

**Towards General Performance Bounds of the
Distributed Sensor Coverage Problem**

by

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Towards General Performance Bounds of the Distributed Sensor Coverage Problem

Thesis directed by Professor Jason Marden

We study a class of resource allocation problems known as distributed sensor coverage problems, whereby sensors are distributed to regions in space with the goal of maximizing the detection of high value events. Computing an optimized allocation of sensors onto regions would become intractable for a centralized controller in large scale systems and would require continuous communication to all agents in the system, which may not be feasible. We analyze the alternate approach of a distributed sensor coverage problem, where each sensor is responsible for selecting a region to search. When each sensor acts in their own interests to select a resource, degradation in system performance can occur. Game Theory is used as a mathematical framework to model and study the behavior of large groups of agents in the sensor coverage problem. We study the lower bound on performance of the distributed sensor coverage problem and conjecture that this bound is $\frac{e}{e+1}$. We support this claim with provable properties about the structure of games bounded by this price of anarchy, and through empirical simulations.

Dedication

To my wife, Shawnasie. For all her love and support.

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Chapter 1

Introduction

Resource allocation is a general set of problems whereby a centralized controller attempts to allocate agents to a set of resources. One such example is network routing, such as cars traveling on a finite set of roads or information being routed on a computer network. Ideally, the goal of a centralized controller is to assign allocations to each agent in order to optimize some global objective. In traffic routing, for example in Figure 1.1, the controller would assign a complete route for every vehicle on the road in such a way to minimize traffic congestion or commute times. If the number of cars on the road is small, then a centralized, global planner can find the optimal routing allocation for every car. However, if the number of cars grows large, then finding an optimal allocation quickly becomes intractable. There is also the problem that a centralized controller would be required to have communication to all of the agents in order to send allocation commands to them. This is problematic, since many large scale system this may not be feasible.

Instead of centralized control, we instead propose a decentralized control architecture. In the example of traffic routing, we can devise a distributed control policy whereby drivers individually choose their own routes and attempt to optimize only their individual commute times. Normally, the shortest route might provide the fastest commute, but if that particular route has a large traffic jam, then the driver could choose a longer route that would provide a faster commute than any alternative. This illustrates the fact that in such systems, the individual payoff or cost, which we will call a **utility**, is dependent not only on the player's individual

action, but also on the actions of others. We say that when agents choose an action in order to optimize their utility, those agents are acting **self interested**. In situations where agents act self interested, there is no guarantee the global system objective is optimized. In fact, situations arise where if every player acts self interested, then the global system performance can suffer dramatic losses compared to the centralized optimal. This phenomenon was famously illustrated in what is called the tragedy of the commons [6]. Insight into conditions for which this occurs can help engineers design distributed systems which avoid and minimize this phenomenon. Characterizing the degradation of global system performance due to decentralized decision making is referred to as **price of anarchy** [8] which is an area of current research and the focus of this work.

Examples of systems such as traffic routing are referred to as social systems because they are used to describe and model elements of human behavior, i.e. the driver. A system designer has no control on the utilities of the drivers, so instead the focus is on offering incentives to alter the driver's decision, such as tolling or adding strategically placed new roads to reduce congestion. **Game Theory** was developed as a mathematical framework to model and study the behavior of large groups of agents interacting together [18, 23]. Game theory has a long history of modeling social behaviors in economics such as auctions and investment coalitions.

There has been a recent interest in applying these same tools of game theory to modeling and designing engineering systems. In these systems, the designer can carefully craft agent utilities and decision policies in such a way that desirable global behavior results. Examples of these systems include weapon target assignment [2], missile guidance [7], network coding [11], and sensor coverage [3, 15].

We turn our attention to modeling and studying the distributed sensor coverage problem under a game theoretic framework. Consider the problem where a set of areas that need to be searched. We will conduct this search with a collection of sensors that can individually be positioned at any search area. Each sensor has a detection probability, which is the probability the sensor will detect an event, and every search area has assigned a event value. We seek to position each sensor to a search area with the goal of maximizing detection of high value events

within that region. This type of problem is what we call the **sensor coverage problem**. In the traditional study of the sensor coverage problem, a global planner would allocate each sensor to a search region. However, this can be problematic since the global planner must have communication to each of the sensors and have complete information of the entire system state.

Instead we decentralize the sensor coverage problem whereby every sensor has a local control policy by which to make its own search selection. In this distributed setting, we can model this system as a game, where sensor are the players, allowing us to use the rich set of tool developed under the game theoretic framework. This thesis is dedicated to bounding the global performance loss when players act self-interested in a decentralized control paradigm.

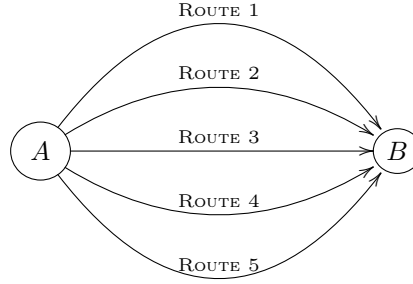


Figure 1.1: A graph representing the possible routes of cars traveling from point A to B.

1.1 Previous Work

Since computing optimized centralized solutions for resource allocation problem becomes intractable for large scale problems, previous work has focused on efficient approximation algorithms to this approach. However, these results only guarantee an allocation that is a factor of $1 - \frac{1}{e}$ of the optimal, or approximately 63% [5]. If we consider the distributed resource allocation problem, then there is a gap in the literature with respect to performance bounds for these algorithms. [22] showed that if the system meets certain basic properties, then we can guarantee the solution is within $\frac{1}{2}$ of the optimal. However, this is a general result, and it is not known under what set of conditions can this 50% of optimal be improved on. Several recent works have attempted to address this question, by focusing on specific problem sets such as network

routing [19] or network coding [10]. Only limited work has attempted to address distributed performance guarantees in the sensor coverage problem. [14] analyzed the subset of distributed resource allocation games where all the players are equal, sometimes called anonymous players, using the Shapley value utility design (Section 2.1.4). However the assumption of anonymous players is very restrictive, and leaves the more general question of the effect on performance when including all non-anonymous players. Furthermore, Shapley value is computationally intractable for large scale systems, limiting its use in practical systems. [4] studied the sensor coverage problem with non-anonymous players using the wonderful life utility (Section 2.1.3), and showed that when considering problems consisting of only 2 sensors, the performance is within 80% of the optimal. Also presented was a conjecture of 79% for games with greater than 2 sensors. This leaves open the problem of performance bounds in the general n-player sensor coverage problem, and if there exist any rigorous design methodologies for utility functions that result in optimized performance bounds.

1.2 Our Contributions

This thesis is dedicated to studying the distributed sensor coverage using the wonderful life utility design. We discover a set of games $\mathcal{H}$ in (Definition 4.2.1) that has the following properties

- $v_2 = v_3 = \dots = v_n = 1$
- $p_1 = 1$
- $p_2 = p_3 = \dots = p_n = 1 - v_1^{\frac{1}{1-n}}$
- $a^{\text{ne}} = (x_n, x_1, x_1, \dots)$
- $a^{\text{opt}} = (x_1, x_2, \dots, x_n)$.

We show that the worst case performance for $\mathcal{H}$ is $\frac{e}{e+1} \approx 73\%$, which is lower than conjectured in [4]. We then study the set of games $\mathcal{F}$ such that $\mathcal{H} \subset \mathcal{F}$ which has the following properties

- $p_1 = 1$
- $a_1^{\text{ne}} = x_n$
- $a_1^{\text{opt}} = x_1$
- $v_n = 1$
- $W_r(\{a\}_r) = v_r - 1, \forall r \neq x_n.$

We prove that the performance of any game is bounded below by a game in $\mathcal{F}$. This shows that games with the worst performance have a single player with detection probability of 1, and this player is allocated to the resource with the lowest value in the equilibrium. This contrasts with the optimum allocation where a player of detection 1 must select the highest value resource or it can't be an optimum. This shows that poor performance is driven, in part, by the greatest separation in resource selection between the optimum and equilibrium allocations for a player of detection probability 1, which we later define as the pivot player in Definition 3.3.2. We show that the pivot player also bounds the welfare at all other resources in the equilibrium and hence bounds the relative amount of welfare at all resources. We conjecture the performance of all single selection distributed sensor coverage problems is bounded below by $\frac{e}{e+1}$. This result is much improved over the general 50% result presented in [22]. Finally, simulations are used as empirical evidence to support the claim of 73%.

Chapter 2

Background

Let $N = \{1, \dots, n\}$ consist of a set of players and for each player $i \in N$, $\mathcal{A}_i$ represents the set of available actions for player i . The game consists of a set of resources $\mathcal{R}$, such that $\mathcal{A}_i = \mathcal{R}$. This implies each player action consists of selecting a single resource in $\mathcal{R}$. When each player selects an action, the players become allocated to the resources. $\mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$ is the set of all joint allocations and each $a = (a_1, a_2, \dots, a_n)$ represents a joint allocation of all agents to resources. Each player has a utility function $U_i : \mathcal{A} \rightarrow \mathbb{R}^+$. Every possible action results in a utility for the player, and serves as a criteria for determining if one particular action is better than another. As we can see from this definition, every player's utility depends not only on its own resource selection, but also on the action selection of every other player.

2.1 Distributed Welfare Games

Player utility does not quantify global system performance. In order to quantify global system performance, we introduce global performance measure called a **welfare function**. A welfare function $W : \mathcal{A} \rightarrow \mathbb{R}^+$, defines the global performance of any allocation. We restrict our attention to the class of **separable** welfare function of the form

$$W(a) = \sum_{r \in \mathcal{R}} W_r(\{a\}_r),$$

where $W_r : 2^N \rightarrow \mathbb{R}^+$ is the welfare function for resource $r \in \mathcal{R}$, and $\{a\}_r \subseteq N$ represents the set of players selecting resource r using allocation $a \in \mathcal{A}$. That is

$$\{a\}_r \equiv \{i \in N : a_i = r\}.$$

Definition 2.1.1. A resource welfare function $W_r : 2^N \rightarrow \mathbb{R}^+$ is called **submodular** if for all $S \subseteq T \subseteq N$ and any agent $i \in N$,

$$W_r(S \cup \{i\}) - W_r(S) \geq W_r(T \cup \{i\}) - W_r(T). \quad (2.1)$$

Submodular welfare functions are thought of as the idea of diminishing returns: as players are added to a resource, the amount of welfare available to subsequent players gets reduced. Submodular welfare functions naturally occur in certain game types such as the sensor coverage problem [12], and they are required to show certain performance guarantees [22]. Now consider the tuple

$$G = (N, \mathcal{R}, \{\mathcal{A}_i\}_{i \in N}, \{W_r\}_{r \in \mathcal{R}}, \{U_i\}_{i \in N}).$$

G is said to be a single selection distributed welfare game, and we can define $\mathcal{G}$ as the set of all single selection resource allocation games.

