

Properties of the  $\mathfrak{g}$ -Invariant Bilinear Form on  
the Spin Representations of the Simple Lie  
Algebras of Type  $D_n$  and  $B_n$

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### Abstract

The spin modules for the Lie algebras of type  $D_n$  and  $B_n$  are constructed through minuscule systems. It is shown that the unique (up to multiplication by a nonzero scalar)  $\mathfrak{g}$ -invariant bilinear form on the modules is nondegenerate. By defining an equation for the height of weights of the modules, it is shown that the bilinear form is symplectic when  $n = 1$  or  $2 \pmod{4}$ , and orthogonal when  $n = 0$  or  $3 \pmod{4}$ .

# 1 Introduction

Lie algebras were originally introduced by S. Lie for the study of Lie groups. The focus of this paper is on the finite dimensional simple Lie algebras over the complex field, which were classified between 1890–1900 with the work of E. Cartan and W. Killing. We construct the spin modules on the Lie algebras of type  $D_n$  and  $B_n$ , and our main result defines an equation for the height of weights on these modules. We then use these equations to prove that the  $\mathfrak{g}$ -invariant bilinear form on the modules alternates between orthogonal and symplectic depending on the choice of  $n$ . For the simple Lie algebra of type  $B_n$ , the bilinear form on the spin module is orthogonal when  $n = 0$  or  $3 \pmod{4}$  and symplectic when  $n = 1$  or  $2 \pmod{4}$ . For the simple Lie algebra of type  $D_n$ , there are two spin modules, and the bilinear form on both is orthogonal when  $n = 0 \pmod{4}$  and symplectic when  $n = 2 \pmod{4}$ . Both the equations and the proofs are original to the best knowledge of the author.

The strategy of the paper is as follows. In Section 1 we introduce general results in Lie algebra. We define the simple Lie algebras  $B_n$  and  $D_n$  through matrices under matrix multiplication, as subalgebras of the general linear algebra  $\mathfrak{gl}(n, \mathbb{C})$ , and derive their root systems and bases via the root space decomposition. We then show that these algebras can be constructed using only generators, relations, and their associated simple roots.

In Section 2, we begin by introducing general results of modules. We then introduce minuscule systems and show that they have the structure of weights of irreducible modules. We use these to construct the spin modules for the Lie algebras  $D_n$  and  $B_n$ , based on the method introduced in [1].

In Section 3, we define a  $\mathfrak{g}$ -invariant bilinear form on the spin modules, and show that it is unique up to multiplication by a nonzero scalar. Using the construction of the spin modules from Section 2, we then provide an equation for the height of weights of these modules. Finally, we use these equations to prove that the bilinear form alternates from orthogonal to symplectic as discussed above.

**Definition 1.1.** A *Lie algebra* is a  $k$ -vector space  $\mathfrak{g}$  equipped with a  $k$ -bilinear map called the *Lie bracket*,

$$\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \quad (x, y) \mapsto [xy]$$

satisfying the conditions:

$$\begin{aligned} [xx] &= 0 \text{ for all } x \in \mathfrak{g}, \\ [x[yz]] + [z[xy]] + [y[zx]] &= 0 \text{ for all } x, y, z \in \mathfrak{g}. \end{aligned}$$

These are known as *antisymmetry* and the *Jacobi identity* respectively.

A Lie algebra  $\mathfrak{g}$  is said to be *abelian* if  $[xy] = 0$  for all  $x, y \in \mathfrak{g}$ .

**Definition 1.2.** Let  $A$  be an associative  $k$ -algebra and let  $x, y \in A$ . We define a new  $k$ -bilinear operation  $[\ , \ ]$  such that  $[x, y] := xy - yx$  under the usual operations on the right.

**Lemma 1.3.** Let  $M_n(k)$  denote the associative  $k$ -algebra of all  $n \times n$  matrices under matrix multiplication, and let  $\mathfrak{gl}(V)$  denote the algebra of all  $k$ -linear endomorphisms of a  $k$ -vector space  $V$  under composition of functions.

(i) The algebra  $M_n(k)$  endowed with the operation  $[\ , \ ]$  from Definition 1.2 is a Lie algebra over  $k$ , denoted  $\mathfrak{gl}(n, k)$ .

(ii) The algebra  $\mathfrak{gl}(V)$  endowed with  $[\ , \ ]$  is a Lie algebra over  $k$ .

*Proof.* (i) Let  $x, y, z \in M_n(k)$ . Then  $[x, y] = xy - yx \in M_n(k)$  so the set is closed under the operation. We have:  $[x, x] = xx - xx = 0$ , and

$$\begin{aligned} & [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\ &= x(yz - zy) - (yz - zy)x + y(zx - xz) - (zx - xz)y \\ &\quad + z(xy - yx) - (xy - yx)z \\ &= xyz - xzy - yzx + zyx + yzx - yxz - zxy - xzy \\ &\quad + zxy - zyx - xyz + yxz \\ &= (xyz - xzy) + (xyx - xzy) + (yxz - yxz) + (yzx - yzx) \\ &\quad + (zxy - zxy) + (zyx - zyx) \\ &= 0 \end{aligned}$$

satisfying both conditions of Definition 1.1.

The proof for (ii) is identical when considering multiplication as composition of functions.  $\square$

**Definition 1.4.** A subspace  $\mathfrak{h}$  of a Lie algebra  $\mathfrak{g}$  is called a *subalgebra* of  $\mathfrak{g}$  if  $[xy] \in \mathfrak{h}$  whenever  $x, y \in \mathfrak{h}$ . We call  $\mathfrak{h}$  an *ideal* if  $[xy] \in \mathfrak{h}$  for all  $x \in \mathfrak{h}$ ,  $y \in \mathfrak{g}$ .

**Lemma 1.5.** Let  $h_1, \dots, h_k : V \rightarrow V$  be diagonalizable linear transformations such that  $h_1, \dots, h_k$  commute. Then the maps  $h_1, \dots, h_k$  can be simultaneously diagonalized.

*Proof.* This is a general result in linear algebra, and is proved in [3, Lemma 16.7].  $\square$

**Definition 1.6.** Let  $\mathfrak{g}$  be a Lie algebra. We say that an element  $h \in \mathfrak{g}$  is *semisimple* if it can be represented by a diagonal matrix.

**Definition 1.7.** Let  $\mathfrak{g}$  and  $\mathfrak{h}$  be Lie algebras over  $k$ . A Lie algebra *homomorphism* is a  $k$ -linear map

$$\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$$

such that  $\varphi([xy]) = [\varphi(x)\varphi(y)]$  for all  $x, y \in \mathfrak{g}$ . A Lie algebra *isomorphism* is a bijective homomorphism.

**Definition 1.8.** Let  $\mathfrak{g}$  be a Lie algebra. We define the *adjoint homomorphism* as the linear map

$$ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}) \quad ad(x)(y) := [x, y]$$

**Lemma 1.9.**

- (i) Suppose  $h$  is a semisimple element of  $\mathfrak{g}$ . Then  $\text{adh}$  is semisimple.  
(ii) Suppose the elements  $h_1, \dots, h_n$  form an abelian subalgebra of a Lie algebra  $\mathfrak{g}$ . Then the maps  $\text{adh}_1, \dots, \text{adh}_n$  form an abelian subalgebra of  $\mathfrak{gl}(\mathfrak{g})$ .

*Proof.* The proof for this is contained in [4, §6.1].  $\square$

**Proposition 1.10.** Let  $s \in \mathfrak{gl}(n, k)$ . The set  $\mathfrak{h} = \{x \mid sx = -x^t s, x \in \mathfrak{gl}(n, k)\}$  endowed with  $[\ , \ ]$  is a subalgebra of  $\mathfrak{g}$ ;  $x^t$  denotes the transpose of  $x$ .

*Proof.* The set  $\mathfrak{h}$  is a subspace of  $\mathfrak{gl}(n, k)$ . Satisfaction of the two axioms of Definition 1.1 is inherited from  $\mathfrak{gl}(n, k)$ . Therefore, it is sufficient to show that  $\mathfrak{h}$  is closed under  $[\ , \ ]$ .

Let  $x, y \in \mathfrak{h}$ . Then

$$\begin{aligned} s[x, y] &= s(xy - yx) \\ &= sxy - syx \\ &= -x^t sy + y^t sx \\ &= x^t y^t s - y^t x^t s \\ &= (x^t y^t - y^t x^t)s \\ &= ((yx)^t - (xy)^t)s \\ &= (yx - xy)^t s \\ &= -[x, y]^t s. \end{aligned}$$

$\square$

**Definition 1.11.** Let  $I_l$  be the  $l \times l$  identity matrix. Let  $s_d$  and  $s_b$  be matrices of dimension  $2l \times 2l$  and  $2l + 1 \times 2l + 1$  respectively, such that

$$s_d = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix} \text{ and } s_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}.$$

- (i) The subalgebra  $\mathfrak{so}(2l, \mathbb{C})$  of  $\mathfrak{gl}(2l + 1, \mathbb{C})$  is defined as the following:

$$\mathfrak{so}(2l, \mathbb{C}) = \{x \in \mathfrak{gl}(2l, k) \mid x^t s_d = -s_d x\},$$

and is known as the *orthogonal Lie algebra* of type  $D_n$ .

- (ii) The subalgebra  $\mathfrak{so}(2l + 1, \mathbb{C})$  of  $\mathfrak{gl}(2l + 1, \mathbb{C})$  is defined as the following:

$$\mathfrak{so}(2l + 1, \mathbb{C}) = \{x \in \mathfrak{gl}(2l + 1, k) \mid x^t s_b = -s_b x\},$$

and is known as the *orthogonal Lie algebra* of type  $B_n$ .

**Lemma 1.12.** *We have:*

$$\mathfrak{so}(2l, \mathbb{C}) = \left\{ \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} \mid p = -p^t \text{ and } q = -q^t \right\}, \quad (1)$$

$$\mathfrak{so}(2l+1, \mathbb{C}) = \left\{ \begin{pmatrix} 0 & -b^t & -c^t \\ b & m & p \\ c & q & -m^t \end{pmatrix} \mid p = -p^t \text{ and } q = -q^t \right\}. \quad (2)$$

*Proof.* For (1),  $s_d x = -x^t s_d$  implies

$$\begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix} \begin{pmatrix} m & p \\ q & n \end{pmatrix} = \begin{pmatrix} -m^t & -q^t \\ -p^t & -n^t \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\begin{pmatrix} q & n \\ m & p \end{pmatrix} = \begin{pmatrix} -q^t & -m^t \\ -n^t & -p^t \end{pmatrix},$$

showing that  $q = -q^t, p = -p^t$ , and  $n = -m^t$  as required.

For (2),  $s_b x = -x^t s_b$  implies

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} a & d & e \\ b & m & p \\ c & q & n \end{pmatrix} = \begin{pmatrix} -a^t & -b^t & -c^t \\ -d^t & -m^t & -q^t \\ -e^t & -p^t & -n^t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & d & e \\ c & q & n \\ b & m & p \end{pmatrix} = \begin{pmatrix} -a^t & -c^t & -b^t \\ -d^t & -q^t & -m^t \\ -e^t & -n^t & -p^t \end{pmatrix},$$

showing as required that  $c = -d^t, b = -e^t, q = -q^t, p = -p^t, m = -n^t$ , and  $a = -a^t = 0$  as it is a  $1 \times 1$  matrix.  $\square$

**Definition 1.13.** A Lie algebra is said to be *simple* if it has no ideals other than 0 and itself, and it is not abelian. A Lie algebra is *semisimple* if it can be written as a direct sum of simple Lie algebras.

