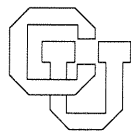Decomposing Symmetric Exchanges
In Matroid Bases

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CU-CS-053-74 September 1974

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Abstract

Let B, B' be bases of a matroid, with $X \subset B$, $X' \subset B'$. Sets X, X' are a <u>symmetric exchange</u> if $(B-X) \cup X'$ and $(B'-X') \cup X$ are bases. Sets X, X' are a <u>strong serial B-exchange</u> if there is a bijection $f: X \to X'$, where for any ordering of the elements of X, say $x_i, i=1, \ldots, m$, bases are formed by the sets $B_0 = B$, $B_i = (B_{i-1} - x_i) \cup f(x_i)$, for $i=1, \ldots, m$. Any symmetric exchange X, X' can be decomposed by partitioning

$$X = \bigcup_{i=1}^{m} Y_i, \quad X' = \bigcup_{i=1}^{m} Y_i',$$

where (1) bases are formed by the sets $B_0 = B$, $B_i = (B_{i-1} - Y_i) \cup Y_i'$; (2) sets Y_i, Y_i' are a strong serial B_{i-1}-exchange; (3) properties analogous to (1) and (2) hold for base B' and sets Y_i', Y_i.

1. Introduction

Exchanges in matroid bases are interesting mathematically, since matroids (combinatorial pregeometries [3]) are so common, and algorithmically, since exchanges occur in many graph and network flow algorithms [4]. This note presents a theorem on the structure of symmetric exchanges. Recently, several derivations of the "symmetric subset exchange axiom" have been given [2,5,7]. Although two derivations are constructive [2,5], they do not offer immediate insight into the structure of symmetric exchanges. Our theorem shows how a symmetric exchange can be decomposed into smaller symmetric exchanges that can be executed "element-by-element".

2. Definitions and Related Results

This section describes several types of exchanges for bases of a matroid. In what follows, let B and B' be bases of a given matroid. For example, Figures 1-2 show several bases of the graphic matroid on four nodes. These bases consist of three solid arcs or three dotted arcs.

Let X and X' be sets of elements of M. The ordered pair X,X' is a <u>B-exchange</u> if $X \subseteq B$ and $(B-X) \cup X'$ is a base. When X and X' both consist of single elements, say x and x', we say x,x' is a B-exchange. For example in Figure 1, let B be the first base shown, B = {1,2,6}. The first and last bases illustrate the B-exchange {1,2}, {4,5}; two other B-exchanges, 2,4 and 1,5, are also illustrated. Note any matroid satisfies the "basis exchange axiom":

Let B and B' be bases of M. For any element $x \varepsilon B$, there is an element $x' \varepsilon B'$ such that x,x' is a B-exchange.

The ordered pair of sets X,X' is a <u>symmetric exchange</u> for B,B' if X,X' is a B-exchange and X',X is a B'-exchange, or equivalently, if $X \subseteq B$, $X' \subseteq B'$, and both $(B-X) \cup X'$ and $(B'-X') \cup X$ are bases. In Figure 1 for B= {1,2,6} and B'={4,5,6}, the pair 2,4 is a symmetric exchange. Note any matroid satisfies the "symmetric subset exchange axiom" [2,5,7]:

Let B and B' be bases of M. For any set $X \subseteq B$, there is a set $X' \subseteq B'$ such that X,X' is a symmetric exchange.

The sequence of ordered pairs of sets, X_i, X_i', i=1,...,m, is a <u>serial B-exchange</u> if, for i=1,...,m, bases are formed by the sets $B_0 = B$, $B_i = (B_{i-1} - X_i) \cup X_i'$; and further, $X_i \subseteq B_{i-1}$. Thus the exchanges X_i, X_i' can be made one after another. Figure 1 shows a serial B-exchange, 2,4;1,5.

The ordered pair of sets X,X' is a <u>strong serial B-exchange</u> if there is a bijection $f: X \to X'$, where for any ordering of the elements of X, say $x_i, i=1,\ldots,m$, the sequence of pairs $x_i, f(x_i), i=1,\ldots,m$, is a serial B-exchange. Thus the exchanges $x,f(x)$ can be executed in any order. Figure 1 shows a strong serial B-exchange, $\{1,2\}$, $\{5,4\}$ (where $f(1)=5$, $f(2)=4$).

