembering that m has a different sign on the different Riemann surfaces, then using an identity transformation these integrals can be represented in the form

$$W_\nu'' = \int_{k_3}^{\infty} F_\nu(\lambda) e^{\gamma'(\lambda - k_3)\sqrt{\lambda - k_3}} d\lambda; \quad F_\nu(\lambda) = \frac{2\tau_F\sqrt{\lambda + k_3}\lambda f_\nu(\lambda)}{(\lambda^2 - k^2a_1^2)(1 - \tau^4)} e^{\gamma''}$$

where

$$f_1(\lambda) = -\lambda^2 H_0(\lambda\rho)/k^2; \quad f_3(\lambda) = -k(k\cos\phi + n\sin\phi)f_1(\lambda)/\lambda^2;$$

$$\gamma' = i\rho - z/\sqrt{1 - \tau^2}; \quad f_2(\lambda) = -\lambda n H_1(\lambda\rho)/k^2;$$

$$f_4(\lambda) = -k(k\cos\phi + n\sin\phi)f_2(\lambda)/\lambda^2; \quad \gamma'' = \gamma'(k_3 + \lambda) - zn.$$

If the asymptotic representation for the Hankel function, which will be valid over the whole contour of integration, is used, then it can be seen at once that the function $F_\nu(\lambda)$ is slowly varying over the whole contour and thus can be expanded into a series about the point $\lambda = k_3$, and the usual steepest descent method employed to evaluate W_ν'':

$$W_n'' \sum_{\sigma=0}^{\infty} g_\sigma^\nu \phi_\sigma, \quad \phi_\sigma = \int_0^{\infty} u^{\nu + \frac{1}{2}} e^{\gamma' u} du/k^{\nu + \frac{3}{2}} = \frac{(2n+1)!!\sqrt{\pi}}{2^{\nu+1}(-k\gamma')^{\nu + \frac{3}{2}}} \tag{46}$$

For the final solution to the problem it remains to calculate the coefficients g_σ^ν which determine the expansion of $F_\nu(\lambda)$ in the neighborhood of $\lambda = k_3$. Since $F_\nu(\lambda)$ has a rather complicated form, differentiating it analytically is considerably difficult, and it is most reasonable to do so numerically, to do which we must know its value at several points [21]. It is convenient to choose all these points to lie on the contour, with $\lambda = k_3 + kj\delta$, where $\delta = - p/k\gamma'$, and p is some positive number. It is most convenient to choose p from the condition that the rapidly varying function $e^{\gamma'(\lambda-k_3)}$ has decreased by e^{-jp} at these points, i.e., such a value of p must be chosen that all points lie in the vicinity of $\lambda = k_3$ in the most important range of integration. Applying Newton's formula for forward extrapolation [21], we obtain an expression for $F_\nu(\lambda)$ in terms of central differences [22]:

$$F_\nu(\lambda) = \Delta_\nu^{(o)} + \frac{\lambda - k_3}{1!k\delta} \Delta_\nu^{(1)} + \frac{(\lambda-k_3)(\lambda-k_3-k\delta)}{2!(k\delta)^2} \Delta_\nu^{(2)} + \ldots,$$

$$\Delta_\nu^{(\sigma)} = \sum_{j=o}^{\sigma} (-1)^{j-\sigma} C_\sigma^j F_\nu(k_3 + kj\delta),$$

from which an expression for $F_\nu(\lambda)$ and its derivatives can be immediately obtained in terms of central differences. If we now express the coefficients g_σ^ν in terms of central differences and regroup terms with identical central differences, then we obtain a simple expression for computing the integrals W_ν''

$$W_\nu'' \sim \frac{\sqrt{\pi}}{2(-k\gamma')^{3/2}} \sum_{\sigma=0}^{\infty} \Delta_\nu^{(\sigma)} c_\sigma'/p^\sigma \tag{47}$$

where the first few coefficients c_σ' are equal to:

$$c_o' = 1;\ c_1' = 3/2;\ c_2' = 15/8 - 3\delta/4;\ c_3' = 7/3 - 15\delta/8 + \delta^2/2$$

and if other coefficients are needed they can be calculated without difficulty. Thus, the asymptotic expansions obtained here for the integrals W_ν'' completely solve the problem of computing the electric fields of a dipole or an antenna in space.

13. Limiting Electromagnetic Field Patterns

The rigorous expansions constructed in the foregoing sections to evaluate the electromagnetic field allow this to be done in all space; however these computations can be considerably simplified if the observation point is allowed to tend to infinity in some fixed direction. The limiting field patterns thus obtained can be used in many cases to predict the possible disturbance to the field characteristics, due to the finite conductivity of the earth.