It is well known that the orthogonal algebras are simple. Formal proofs of this may be found in [4, §11] or [3, §10].

**Definition 1.14.** Let  $\mathfrak{g}$  be a Lie algebra. A Lie subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  is said to be a *Cartan subalgebra* if  $\mathfrak{h}$  is abelian, every element  $h \in \mathfrak{h}$  is semisimple, and moreover  $\mathfrak{h}$  is maximal with these properties.

**Proposition 1.15.** *Let  $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ . The subspace of  $\mathfrak{g}$  consisting of all diagonal matrices in  $\mathfrak{g}$  is a Cartan subalgebra of  $\mathfrak{g}$ .*

*Proof.* By Definition 1.14 we must show (i) that every element of  $\mathfrak{h}$  is semisimple, (ii) that  $\mathfrak{h}$  is abelian, (iii) that it is a subalgebra of  $\mathfrak{g}$ , and (iv) that  $\mathfrak{h}$  is maximal with these properties. Since  $\mathfrak{h}$  consists entirely of diagonal matrices, every element is obviously semisimple by Definition 1.6, satisfying (i).

Since multiplication of diagonal matrices is commutative, for  $h_1, h_2 \in \mathfrak{h}$  we have  $[h_1, h_2] = h_1 h_2 - h_2 h_1 = 0$ , showing that it is abelian and satisfying (ii). Furthermore, this satisfies (iii) by Definition 1.4, since  $0 \in \mathfrak{h}$ .

Now for (iv), suppose that  $\mathfrak{h}$  is not maximal and is contained in a maximal Cartan subalgebra  $H$ . Let  $a \in H$  and  $h \in \mathfrak{h}$ . Since  $H$  is abelian, we have  $[h, a] = ha - ah = 0$  which implies  $ha = ah$ . Suppose  $a$  is a matrix with entries  $(a_{i,j})$ , and  $h$  has diagonal entries  $(h_0, \dots, h_{n-1})$ . Then  $ha = (h_j a_{i,j})$  and  $ah = (h_i a_{i,j})$ . Since this is true for all  $h \in \mathfrak{h}$ , this implies that  $i = j$  and  $a$  is a diagonal matrix. Therefore,  $H = \mathfrak{h}$ , completing the proof.  $\square$

Let  $\mathfrak{g}$  be a Lie algebra and let  $\mathfrak{h}$  be the Cartan subalgebra of  $\mathfrak{g}$ . We denote the dual space of  $\mathfrak{h}$  by  $\mathfrak{h}^*$ . Since every element  $h \in \mathfrak{h}$  is diagonalizable and  $\mathfrak{h}$  is abelian, by Lemma 1.9 the maps  $\text{ad}\mathfrak{h} = \{\text{ad}h \mid h \in \mathfrak{h}\}$  are both abelian and diagonalizable. Therefore, as a consequence of Lemma 1.5, the map

$$\text{ad} : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{g})$$

induces the following decomposition on  $\mathfrak{g}$ :

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where  $\Phi$  is the set of  $\alpha \in \mathfrak{h}^*$  such that  $\alpha \neq 0$  and  $\mathfrak{g}_\alpha \neq 0$ ,

$$\mathfrak{g}_\alpha = \{v \in \mathfrak{g} \mid [h, v] = \alpha(h)v \text{ for all } h \in \mathfrak{h}\},$$

and

$$\mathfrak{g}_0 = \{z \in \mathfrak{g} \mid [h, z] = 0 \text{ for all } h \in \mathfrak{h}\}.$$

It is a well known result that  $\mathfrak{g}_0 = \mathfrak{h}$ , and this is proved in [4, Proposition 8.2]. We call this the *Cartan decomposition*. We say that an element  $\alpha \in \Phi$  is a *root* of  $\mathfrak{g}$  with associated *root space*  $\mathfrak{g}_\alpha$ .

There is a natural isomorphism between a finite dimensional euclidean dot product space  $V$  with its dual space  $V^*$ , given by

$$\varphi : V \rightarrow V^* \quad v \mapsto \langle v, - \rangle.$$

The kernel of the map is zero since the dot product is nondegenerate. Since  $\dim(V) = \dim(V^*)$ , it follows immediately from the first isomorphism theorem of linear algebra that the map is an isomorphism.

By imposing a dot product on the elements of  $\mathfrak{h}$ , we obtain the following isomorphism:

$$\varphi : \mathfrak{h} \rightarrow \mathfrak{h}^* \quad 2 \frac{(e_i, e_j)}{(e_j, e_j)} \equiv \langle \varepsilon_i, \varepsilon_j \rangle,$$

where  $e_i, e_j$  and  $\varepsilon_i, \varepsilon_j$  are the standard basis elements of  $\mathfrak{h}$  and the standard basis elements of  $\mathfrak{h}^*$  with respect to  $\mathfrak{h}$ , respectively. We omit the details of

this derivation, but the reader is referred to [3, §10.5] or [4, §8.5] for further discussion. It is a major result that the Cartan decomposition produces roots that form a root system as defined below in Definition 1.17, and we refer the reader to [4, §8] for further discussion.

**Definition 1.16.** Let  $E$  be a finite-dimensional euclidean space with the regular bilinear dot product written  $(-, -)$ . Let  $\lambda, \alpha \in E$ . We define

$$\langle \lambda, \alpha \rangle := \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha,$$

where  $\langle \lambda, \alpha \rangle$  is only linear in the first variable. We let  $\sigma_\alpha$  denote the reflection in the hyperplane normal to  $\alpha$ . That is, for some  $\lambda \in E$ ,

$$\sigma_\alpha(\lambda) := \lambda - \langle \lambda, \alpha \rangle \alpha.$$

**Definition 1.17.** A subset  $\Phi$  of a euclidean dot product space  $E$  is called a *root system* in  $E$  if the following axioms are satisfied:

- (R1)  $\Phi$  is finite, spans  $E$ , and does not contain 0;
- (R2) if  $\alpha \in \Phi$ , the only multiples of  $\alpha$  in  $\Phi$  are  $\pm\alpha$ ;
- (R3) if  $\alpha \in \Phi$ ,  $\sigma_\alpha$  is invariant under  $\Phi$ ;
- (R4) if  $\alpha, \beta \in \Phi$ , then  $\langle \beta, \alpha \rangle \in \mathbb{Z}$ .

**Definition 1.18.** A subset  $\Delta$  of  $\Phi$  is called a *base* if:

- (B1)  $\Delta$  is a basis of  $E$ ;
- (B2) each root  $\beta \in \Phi$  can be written as  $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$  with  $k_\alpha \in \mathbb{Z}$ , where the coefficients  $k_\alpha$  are all nonnegative or all nonpositive.

The elements of  $\Delta$  are called *simple roots*.

**Definition 1.19.** We say that a root  $\alpha \in \Phi$  is *positive with respect to  $\Delta$* , if the coefficients given in (B2) of Definition 1.18 are positive, and otherwise it is *negative with respect to  $\Delta$* .

Though it is not obvious, every root system has a base. However, a base is not unique. This is not proved and the reader is referred to [3, Theorem 11.10].

**Proposition 1.20.** Let  $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$ . A root system of  $\mathfrak{g}$  is given by the following set:

$$\Phi = b \cup c \cup m \cup n \cup p \cup q,$$

where

$$\begin{aligned} b &= \{\varepsilon_i \in \mathfrak{h}^* \mid 0 \leq i \leq l-1\} \\ c &= \{-\varepsilon_i \in \mathfrak{h}^* \mid 0 \leq i \leq l-1\} \\ m &= \{(\varepsilon_i - \varepsilon_j) \in \mathfrak{h}^* \mid 0 \leq i < j \leq l-1\} \\ n &= \{(\varepsilon_i - \varepsilon_j) \in \mathfrak{h}^* \mid 0 \leq j < i \leq l-1\} \\ p &= \{(\varepsilon_i + \varepsilon_j) \in \mathfrak{h}^* \mid 0 \leq i < j \leq l-1\} \\ q &= \{-(\varepsilon_i - \varepsilon_j) \in \mathfrak{h}^* \mid 0 \leq i < j \leq l-1\}. \end{aligned}$$



*Proof.* This is derived in detail in [3, §12.3].  $\square$

**Proposition 1.21.** *Let  $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$ . The root system of  $\mathfrak{g}$  is given by the following set:*

$$\Phi = m \cup n \cup p \cup q,$$

where

$$\begin{aligned} m &= \{(\varepsilon_i - \varepsilon_j) \in \mathfrak{h}^* \mid 0 \leq i < j \leq l-1\} \\ n &= \{(\varepsilon_i - \varepsilon_j) \in \mathfrak{h}^* \mid 0 \leq j < i \leq l-1\} \\ p &= \{(\varepsilon_i + \varepsilon_j) \in \mathfrak{h}^* \mid 0 \leq i < j \leq l-1\} \\ q &= \{-(\varepsilon_i - \varepsilon_j) \in \mathfrak{h}^* \mid 0 \leq i < j \leq l-1\}. \end{aligned}$$

*Proof.* This is derived in detail in [3, §12.4].  $\square$

**Proposition 1.22.** *Let  $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$ , and let  $\Phi$  be the corresponding root system as in Proposition 1.20. A base for  $\Phi$  is given by the following set:*

$$\Delta = \{\alpha_i \in \mathfrak{h}^* \mid 0 \leq i \leq l-1\},$$

where

$$\begin{aligned} \alpha_i &= \varepsilon_i - \varepsilon_{i+1} \quad \text{for } 0 \leq i \leq l-2, \text{ and} \\ \alpha_{l-1} &= \varepsilon_{l-1}. \end{aligned}$$

The positive roots with respect to  $\Phi$  are the elements of the sets  $b, p, m \subseteq \Phi$  of Proposition 1.20.

*Proof.* We must show that axiom (B1) of Definition 1.18 is satisfied, and that the elements of  $b, p$ , and  $m$  can be written as nonnegative integer linear combinations of  $\Delta$ . We first observe that the elements of  $\Delta$  are linearly independent and that  $\dim(\Delta) = l = \dim(\mathfrak{h}^*)$ . This satisfies (B1). Now, the elements of  $b, m$  and  $p$  can be written as follows, respectively:

$$\begin{aligned} \varepsilon_i &= (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \cdots + (\varepsilon_{l-2} - \varepsilon_{l-1}) + \varepsilon_{l-1} \\ &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-1} + \alpha_l \\ \varepsilon_i - \varepsilon_j &= (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \cdots + (\varepsilon_{j-2} - \varepsilon_{j-1}) + (\varepsilon_{j-1} - \varepsilon_j) \\ &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \\ \varepsilon_i + \varepsilon_j &= (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-2} + \alpha_{l-1}) + (\alpha_j + \alpha_{j+1} + \cdots + \alpha_{l-2} + \alpha_{l-1}) \end{aligned}$$

This shows that the elements of the sets  $b, p, m$  are positive roots by Definition 1.19. As the elements of the sets  $c, n, q$  are all negatives of those of  $b, p, m$ , this satisfies condition (B2), completing the proof.  $\square$

**Proposition 1.23.** Let  $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$ , and let  $\Phi$  be the corresponding root system as in Proposition 1.21. A base for  $\Phi$  is given by the following set:

$$\Delta = \{\alpha_i \in \mathfrak{h}^* \mid 0 \leq i \leq l-1\},$$

where

$$\begin{aligned} \alpha_i &= \varepsilon_i - \varepsilon_{i+1} \text{ for } 0 \leq i \leq l-2, \text{ and} \\ \alpha_{l-1} &= \varepsilon_{l-2} + \varepsilon_{l-1}. \end{aligned}$$

The positive roots with respect to  $\Phi$  are the elements of the sets  $b, m \subseteq \Phi$  of Proposition 1.21.