3. The Decomposition Theorem

This section derives the following decomposition theorem.

<u>Theorem</u>: Let X,X' be a symmetric exchange for bases B,B' of a matroid. Sets X,X' can be partitioned, $X = \bigcup\limits_{i=1}^{m} Y_i$, $X' = \bigcup\limits_{i=1}^{m} Y_i'$, where

(1) the sequence Y_i, Y_i', $i=1,\ldots,m$, is a serial B-exchange;

(2) the sequence $Y_i', Y_i, i=1,\ldots,m$, is a serial B'-exchange;

(3) the exchanges made in (1)-(2), Y_i, Y_i' and Y_i', Y_i, are strong serial exchanges.

Figure 2 illustrates the theorem, for $B=X=\{1,2,3\}$ and $B'=X'=\{4,5,6\}$. The symmetric exchange X,X' is decomposed into the strong serial exchange 3,6, followed by the strong serial exchange $\{1,2\}$, $\{4,5\}$ (see Figure 1).

To derive the theorem, we first refine the basis exchange axiom.

<u>Lemma 1</u>: Let X,X' be a B-exchange. For any element $x \in X$, there is an element $x' \in X'$ such that x,x' is a B-exchange.

<u>Proof</u>: By hypothesis, B and $(B-X) \cup X'$ are bases. So for any element $x \in X$, there is an element $x' \in (B-X) \cup X'$ such that x,x' is a B-exchange. Also, $x' \notin B-X$, since $B-X \subset B-x$. Hence $x' \in X'$.

Next we investigate when exchanges can be made sequentially.

Lemma 2: Let x, x', y be elements, where x, x' is a B-exchange, and y, x' is not a B-exchange. For any element y', the sequence $x, x'; y, y'$ is a serial B-exchange if and only if y, y' is a B-exchange.

Proof: First we prove the "only if" implication. Suppose $x, x'; y, y'$ is a serial B-exchange. So $\{x, y\}, \{x', y'\}$ is a B-exchange. By Lemma 1, there is an element $z \in \{x', y'\}$ such that y, z is a B-exchange. By hypothesis, $z \neq x'$. Thus $z = y'$, and y, y' is a B-exchange.

The "if" implication follows similarly, by considering the exchange $\{x', y\}, \{x, y'\}$ for base $(B-x) \cup x'$. ∎

The next result is the basis of the theorem.

Lemma 3: Let X, X' be a symmetric exchange for bases B, B'. There are sets Y, Y', where $Y \subset X$, $Y' \subset X'$, such that Y, Y' is a strong serial B-exchange, and Y', Y is a strong serial B'-exchange.

Proof: Choose any element $y_1 \in X$. Apply Lemma 1 to get an element $y_1' \in X'$ such that y_1, y_1' is a B-exchange. Then apply Lemma 1 to get an element $y_2 \in X$ such that y_1', y_2 is a B'-exchange. Continue in this manner, until an element y_i or y_i' repeats. This construction shows there is a sequence of distinct elements, $y_i, y_i', i=1, \ldots, m$, such that $y_i \in X; y_i' \in X'; y_i, y_i'$ is a B-exchange; and y_i', y_{i+1} is a B'-exchange (We use arithmetic module m, so $y_{m+1} = y_1$).

Choose such a sequence $y_i, y_i', i=1, \ldots, m$, with m as small as possible. Set $Y = \{y_1, \ldots, y_m\}$ and $Y' = \{y_1', \ldots, y_m'\}$. Now we show the bijection y_i, y_i' makes Y, Y' a strong serial B-exchange.

First note if y_i, y_j' is a B-exchange, then $i=j$. For if $i \neq j$, the

sequence $y_{j+1}, y'_{j+1}, \ldots, y_{i-1}, y'_{i-1}, y_i, y'_j$ contradicts the minimality of m.

Now suppose $w, w'; x, x'; z, z';$ are pairs of corresponding elements. We show they give a serial B-exchange, i.e., bases are formed by the sets $B_1 = (B-w) \cup w'$, $B_2 = (B_1 - x) \cup x'$, $B_3 = (B_2 - z) \cup z'$. The B-exchanges involving elements $w, x,$ and z with elements of Y' are $w, w'; x, x'; z, z';$ and no others, by the above remark. Thus B_1 is a base. Furthermore, the B_1-exchanges involving elements x and z with elements of Y' are $x, x'; z, z';$ and no others, by Lemma 2. So B_2 is a base. Similarly, we see B_3 is a base.