First of all, let us consider the case when $z \to \infty$ and $\sin \psi = z/R \to \text{const} \neq 0$. In this case it should be noted that the field pattern has only one component each of electric and magnetic field, if we deal in spherical coordinates, namely E_θ and H_ϕ. If the contour of integration is split into four as shown in Figure 6,

$$\pi_{oz} = \int_{i\infty}^{E_1} \phi(\lambda)d\lambda + \int_{E_1}^{E_2} \phi(\lambda)d\lambda + \int_{E_2}^{i\infty} \phi(\lambda)d\lambda + \int_{[k_3]} \phi(\lambda)d\lambda,$$

then it can be shown that as $z \to \infty$ for $\mathrm{Im}\ \tau \neq 0$, the first term and the last two terms vanish for arbitrary $E_1 > 0$ and $E_2 > 0$, in which the contour from E_1 to E_2 can intersect the real axis on the interval from $\lambda = o$ to $\lambda = k$. If the asymptotic representation for the Hankel function is used, then it can be shown that the saddle point lies on a segment of the real axis at $\lambda = k \cos \psi$, which corresponds physically to the angle of the reflected ray according to

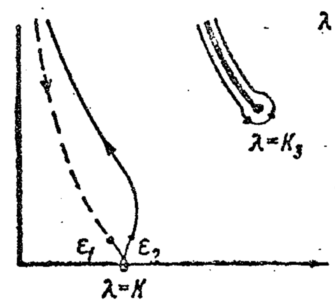

Figure 6 - Deformation of the Integration Contour to Obtain the Limiting Field Patterns

geometrical optics. Carrying out the intermediate arguments, we obtain limiting directivity patterns for the vertical dipole to within the constant:

$$2\omega\mu e^{ikR_0}/R_0, \text{ where } R_0 = \sqrt{\rho^2 + z^2},$$

$$[E_\theta] = -[H_\phi] = [\cos(kh_1 \sin\psi) + e^{ikh_1 \sin\psi}\Omega]\cos\psi, \tag{48}$$

where

$$\Omega = \frac{1}{1-\tau^4}\left[\tau^4 - \frac{a_1}{\sin\psi + a_1} - \frac{\tau^3 \sin\psi}{\sqrt{1 - r^2\cos^2\psi} + a_2}\right].$$

The directivity pattern of a vertical antenna is obtained in analogous fashion:

$$[E^a_\theta] = -[H^a_\phi] = \cos(kh \sin\psi)\cos\phi + \sin(kh \sin\psi)\sin\phi\sin\psi + e^{ikh\sin\psi}(\cos\phi - i\sin\phi\sin\psi)\,\Omega/\cos\psi \Big|_{h_2}^{h_1}. \tag{49}$$

If we substitute z = const and $\rho \to \infty$ in these limiting relations, we obtain to a first approximation that $[E_\phi]$ and $[E^a_\theta]$ are zero. To perform the limit along the surface of the earth, it must be recognized that the field is of different order, and the electric field vector will consist of both horizontal and vertical components. To obtain the limiting pattern in this case it is not sufficient to put $\psi = 0$, since in the derivation of (48) and (49) terms of order $1/k\rho^2$ were neglected. In this case only the first two terms from the first equation of this section give non-vanishing terms in the limit, which allows us to write down a final limiting pattern for the dipole to within a factor of $\omega\mu e^{ik\rho}/k\rho^2$:

$$(E_z) = i[1 - kh\,kz'\,\tau^2(1 - \tau^2) - ik(z' + h)\,\tau\sqrt{1 - \tau^2}]/\tau^2(1 - \tau^2);$$

$$(E_\rho) = i[1 - ikh\tau\sqrt{1 - \tau^2}]/\tau\sqrt{1 - \tau^2}$$

and an analogous relation for the vertical antenna:

$$(E_z^a) = i\Big\{(1 - ikz'\,\tau\sqrt{1 - \tau^2})\cos\phi + (kh\cos\phi - \sin\phi)\,x$$

$$[- kz'\tau^2(1 - \tau^2) - \tau i\sqrt{1 - \tau^2}]\Big\}/\tau^2(1 - \tau^2)\Big|_{h_2}^{h_1},$$

$$(E_\rho^a) = i\Big\{\cos\phi - i\tau\sqrt{1 - \tau^2}(kh\cos\phi - \sin\phi)\Big\}/\tau\sqrt{1 - \tau^2}\;\Big|_{h_2}^{h_1}.$$

A study of these limiting patterns allows the possible disturbance to the phase structure of the electro-magnetic field above a real path to be determined.

Conclusion

In concluding this study we should mention the effects which are to be expected in a study of the field of a vertical antenna as compared to the field of a dipole. If the length of the antenna is chosen to guarantee radiation directed along the surface of the earth, then in view of the change in the directivity diagram, such an antenna will be less able to receive energy from the upper regions of space to the earth's surface than will the dipole. The phase structure of the field of such an antenna will depend more strongly on the conductivity of the earth, i.e., the vertical antenna will possess a more clearly defined "phase memory".

At present, the authors have carried out calculations of the limiting patterns of dipoles and antennas and performed calculations of the electromagnetic field structure on a computer. The results of these computations have already been partially published [23].

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