*Proof.* We proceed as in Proposition 1.22. The elements of  $\Delta$  are linearly independent and  $\dim(\Delta) = l = \dim(\mathfrak{h}^*)$ , satisfying condition (B1) of Definition 1.18. The elements of the sets  $b$  and  $m$  can be expressed as follows, respectively:

$$\begin{aligned} \varepsilon_i - \varepsilon_j &= (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \cdots + (\varepsilon_{j-2} - \varepsilon_{j-1}) + (\varepsilon_{j-1} - \varepsilon_j) \\ &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + \alpha_{j-1} \\ \varepsilon_i + \varepsilon_j &= ((\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \cdots + (\varepsilon_{j-2} - \varepsilon_{j-1}) + (\varepsilon_{j-1} - \varepsilon_j)) \\ &\quad + ((\varepsilon_j - \varepsilon_{j+1}) + (\varepsilon_{j+1} - \varepsilon_{j+2}) + \cdots + (\varepsilon_{l-2} - \varepsilon_{l-1}) + (\varepsilon_{l-1} - \varepsilon_l)) \\ &= (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-2} + \alpha_{j-1}) + (\alpha_j + \alpha_{j+1} + \cdots + \alpha_{l-2} + \alpha_{l-1}), \end{aligned}$$

which satisfies Definition 1.19. As the elements of the sets  $n$  and  $p$  are negatives of the elements of  $b$  and  $m$ , this satisfies condition (B2) of Definition 1.18, and completes the proof.  $\square$

**Definition 1.24.**

(i) Let  $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$ .

We denote  $\Phi_B$  to be the root system of  $\mathfrak{g}$  as in Proposition 1.20.

We denote  $\Delta_B$  to be the base of  $\Phi_B$ , with the fixed order of elements as in Proposition 1.22.

(ii) Let  $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$ .

We denote  $\Phi_D$  to be the root system of  $\mathfrak{g}$  as in Proposition 1.21.

We denote  $\Delta_D$  to be the base of  $\Phi_D$ , with the fixed order of elements as in Proposition 1.23.

**Definition 1.25.** Let  $\Delta$  be a base for a root system  $\Phi$ . Fix an order on the elements of  $\Delta$ , say  $(\alpha_0, \dots, \alpha_{l-1})$ . The *Cartan matrix* of  $\Delta$  is defined as the  $l \times l$  matrix with  $(i, j)$ th entry  $\langle \alpha_i, \alpha_j \rangle$ .

**Lemma 1.26.** (i) The Cartan matrix of  $\Delta_B$  is

$$\begin{pmatrix} 2 & -1 & \cdots & 0 & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

(ii) The Cartan matrix of  $\Delta_D$  is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 2 & -1 & 0 & 0 \\ 0 & \ddots & -1 & 2 & -1 & -1 \\ \vdots & \ddots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}.$$

*Proof.* We must show that the  $(i, j)$ th entry of each matrix is equal to  $\langle \alpha_i, \alpha_j \rangle$  in its respective algebra, thereby satisfying Definition 1.25. Given our selected bases in Propositions 1.22 and 1.23 respectively, the majority of calculations are the same.

Since  $\langle \alpha_i, \alpha_i \rangle = \frac{2(\alpha_i, \alpha_i)}{(\alpha_i, \alpha_i)} = 2$ , the diagonal entries are justified.

With two exceptions, we have the following hold for both algebras:

$$\begin{aligned} \langle \alpha_i, \alpha_j \rangle &= 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \\ &= 2 \frac{(\varepsilon_i - \varepsilon_{i+1}, \varepsilon_j - \varepsilon_{j+1})}{(\varepsilon_j - \varepsilon_{j+1}, \varepsilon_j - \varepsilon_{j+1})} \\ &= -1 \text{ if } |i - j| = 1, \text{ and } 0 \text{ otherwise.} \end{aligned}$$

The first exception is for  $\alpha_{l-2}$  and  $\alpha_{l-3} \in \Delta_B$ , as

$$\begin{aligned} \langle \alpha_{l-2}, \alpha_{l-1} \rangle &= \langle \varepsilon_{l-2} - \varepsilon_{l-1}, \varepsilon_{l-1} \rangle \\ &= 2 \frac{(\varepsilon_{l-2} - \varepsilon_{l-1}, \varepsilon_{l-1})}{(\varepsilon_{l-1}, \varepsilon_{l-1})} \\ &= 2 \frac{-1}{1} = -2, \end{aligned}$$

and the second exception is for  $\alpha_{l-1}, \alpha_j \in \Delta_D$ , as

$$\begin{aligned} \langle \alpha_{l-1}, \alpha_{l-3} \rangle &= \langle \varepsilon_{l-2} + \varepsilon_{l-1}, \varepsilon_{l-3} - \varepsilon_{l-2} \rangle \\ &= 2 \frac{(\varepsilon_{l-2} + \varepsilon_{l-1}, \varepsilon_{l-3} - \varepsilon_{l-2})}{(\varepsilon_{l-3} - \varepsilon_{l-2}, \varepsilon_{l-3} - \varepsilon_{l-2})} \\ &= 2 \frac{-1}{2} = -1 = \langle \alpha_{l-3}, \alpha_{l-1} \rangle, \end{aligned}$$

showing that all the entries are as required by Definition 1.25.  $\square$

We have thus far considered Lie algebras, particularly the orthogonal Lie algebras, as vector spaces consisting of matrices. Though this is a convenient method to intuitively derive preliminary results, a more comprehensive and applicable way to describe them is through generators and relations, as is introduced in the following theorem.

**Theorem 1.27** (Serre's Theorem). *Fix a root system  $\Phi$ , with base*

$$\Delta = \{\alpha_1, \dots, \alpha_{l-1}\}.$$

*Let  $\mathfrak{g}$  be the Lie algebra generated by the  $3l$  elements  $\{e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i} \mid 0 \leq i \leq l-1\}$  subject to the following relations,*

- (S1)  $[h_{\alpha_i}, h_{\alpha_j}] = 0$  for all  $i, j$ ;
- (S2)  $[h_{\alpha_i}, e_{\alpha_j}] = \langle \alpha_j, \alpha_i \rangle e_{\alpha_j}$  and  $[h_{\alpha_i}, f_{\alpha_j}] = -\langle \alpha_j, \alpha_i \rangle f_{\alpha_j}$  for all  $i, j$ ;
- (S3)  $[e_{\alpha_i}, f_{\alpha_j}] = \delta_{ij} h_{\alpha_i}$ ;
- (S4)  $[e_{\alpha_i}, [e_{\alpha_i}, \dots [e_{\alpha_i}, e_{\alpha_j}] \dots]] = 0$ ;
- (S5)  $[f_{\alpha_i}, [f_{\alpha_i}, \dots [f_{\alpha_i}, f_{\alpha_j}] \dots]] = 0$ ;

where  $\delta$  is the Kronecker delta. Then  $\mathfrak{g}$  is a (finite dimensional) semisimple algebra with Cartan subalgebra  $\mathfrak{h}$  spanned by the  $h_{\alpha_i}$ , with corresponding root system  $\Phi$ .

*Proof.* This is proved in [4, §18.3]. □

**Definition 1.28.** Let  $\{e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i} \mid \alpha_i \in \Delta\}$  be the generating elements of the Lie algebra  $\mathfrak{g}$  as defined in Theorem 1.27. We call these elements *Serre generators*.

## 2 Modules

In this section we define and construct the spin modules for the simple Lie algebras  $B_n$  and  $D_n$ . We begin by defining modules and representations and introducing standard results associated with these structures. We then introduce the concept of a minuscule system, which is not conventional and is introduced in [1]. We show that a minuscule system imposes the structure of a module on a vector space. Next, we introduce classification theorems of finite dimensional irreducible modules. Combining these results, we are able to explicitly construct the required spin modules, and explicitly observe their interaction with the Serre generators, which allows further study of their structures in Section 3.

**Definition 2.1.** Let  $\mathfrak{g}$  be a Lie algebra over  $k$  and let  $V$  be a finite-dimensional  $k$ -vector space. A *representation* of  $\mathfrak{g}$  is a Lie algebra homomorphism

$$\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(V).$$

**Definition 2.2.** Let  $\mathfrak{g}$  be a Lie algebra over  $k$ . A  $\mathfrak{g}$ -module  $V$  is a finite-dimensional  $k$ -vector space  $V$  with a map

$$\mathfrak{g} \times V \rightarrow V \quad (x, v) \mapsto x \cdot v$$

satisfying:

$$\begin{aligned}(\lambda x + \mu y) \cdot v &= \lambda(x \cdot v) + \mu(y \cdot v), \\ x \cdot (\lambda v + \mu w) &= \lambda(x \cdot v) + \mu(x \cdot w), \\ [xy] \cdot v &= x \cdot (y \cdot v) - y \cdot (x \cdot v),\end{aligned}$$

for all  $x, y \in \mathfrak{g}$ ,  $v, w \in V$ , and  $\lambda, \mu \in k$ .

Given a representation,  $\varphi$ , we can make a corresponding  $\mathfrak{g}$ -module by defining  $x \cdot v := \varphi(x)(v)$  for  $x \in \mathfrak{g}, v \in V$ . Conversely, given a  $\mathfrak{g}$ -module  $V$  and a basis of  $V$ , we can use the same equation to define a representation of  $\mathfrak{g}$ .

**Definition 2.3.** Let  $V$  be a  $\mathfrak{g}$ -module. A *submodule* of  $V$  is a subspace  $W$  of  $V$  such that  $x \cdot w \in W$  for all  $x \in \mathfrak{g}, w \in W$ .

**Definition 2.4.** Let  $V$  be a  $\mathfrak{g}$ -module. We say that  $V$  is *simple* or *irreducible* if it is non-zero and it has no submodules other than 0 and  $V$ .

**Definition 2.5.** Let  $V$  and  $W$  be  $\mathfrak{g}$ -modules. A *module homomorphism* from  $V$  to  $W$  is a  $k$ -linear map

$$\theta : V \rightarrow W$$

such that  $\theta(x \cdot v) = x \cdot \theta(v)$  for all  $v \in V, x \in \mathfrak{g}$ . A *module isomorphism* is a bijective module homomorphism.

Let  $\mathfrak{g}$  be a simple Lie algebra with Cartan subalgebra  $\mathfrak{h}$ , and suppose that the  $k$ -vector space  $V$  is a finite dimensional  $\mathfrak{g}$ -module. Since every element of  $\mathfrak{h}$  is semisimple by Definition 1.14, by Lemma 1.5 the map

$$\mathfrak{h} : V \rightarrow V$$

induces a decomposition of  $V$  into simultaneous eigenspaces for  $\mathfrak{h}$  as follows:

$$V = \bigoplus_{\lambda \in \Psi} V_{\lambda},$$

where  $\Psi$  is the set of  $\lambda \in \mathfrak{h}^*$  such that  $\lambda \neq 0$  and  $V_{\lambda} \neq 0$ , and

$$V_{\lambda} = \{v \in V \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

**Definition 2.6.** Maintain the notation of the previous paragraph. The decomposition of  $V$  is known as the *weight space decomposition*. The elements  $\lambda \in \Psi$  are known as *weights* with corresponding weight space  $V_{\lambda}$ . The elements of  $V_{\lambda}$  are known as *weight vectors* of weight  $\lambda$ .