Consider a game $G \in \mathcal{G}$ where we select a single player i at random from N and allow player i to select an action based on the follow strategy

$$a_i = \operatorname{argmax}_{\tilde{a}_i \in \mathcal{A}_i} U_i(\tilde{a}_i, a_{-i}), \quad (2.2)$$

where a_{-i} is the joint allocation of all players other player i . We say players following this strategy are **self-interested**. Suppose we continue the process of randomly selecting a player and following the strategy in Equation 2.2. Adding player action dynamics such as this in a game is called **learning**. Now consider a case that after many iterations, every player in the game has a strategy to remain on the same action. In this case, every player cannot increase their utility by changing actions, as long as all other players remain at the same resource. When this situation occurs, we say the game has reached an **equilibrium**. The above example illustrates

how an equilibrium could occur in such a setting. The idea of an equilibrium is an important solution concept in game theory [17]. A natural question is whether given a game, following a fixed strategy such as that described in Equation 2.2, will always result in an equilibrium. The work of John Forbes Nash in [16] showed that in any game with a finite action set, there exists at least one equilibrium allocation. These equilibrium allocations are commonly referred to as **Nash equilibrium**.

Definition 2.1.2. An joint allocation $a^{\text{ne}} \in \mathcal{A}$ is said to be a pure Nash equilibrium if for all players $i \in N$, and all $\tilde{a}_i \in \mathcal{A}_i$

$$U_i(a_i^{\text{ne}}, a_{-i}^{\text{ne}}) \geq U_i(\tilde{a}_i, a_{-i}^{\text{ne}}).$$

An allocation can be thought of as being a Nash equilibrium when no player has any unilateral incentive to deviate from their current resource selection. We will henceforth refer to pure Nash equilibrium as just a Nash equilibrium or just equilibrium. It is important to consider the existence of a Nash equilibrium when designing game utilities.

2.1.1 Utility Design

Let $S \subseteq N$ be the set of players that share the same resource $r \in \mathcal{R}$. Within the context of distributed welfare games, the utility function is constructed to distribute the available resource welfare to each of the players such that

$$U_i(a_i = r, a_{-i}) = f_r(i, S),$$

where $f_r : N \times 2^N \rightarrow \mathbb{R}^+$ is the **welfare sharing protocol** for player $i \in S$ [13]. Hence, a player's utility is local as it only depends on S . Ideally we want to design utility functions for each player in such a way that results in global allocations that garner maximum possible welfare. Currently there is no systematic design methodologies for utility functions that lead to optimized welfare performance [15]. However, there are important properties of utility functions one must consider. First, and most important, is whether the utility guarantees the existence of a Nash

equilibrium. This ensures the game has a stable operating point, because if no equilibrium exists, players could continually switch resources. Other factors the game designer must consider is informational dependency and budget balanced utility functions [15]. Informational dependency is the amount of information about the game a player needs to compute it's utility. High informational dependency can cause implementation problems, communication between every player may not be feasible, and high computational complexity in tracking and computing utilities in vary large scale games can cause delays.

We review commonly suggested utility functions and their properties.

2.1.2 Equal Share

The simplest sharing protocol, equal share is based on the notion that any welfare available at a resource is equally distributed among players that select that resource. More formally, let $S \subseteq N$ be the set of all players the select the same resource r , then the equal share protocol for player $i \in S$ is

$$f_r^{\text{ES}}(i, S) = \frac{W_r(S)}{|S|}. \quad (2.3)$$

Only the number of players selecting a resource is required for computing equal share protocol, and therefore information dependency is very low. However, the equal share protocol, in general, does not guarantee the existence of a Nash equilibrium. Only if extra restrictions on allowable player sets are enforced, then can one guarantee a Nash equilibrium to exist when using equal share utility design [15].

2.1.3 Marginal Contribution

The marginal contribution (MC) sharing protocol computes a player utility by the net welfare increase that player brings to the resource. For any set of players $S \subseteq N$ that select the same resource r , then the marginal contribution protocol for player $i \in S$ is

$$f_r^{\text{MC}}(S) = W_r(S) - W_r(S \setminus \{i\}). \quad (2.4)$$

When used as a utility design, this protocol is commonly referred to as the **wonderful life utility** because it can be thought of as comparing the welfare at a resource with and without a player and shows how much a player adds to the welfare at a resource. Marginal contribution has shown promise as a sharing protocol for distributed engineering systems. It guarantees the existence of a Nash equilibrium and has moderate informational dependency [15]. The research in this thesis is devoted to the analysis of the marginal contribution protocol on the distributed sensor coverage problem.

2.1.4 Shapley Value

The Shapley value [20] comes from the idea that any group of players sharing the same resource form a coalition to distribute the available welfare to each player. Formally, for any set of players $S \subseteq N$ that select the same resource r , then the Shapley sharing protocol for player $i \in S$ is

$$f_r^{SV}(S) = \sum_{T \subseteq S \setminus \{i\}} \frac{|T|! (|S| - |T| - 1)!}{|S|!} (W_r(S \cup \{i\}) - W_r(S)). \quad (2.5)$$

The Shapley value can be thought of as the average marginal contribution for player i out of all possible combinations of coalitions that can be formed with i . The Shapley value has many important properties as a sharing protocol including the existence of a Nash equilibrium [15]. However, this utility function has high informational dependence and for large scale games can be intractable to compute for every player.

2.2 Performance Metrics

There are many different utility functions that can be considered by a system designer, but let us only consider utility functions that guarantee the existence of a Nash equilibrium. There can exist many equilibrium allocations in any given game, and any equilibrium allocation is not, in general, an optimum. Ideally, we would want to design the game such that any resulting equilibrium achieves a welfare that is as close to optimum as possible. But how do we evaluate the performance of a game design?

2.2.1 Price of Stability

The price of stability (PoS) forms a best case scenario measure of a game and is formed as a ratio of best equilibrium allocation with the optimum allocation. Let $\mathcal{G}$ be a set of games, then we define the price of stability as

$$\text{PoA}(\mathcal{G}) \equiv \inf_{G \in \mathcal{G}} \left(\max_{a^{\text{ne}} \in \text{NE}(G)} \frac{W(a^{\text{ne}}; G)}{W(a^{\text{opt}}; G)} \right), \quad (2.6)$$

where $\text{NE}(G)$ is the set of all equilibrium allocations for game G and

$$a^{\text{opt}} = \underset{a \in \mathcal{A}}{\text{argmax}} W(a). \quad (2.7)$$

We can think of the price of stability as the best possible performance one could expect from a game. If the price of stability is less than 1, then there exists a game where the optimum allocation cannot be an equilibrium [15].

Theorem 2.2.1. *Let $\mathcal{G}$ be the set of single selection resource allocation games using the marginal contribution utility design. Then $\text{PoS}(\mathcal{G}) = 1$.*

2.2.2 Price of Anarchy

The price of anarchy (PoA) forms a worst case scenario analysis of a game and is formed as a ratio of the worst equilibrium allocation with the optimum allocation. Let $\mathcal{G}$ be a set of games, then we define the price of anarchy as

$$\text{PoA}(\mathcal{G}) \equiv \inf_{G \in \mathcal{G}} \left(\min_{a^{\text{ne}} \in \text{NE}(G)} \frac{W_G(a^{\text{ne}})}{W_G(a^{\text{opt}})} \right). \quad (2.8)$$

When considering how to design and evaluate utility functions, optimizing the price of anarchy is promising as a design criteria. Constructing a utility function by optimizing price of anarchy would ensure a that a game undergoing learning dynamics is guaranteed a minimum level of performance regardless of game specifics. While the price of anarchy is formally defined in Equation 2.8, with a slight abuse of notation, for a specific game $G \in \mathcal{G}$ with allocations a^{ne} and a^{opt} we refer to $\frac{W(a^{\text{ne}})}{W(a^{\text{opt}})}$ as the game price of anarchy. This notation is used when proving

different game properties. In what follows, we define a sensor coverage problem, and study this problem as a distributed welfare game in Chapter 3, then conjecture the price of anarchy for the set of all single selection sensor coverage problems using the wonderful life utility function in Chapter 4. The hope is that proving the PoA for this set of games can give insight that might assist in determining the price of anarchy for more general sets of games. Designing utility functions that result in an optimized price of anarchy is an open problem, and this thesis is aimed at answering this question.

Chapter 3

Sensor Coverage Problem

The goal of the sensor coverage problem is to allocate sensors to search sectors in such a way to maximize detecting high value events. If we consider a situation where there is no global planner allocating sensors to search regions, then every sensor is responsible for choosing which sector to search. Such a situation is a distributed sensor coverage problem. We can pose this distributed sensor coverage problem as a resource allocation game where the sensors are players, and the search sectors are resources, and each player can choose an action by selecting a particular resource. We will study this problem as an instance of a distributed welfare game discussed in Section 2.1.

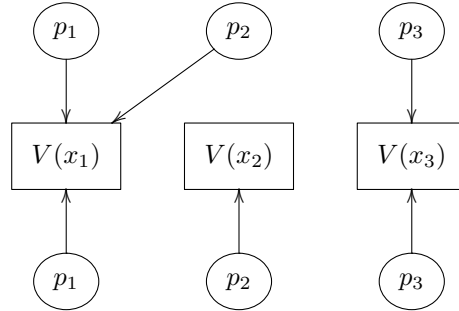


Figure 3.1: Example of a 3 player game resource allocation game. The top represents an equilibrium allocation and the bottom represents the optimal allocation.

3.1 Our Model

Let N be the set of sensors that represent the players. Each player $i \in N$ has a detection probability p_i , which is the probability of detecting an event that occurs in the sector that the sensor is allocated. Let $\mathcal{R}$ be the set of search sectors that represent the resources. Each resource $r \in \mathcal{R}$ has a value v_r , which encodes the value of a potential event, if detected.