This argument indicates how an induction can be made to show Y, Y' is a strong serial B-exchange. We show Y', Y is a strong serial B'-exchange in a similar manner, using the bijection y'_i, y_{i+1}. ∎

Now we derive the decomposition theorem.

Proof of theorem: Let X, X' be a symmetric exchange for bases B, B'. Apply Lemma 3, getting sets $Y_1 \subset X, Y'_1 \subset X'$. If Y_1, Y'_1 do not exhaust X, X', then $X - Y_1, X' - Y'_1$ is a symmetric exchange for bases $(B - Y_1) \cup Y'_1$, $(B' - Y'_1) \cup Y_1$. Apply Lemma 3 again. Continuing this way, we derive the desired decomposition of X, X'. ∎

4. Discussion

The decompositon of the theorem is not unique. However, it does give information about the structure of symmetric exchanges. Consider an exchange X,X' that cannot be further decomposed, i.e., $m=1$ in the theorem. Such exchanges are the "building blocks" of symmetric exchanges. If $|X|=1$, then X,X' is a symmetric exchange of two elements. The case $|X|=2$ is illustrated by the second exchange in Figure 2, $\{1,2\}$, $\{4,5\}$. Similar exchanges for arbitrary cardinalities $|X|=n$ can be constructed. For example, in the graphic matroid on nodes $1,2,\ldots,2n$, take

$$X = \{(2k,2k-1) \mid 1 \le k \le n\},$$
$$X' = \{(2k,2k+1) \mid 1 \le k \le n\},$$
$$B = \{(i,2n-i) \mid 1 \le i < n\} \cup X,$$
$$B' = \{(i,2n+2-i) \mid 1 < i \le n\} \cup X'.$$

A special case of the theorem occurs when the symmetric exchange X,X' is actually B,B'. In this case, it is interesting to ask if the index m can be made to take on the extreme values, $m=1$ and $m=r$ (where r is the rank of the matroid). Clearly $m=1$ cannot always be achieved. (This is illustrated by the bases in the first graph of Figure 2.) For $m=r$, each exchange Y_i,Y_i' is a symmetric exchange of two elements, y_i,y_i'. This is achieved for the bases in Figure 2 by the exchanges $1,6$; $2,4$; $3,5$. The bound $m=r$ can be achieved in other special cases, such as partition matroids, matching and transversal matroids, and matroids of rank less than 4. In general, however, it is not known if this bound can be achieved.