The concept of a *minuscule system* in the following definition is not standard, and is introduced in [1].

**Definition 2.7.** Let  $\Delta$  be the set of simple roots of a simple Lie algebra  $\mathfrak{g}$ . Suppose that  $\Psi$  is the set of elements dual to the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ , such that for all  $\alpha \in \Delta$  and  $\lambda \in \Psi$ ,

- (i)  $\langle \lambda, \alpha \rangle \in \{-1, 0, 1\}$ , and
- (ii)  $\langle \lambda, \alpha \rangle = -1$  if and only if  $\lambda + \alpha \in \Psi$ ,  $\langle \lambda, \alpha \rangle = 1$  if and only if  $\lambda - \alpha \in \Psi$ , and  $\langle \lambda, \alpha \rangle = 0$  if and only if  $\lambda \pm \alpha \notin \Psi$ .

We say that  $\Psi$  is a *minuscule system with respect to the simple system  $\Delta$* .

**Definition 2.8.** Let  $\Psi$  be a minuscule system with respect to the simple system  $\Delta$  and let  $\lambda \in \Psi$ . We define  $b_\lambda$  to be the vector indexed by  $\lambda$ , and we define  $V_\Psi$  to be the  $k$ -vector space with basis  $b_\Psi$ , where

$$b_\Psi = \{b_\lambda \mid \lambda \in \Psi\}.$$

**Definition 2.9.** We define the action of the Serre generators on  $V_\Psi$  by specifying a  $k$ -linear endomorphism on the elements of  $b_\Psi$ , such that for all  $\lambda \in \Psi$ ,

$$\begin{aligned} e_{\alpha_i} \cdot b_\lambda &= \begin{cases} b_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in \Psi; \\ 0 & \text{otherwise.} \end{cases} \\ f_{\alpha_i} \cdot b_\lambda &= \begin{cases} b_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in \Psi; \\ 0 & \text{otherwise.} \end{cases} \\ h_{\alpha_i} \cdot b_\lambda &= \langle \lambda, \alpha_i \rangle b_\lambda. \end{aligned}$$

These maps impose the structure of a  $\mathfrak{g}$ -module on  $V_\Psi$ , where  $\mathfrak{g}$  has base  $\Delta$ . This is because by construction,  $V_\Psi$  has a basis of weight vectors for the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . However, this is not a sufficient condition, as we must also show that the axioms of Definition 2.2 are satisfied. That is, we must check that  $(x \cdot y - y \cdot x) \cdot v = [x, y] \cdot v$  for all  $x, y \in \mathfrak{g}$  and all  $v \in V_\Psi$ . It is enough to show that this holds for all Serre generators of  $\mathfrak{g}$  and all basis elements of  $V_\Psi$ , because every element in  $\mathfrak{g}$  can be expressed as iterated brackets and linear combinations of the Serre generators, and every element in  $V_\Psi$  can be expressed as linear combinations of its basis elements. It then follows by induction, bilinearity in  $[\ , \ ]$ , and linearity in  $V_\Psi$  that this holds for all elements of  $\mathfrak{g}$  and  $V_\Psi$ . The calculations for this are done in [1], and we introduce this result in the following theorem.

**Theorem 2.10.** *Let  $\Psi$  be a minuscule system with respect to the simple system  $\Delta$ , and let  $\mathfrak{g}$  be the corresponding simple Lie algebra. The  $\mathbb{C}$ -vector space  $V_\Psi$  has the structure of a  $\mathfrak{g}$ -module, where the Serre generators act on  $V_\Psi$  as in Definition 2.9.*

*Proof.* This is proved in [1, §3]. □

**Definition 2.11.** Let  $\Psi$  be a minuscule system with respect to the simple system  $\Delta$ . We define a partial order  $\leq$  on  $\Psi$  such that  $\mu \leq \lambda$  if and only if  $\lambda - \mu = \sum_{\alpha_i \in \Delta} a_i \alpha_i$ , where  $a_i \in \mathbb{Z}$  and  $a_i \geq 0$ . We say that  $\Psi$  has *highest weight*  $\lambda_h$  if  $\mu \leq \lambda_h$  for all  $\mu \in \Psi$ . Similarly, we say that  $\Psi$  has *lowest weight*  $\lambda_l$  if  $\lambda_l \leq \mu$  for all  $\mu \in \Psi$ .

**Definition 2.12.** Let  $\mathfrak{h}$  be the Cartan subalgebra of the simple algebra  $\mathfrak{g}$  with base  $\Delta$ , spanned by elements  $\{h_{\alpha_i} \mid \alpha_i \in \Delta\}$  as defined in Theorem 1.27. Let  $\{\omega_i \mid \alpha_i \in \Delta\}$  be the basis of  $\mathfrak{h}^*$  dual to  $\{h_{\alpha_i} \mid \alpha_i \in \Delta\}$ . That is,  $\omega_i(h_{\alpha_j}) = \delta_{ij}$ . The weights  $\omega_i$  are known as *fundamental weights*.

We can now introduce a well known result which classifies all finite dimensional irreducible modules.

**Proposition 2.13.** (i) *Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$ . If  $\lambda$  is a nonnegative  $\mathbb{Z}$ -linear combination of the fundamental weights  $\omega_i$  then up to isomorphism there is a unique finite dimensional irreducible  $\mathfrak{g}$ -module  $L(\lambda)$  where  $v_\lambda$  is of weight  $\lambda$  and is the unique nonzero highest weight vector of  $L(\lambda)$ . The modules  $L(\lambda)$  are pairwise nonisomorphic and exhaust all finite dimensional irreducible modules of  $\mathfrak{g}$ .*

(ii) *Suppose that  $V$  is a finite dimensional  $\mathfrak{g}$ -module containing a nonzero highest weight vector  $v_\lambda$  of weight  $\lambda$ , and that  $\dim(V) = \dim(L(\lambda))$ . Then  $V$  is isomorphic to  $L(\lambda)$ .*

*Proof.* Part (i) is a special case of [5, Theorem 10.21]. Part (ii) is proved in [1, Proposition 1.4 (ii)].  $\square$

We are now able to define the spin modules.

**Definition 2.14.** Maintain the notation of Proposition 2.13.

The spin module for the simple Lie algebra of type  $B_n$  (for  $n \geq 2$ ), is the algebra  $L(\omega_{n-1})$ . By [6, 60.E.2], it has dimension  $2^n$ .

The two spin modules for the simple Lie algebra of type  $D_n$  (for  $n \geq 4$ ), are the modules  $L(\omega_{n-2})$  and  $L(\omega_{n-1})$ . By [6, 60.E.2], each of the modules has dimension  $2^{n-1}$ .

According to [5, Proposition 10.17], the fundamental weights can be obtained by the formula

$$\omega_i = \sum_j (A^{-1})_{ji} \alpha_j,$$

where  $(A^{-1})_{ji}$  is the  $(j, i)$ th entry of the inverse of the Cartan matrix associated with a fixed base  $\Delta$ . This motivates the following result.

**Lemma 2.15.** *Let  $\mathfrak{h}$  be the Cartan subalgebra of the simple Lie algebra  $\mathfrak{so}(2l+1, \mathbb{C})$  with base  $\Delta_B$ , as defined in 1.27. The fundamental weights of  $\mathfrak{h}$  are given by the following:*

$$\begin{aligned} \omega_i &= \sum_{j=0}^i (j+1)\alpha_j + \sum_{k=1}^{n-i-1} (i+1)\alpha_{i+k} \\ &= \alpha_0 + 2\alpha_1 + \cdots + (i+1)\alpha_i + (i+1)\alpha_{i+1} + \cdots + (i+1)\alpha_{n-1} \\ &= \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{i-1} \end{aligned}$$

for  $0 \leq i < l-1$ , and

$$\begin{aligned}\omega_{n-1} &= \frac{1}{2} \sum_{i=0}^{n-1} (i+1)\alpha_i \\ &= \frac{1}{2}\alpha_0 + \alpha_1 + \cdots + \frac{n}{2}\alpha_{n-1} \\ &= \frac{1}{2}\varepsilon_0 + \frac{1}{2}\varepsilon_1 + \cdots + \frac{1}{2}\varepsilon_{n-1}.\end{aligned}$$

*Proof.* The inverse of the Cartan matrix associated with  $\Delta_B$  is given by

$$\begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & 2l & 2l \\ \frac{1}{2} & 1 & \frac{3}{2} & \cdots & \frac{2l-1}{2} & \frac{2l-1}{2} \end{pmatrix},$$

which yields the identity matrix when multiplied by the Cartan matrix in Lemma 1.26 (i). The fundamental weight  $\omega_i$  is obtained by summing all of the simple roots  $\alpha_i \in \Delta_B$  with respect to the fixed order, with coefficients from the  $i$ th row. This induces the above equations, completing the proof.  $\square$

**Lemma 2.16.** *Let  $\mathfrak{h}$  be the Cartan subalgebra of the simple Lie algebra  $\mathfrak{so}(2l, \mathbb{C})$  with base  $\Delta_D$ , as defined in 1.27. The fundamental weights of  $\mathfrak{h}$  are given by the following:*

$$\begin{aligned}\omega_i &= \sum_{j=0}^i (j+1)\alpha_j + \sum_{k=1}^{l-i-3} (i+1)\alpha_{i+k} + \frac{1}{2} \sum_{j=l-2}^{l-1} (i+1)\alpha_j \\ &= \alpha_0 + 2\alpha_1 + \cdots + (i+1)\alpha_i + (i+1)\alpha_{i+1} + \cdots + (i+1)\alpha_{l-3} + \frac{i+1}{2}\alpha_{l-2} + \frac{i+1}{2}\alpha_{l-1} \\ &= \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{i-1}\end{aligned}$$

for  $0 \leq i < l-2$ , and

$$\begin{aligned}\omega_{l-2} &= \frac{1}{2} \sum_{i=0}^{l-3} (i+1)\alpha_i + (l)\alpha_{n-1} + (l-1)\alpha_{n-1} \\ &= \frac{1}{2}\alpha_0 + \alpha_1 + \cdots + \frac{l-2}{2}\alpha_{l-3} + (l)\alpha_{l-2} + (l-1)\alpha_{l-1} \\ &= \frac{1}{2}\varepsilon_0 + \frac{1}{2}\varepsilon_1 + \cdots + \frac{1}{2}\varepsilon_{l-3} + \frac{1}{2}\varepsilon_{l-2} - \frac{1}{2}\varepsilon_{l-1} \\ \omega_{l-1} &= \frac{1}{2} \sum_{i=0}^{l-3} (i+1)\alpha_i + (l-1)\alpha_{n-1} + (l)\alpha_{n-1} \\ &= \frac{1}{2}\alpha_0 + \alpha_1 + \cdots + \frac{l-2}{2}\alpha_{l-3} + (l-1)\alpha_{l-2} + (l)\alpha_{l-1} \\ &= \frac{1}{2}\varepsilon_0 + \frac{1}{2}\varepsilon_1 + \cdots + \frac{1}{2}\varepsilon_{l-3} + \frac{1}{2}\varepsilon_{l-2} + \frac{1}{2}\varepsilon_{l-1}.\end{aligned}$$