The welfare function is a global measure of performance for an allocation and represents a system level objective. In sensor coverage, we wish to quantify the probability of detecting high value events from within search regions. Conceptually, we would expect our welfare function to increase when player detection probability increases, or when a player selects a resource of higher value. We construct the resource welfare function $W_r : 2^N \rightarrow \mathbb{R}^+$ as

$$W_r(S) = \left(1 - \prod_{k \in S} (1 - p_k)\right) v_r. \quad (3.1)$$

Where $r \in \mathcal{R}$ is any resource, and $S \subseteq N$ is the set of players that select r . This resource welfare can be thought of as the expected value of detecting the event. For convenience we define a joint detection probability which represents the probability at least one player will detect the event on that resource as

$$p_r(S) = \left(1 - \prod_{k \in S} (1 - p_k)\right). \quad (3.2)$$

Now consider a subset players $S \subseteq N$ such that every player in S selects resource r . Instead of computing the joint detection probability and finding resource welfare with Equation 3.1, we can equivalently write the resource welfare, for any ordering of the players in S as

$$W_r(S) = p_1 v_r + p_2 (1 - p_1) v_r + \cdots + p_{|S|} (1 - p_1) (1 - p_2) \cdots (1 - p_{|S|-1}) v_r. \quad (3.3)$$

We also define the quantity

$$Z_r(S) = (1 - p_r(S)) v_r = v_r - W_r(S), \quad (3.4)$$

where $Z_r(S)$ can be thought of as the amount of resource value remaining and available for other potential players at resource r . We restrict our attention to modeling a global welfare for all

resources by summing the welfare in all the individual resources. Such global welfare functions are considered separable, and are defined as

$$W(a) = \sum_{r \in \mathcal{R}} W_r(S). \quad (3.5)$$

Since no systematic procedure exists for designing utility functions for specific game performance objectives, we choose the marginal contribution (MC) utility design for our analysis of the distributed sensor coverage problem. Generally for MC we construct the utility function as

$$\begin{aligned} U_i(a_i = r, a_{-i}) &= f_r^{\text{MC}}(i, \{a\}_r) \\ &= W_r(\{a\}_r) - W_r(\{a_{-i}\}_r). \end{aligned}$$

Where $a_{-i} \in \mathcal{A}$ is the joint allocation of all players other than player i , $\{a\}_r \subseteq N$ is the set of player that select resource r in allocation a , and $a = (a_i, a_{-i})$.

3.2 Existing Performance Bounds

Vetta [22] showed that if the following properties are met, then we are guaranteed a lower bound on the system performance.

Theorem 3.2.1. *If $G \in \mathcal{G}$ is a distributed welfare game where for each resource $r \in \mathcal{R}$*

- *The welfare function W_r is submodular,*
- *For each set of players $S \subseteq N$ and player $i \in S$, the distribution rule satisfies $f_r(i, S) \geq W_r(S) - W_r(S \setminus \{i\})$,*
- *For each set of players $S \subseteq N$, the distribution rule satisfies $\sum_{i \in S} f_r(i, S) \leq W_r(S)$,*

then if an equilibrium exists the price of anarchy is greater than or equal to $\frac{1}{2}$.

So since our model of the sensor coverage problem meets these criteria, we are guaranteed that any equilibrium allocation will be within $\frac{1}{2}$ of the optimal. We now seek to study whether this bound is tight for the sensor coverage problem, or there exists a greater bound.

3.3 Preliminary Analysis

It is interesting to note that a sensor's utility can be equivalently defined using Equation (3.4) as

$$U_i(a_i = r, a_{-i}) = p_i Z_r(\{a\}_r). \quad (3.6)$$

With this form we can construct a new definition for a Nash equilibrium.

Claim 3.3.1. An joint allocation $a^{\text{ne}} \in \mathcal{A}$ is said to be a Nash equilibrium if for every player $i \in N$, such that $a_i = r$, and all $\tilde{r} \in \mathcal{R}$

$$Z_r(\{a_{-i}^{\text{ne}}\}_r) \geq Z_{\tilde{r}}(\{a_{-i}^{\text{ne}}\}_{\tilde{r}})$$

Proof. From Definition 2.1.2 we know that the allocation $a^{\text{ne}} \in \mathcal{A}$ is a Nash equilibrium if for every player $i \in N$ and resource $\tilde{r} \in \mathcal{R}$

$$U_i(a_i^{\text{ne}} = r, a_{-i}^{\text{ne}}) \geq U_i(\tilde{a}_i = \tilde{r}, a_{-i}^{\text{ne}}).$$

We can use Equation (3.6) to get

$$\begin{aligned} p_i Z_r(\{a_{-i}^{\text{ne}}\}_r) &\geq p_i Z_{\tilde{r}}(\{a_{-i}^{\text{ne}}\}_{\tilde{r}}) \\ \implies Z_r(\{a_{-i}^{\text{ne}}\}_r) &\geq Z_{\tilde{r}}(\{a_{-i}^{\text{ne}}\}_{\tilde{r}}). \end{aligned}$$

Where $\tilde{r}$ is the resource selected by player i in the allocation $\tilde{a}_i$. □

This fact gives important insight that a player's incentive to deviate to another resource is entirely dependent on the relative amount of resource value remaining at every resource and not their individual detection probability. We can also use this concept to define a special player type.

Definition 3.3.2. We say that any player $i \in N$ is a **pivot player** if $i = \underset{i \in N}{\operatorname{argmin}} (Z_x(\{a_{-i}^{\text{ne}}\}_x))$, where x is the resource selected by player i in the equilibrium allocation a^{ne} , and we denote a_{-i}^{ne} as the equilibrium allocation where player i has been removed.

The pivot player can be thought of as the player that has the minimum amount of resource value left when removed from its respective resource. In any game, there exists at least one pivot player and the pivot player is important in that it bounds the amount of value left over at every other resource in order for an allocation to be considered an equilibrium.

Claim 3.3.3. Let $i \in N$ be the pivot player such that $a_i^{\text{ne}} = x$ in game $G \in \mathcal{G}$. Then for every resource $\tilde{r} \neq x$ we have $W_{\tilde{r}}(\{a^{\text{ne}}\}_{\tilde{r}}) \geq v_{\tilde{r}} - Z_x(\{a_{-i}^{\text{ne}}\}_x)$.

Proof. From Claim 3.3.1 we know for every $\tilde{r} \neq x$

$$Z_x(\{a_{-i}^{\text{ne}}\}_x) \geq Z_{\tilde{r}}(\{a_{-i}^{\text{ne}}\}_{\tilde{r}}).$$

Using Definition 3.4 and the fact that $\{a_{-i}^{\text{ne}}\}_{\tilde{r}} = \{a^{\text{ne}}\}_{\tilde{r}}$ since $a_i \neq \tilde{r}$

$$\begin{aligned} Z_x(\{a_{-i}^{\text{ne}}\}_x) &\geq v_{\tilde{r}} - W_{\tilde{r}}(\{a^{\text{ne}}\}_{\tilde{r}}) \\ \implies W_{\tilde{r}}(\{a^{\text{ne}}\}_{\tilde{r}}) &\geq v_{\tilde{r}} - Z_x(\{a_{-i}^{\text{ne}}\}_x). \end{aligned}$$

□

Chapter 4

Results

4.1 A Previous Conjecture

Doroudi [4] presented two important insights into the problem of determining the price of anarchy for the single selection sensor coverage problem using the marginal contribution utility design. First was finding the price of anarchy of $\frac{4}{5}$ for the restricted class of games for which there is only two players. For games with more than 2 players, [4] conjectured that the price of anarchy occurred within a set of games that has the following properties

- $p_1 = 1$
- $v_n = 1$
- $a^{\text{ne}} = (x_n, x_1, x_2, \dots, x_{n-1})$
- $a^{\text{opt}} = (x_1, x_2, x_3, \dots, x_n)$.

Where $a = (x_1, x_2, \dots)$ means $a_1 = x_1$, $a_2 = x_2$, etc. This set of games can be thought of as where the players are all 'spread out' in both the equilibrium and optimal allocations. There was no explicit minimum for the price of anarchy found in [4] for this set of games. To support his claim, Doroudi studied this class of games for when there only 3 players, and numerically optimized the remaining parameters of the 3 player game under the constraints to maintain a^{ne} as an equilibrium allocation which gave a PoA ≈ 0.7942 with $p_2 \approx 0.5295$, $p_3 \approx 0.1458$, $v_1 \approx 2.1254$,

and $v_2 \approx 1.1707$. However, consider a different 3 player game of a different type such that it has the following properties

- $p_1 = 1, p_2 = \frac{1}{3}, p_3 = \frac{1}{3}$
- $v_1 = 2.25, v_2 = 1, v_3 = 1$
- $a^{\text{ne}} = (x_3, x_1, x_1)$
- $a^{\text{opt}} = (x_1, x_2, x_3)$.

Now it is simple to show that for this new game that $\frac{W(a^{\text{ne}})}{W(a^{\text{opt}})} = \frac{24}{31} \approx 0.7742 < 0.7942$. Which proves that the conjecture in [4] is false. The structural form of the above counter example leads us to the following conjecture.

4.2 A New Conjecture

Definition 4.2.1. Let us consider a new class of games $\mathcal{H} \subset \mathcal{G}$ defined by the following properties

- $v_2 = v_3 = \dots = v_n = 1$
- $p_1 = 1$
- $p_2 = p_3 = \dots = p_n = 1 - v_1^{\frac{1}{1-n}}$
- $a^{\text{ne}} = (x_n, x_1, x_1, \dots)$
- $a^{\text{opt}} = (x_1, x_2, \dots, x_n)$.