Acknowledgment

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Figure 1. A strong serial B-exchange
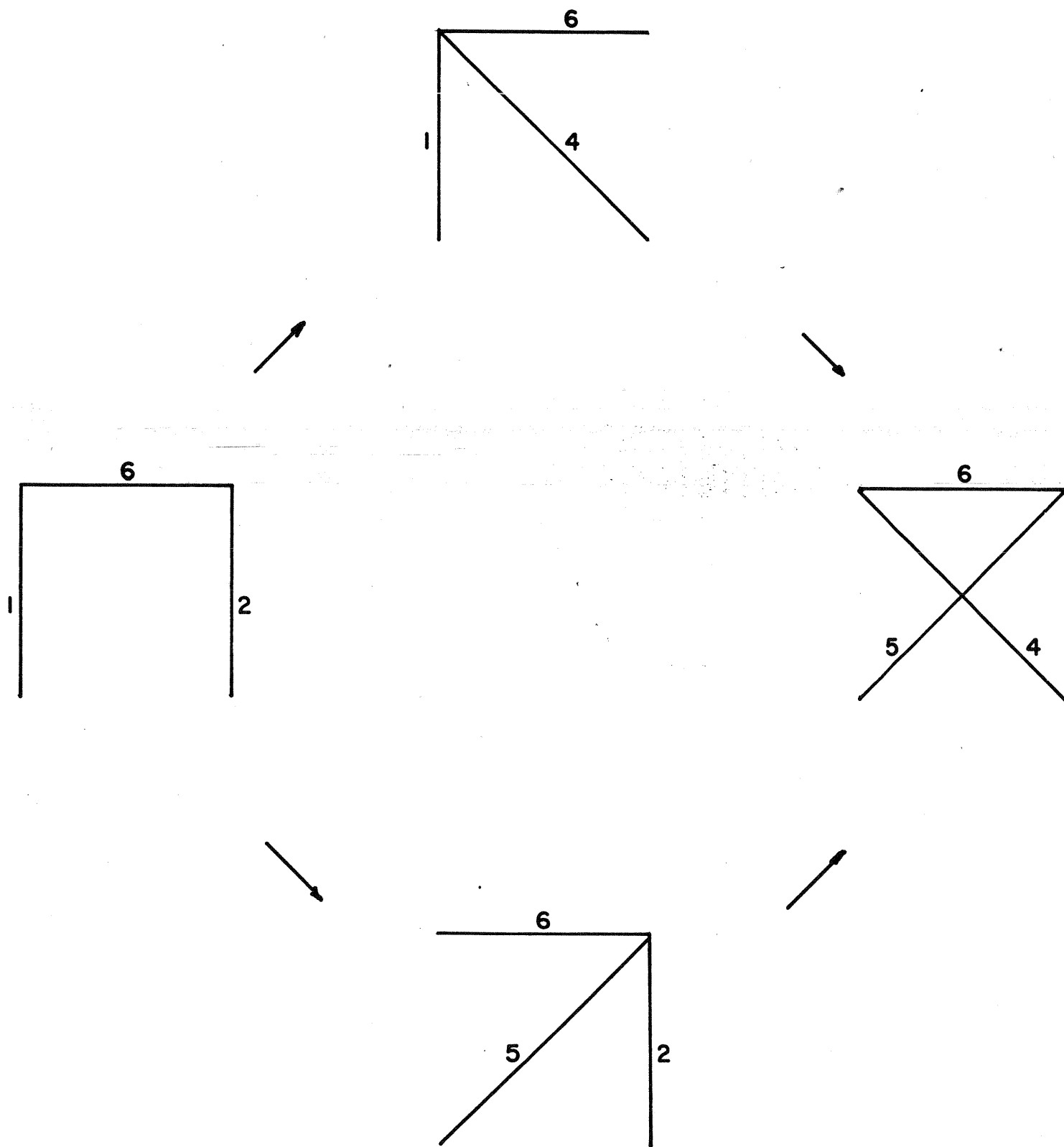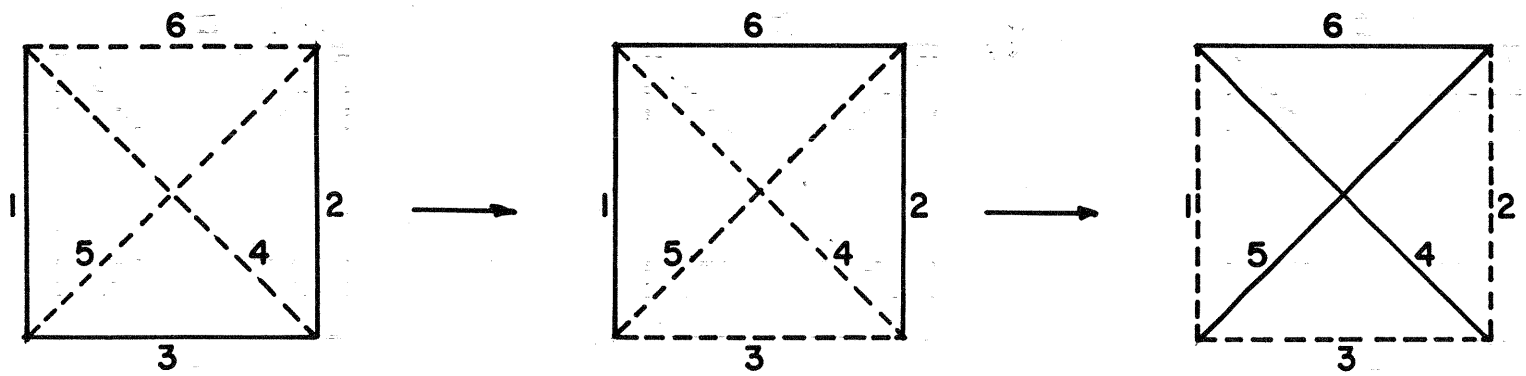
Figure 2. Decomposition of a symmetric exchange

Fig. I

Fig. 2