*Proof.* The inverse of the Cartan matrix with respect to  $\Delta_D$  is

$$\begin{pmatrix} 1 & 1 & \dots & \dots & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & \dots & 2 & 1 & 1 \\ 1 & 2 & 3 & \dots & 3 & \frac{3}{2} & \frac{3}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & l-2 & \frac{l-2}{2} & \frac{l-2}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & \dots & \frac{l-2}{2} & l & l-1 \\ \frac{1}{2} & 1 & \frac{3}{2} & \dots & \frac{l-2}{2} & l-1 & l \end{pmatrix},$$

which yields the identity matrix when multiplied by the Cartan matrix in Lemma 1.26 (ii). The the rest of the proof follows the same argument as in Lemma 2.15.  $\square$

**Lemma 2.17.** *Let  $\mathfrak{h}^*$  be the dual space of the Cartan subalgebra  $\mathfrak{h}$ , of the simple Lie algebra of type  $B_n$ . Define  $\Psi_B$  to be the subset of  $\mathfrak{h}^*$  consisting of all vectors of the form*

$$\pm \frac{1}{2}\varepsilon_0 \pm \frac{1}{2}\varepsilon_1 \cdots \pm \frac{1}{2}\varepsilon_{n-2} \pm \frac{1}{2}\varepsilon_{n-1}.$$

*Then  $\Psi_B$  is a minuscule system with respect to  $\Delta_B$ .*

*Proof.* We must show that the axioms of Definition 2.7 are satisfied. Suppose first that  $\alpha_i \in \Delta_B$  for some  $0 \leq i < n-1$ , and let  $\lambda \in \Psi_B$ . Write  $\lambda = \sum_{i=0}^{n-1} c_i \varepsilon_i$ . We note that  $c_i - c_{i+1} \in \{-1, 0, 1\}$ , as  $c_i, c_j \in \{\pm \frac{1}{2}\}$ . The proof is a case by case check according to the values of  $c_i$  and  $c_{i+1}$ . There are three cases to check.

The first possibility is that  $c_i = c_{i+1}$ . This implies that  $\langle \lambda, \alpha_i \rangle = 0$ , as  $\alpha_i$  is orthogonal to  $\lambda$ . The coefficients of  $\varepsilon_i$  and  $\varepsilon_j$  in  $\lambda \pm \alpha_i$  differ by  $\frac{3}{2}$  in absolute value, which means that  $\lambda \pm \alpha_i \notin \Psi_B$ , satisfying the conditions of Definition 2.7.

The second possibility is that  $c_i = -\frac{1}{2} = -c_{i+1}$ . This implies that  $\langle \lambda, \alpha_i \rangle = -1$ . The coefficients of  $\varepsilon_i$  and  $\varepsilon_{i+1}$  in  $\lambda + \alpha_i$  differ by 1, and differ by  $-2$  in  $\lambda - \alpha_i$ . This means that  $\lambda + \alpha_i \in \Psi_B$  and  $\lambda - \alpha_i \notin \Psi_B$ , satisfying the conditions of Definition 2.7.

The third possibility is that  $c_i = \frac{1}{2} = -c_{i+1}$ . An analysis like the previous paragraph shows that  $\langle \lambda, \alpha_i \rangle = 1$  and  $\lambda - \alpha_i \in \Psi_B$ ,  $\lambda + \alpha_i \notin \Psi_B$ . This satisfies the conditions of Definition 2.7.

It remains to show that Definition 2.7 is satisfied in the case  $i = n-1$ . There are two cases to check, according to the value of  $c_{n-1}$ . If  $c_{n-1} = \frac{1}{2}$ , then  $\langle \lambda, \alpha_i \rangle = 1$ , and  $\lambda - \alpha_i$  has  $-\frac{1}{2}$  as the coefficient of  $\varepsilon_{n-1}$ , whereas  $\lambda + \alpha_{n-1}$  has coefficient of  $\frac{3}{2}$  of  $\varepsilon_{n-1}$ , satisfying the conditions of Definition 2.7. For the case of  $c_{n-1}$ , a similar argument shows that Definition 2.7 is satisfied, completing the proof.  $\square$

**Lemma 2.18.** *Let  $\mathfrak{h}^*$  be the dual space of the Cartan subalgebra  $\mathfrak{h}$  of the simple Lie algebra of type  $D_n$ . Let  $\Psi_D^+$  and  $\Psi_D^-$  be the subsets of  $\mathfrak{h}^*$  consisting of all vectors of the form*

$$\pm \frac{1}{2}\varepsilon_0 \pm \frac{1}{2}\varepsilon_1 \cdots \pm \frac{1}{2}\varepsilon_{n-2} \pm \frac{1}{2}\varepsilon_{n-1},$$

with an even and odd number of occurrences of  $-\frac{1}{2}\varepsilon_i$  respectively. Then  $\Psi_D^+$  and  $\Psi_D^-$  are minuscule systems with respect to the simple system  $\Delta_D$ .

*Proof.* We prove this for  $\Psi_D^+$  and  $\Psi_D^-$  respectively. For the proof of  $\Psi_D^+$ , suppose first that  $\alpha_i \in \Delta_D$  for some  $0 \leq i < n-1$ . Write  $\lambda = \sum_{i=0}^{n-1} c_i \varepsilon_i$ . The proof follows an argument similar to Lemma 2.17.

It remains to show that Definition 2.7 is satisfied when  $i = n-1$ . There are three cases to check, according to the values  $c_{n-2}$  and  $c_{n-1}$ . The first possibility is that  $c_{n-2} = -c_{n-1}$ . This implies that  $\langle \lambda, \alpha_{n-1} \rangle = 0$ . The coefficients  $c_{n-2}$  and  $c_{n-1}$  of  $\lambda \pm \alpha_{n-1}$  differ by  $\frac{3}{2}$  in absolute value, satisfying the conditions of Definition 2.7.

The second possibility is that  $c_{n-2} = c_{n-1} = -\frac{1}{2}$ . This implies that  $\langle \lambda, \alpha_i \rangle = -1$ . The coefficients  $c_{n-2}, c_{n-1}$  in  $\lambda + \alpha_{n-1}$  are both  $\frac{1}{2}$ , and the same coefficients in  $\lambda - \alpha_{n-1}$  are both  $-\frac{3}{2}$ , which satisfies the conditions of Definition 2.7.

The third possibility is that  $c_{n-2} = c_{n-1} = \frac{1}{2}$ . This follows a similar argument to the case  $c_{n-2} = c_{n-1} = -\frac{1}{2}$ , satisfying Definition 2.7 and completing the proof for  $\Psi_D^+$ .

We now prove this for  $\Psi_D^-$ . Suppose first that  $\alpha_i \in \Delta_D$  for some  $0 \leq i \leq n-3$  or  $i = n-1$ . The proof follows a similar argument to the first three cases of Lemma 2.17.

Now, suppose that  $i = n-2$ . The proof follows the same argument as above for  $\Psi_D^+$  in the case  $i = n-1$ , satisfying the conditions of Definition 2.7 and completing the proof.  $\square$

To construct the spin modules, we desire the sets  $\Psi_B$ ,  $\Psi_D^+$ , and  $\Psi_D^-$  to be closed under negation, and therefore introduce the following Lemma.

**Lemma 2.19.**

- (i) The set  $\Psi_B$  is closed under negation.
- (ii) The sets  $\Psi_D^+$  and  $\Psi_D^-$  are closed under negation if and only if they are subsets of dual spaces of dimension  $2n$ .

*Proof.* For part (i), it is clear that the negative of any vector of  $\Psi_B$  is also of the form  $\pm \frac{1}{2}\varepsilon_1 \pm \dots \pm \frac{1}{2}\varepsilon_n$ .

For part (ii), suppose  $\Psi_D^+$  is a subset of a dual space of dimension  $n$ , and  $\lambda \in \Psi_D^+$ . By Definition 2.18,  $\lambda$  has an even number of negative coefficients, say  $2x$ , and  $n-2x$  positive coefficients. Therefore, if  $n$  is odd,  $-\lambda$  will contain an odd number of negative coefficients. The same argument applies to  $\Psi_D^-$ , completing the proof.  $\square$

**Corollary 2.20.** The set  $\Psi_B$  has highest weight

$$\lambda = \frac{1}{2} \sum_{i=0}^{n-1} \varepsilon_i.$$

*Proof.* To satisfy Definition 2.11, we must show that the difference between  $\lambda$  and any other weight in  $\Psi_B$  is a nonnegative integer linear combination of

simple roots. Since every coefficient in  $\lambda$  is positive, its difference with another weight will be a linear combination of the vectors  $\varepsilon_i, 0 \leq i \leq n-1$  with coefficients 1 or 0. This is obviously a linear combination of the positive roots of the set  $b \subset \Phi_B$ , defined in Proposition 1.20 and derived in Proposition 1.22. By Definition 1.19, these are in turn nonnegative integer linear combinations of simple roots, which satisfies Definition 2.11 and completes the proof.  $\square$

**Corollary 2.21.** *The sets  $\Psi_D^+$  has highest weight*

$$\lambda = \frac{1}{2} \sum_{i=0}^{n-1} \varepsilon_i.$$

*Proof.* Suppose that  $\mu \in \Psi_D^+$ . To satisfy Definition 2.11, we must show that  $\lambda - \mu$  is a nonnegative integer linear combination of simple roots. Since we require  $\Psi_D^+$  to be closed under negation,  $n$  is even by Lemma 2.19. Furthermore, since  $\lambda$  consists entirely of components with coefficient  $\frac{1}{2}$ , this implies that  $\lambda - \mu$  has an even number of components with positive coefficient 1 and an even number of components with coefficient 0. This is clearly a nonnegative linear combination of the positive roots of the set  $b \subset \Phi_D$ , defined in Proposition 1.21 and derived in Proposition 1.23. Since by Definition 1.19 the elements of  $b$  are in turn a nonnegative linear combination of simple roots, this satisfies Definition 2.11 and completes the proof.  $\square$

**Corollary 2.22.** *The set  $\Psi_D^-$  has highest weight*

$$\lambda = \frac{1}{2} \sum_{i=0}^{n-2} \varepsilon_i - \frac{1}{2} \varepsilon_{n-1}.$$

*Proof.* Suppose that  $\mu \in \Psi_D^-$ . To satisfy Definition 2.11, we must show that  $\lambda - \mu$  is a nonnegative integer linear combination of simple roots. Since we require  $\Psi_D^+$  to be closed under negation, by Lemma 2.19  $n$  is even. There are two cases to check, according to the coefficient of  $\varepsilon_{n-1}$  in  $\mu$ .

Suppose first that the coefficient is  $\frac{1}{2}$ . By Definition 2.18, this implies that there is an odd number of components  $\varepsilon_i$  with coefficient  $-\frac{1}{2}$  in  $\mu$ , where  $0 \leq i \leq n-2$ . Since the components  $\varepsilon_i$  of  $\lambda$  have positive coefficients  $\frac{1}{2}$  where  $0 \leq i \leq n-2$ , and  $-\frac{1}{2}$  when  $i = n-1$ ,  $\lambda - \mu$  results in an odd number of coefficients of 1 for  $\varepsilon_i$  where  $0 \leq i \leq n-2$ , and one coefficient of  $-1$  for  $\varepsilon_{n-1}$ , whereas the rest of the coefficients are 0. This is clearly a nonnegative integer linear combination of the positive roots of the sets  $b, m \subset \Phi_D$ , defined in Proposition 1.21 and derived in Proposition 1.23. Since by Definition 1.19 the positive roots are in turn a nonnegative integer linear combination of simple roots, this satisfies Definition 2.11.