Theorem 4.2.2. *Let $\mathcal{H} \subset \mathcal{G}$ be the class of games described in definition 4.2.1. Then $\text{PoA}(\mathcal{H}) = \frac{e}{e+1}$.*

Proof. The class of games described in definition 4.2.1 relies on only two game parameters: v_1 and then number of players n . Let $p' = p_2 = p_3 = \dots = p_n = 1 - v_1^{\frac{1}{1-n}}$. Then we know for any

n-player game $G_n \in \mathcal{H}$

$$\begin{aligned}
\frac{W(a^{\text{ne}})}{W(a^{\text{opt}})} &= \frac{\sum_{r \in \mathcal{R}} W_r(\{a^{\text{ne}}\}_r)}{\sum_{r \in \mathcal{R}} W_r(\{a^{\text{opt}}\}_r)} = \frac{\sum_{r \in \mathcal{R}} p_r(\{a^{\text{ne}}\}_r) v_r}{\sum_{r \in \mathcal{R}} p_r(\{a^{\text{opt}}\}_r) v_r} \\
&= \frac{\left\{ 1 - \left(1 - \left(1 - v_1^{\frac{1}{1-n}} \right) \right)^{n-1} \right\} v_1 + p_1 v_n}{p_1 v_1 + p_2 + \dots + p_n} \\
&= \frac{\left(1 - \frac{1}{v_1} \right) v_1 + 1}{v_1 + p' + \dots + p'} \\
&= \frac{v_1}{v_1 + p'(n-1)}.
\end{aligned}$$

Then this implies that the price of anarchy is

$$\text{PoA}(G_n) = \min_{v_1 \in [1, \infty)} \left(\frac{v_1}{v_1 + p'(n-1)} \right) \quad (4.1)$$

Clearly, the minimum does not occur at the boundary, so therefore must occur when the gradient is equal to zero.

$$\begin{aligned}
&\Rightarrow \frac{\partial \text{PoA}(G_n)}{\partial v_1} = 0 \\
&\Rightarrow \left(\frac{\partial}{\partial v_1} v_1 \right) (v_1 + (n-1)p') - \frac{\partial}{\partial v_1} (v_1 + (n-1)p') (v_1) = 0 \\
&\Rightarrow (v_1 + (n-1)p') - v_1 \left(1 + \frac{1}{v_1^{\frac{n}{n-1}}} \right) = 0 \\
&\Rightarrow (n-1)p' - \frac{v_1}{v_1^{\frac{n}{n-1}}} = 0 \\
&\Rightarrow (n-1) - \frac{n-1}{v_1^{\frac{1}{n-1}}} - \frac{1}{v_1^{\frac{1}{n-1}}} = 0 \\
&\Rightarrow \frac{n}{v_1^{\frac{1}{n-1}}} = (n-1) \\
&\Rightarrow v_1^{\frac{1}{n-1}} = \frac{n}{n-1} \\
&\Rightarrow v_1 = \left(\frac{n}{n-1} \right)^{n-1}
\end{aligned}$$

Substituting back into the original equation, we get

$$\text{PoA}(G_n) = \min_{v_1 \in [1, \infty)} \left(\frac{v_1}{v_1 + p'(n-1)} \right) = \frac{\left(\frac{n}{n-1} \right)^{n-1}}{\left(\frac{n}{n-1} \right)^{n-1} + \frac{n-1}{n}} \quad (4.2)$$

To show that the price of anarchy is $\frac{e}{e+1}$, we must show that the sequence $\text{PoA}(G_n)$ is monotonically decreasing and convergent to $\frac{e}{e+1}$.

Monotonic Decreasing Let $t_n = \left(\frac{n}{n-1}\right)^n$, this implies

$$\text{PoA}(G_n) = \frac{\left(\frac{n}{n-1}\right)^{n-1}}{\left(\frac{n}{n-1}\right)^{n-1} + \frac{n-1}{n}} = \frac{\left(\frac{n}{n-1}\right)^n}{\left(\frac{n}{n-1}\right)^n + 1} = \frac{t_n}{t_n + 1}$$

Now let $a = n^2$ and $b = n^2 - 1$. Clearly, $a > b$. Now by the identity for factoring difference of powers

$$\begin{aligned} a^n - b^n &= (a - b) \sum_{k=0}^{n-1} a^k b^{n-k-1} \\ \implies n(a - b)a^{n-1} &> a^n - b^n > n(a - b)b^{n-1} \\ \implies (n^2)^n - (n^2 - 1)^n &\geq n(n^2 - 1)^{n-1} \\ \implies n^{n+1}n^n = n(n^2)^n &\geq n(n^2 - 1)^n + n^2(n^2 - 1)^{n-1} \\ &> n(n^2 - 1)^n + (n^2 - 1)(n^2 - 1)^{n-1} \\ &= (n + 1)(n^2 - 1)^n = (n + 1)((n + 1)(n - 1))^n \\ &= (n + 1)^{n+1}(n - 1)^n \\ \implies \left(\frac{n}{n-1}\right)^n &> \left(\frac{n+1}{n}\right)^{n+1} \\ \implies t_n &> t_{n+1} \\ \implies \frac{t_n}{t_n + 1} &> \frac{t_{n+1}}{t_{n+1} + 1} \\ \text{PoA}(G_n) &> \text{PoA}(G_{n+1}). \end{aligned}$$

Therefore $\text{PoA}(G_n)$ is monotonically decreasing.

Convergence Let $v_1^n = \left(\frac{n}{n-1}\right)^{n-1}$, this implies

$$\begin{aligned} \text{PoA}(G_n) &= \frac{\left(\frac{n}{n-1}\right)^{n-1}}{\left(\frac{n}{n-1}\right)^{n-1} + \frac{n-1}{n}} = \frac{v_1^n}{v_1^n + \frac{n-1}{n}} \\ \implies \lim_{n \rightarrow \infty} \text{PoA}(G_n) &= \frac{\lim_{n \rightarrow \infty} v_1^n}{\lim_{n \rightarrow \infty} v_1^n + \lim_{n \rightarrow \infty} \left(\frac{n-1}{n}\right)} = \frac{\lim_{n \rightarrow \infty} v_1^{n+1}}{\lim_{n \rightarrow \infty} v_1^{n+1} + 1} \\ &= \frac{e}{e + 1} \end{aligned}$$

Since $v_1^{n+1} = (1 + \frac{1}{n})^n$ and $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n \equiv e$. Therefore $\text{PoA}(G_n)$ is monotonically decreasing and converges to $\frac{e}{e+1}$, which implies $\text{PoA}(\mathcal{H}) = \frac{e}{e+1}$.

□

We conjecture that the price of anarchy of all single selection sensor coverage problems is $\frac{e}{e+1}$. To support this claim we will prove that for any game, there exists a new game with non-decreasing price of anarchy that possess some of the same properties as the set of games described by $\mathcal{H}$. First we show an important property about the effect of increasing the detection probability of the pivot player on the price of anarchy.

Lemma 4.2.3. *Given any game $G \in \mathcal{G}$, let $i \in N$ be the pivot player, with $p_i \neq 1$, such that $a_i^{\text{ne}} = x$ and $a_i^{\text{opt}} = y$. Also, let i be alone in the equilibrium allocation. If $\frac{v_x}{Z_y(\{a_{-i}^{\text{opt}}\}_y)} \leq \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})}$, there exists a game $H \in \mathcal{G}$ such that $p_i = 1$ and $\text{PoA}(H) \leq \text{PoA}(G)$.*

Proof. Let $S \subseteq N$ be the set of players that select resource y in the optimum. If player i is alone on resource y in the optimum, then we merely use Lemma A.0.1 to increase $p_i = 1$. Now, let's consider the case where there are two or more players on y . Let $j \in S$ be the player such that $Z_y(\{a_{-j}^{\text{opt}}\}_y) \leq Z_y(\{a_{-k}^{\text{opt}}\}_y)$ for all $k \neq i \in S$. We write the welfare in the optimum at resource y as

$$\begin{aligned} W_y(\{a^{\text{opt}}\}_y) &= p_y(\{a_{-i,j}^{\text{opt}}\}_y) v_y + p_j Z_y(\{a_{-i,j}^{\text{opt}}\}_y) + p_i(1 - p_j) Z_y(\{a_{-i,j}^{\text{opt}}\}_y) \\ &= p_y(\{a_{-i,j}^{\text{opt}}\}_y) v_y + p_i Z_y(\{a_{-i,j}^{\text{opt}}\}_y) + p_j(1 - p_i) Z_y(\{a_{-i,j}^{\text{opt}}\}_y) \end{aligned}$$

Where $a_{-i,j}^{\text{opt}}$ is the optimum allocation with players i and j removed. Let $Z' = \max_{r \neq y} Z_r(\{a^{\text{opt}}\}_r)$ be the largest value available at any other resource in the optimum. We can write

$$\frac{W(a^{\text{ne}})}{W(a^{\text{opt}})} = \frac{\sum_{r \neq x} W_r(\{a^{\text{ne}}\}_r) + p_i v_x}{\sum_{r \neq y} W_r(\{a^{\text{opt}}\}_r) + p_y(\{a_{-i,j}^{\text{opt}}\}_y) v_y + p_j Z_y(\{a_{-i,j}^{\text{opt}}\}_y) + p_i(1 - p_j) Z_y(\{a_{-i,j}^{\text{opt}}\}_y)}.$$

Now since we know that $(1 - p_j) Z_y(\{a_{-i,j}^{\text{opt}}\}_y) = Z_y(\{a_{-i}^{\text{opt}}\}_y)$ and $\frac{v_x}{Z_y(\{a_{-i}^{\text{opt}}\}_y)} \leq \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})}$, we

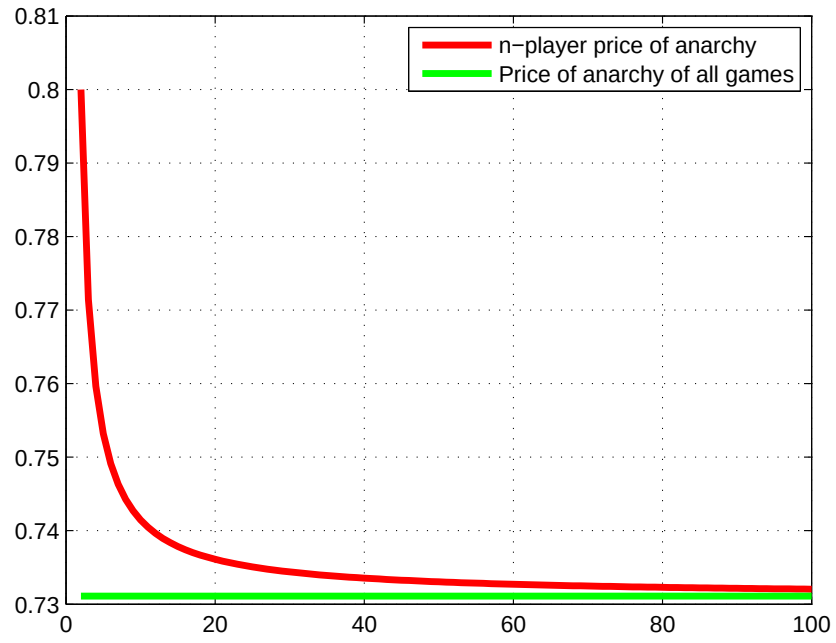


Figure 4.1: The conjectured price of anarchy for n-player games, $\frac{1}{1+(1-\frac{1}{n})^n}$, is shown in red, and all games $\frac{e}{e+1}$ is shown in green.

can use Lemma A.0.1 to increase p_i to $p' \in [p_i, 1]$ such that $(1 - p') Z_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right) = Z'$ and

$$\begin{aligned} \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})} &\geq \frac{\sum_{r \neq x} W_r(\{a^{\text{ne}}\}_r) + p' v_x}{\sum_{r \neq y} W_r(\{a^{\text{opt}}\}_r) + p_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right) v_y + p_j Z_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right) + p' (1 - p_j) Z_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right)} \\ &= \frac{\sum_{r \neq x} W_r(\{a^{\text{ne}}\}_r) + p' v_x}{\sum_{r \neq y} W_r(\{a^{\text{opt}}\}_r) + p_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right) v_y + p' Z_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right) + p_j (1 - p') Z_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right)} \\ &= \frac{\sum_{r \neq x} W_r(\{a^{\text{ne}}\}_r) + p' v_x}{\sum_{r \neq y} W_r(\{a^{\text{opt}}\}_r) + p_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right) v_y + p' Z_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right) + p_j Z'}. \end{aligned}$$

Since we know by definition of Z' , there exists another action $a'_j \neq y$ such that $W(a'_j, a_{-j}^{\text{opt}}) = W(a_j^{\text{opt}}, a_{-j}^{\text{opt}})$ in the denominator in the last line of the equation above. Denoting $a' = (a'_j, a_{-j}^{\text{opt}})$ we have

$$\frac{W(a^{\text{ne}})}{W(a^{\text{opt}})} \geq \frac{\sum_{r \neq x} W_r(\{a^{\text{ne}}\}_r) + p' v_x}{\sum_{r \neq y} W_r(\{a'\}_r) + p_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right) v_y + p' Z_y \left(\{a_{-i,j}^{\text{opt}}\}_y \right)}.$$

Noting that $\frac{v_x}{Z_y(\{a_{-i}^{\text{opt}}\}_y)} \leq \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})} \implies \frac{v_x}{Z_y(\{a_{-i,j}^{\text{opt}}\}_y)} \leq \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})}$ allows us to repeat the process above with every other player in S until player i is alone on resource y . Then we use Lemma A.0.1 to raise $p' = 1$ and get

$$\frac{W(a^{\text{ne}})}{W(a^{\text{opt}})} \geq \frac{\sum_{r \neq x} W_r(\{a^{\text{ne}}\}_r) + v_x}{\sum_{r \neq y} W_r(\{a'\}_r) + v_y}.$$

After allowing a' to reallocate to a new optimum, and thereby increasing the denominator in the equation above while leaving the equilibrium unchanged, we now have a new game $H \in \mathcal{G}$ and we know that $PoA(H) \leq PoA(G)$. $\square$

Theorem 4.2.4. *Let $G_n \in \mathcal{G}$ be any n -player single selection sensor coverage game using marginal contribution utility design. Then there exists a game $H \in \mathcal{G}$ with the following properties*

- $p_1 = 1$
- $a_1^{\text{ne}} = x_n$
- $a_1^{\text{opt}} = x_1$
- $v_n = 1$.

such that $PoA(H) \leq PoA(G_n)$.

Proof. Let $i \in N$ denote the pivot player of game G_n with $Z_x(\{a_{-i}^{ne}\}_x) = Y$, where x is the resource selected by player i in the equilibrium allocation. First, consider the case where there exists at least one player $j \in N$ with detection probability of $p_j = 1$ that selects resource q in the equilibrium. We can use Lemma A.0.4 to reduce the value of the resource selected by that player in the equilibrium allocation to Y . Since the detection probability is 1, then regardless of the value of the resource, no other player would ever have any incentive to move to that resource. We know from definition of the pivot player, the lowest available value at any resource is Y , so the allocation remains an equilibrium. We can repeat this for any number of players that have detection probability of 1. Since any player that has probability 1 now has $Z_q(\{a_{-j}^{ne}\}_q) = Y$, they are all a valid pivot players. So that set of games now has the pivot player with a detection probability of $p_i = 1$.

Now, without loss of generality, let the new set of resource values be $v_1 \geq v_2 \geq \dots \geq v_n$. Regardless of the value of p_i , if the pivot player is alone and no other resource is unallocated, then by definition the pivot is already on resource x_n . If the player is not on resource x_n then since the number of players equals the number of resources there exists at least one resource that is empty. We also know that any empty resource has value less than Y , or the pivot player would have incentive to deviate to that resource and would not be an equilibrium allocation. We can raise the value of all of these empty resources to have a value of Y . This will not change welfare or allocation in the equilibrium, and the welfare in the optimum is non-decreasing. Let $x \in \mathcal{R}$ be the resource selected by the pivot player in the equilibrium allocation. We can write the sum of the welfare at resources x and x_n as $W_x(\{a^{ne}\}_x) + W_{x_n}(\{a^{ne}\}_{x_n}) = p_i Y + W_x(\{a_{-i}^{ne}\}_x) + 0$, since resource x_n is empty and therefore garners a welfare of 0. We can now move the pivot player to the empty resource $a'_i = x_n$ and since we now have $v_n = Y$, this implies, under the new allocation $a' = (a'_i, a_{-i}^{ne})$, that $W_x(\{a'\}_x) + W_{x_n}(\{a'\}_{x_n}) = W_x(\{a_{-i}^{ne}\}_x) + p_i Y = W_x(\{a^{ne}\}_x) + W_{x_n}(\{a^{ne}\}_{x_n})$. Which shows the welfare in the equilibrium is unchanged. With

Lemma B.0.6, we can now scale all the resources such that $v_n = 1$.

We now have the pivot player on resource x_n and $v_n = 1$. All we now have to consider is the case where the pivot player doesn't have detection probability of 1. First suppose that all games $G_{n-1} \in \mathcal{G}$ is bounded below by a game with the pivot player $p_i = 1$. Now if $\frac{1}{Z_y(\{a_{-i}^{\text{opt}}\}_y)} > \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})}$ then we can use Lemma A.0.2 to reduce the detection probability of the pivot player to $p_i = 0$, which is equivalent to a game with $n - 1$ players. Which by assumption is bound below by a game of $n - 1$ players with $p_i = 1$. And we know from [4] that every 2-player game is bounded below by a game with $p_i = 1$. Now all we have to consider is the case where $\frac{1}{Z_y(\{a_{-i}^{\text{opt}}\}_y)} \leq \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})}$. We can use Lemma 4.2.3 to increase the pivot player to 1, which completes the proof. $\square$

The above theorem proves the important property that the price of anarchy occurs when the pivot player has detection probability of 1, and is allocated to the resource, x_n , that has the least available value. We also know that any optimum allocation must have the pivot player allocated to the highest value resource, x_1 , or it can't, by definition, be an optimal allocation. We now can show, using Claim 3.3.3, how this condition can be used to bound the welfare garnered in the equilibrium allocation at other resources.

Theorem 4.2.5. *Let $G_n \in \mathcal{G}$ be a n -player single selection sensor coverage problem using marginal contribution utility design with the following properties*

- $p_1 = 1$
- $a_1^{\text{ne}} = x_n$
- $a_1^{\text{opt}} = x_1$
- $v_n = 1$.

Then there exists a game $H \in \mathcal{G}$ such that for any resource $r \neq x_n \in \mathcal{R}$ and joint probability $p_r(\{a^{\text{ne}}; H\}_r) \neq 1$, $W_r(\{a^{\text{ne}}; H\}_r) = v_r - 1$ and $PoA(H) \leq PoA(G)$.

Proof. For all resources $r \neq x_n$, let $S \subseteq N$ be the set of players that select resource r in the equilibrium allocation. If there exists a player $i \in S$, selecting resource y in the optimum such that $\frac{Z_r(\{a_{-i}^{\text{ne}}\}_r)}{Z_y(\{a_{-i}^{\text{opt}}\}_y)} \geq \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})}$, then we can use Lemma A.0.2 to reduce the detection probability of player i such that $W_r(\{a^{\text{ne}}\}_r) = v_r - 1$. If this condition is not met, then we know that every player $k \in S$ has $\frac{Z_r(\{a_{-i}^{\text{ne}}\}_r)}{Z_y(\{a_{-i}^{\text{opt}}\}_y)} < \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})}$. If there is only one player allocated to resource r , then we can use Lemma 4.2.3 to increase the detection probability to 1. Consider the case where there are more than one player on resource r , which would imply there exists at least one empty resource in the equilibrium, and any empty resource must have value of 1. Let $j \in S$ be the player such that $Z_r(\{a_{-j}^{\text{ne}}\}_r) \leq Z_r(\{a_{-k}^{\text{ne}}\}_r)$ for all $k \in S$. Also let $l \in S$ be the player such that $Z_r(\{a_{-l}^{\text{ne}}\}_r) \geq Z_r(\{a_{-k}^{\text{ne}}\}_r)$ for all $k \in S$. Then we can use Lemma 4.2.3 to increase the detection probability of player l such that $Z_r(\{a_{-j}^{\text{ne}}\}_r) = 1$. There now exists another action $a'_j \neq r$ with $a' = (a'_j, a_{-j}^{\text{ne}})$ such that $W(a'_j, a_{-j}^{\text{ne}}) = W(a_j^{\text{ne}}, a_{-j}^{\text{ne}})$. Once on this new resource, we can use Lemma 4.2.3 to increase the detection probability to $p_j = 1$, and we have $W_r(\{a'\}_r) = v_r - 1$. After letting the a^{opt} reallocate to the new optimum allocation, we have a game $H \in \mathcal{G}$, such that the welfare at all resources in the equilibrium at resources $r \in \mathcal{R}$ such that $p_r(\{a^{\text{ne}}\}_r) \neq 1$, is $W_r(\{a^{\text{ne}}\}_r) = v_r - 1$. $\square$

Chapter 5

Simulations

We seek to empirically test our hypothesis that the price of anarchy over all single selection distributed sensor coverage problems is $\frac{e}{e+1}$ by simulating many randomly sampled games. To simulate a n -player game we first sample the detection probabilities $\mathbf{p} = (p_1, p_2, \dots, p_n)^\top \in [0, 1]^n$ where each detection probability is drawn uniformly as $p_i \sim \text{Unif}(0, 1)$. We then follow a similar procedure to sample resource values $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ where each resource value is sampled according to $v_i \sim \text{Unif}(0, 1)$. After sorting the values so we have $v_1 \geq v_2 \geq \dots \geq v_n$, we can then scale the by $\frac{1}{v_n}$, so for every game sampled has $v_n = 1$.