It remains to check the case where the coefficient of  $\varepsilon_{n-1}$  in  $\mu$  is  $-\frac{1}{2}$ . In this case,  $\mu$  contains an even number of occurrences of  $\frac{1}{2}\varepsilon_i$ , such that  $0 \leq i \leq n-2$ . Therefore, the components of  $\lambda - \mu$  have an even number of coefficients of 1, whereas all other coefficients are 0. Thus, this is clearly a nonnegative integer

linear combination of the positive roots in the set  $b \in \Phi_D$ , defined in Proposition 1.21. The rest of the proof follows the same argument as the previous case, thereby satisfying Definition 2.11 as required.  $\square$

We now have all the tools to explicitly construct the modules.

**Theorem 2.23.**

- (i) *The module  $V_{\Psi_B}$  is the spin module of type  $B_n$ .*
- (ii) *The module  $V_{\Psi_B}$  is irreducible.*

*Proof.* By Theorem 2.10,  $V_{\Psi_B}$  is a module for the simple Lie algebra of type  $B_n$ . It has dimension  $2^n$  because it consists of  $n$  vector components each with a positive or negative coefficient of  $\frac{1}{2}$ . By Corollary 2.20, the highest weight vector of  $V_{\Psi_B}$  is equal to the highest weight vector of the spin module defined in Definition 2.14 and derived in Lemma 2.15. Therefore, by Proposition 2.13 (ii), the modules are isomorphic. Furthermore, by Proposition 2.13 (i),  $V_{\Psi_B}$  is irreducible.  $\square$

**Theorem 2.24.**

- (i) *The modules  $V_{\Psi_D^+}$  and  $V_{\Psi_D^-}$  are the two spin modules of type  $D_n$ .*
- (ii) *The modules  $V_{\Psi_D^+}$  and  $V_{\Psi_D^-}$  are irreducible.*
- (iii) *The modules  $V_{\Psi_D^+}$  and  $V_{\Psi_D^-}$  are nonisomorphic.*

*Proof.* The dimension of  $V_{\Psi_D^+}$  and  $V_{\Psi_D^-}$  is  $\frac{2^n}{2} = 2^{n-1}$  as they each have an odd and even number of occurrences of  $-\frac{1}{2}\varepsilon_i$ , respectively. The proof for parts (i) and (ii) follows the same argument as Theorem 2.23. By Proposition 2.13 (i), the modules are nonisomorphic.  $\square$

We note that the spin modules  $V_{\Psi_D^+}$  and  $V_{\Psi_D^-}$  are not defined for  $D_n$  when  $n$  is odd, as by Lemma 2.19, they are not closed under negation.

### 3 Main Result

In this section we define a bilinear form on the spin modules constructed in the previous section. We show that this bilinear form is  $\mathfrak{g}$ -invariant, and is the unique  $\mathfrak{g}$ -invariant bilinear form up to multiplication by a nonzero scalar on these modules. We then introduce our original results. We define equations for the height of weights of the spin modules and use them to prove that the bilinear form alternates between orthogonal and symplectic depending on the dimension of its underlying Lie algebra.

**Definition 3.1.** A *bilinear form* on a  $k$ -vector space  $V$  is a map

$$B : V \times V \rightarrow k$$

that is linear in both components.

**Definition 3.2.** Let  $B$  be a bilinear form on a  $k$ -vector space  $V$ . The *radical* of  $B$  is the set  $\text{rad}(B) = \{v \in V \mid B(v, w) = 0 \text{ for all } w \in V\}$ . A bilinear form  $B$  is *nondegenerate* if  $\text{rad}(B) = 0$ .

**Example 3.3.** Let  $V = \mathbb{R}^n$  be the euclidean vector space of dimension  $n$ . The usual dot product is a bilinear form on  $V$ . Furthermore, it is nondegenerate as the only element orthogonal to every vector is 0.

**Definition 3.4.** Let  $\mathfrak{g}$  be a Lie algebra over a field  $k$  with characteristic  $\neq 2$ , and let  $B$  be a bilinear form on the  $\mathfrak{g}$ -module  $V$ .

(i) The bilinear form  $B$  is said to be  *$\mathfrak{g}$ -invariant* if  $B(g \cdot v, w) + B(v, g \cdot w) = 0$  for all  $g \in \mathfrak{g}, v, w \in V$ .

(ii) The bilinear form  $B$  is said to be *orthogonal* if it is nondegenerate and  $B(v, w) = B(w, v)$  for all  $v, w \in V$ .

(iii) The bilinear form  $B$  is said to be *symplectic* if it is nondegenerate and  $B(v, w) = -B(w, v)$  for all  $v, w \in V$ .

**Lemma 3.5.** Suppose that the  $k$ -vector space  $V$  is a  $\mathfrak{g}$ -module, and suppose  $B$  is a  $\mathfrak{g}$ -invariant bilinear form on  $V$ . Then  $\text{rad}(B)$  is a submodule of  $V$ .

*Proof.* It is clear that  $\text{rad}(B)$  is a subspace of  $V$ , because  $B$  is linear in the first entry. Therefore, the proof reduces to showing that the subspace is invariant under the action of the elements of  $\mathfrak{g}$ .

Suppose that  $r \in \text{rad}(B), g \in \mathfrak{g}$ , and  $w \in V$ . Then  $B(g \cdot r, w) = -B(r, g \cdot w) = 0$ , completing the proof.  $\square$

**Lemma 3.6.** Let  $b_\Psi$  be a basis for the  $\mathfrak{g}$ -module  $V_\Psi$  over  $k$ , and let  $S$  be the set of generators for  $\mathfrak{g}$ . Suppose  $B$  is a bilinear form on  $V_\Psi$  such that

$$B(s \cdot b_\lambda, b_\mu) + B(b_\lambda, s \cdot b_\mu) = 0 \text{ for all } s \in S, b_\lambda, b_\mu \in b_\Psi.$$

Then  $B$  is  $\mathfrak{g}$ -invariant.

*Proof.* We show that the above property holds on the bracket of generators of  $\mathfrak{g}$ . Since every element of  $\mathfrak{g}$  can be expressed by iterated linear combinations and brackets of the elements of  $S$ , and every element of  $V_\Psi$  can be expressed as a linear combination of the elements of  $b_\Psi$ , it follows by induction and bilinearity in  $B$ , and  $[\cdot, \cdot]$ , that  $B$  is  $\mathfrak{g}$ -invariant. Let  $s, t \in S$ , and  $u, v \in b_\Psi$ . We have:

$$\begin{aligned} & B([s, t] \cdot v, w) + B(v, [s, t] \cdot w) \\ &= B((s \cdot t - t \cdot s) \cdot v, w) + B(v, (s \cdot t - t \cdot s) \cdot w) \\ &= B(s \cdot t \cdot v, w) - B(v, t \cdot s \cdot w) + B(v, s \cdot t \cdot w) - B(t \cdot s \cdot v, w) \\ &= B(s \cdot t \cdot v, w) - B(v, t \cdot s \cdot w) + B(v, s \cdot t \cdot w) - B(t \cdot s \cdot v, w) \\ &= B((s \cdot t \cdot v, w) + B(t \cdot v, s \cdot w)) - (B(t \cdot v, s \cdot w) + B(v, t \cdot s \cdot w)) \\ &\quad + (B(v, s \cdot t \cdot w) + B(s \cdot v, t \cdot w)) - (B(s \cdot v, t \cdot w) + B(t \cdot s \cdot v, w)) \\ &= 0, \end{aligned}$$

completing the proof.  $\square$

**Definition 3.7.** Let  $\mu$  be an element of one of the sets  $\Psi_B, \Psi_D^+$ , or  $\Psi_D^-$ , with highest weight  $\lambda_h$ . We define the *height* of  $\mu$ , denoted  $|\mu|$ , such that

$$|\mu| = \sum_{\alpha_i \in \Delta} a_i, \text{ where } \mu + \lambda_h = \sum_{\alpha_i \in \Delta} a_i \alpha_i.$$

Definition 3.7 introduces unconventional notation for the height of  $\mu$  for convenience, which is well defined in this special case due to Lemma 2.19.

**Corollary 3.8.** *Maintain the notation of Definition 3.7. The height of  $-\mu$  is given by*

$$|-\mu| = |\lambda_h| - |\mu|$$

*Proof.* By Definition 3.7,  $|-\mu| = \sum_{\alpha_i \in \Delta} a_i$  such that

$$\begin{aligned} \sum_{\alpha_i \in \Delta} a_i \alpha_i &= -\mu + \lambda_h \\ &= (\lambda_h + \lambda_h) - (\mu + \lambda_h), \end{aligned}$$

implying

$$\sum_{\alpha_i \in \Delta} a_i = |\lambda_h| - |\mu|$$

as required.  $\square$

**Definition 3.9.** Let  $\Psi$  be a minuscule system with respect to the simple system  $\Delta$ . We define a bilinear form on  $V_\Psi$  by specifying its effect on the elements of the basis  $b_\Psi$  of  $V_\Psi$ , such that for all  $b_\lambda, b_\mu \in b_\Psi$ ,

$$B_{\mathfrak{g}}(b_\lambda, b_\mu) := \begin{cases} (-1)^{|\lambda|} & \text{if } \lambda = -\mu; \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 3.10.** *The bilinear form  $B_{\mathfrak{g}}$  is  $\mathfrak{g}$ -invariant.*

*Proof.* Let  $S$  be the set of Serre generators of  $\mathfrak{g}$ . By Lemma 3.6, it is sufficient to show that the property  $B_{\mathfrak{g}}(s \cdot b_\lambda, b_\mu) + B_{\mathfrak{g}}(b_\lambda, s \cdot b_\mu) = 0$  is satisfied for all  $s \in S$  and  $b_\lambda, b_\mu \in b_\Psi$ . We prove this in three parts, by showing that the condition is satisfied for all  $h_{\alpha_i}, e_{\alpha_i}$ , and  $f_{\alpha_i} \in S$ , respectively.

The first part reduces to a cases by case check on the value of  $\lambda$  relative to  $\mu$ . There are two cases to check. First, suppose that  $\lambda \neq -\mu$ . We then have

$$\begin{aligned} B_{\mathfrak{g}}(h_{\alpha_i} \cdot b_\lambda, b_\mu) + B_{\mathfrak{g}}(b_\lambda, h_{\alpha_i} \cdot b_\mu) &= \langle \lambda, \alpha_i \rangle B_{\mathfrak{g}}(b_\lambda, b_\mu) + \langle \mu, \alpha_i \rangle B_{\mathfrak{g}}(b_\lambda, b_\mu) \\ &= \langle \lambda, \alpha_i \rangle (0) + \langle \mu, \alpha_i \rangle (0) \\ &= 0, \end{aligned}$$

as required by Definition 3.9. For the second case, assume that  $\lambda = -\lambda$ . We have

$$\begin{aligned} B_{\mathfrak{g}}(h_{\alpha_i} \cdot b_{\lambda}, b_{-\lambda}) &= \langle \lambda, \alpha_i \rangle B_{\mathfrak{g}}(b_{\lambda}, b_{-\lambda}) \\ &= -\langle -\lambda, \alpha_i \rangle B_{\mathfrak{g}}(b_{\lambda}, b_{-\lambda}) \\ &= -B_{\mathfrak{g}}(b_{\lambda}, h_{\alpha_i} \cdot b_{-\lambda}), \end{aligned}$$

Thus,

$$B_{\mathfrak{g}}(h_{\alpha_i} \cdot b_{\lambda}, b_{-\lambda}) + B_{\mathfrak{g}}(b_{\lambda}, h_{\alpha_i} \cdot b_{-\lambda}) = 0$$

as required by Definition 3.9.