Once the game parameters have been sampled, we need to find both the optimal welfare. This is performed by exhaustively searching all possible allocations $a \in \mathcal{A}$ until we have found $a^{\text{opt}} \in \mathcal{A}$ such that

$$W(a^{\text{opt}}) \geq W(a).$$

Once the optimal welfare is found, we then turn our attention to finding an equilibrium welfare. In order to find the welfare, we must first find a valid equilibrium allocation. The process of numerically computing the equilibrium is commonly known as game learning. Many efficient learning algorithms exist in the literature such as regret matching and fictitious play [9], however we exhaustively search all possible equilibrium allocations until we find

$$a^* = \underset{a^{\text{ne}} \in \text{NE}(G)}{\text{argmin}} W(a^{\text{ne}}).$$

5.1 Results

We simulated games between 2 and 5 players, each with 10000 game samples. Table 5.1 summarizes the results of the simulations including minimum and average price of anarchy. As can be seen from the table, the minimum empirical PoA is less than the price of anarchy as conjectured in Theorem 4.2.2. Also it should be noted that the average price of anarchy is close to 1, and we can see how most of the mass of the games samples are distributed in Figures 5.2-5.5.

Number of players	Conjectured PoA	Worst Observed PoA	Maximum PoA	Average Observed PoA
2	0.8000	0.8064	1	0.9848
3	0.7714	0.8032	1	0.9670
4	0.7596	0.8034	1	0.9524
5	0.7532	0.7813	1	0.9413

Table 5.1: Simulation results.

However, since we know from Lemma 4.2.3 that the price of anarchy occurs only in games where $p_1 = 1$, we now repeat the simulations above, but now only consider games with $p_1 = 1$. The results are shown in Table 5.2 and Figures 5.7-5.10.

Number of players	Conjectured PoA	Worst Observed PoA	Maximum PoA	Average Observed PoA
2	0.8000	0.8018	1	0.9745
3	0.7714	0.7838	1	0.9414
4	0.7596	0.7751	1	0.9229
5	0.7532	0.7841	1	0.9141

Table 5.2: Simulation results with $p_1 = 1$.

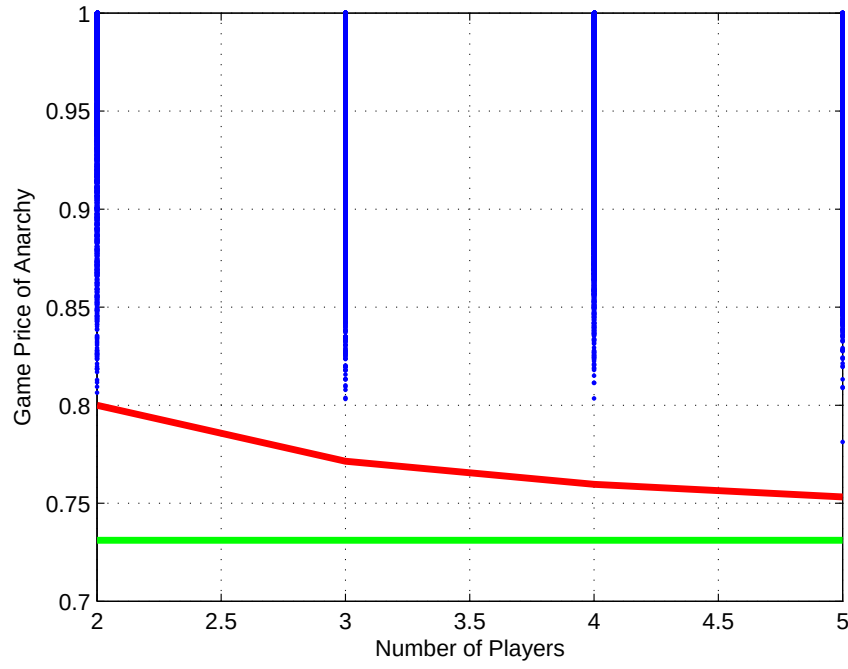


Figure 5.1: Price of anarchy for game samples with different number of players is shown in blue. The red line represents $\frac{1}{1+(1-\frac{1}{n})^n}$, the conjectured price of anarchy for n-player games, and the green line represents $\frac{e}{e+1}$.

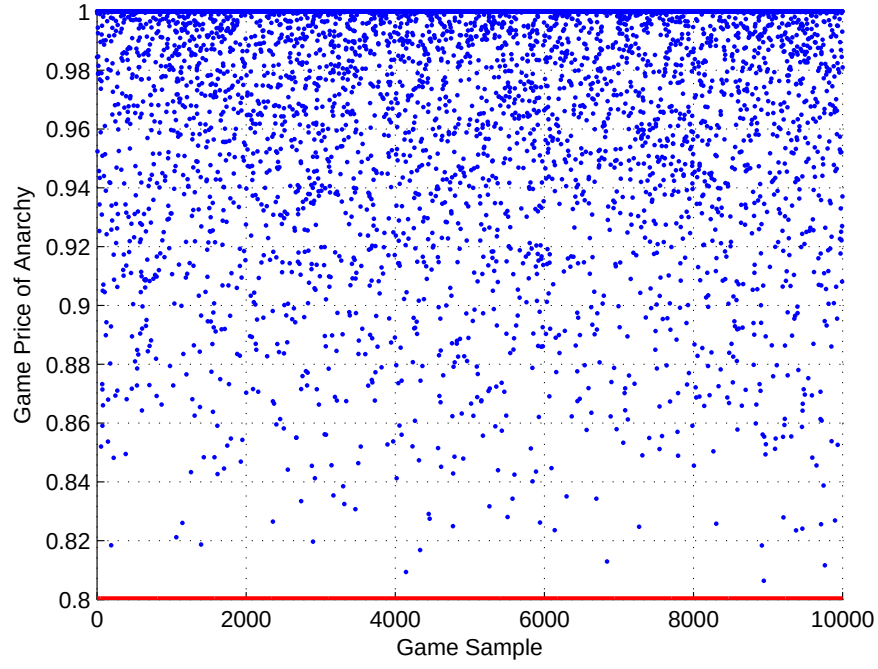


Figure 5.2: Price of anarchy of 10000 samples for 2 player sensor coverage games. The red line indicates the lower bound conjectured in Theorem 4.2.2.

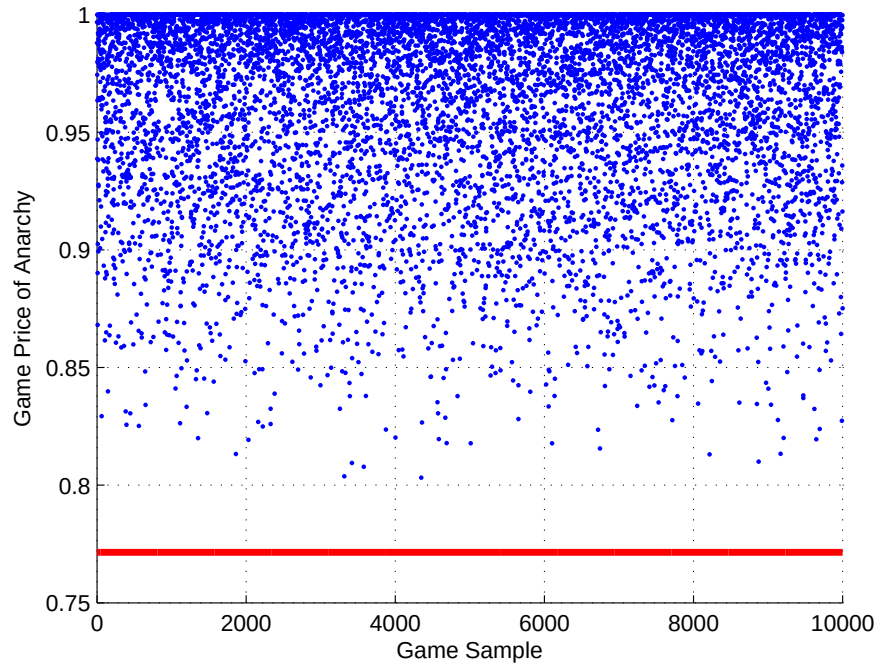


Figure 5.3: Price of anarchy of 10000 samples for 3 player sensor coverage games. The red line indicates the lower bound conjectured in Theorem 4.2.2.

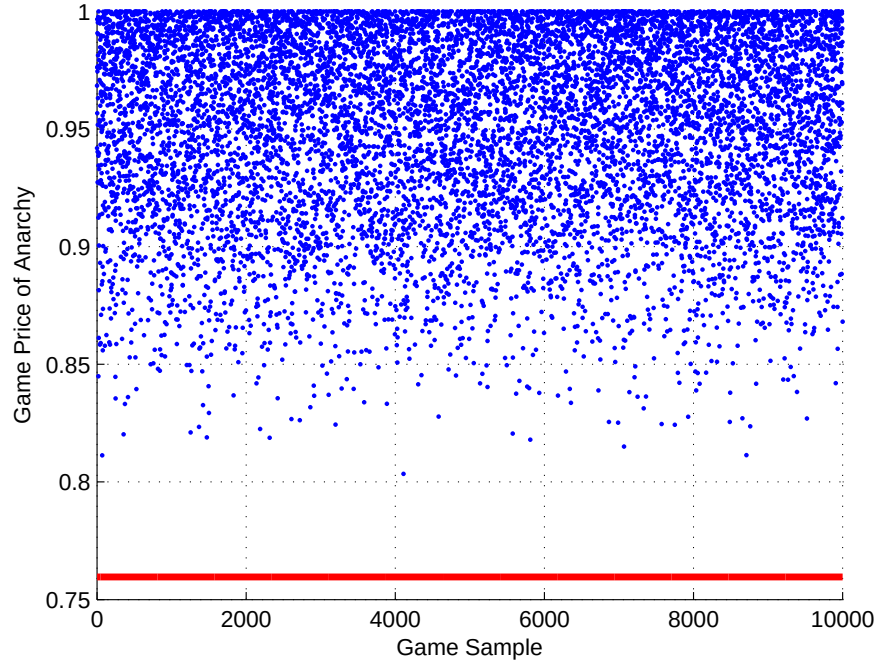


Figure 5.4: Price of anarchy of 10000 samples for 4 player sensor coverage games. The red line indicates the lower bound conjectured in Theorem 4.2.2.

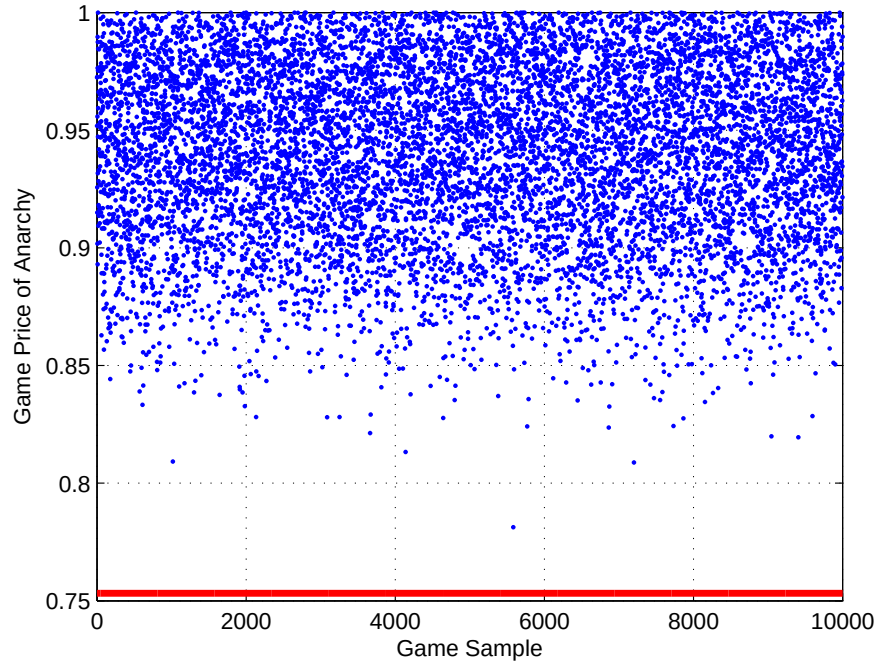


Figure 5.5: Price of anarchy of 10000 samples for 5 player sensor coverage games. The red line indicates the lower bound conjectured in Theorem 4.2.2.

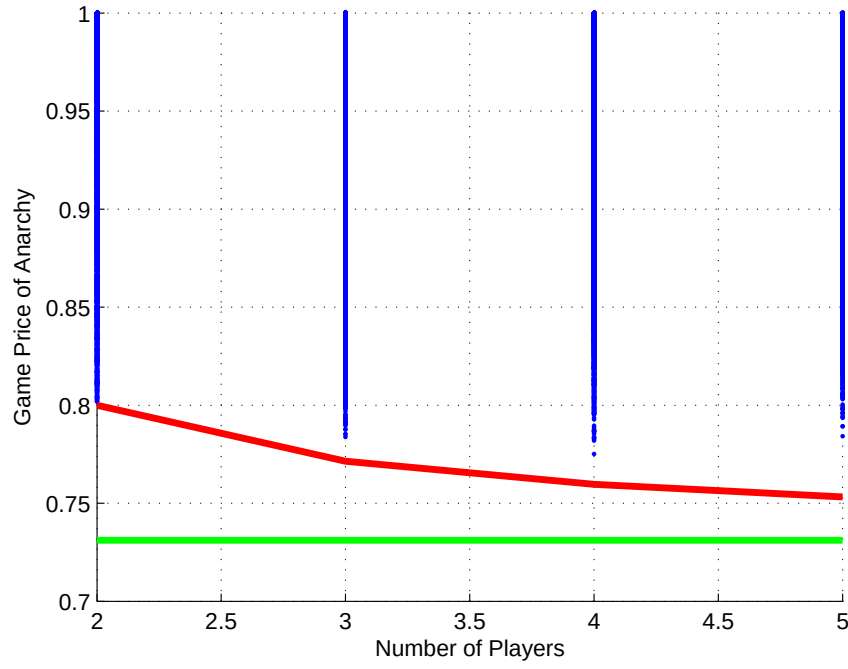


Figure 5.6: Price of anarchy for game samples with different number of players is shown in blue. The red line represents $\frac{1}{1+(1-\frac{1}{n})^n}$, the conjectured price of anarchy for n-player games, and the green line represents $\frac{e}{e+1}$.

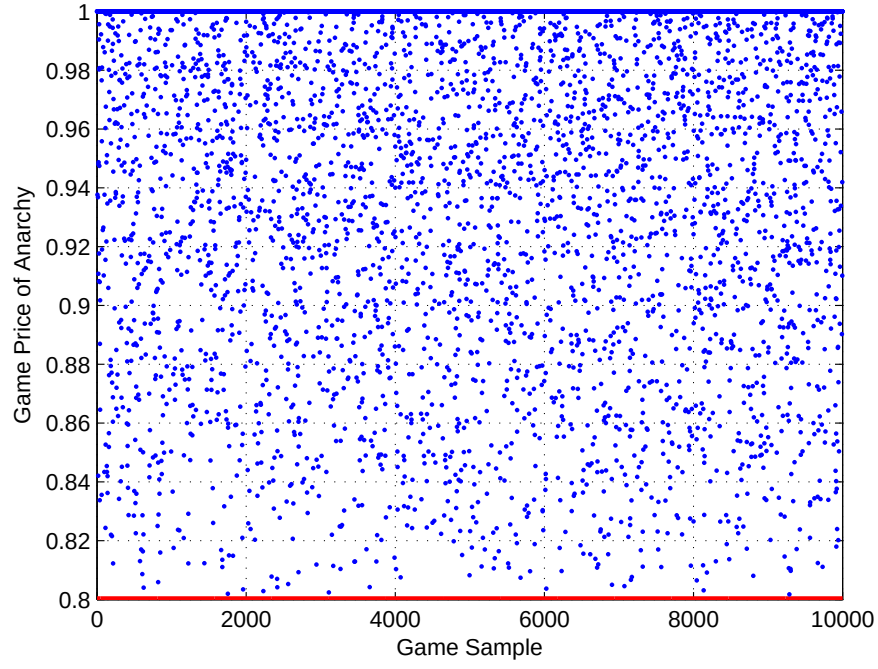


Figure 5.7: Price of anarchy of 10000 samples for 2 player sensor coverage games with $p_1 = 1$. The red line indicates the lower bound conjectured in Theorem 4.2.2.

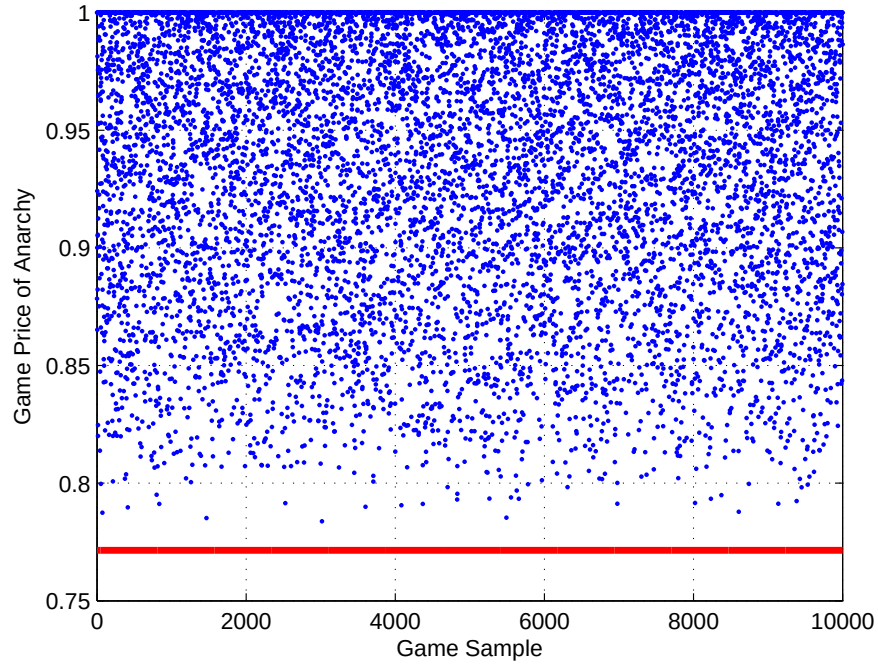


Figure 5.8: Price of anarchy of 10000 samples for 3 player sensor coverage games with $p_1 = 1$. The red line indicates the lower bound conjectured in Theorem 4.2.2.

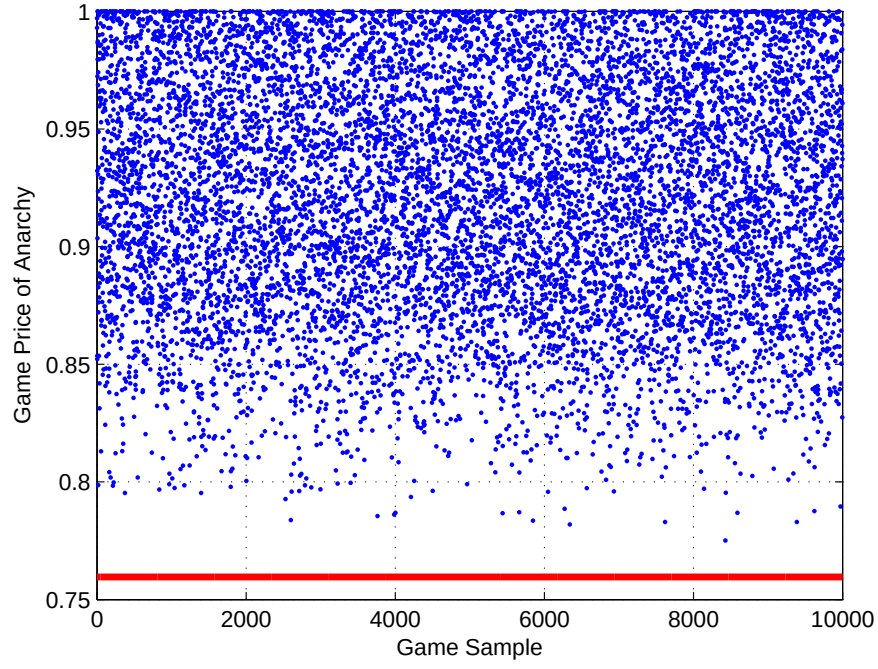


Figure 5.9: Price of anarchy of 10000 samples for 4 player sensor coverage games with $p_1 = 1$. The red line indicates the lower bound conjectured in Theorem 4.2.2.