We proceed to check that the condition is satisfied for all  $e_{\alpha_i} \in S$ . As above, this reduces into two cases according to the values of  $\lambda$  in relation to  $\mu$ . First, suppose for all  $i$  that  $\lambda + \alpha_i \neq -\mu$ . This is equivalent to  $-\lambda \neq \mu + \alpha_i$ . We have

$$\begin{aligned} B_{\mathfrak{g}}(e_{\alpha_i} \cdot b_{\lambda}, b_{\mu}) + B_{\mathfrak{g}}(b_{\lambda}, e_{\alpha_i} \cdot b_{\mu}) &= B_{\mathfrak{g}}(b_{\lambda+\alpha_i}, b_{\mu}) + B_{\mathfrak{g}}(b_{\lambda}, b_{\mu+\alpha_i}) \\ &= 0 \end{aligned}$$

by Definition 3.9. Now, suppose that for all  $i$ ,  $\lambda + \alpha_i = -\mu$ . We then have

$$\begin{aligned} B_{\mathfrak{g}}(e_{\alpha_i} \cdot b_{\lambda}, b_{-(\lambda+\alpha_i)}) &= B_{\mathfrak{g}}(b_{\lambda+\alpha_i}, b_{-(\lambda+\alpha_i)}) \\ &= (-1)^{|\lambda+\alpha_i|} \\ &= (-1)^{|\lambda|+1} \\ &= -(-1)^{|\lambda|} \\ &= -B_{\mathfrak{g}}(b_{\lambda}, b_{-\lambda}) \\ &= -B_{\mathfrak{g}}(b_{\lambda}, e_{\alpha_i} \cdot b_{-(\lambda+\alpha_i)}), \end{aligned}$$

as required by Definition 3.9. The proof for the elements  $f_{\alpha_i}$  follows the same argument as for  $e_{\alpha_i}$ , which completes the proof.  $\square$

**Proposition 3.11.** *The bilinear form  $B_{\mathfrak{g}}$  is nondegenerate on  $V_{\Psi_B}, V_{\Psi_D^+}$ , and  $V_{\Psi_D^-}$ .*

*Proof.* Let  $V_{\Psi}$  denote  $V_{\Psi_B}, V_{\Psi_D^+}$  or  $V_{\Psi_D^-}$ , and suppose for a contradiction that  $\text{rad}(B_{\mathfrak{g}}) \neq 0$ . By Proposition 3.10,  $B_{\mathfrak{g}}$  is  $\mathfrak{g}$ -invariant. Therefore, by Lemma 3.5  $\text{rad}(B_{\mathfrak{g}})$  is a submodule of  $V_{\Psi}$ . However, by Theorems 2.24 and 2.23,  $V_{\Psi}$  is irreducible, which by assumption implies that  $\text{rad}(B_{\mathfrak{g}}) = V_{\Psi}$ . By Definitions 2.17 and 2.18,  $V_{\Psi}$  is not zero, and by Corollary 2.19 it is closed under negation. Therefore, there exists an element  $b_{\lambda} \in V_{\Psi}$  such that  $B_{\mathfrak{g}}(b_{\lambda}, b_{-\lambda}) = (-1)^{|\lambda|} \neq 0$ , which is a contradiction that implies  $\text{rad}(B_{\mathfrak{g}}) = 0$ , as required by Definition 3.11.  $\square$

**Theorem 3.12.** *The bilinear form  $B_{\mathfrak{g}}$  is the unique  $\mathfrak{g}$ -invariant bilinear form on  $V_{\Psi}$  up to multiplication by a nonzero scalar.*

*Proof.* We prove this in two steps, first by showing  $\mathfrak{g}$  invariance imposes the condition  $B_{\mathfrak{g}}(b_{\lambda}, b_{\mu}) = 0$  unless  $\lambda = -\mu$ . We then show that  $B_{\mathfrak{g}}$  alternates sign as

the weights increase or decrease by adding or subtracting a simple root, which satisfies Definition 3.9.

Suppose  $B$  is a  $\mathfrak{g}$ -invariant bilinear form on  $V_\Psi$ . Then we must have

$$B_{\mathfrak{g}}(h_{\alpha_i} \cdot b_\lambda, b_\mu) + (b_\lambda, h_{\alpha_i} \cdot b_\mu) = 0 \text{ for all } h_{\alpha_i} \in S \text{ and } b_\lambda, b_\mu \in b_\Psi.$$

This implies

$$\begin{aligned} 0 &= \langle \lambda, \alpha_i \rangle B_{\mathfrak{g}}(b_\lambda, b_\mu) + \langle \mu, \alpha_i \rangle B_{\mathfrak{g}}(b_\lambda, b_\mu) \\ &= \langle \lambda + \mu, \alpha_i \rangle B_{\mathfrak{g}}(b_\lambda, b_\mu). \end{aligned}$$

Since the dot product  $\langle \cdot, \cdot \rangle$  is nondegenerate and the elements  $\alpha_i$  form a basis, this implies

$$0 = B_{\mathfrak{g}}(b_\lambda, b_\mu),$$

or

$$\begin{aligned} 0 &= \lambda + \mu \\ \lambda &= -\mu, \end{aligned}$$

as required.

Now, suppose that  $\lambda - \alpha_i \in \Psi$  where  $\alpha_i \in \Delta$ . The action of  $e_{\alpha_i} \in S$  gives the following:

$$\begin{aligned} B_{\mathfrak{g}}(b_\lambda, b_{-\lambda}) &= B_{\mathfrak{g}}(e_{\alpha_i} \cdot b_{\lambda - \alpha_i}, b_{-\lambda}) \\ &= -B_{\mathfrak{g}}(b_{\lambda - \alpha_i}, e_{\alpha_i} \cdot b_{-\lambda}) \\ &= -B_{\mathfrak{g}}(b_{\lambda - \alpha_i}, b_{-(\lambda - \alpha_i)}). \end{aligned}$$

By Lemmas 2.20, 2.21, and 2.22, every element  $\mu \in \Psi$  can be written as the highest weight  $\lambda$  minus a nonnegative integer linear combination of positive roots,  $c_i \in \mathbb{Z}$ . Therefore, it follows by induction that

$$B_{\mathfrak{g}}(b_\mu, b_{-\mu}) = (-1)^{c_i} B_{\mathfrak{g}}(b_\lambda, b_{-\lambda}) = (-1)^{|\mu|} B_{\mathfrak{g}}(b_\lambda, b_{-\lambda}) \text{ for all } \mu \in \Psi,$$

as required.  $\square$

The following results are original.

**Definition 3.13.** Let  $\Psi$  be a minuscule system with respect to the simple system  $\Delta$ , and let  $\lambda \in \Psi$ . We define the following:

$$\begin{aligned} b_{\lambda_i} &:= \begin{cases} i & \text{if } \varepsilon_{n-i} \text{ in } \lambda \text{ has coefficient of positive } \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases} \\ d_{\lambda_i} &:= \begin{cases} i & \text{if } \varepsilon_{n-i-1} \text{ in } \lambda \text{ has coefficient of positive } \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$



**Proposition 3.14.** *Let  $\lambda \in \Psi_B$ .*

*The height of  $\lambda$  is given by the function  $f(\lambda) = \sum_{i=1}^n b_{\lambda_i}$ .*

*Proof.* Let  $\lambda \in \Psi_B$ , and let  $k$  equal the height of  $\lambda$ . We prove our claim by induction on  $k$ . Let  $\lambda_l \in \Psi$  denote the lowest weight. For the base case  $k = 0$ , we show that that  $f(\lambda_l) = 0$ . The weight  $\lambda_l = -\frac{1}{2} \sum_{i=0}^{n-1} \varepsilon_i$ . Since every coefficient of  $\varepsilon_i$  in  $\lambda_l$  is negative, by Definition 3.13 every number  $b_{\lambda_i} = 0$ , and therefore  $f(\lambda) = 0 = k$ , as required.

For the inductive step, suppose that for some  $k > 0$ ,  $k$  equals the height of  $\lambda$ , and that  $f(\lambda) = k$ . We wish to show that there is an  $\alpha_i \in \Delta$  such that  $\lambda - \alpha_i \in \Psi_B$ , and such that  $f(\lambda - \alpha_i) = k - 1$ .

By Definition 2.17,  $\lambda = \pm \sum_{i=0}^{n-1} \varepsilon_i$ . This is a case by case check according to the sign of coefficients of the component vectors of  $\lambda$ . There are two cases to check.

First, suppose that the sign of all of the coefficients is positive. Then subtracting the root  $\alpha_{n-1}$  from  $\lambda$  yields

$$\left(\frac{1}{2}\varepsilon_1 + \cdots + \frac{1}{2}\varepsilon_{n-1}\right) - \varepsilon_{n-1} = \frac{1}{2}\varepsilon_1 + \cdots + \frac{1}{2}\varepsilon_{n-2} - \frac{1}{2}\varepsilon_{n-1} \in \Psi_B.$$

Since  $f$  is defined in terms of the value of coefficients of components, we can restrict our analysis only to those coefficients that change. The only coefficients which changed were those of  $\varepsilon_{n-1}$ . Therefore, by Definition 3.13,

$$b_{\lambda_{n-1}} = 1 \text{ and } b_{(\lambda - \alpha_{n-1})_{n-1}} = 0,$$

and

$$f(\lambda - \alpha_{n-1}) = f(\lambda) - 1 = k - 1,$$

as required.

For the second case, suppose that  $\lambda$  has coefficients of components not all positive. Then there exist components in  $\lambda$  such that  $\frac{1}{2}\varepsilon_i - \frac{1}{2}\varepsilon_{i+1}$  for some  $0 \leq i < n - 2$ , or  $\frac{1}{2}\varepsilon_{n-1}$ . This is a case by case check on each component. There are two cases. The first case for  $\frac{1}{2}\varepsilon_{n-1}$  follows a similar argument as case one above, satisfying  $f$ .