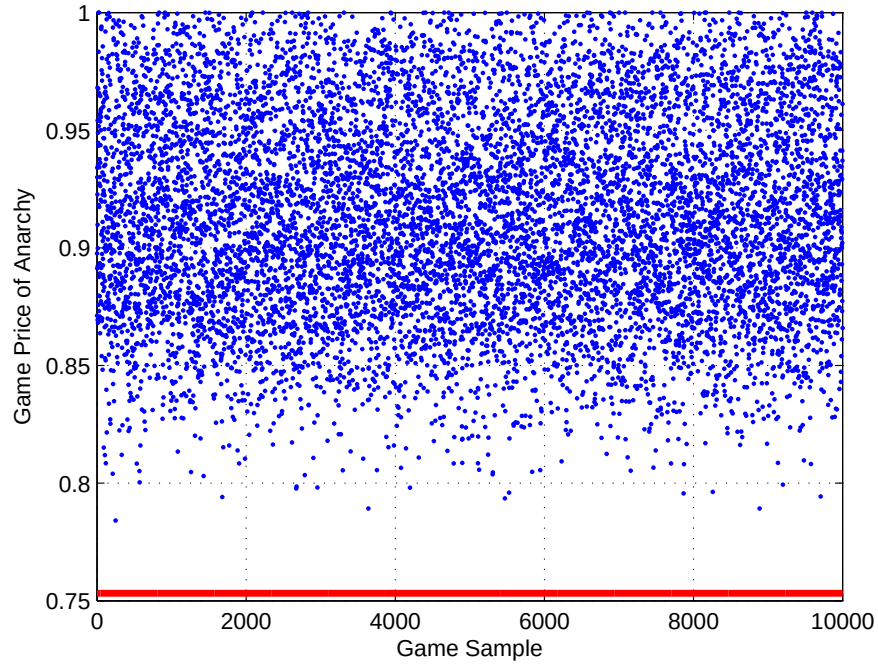


Figure 5.10: Price of anarchy of 10000 samples for 5 player sensor coverage games with $p_1 = 1$. The red line indicates the lower bound conjectured in Theorem 4.2.2.

Chapter 6

Conclusions and Future Work

This thesis studies the distributed sensor coverage problem using the mathematical tools of game theory, and investigates the worst case performance of equilibrium allocations using the wonderful life utility design. We define a class of games $\mathcal{H}$ in Definition 4.2.1, and show the price of anarchy in this set of games is $\frac{e}{e+1} \approx 73\%$, which is below previously conjectured bounds [4]. We then conjecture that $\frac{e}{e+1}$ is the price of anarchy for all single selection sensor coverage games. We support this claim with provable properties about the structure of games bounded by this price of anarchy. We show that the price of anarchy occurs in the set of games that has the following properties

- $p_1 = 1$
- $v_n = 1$
- $a_1^{\text{ne}} = x_n$
- $a_1^{\text{opt}} = x_1$
- $W_r(\{a\}_r) = v_r - 1, \forall r \neq x_n.$

Simulating games, we show that the price of anarchy of all games simulated remained above the proposed bound of $\frac{e}{e+1}$. It is still unproved that the price of anarchy of the set of all games $\mathcal{G}$ is in fact $\frac{e}{e+1}$. However, since the bound proposed in Theorem 4.2.2 is tight, we can claim that $\frac{e}{e+1}$ acts as an upper bound for the price of anarchy for $\mathcal{G}$. That is $\text{PoA}(\mathcal{G}) \leq \frac{e}{e+1}$.

It is important to note the 73% for single selection distributed sensor coverage problem is well above the general bound of 50% as shown in Section 3.2. However, this result is for a specific game type. It is an open problem to determine under what general game constructions can we expect to find a price of anarchy to be above 50%. Ideally, we strive to develop rigorous, systematic tools to design utility functions that optimize guaranteed bounds on distributed systems. Also, it must be noted that any of the results presented are for games with static conditions. How these results generalize to games whereby the structure is dynamic, the continual changing of values of the resources in the sensor coverage problem, for example, is unsolved. Other important questions that remain, are the effect of restrictions on player action sets, limited inter-player communication, and system learning dynamics that are unsynchronized among agents.

Appendix A

Algebraic Properties

We present a series of algebraic properties that will assist us in our work.

Lemma A.0.1. *Let $p \in [0, 1]$ and $a, b, c, d \in \mathbb{R}^+$. Given any $\frac{a}{b}$ such that $\frac{a}{b} \leq \frac{pa+c}{pb+d}$, then*

$$\frac{a+c}{b+d} \leq \frac{pa+c}{pb+d}.$$

Proof. Suppose $\frac{a+c}{b+d} > \frac{pa+c}{pb+d}$, this implies

$$\begin{aligned} \frac{a+c}{b+d} &> \frac{pa+c}{pb+d} \\ \implies (pb+d)(a+c) &> (pa+c)(b+d) \\ \implies pab + pbc + ad + cd &> pab + pad + bc + cd \\ \implies ad(1-p) &> bc(1-p). \end{aligned}$$

If $p = 1$, then we have $0 > 0$, which is a contradiction. If $p \neq 1$, then we have

$$\begin{aligned} \implies ad &> bc \\ \implies pab + ad &> pab + bc \\ \implies a(pb+d) &> b(pa+c) \\ \frac{a}{b} &> \frac{pa+c}{pb+d}. \end{aligned}$$

Which is a contradiction. Therefore $\frac{a+c}{b+d} \leq \frac{pa+c}{pb+d}$. □

Lemma A.0.2. *Let $p \in [0, 1]$ and $a, b, c, d \in \mathbb{R}^+$. Given any $\frac{a}{b}$ such that $\frac{a}{b} \geq \frac{pa+c}{pb+d}$, then*

$$\frac{c}{d} \leq \frac{pa+c}{pb+d}.$$

Proof. Suppose $\frac{c}{d} > \frac{pa+c}{pb+d}$, this implies

$$\frac{c}{d} > \frac{pa+c}{pb+d}$$

$$c(pb+d) > d(pa+c)$$

$$pbc + cd > pad + cd$$

$$pbc > pad.$$

If $p = 0$, then we have $0 > 0$, which is a contradiction. If $p \neq 0$, then we have

$$bc > ad$$

$$pab + bc > pab + ad$$

$$b(pa+c) > a(pb+d)$$

$$\frac{pa+c}{pb+d} > \frac{a}{b}.$$

Which is a contradiction. Therefore $\frac{c}{d} \leq \frac{pa+c}{pb+d}$. □

Lemma A.0.3. Let $a, b, c, d \in \mathbb{R}^+$. Given any $\frac{a+c}{b+d}$ such that $\frac{a}{b} \leq \frac{c}{d}$, then $\frac{a}{b} \leq \frac{a+c}{b+d}$.

Proof. Now suppose $\frac{a}{b} > \frac{a+c}{b+d}$, this implies

$$\implies \frac{a}{b} > \frac{a+c}{b+d}$$

$$\implies a(b+d) > b(a+c)$$

$$\implies ab + ad > ab + bc$$

$$\implies ad > bc$$

$$\implies \frac{a}{b} > \frac{c}{d}$$

Which is a contradiction. Therefore $\frac{a}{b} \leq \frac{a+c}{b+d}$. □

Lemma A.0.4. Let $p \in [0, 1]$, $a, b \in \mathbb{R}^+$, and $v \in [1, \infty)$. Given any $\frac{v+a}{pv+b}$ such that $\frac{v+a}{pv+b} \leq 1$,

then $\frac{v+a}{pv+b} \geq \frac{1+a}{p+b}$.

Proof. From Lemma A.0.3, we know that $\frac{a}{b} \leq 1$. Now suppose that $\frac{v+a}{pv+b} < \frac{1+a}{p+b}$. This implies

$$\implies (p+b)(v+a) < (1+a)(pv+b)$$

$$\implies pv + pa + bv + ab < pv + b + pva + ab$$

$$\implies pa + bv < b + pva$$

$$\implies bv - pva < b - pa$$

$$\implies v(b - pa) < b - pa$$

$$\implies v < 1$$

Which is a contradiction since $v \in [1, \infty) \implies v \not< 1$. □

Appendix B

Game Properties

Lemma B.0.5. *Let $G \in \mathcal{G}$ be a N player, M resource game. Then there exists a N player, N resource game $H \in \mathcal{G}$ such that $PoA(H) \leq PoA(G)$.*

Proof. Case $N > M$: By definition

$$PoA(G) = \frac{W(a^{ne})}{W(a^{opt})} = \frac{\sum_{r \in \mathcal{R}} W_r(\{a^{ne}\}_r)}{\sum_{r \in \mathcal{R}} W_r(\{a^{opt}\}_r)}$$

Now add $N - M$ resources, each with value Y , to game G to create a new game H . No player has incentive to deviate from a^{ne} , so therefore remains an equilibrium allocation. This implies that $p_r(\{a^{ne}\}_r) = 0 \implies W_r(\{a^{ne}\}_r) = 0$ for each of the added resources. The optimum is non-decreasing, so this implies

$$PoA(H) = \frac{\sum_{r \in \mathcal{R}} W_r(\{a^{ne}\}_r)}{\sum_{r \in \mathcal{R}} W_r(\{a^{opt}\}_r)} \leq PoA(G)$$

Case $N < M$: Since each player can choose at most one resource, this implies there is $M - N$ empty resources in both the equilibrium and optimum. Since the optimum is also an equilibrium, that means that the empty resources occur at the $N - M$ least valuable resources in both allocations. This implies that $p_r(\{a^{ne}\}_r) = 0 \implies W_r(\{a^{ne}\}_r) = 0$ for those empty resources. Which implies that

$$PoA(G) = PoA(H)$$

□

Lemma B.0.6. *Let $G \in \mathcal{G}$ be any arbitrary game with price of anarchy $\text{PoA}(G)$. If we scale all resource values in g by the same constant $c \in \mathbb{R} \setminus \{0\}$, then the price of anarchy is unchanged.*

Proof. By definition

$$\text{PoA}(G) = \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})} = \frac{\sum_{r \in \mathcal{R}} W_r(\{a^{\text{ne}}\}_r)}{\sum_{r \in \mathcal{R}} W_r(\{a^{\text{opt}}\}_r)} = \frac{\sum_{r \in \mathcal{R}} p_r(\{a^{\text{ne}}\}_r) v_r}{\sum_{r \in \mathcal{R}} p_r(\{a^{\text{opt}}\}_r) v_r}$$

Now scale each resource in g by the same constant $c \in \mathbb{R} \setminus \{0\}$

$$\frac{\sum_{r \in \mathcal{R}} p_r(\{a^{\text{ne}}\}_r) c v_r}{\sum_{r \in \mathcal{R}} p_r(\{a^{\text{opt}}\}_r) c v_r} = \frac{c \sum_{r \in \mathcal{R}} p_r(\{a^{\text{ne}}\}_r) v_r}{c \sum_{r \in \mathcal{R}} p_r(\{a^{\text{opt}}\}_r) v_r} = \frac{\sum_{r \in \mathcal{R}} p_r(\{a^{\text{ne}}\}_r) v_r}{\sum_{r \in \mathcal{R}} p_r(\{a^{\text{opt}}\}_r) v_r} = \frac{W(a^{\text{ne}})}{W(a^{\text{opt}})} = \text{PoA}(g)$$

□

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