For the second case, suppose  $\lambda$  has components  $\frac{1}{2}\varepsilon_i - \frac{1}{2}\varepsilon_{i+1}$ . We claim that  $\lambda - \alpha_i \in \Psi_B$ . As above, we can restrict our analysis only to components that change. A direct check shows

$$\left(\frac{1}{2}\varepsilon_i - \frac{1}{2}\varepsilon_{i+1}\right) - \left(\frac{1}{2}\varepsilon_i - \frac{1}{2}\varepsilon_{i+1}\right) = -\frac{1}{2}\varepsilon_i + \frac{1}{2}\varepsilon_{i+1} \in \Psi_B, \text{ and}$$

$$b_{\lambda_i} = n - i, \quad b_{\lambda_{i+1}} = 0, \quad b_{(\lambda - \alpha_i)_i} = 0, \quad b_{(\lambda - \alpha_i)_{i+1}} = n - i - 1$$

Thus,  $f(\lambda) = f(\lambda - \alpha_i) - 1 = k - 1$  as required, exhausting all possibilities and completing the proof.  $\square$

**Proposition 3.15.** *Let  $\lambda \in \Psi_B$ . The height of  $-\lambda$  is given by  $f(-\lambda) = \sum_{i=1}^n i - b_{\lambda_i}$ .*

*Proof.* By Corollary 3.8, the height of  $-\lambda$  is given by  $|\lambda| = |\lambda_h| - |\lambda|$ , where  $\lambda_h$  is the highest weight. Therefore, using the equation of Proposition 3.14,

$$\begin{aligned} |\lambda| &= |\lambda_h| - |\lambda| \\ &= \sum_{i=1}^n i - \sum_{i=0}^n b_{\lambda_i} \\ &= \sum_{i=1}^n i - b_{\lambda_i}. \end{aligned}$$

□

**Example 3.16.** Let  $\lambda \in \Psi_B$ , where  $B_n$  is such that  $n = 5$ .

Suppose  $\lambda = -\frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 - \frac{1}{2}\varepsilon_3 - \frac{1}{2}\varepsilon_4 + \frac{1}{2}\varepsilon_5$ .

Then  $b_{\lambda_1} = 0, b_{\lambda_2} = 4, b_{\lambda_3} = 0, b_{\lambda_4} = 0, b_{\lambda_5} = 1$ , and the heights of  $\lambda$  and  $-\lambda$  are

$$|\lambda| = f(\lambda) = \sum_{i=1}^n b_{\lambda_i} = 4 + 1 = 5$$

$$|-\lambda| = f(-\lambda) = \sum_{i=1}^n i - b_{\lambda_i} = 1 + (2 - 4) + 3 + 4 + (5 - 1) = 10,$$

respectively.

**Proposition 3.17.** *Let  $\lambda \in \Psi_D^+$  or  $\Psi_D^-$ .*

*The height of  $\lambda$  is given by  $g(\lambda) = \sum_{i=0}^{n-1} d_{\lambda_i}$ .*

*Proof.* The proof follows a similar argument to Proposition 3.14, and is therefore omitted. □

**Proposition 3.18.** *Let  $\lambda \in \Psi_D^+$  or  $\Psi_D^-$ .*

*The height of  $-\lambda$  is given by  $g(-\lambda) = \sum_{i=0}^{n-1} i - d_{\lambda_i}$ .*

*Proof.* The proof follows an argument similar to Proposition 3.15, and is therefore omitted. □

**Lemma 3.19.** *Let  $n \in \mathbb{Z}$ .*

*If  $n = 0$  or  $3 \pmod{4}$ , then  $\frac{(n)(n+1)}{2} = 0 \pmod{2}$ .*

*If  $n = 1$  or  $2 \pmod{4}$ , then  $\frac{(n)(n+1)}{2} = 1 \pmod{2}$ .*

*Proof.* This is a case by case check on the value of  $n$ . There are four cases to check.

Let  $n = 0 \pmod{4}$ . Then  $n = 4x$  for some  $x \in \mathbb{Z}$ . We then have,

$$\frac{(4x)(4x+1)}{2} = (2x)(4x+1) = 0 \pmod{2}$$

Let  $n = 3 \pmod{4}$ . Then  $n = 4x + 3$ . Thus,

$$\frac{(4x+3)(4x+4)}{2} = (4x+3)(2x+2) = 2(4x+3)(x+1) = 0 \pmod{2}.$$

Let  $n = 1 \pmod{4}$ . Then  $n = 4x + 1$ . Thus,

$$\frac{(4x+1)(4x+2)}{2} = (4x+1)(2x+1) = 1 \pmod{2}$$

as each component of the product is odd.

Finally, let  $n = 2 \pmod{4}$ . Then  $n = 4x + 2$ . Thus,

$$\frac{(4x+2)(4x+3)}{2} = (x+1)(4x+3) = 0 \pmod{2}$$

as above, completing the proof. □

**Lemma 3.20.** *Let  $\lambda \in \Psi_B$ , for the orthogonal Lie algebra of type  $B_n$ .*

(i) *If  $n = 0$  or  $3 \pmod{4}$ , then  $|\lambda| = |-\lambda| \pmod{2}$ .*

(ii) *If  $n = 1$  or  $2 \pmod{4}$ , then  $|\lambda| = 1 + |-\lambda| \pmod{2}$ .*

*Proof.* (i) Let  $n = 0$  or  $3 \pmod{4}$ . By Proposition 3.14 we have

$$\begin{aligned} |\lambda| &= \sum_{i=1}^n p_{\lambda_i} \pmod{2} \\ &= 0 - \sum_{i=1}^n p_{\lambda_i} \pmod{2}, \end{aligned}$$

by Lemma 3.19,

$$\begin{aligned} &= \frac{(n+1)(n)}{2} - \sum_{i=1}^n p_{\lambda_i} \pmod{2} \\ &= \sum_{i=1}^n i - \sum_{i=1}^n p_{\lambda_i} \pmod{2} \\ &= \sum_{i=1}^n i - p_{\lambda_i} \pmod{2}, \end{aligned}$$

by Proposition 3.15,

$$= |-\lambda| \pmod{2}$$

which proves part (i).

(ii) Now, let  $n = 1$  or  $2 \pmod{4}$ . By Proposition 3.14 we have

$$\begin{aligned} |\lambda| &= \sum_{i=1}^n p_{\lambda_i} \pmod{2} \\ &= 0 - \sum_{i=1}^n p_{\lambda_i} \pmod{2}, \end{aligned}$$

by Lemma 3.19,

$$\begin{aligned}
&= 1 + \frac{(n+1)(n)}{2} - \sum_{i=1}^n p_{\lambda_i} \pmod{2} \\
&= 1 + \sum_{i=1}^n i - \sum_{i=1}^n p_{\lambda_i} \pmod{2} \\
&= 1 + \sum_{i=1}^n i - p_{\lambda_i} \pmod{2},
\end{aligned}$$

by Proposition 3.15,

$$= 1 + |-\lambda| \pmod{2},$$

which proves part (ii) and completes the proof.  $\square$

**Lemma 3.21.** *Let  $\lambda \in \Psi_D^+$  or  $\Psi_D^-$ , for the orthogonal Lie algebra of type  $D_n$ .*

*(i) If  $n \equiv 0 \pmod{4}$ , then  $|\lambda| \equiv |-\lambda| \pmod{2}$ .*

*(ii) If  $n \equiv 2 \pmod{4}$ , then  $|\lambda| \equiv 1 + |-\lambda| \pmod{2}$ .*

*Proof.* The proof follows an argument similar to Lemma 3.20, and is therefore omitted.  $\square$

We now have all the tools to prove our main results.

**Theorem 3.22.** *Let  $\mathfrak{g}$  be the simple Lie algebra of type  $B_n$ , and let  $V_{\Psi_B}$  be its corresponding spin module. Let  $B_{\mathfrak{g}}$  be the  $\mathfrak{g}$ -invariant bilinear form on  $V_{\Psi_B}$ .*

*(i) If  $n \equiv 0$  or  $3 \pmod{4}$ ,  $B_{\mathfrak{g}}$  is orthogonal.*

*(ii) If  $n \equiv 1$  or  $2 \pmod{4}$ ,  $B_{\mathfrak{g}}$  is symplectic.*

*Proof.*

(i) Let  $n \equiv 0$  or  $3 \pmod{4}$ . Then by Lemma 3.20 we have

$$\begin{aligned}
B_{\mathfrak{g}}(b_{\lambda}, b_{-\lambda}) &= (-1)^{|\lambda|} \\
&= (-1)^{|\lambda| \pmod{2}} \\
&= (-1)^{|-\lambda| \pmod{2}} \\
&= (-1)^{|-\lambda|} \\
&= B_{\mathfrak{g}}(b_{-\lambda}, b_{\lambda}).
\end{aligned}$$

By Proposition 3.11,  $B_{\mathfrak{g}}$  is nondegenerate on  $V_{\Psi_B}$ . Therefore,  $B_{\mathfrak{g}}$  is orthogonal.

(ii) Let  $n = 1$  or  $2 \pmod{4}$ . Then by Lemma 3.20 we have,

$$\begin{aligned}
B_{\mathfrak{g}}(b_{\lambda}, b_{-\lambda}) &= (-1)^{|\lambda|} \\
&= (-1)^{|\lambda| \pmod{2}} \\
&= (-1)^{1+|\lambda| \pmod{2}} \\
&= (-1)^{1+|\lambda|} \\
&= -(-1)^{|\lambda|} \\
&= -B_{\mathfrak{g}}(b_{-\lambda}, b_{\lambda}).
\end{aligned}$$

By Proposition 3.11,  $B_{\mathfrak{g}}$  is nondegenerate on  $V_{\Psi_B}$ . Therefore,  $B_{\mathfrak{g}}$  is symplectic.  $\square$

**Theorem 3.23.** *Let  $\mathfrak{g}$  be the simple Lie algebra of type  $D_n$ , where  $n$  is even, and let  $V_{\Psi_D^+}$  and  $V_{\Psi_D^-}$  be its corresponding spin modules. Let  $B_{\mathfrak{g}}$  be the  $\mathfrak{g}$ -invariant bilinear form on  $V_{\Psi_D^+}$  and  $V_{\Psi_D^-}$ .*

(i) *If  $n = 0 \pmod{4}$ ,  $B_{\mathfrak{g}}$  is orthogonal.*

(ii) *If  $n = 2 \pmod{4}$ ,  $B_{\mathfrak{g}}$  is symplectic.*

*Proof.*

(i) Let  $n = 0 \pmod{4}$ . Then by Lemma 3.21 we have

$$\begin{aligned}
B_{\mathfrak{g}}(b_{\lambda}, b_{-\lambda}) &= (-1)^{|\lambda|} \\
&= (-1)^{|\lambda| \pmod{2}} \\
&= (-1)^{|\lambda| \pmod{2}} \\
&= (-1)^{|\lambda|} \\
&= B_{\mathfrak{g}}(b_{-\lambda}, b_{\lambda}).
\end{aligned}$$

By Proposition 3.11,  $B_{\mathfrak{g}}$  is nondegenerate on  $V_{\Psi_D^+}$  and  $V_{\Psi_D^-}$ . Therefore,  $B_{\mathfrak{g}}$  is orthogonal.

(ii) Let  $n = 2 \pmod{4}$ . Then by Lemma 3.21 we have

$$\begin{aligned}
B_{\mathfrak{g}}(b_{\lambda}, b_{-\lambda}) &= (-1)^{|\lambda|} \\
&= (-1)^{|\lambda| \pmod{2}} \\
&= (-1)^{1+|\lambda| \pmod{2}} \\
&= (-1)^{1+|\lambda|} \\
&= -(-1)^{|\lambda|} \\
&= -B_{\mathfrak{g}}(b_{-\lambda}, b_{\lambda}).
\end{aligned}$$

By Proposition 3.11,  $B_{\mathfrak{g}}$  is nondegenerate on  $V_{\Psi_D^+}$  and  $V_{\Psi_D^-}$ . Therefore,  $B_{\mathfrak{g}}$  is symplectic.  $\square$

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