Explicit Computation of the Cohomology of a Symbol Algebra

by

Patrick Newberry

B.S., Northern Arizona University, 2005

M.A., University of Colorado at Boulder, 2008

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Explicit Computation of the Cohomology of a Symbol Algebra

Thesis directed by Prof. Alexander Gorokhovsky

We illustrate a link between Alexander-Spanier and cyclic cohomologies, and use this for easier computation of cocycles for a Poisson algebra. In particular, we look at the case where the algebra is the space of pseudodifferential symbols on a manifold.

Dedication

To Angela and Jeannie, who have been my strength.

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Contents

Chapter 1

Background and Definitions

1.1 Hochschild and Cyclic Homology and Cohomology

The following are basic definitions and results [2] of Cyclic theory.

For an algebra A over a commutative ring k, we define the Hochschild homology, $HH_n(A)$ to be the homology of the complex

$$C_*^{Hoch} : \ldots \xrightarrow{b} A^{\otimes(n+1)} \xrightarrow{b} A^{\otimes n} \xrightarrow{b} \ldots \xrightarrow{b} A \xrightarrow{b} 0$$

where $A^{\otimes n}$ is the n˙times tensor product of A with itself over k. The boundary operator b is defined by

$$b = \sum_{i=0}^{n} (-1)^i d_i : A^{\otimes(n+1)} \longrightarrow A^{\otimes n}$$

where d_i, called the face maps, are

$$d_0(a_0, a_1, \ldots, a_n) = (a_0 a_1, a_2, \ldots, a_n)$$

$$d_i(a_0, a_1, \ldots, a_n) = (a_0, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \text{ for } 1 \leq i \leq n-1$$

$$d_n(a_0, a_1, \ldots, a_n) = (a_n a_0, a_1, \ldots, a_{n-1})$$

Also, define the maps $s_i : A^{\otimes(n+1)} \longrightarrow A^{\otimes(n+2)}$, called the degeneracy maps, by

$$\begin{aligned} s_i(a_0, a_1, \ldots, a_n) &= (a_0, a_1, \ldots, a_i, 1, a_{i+1}, \ldots, a_n) \text{ for } 0 \le i \le n-1 \\ s_n(a_0, a_1, \ldots, a_n) &= (a_0, a_1, \ldots, a_n, 1) \\ s_{n+1}(a_0, a_1, \ldots, a_n) &= (1, a_0, a_1, \ldots, a_n) \end{aligned}$$

Finally, we can define $t_n : A^{\otimes(n+1)} \longrightarrow A^{\otimes(n+1)}$, the cyclic operator, by

$$t_n(a_0, a_1, \ldots, a_n) = (-1)^n (a_n, a_0, \ldots, a_{n-1}).$$

The subscript is usually omitted, when it is obvious from context.

Denoting $C_n^\lambda(A) = A^{\otimes(n+1)}/(1-t)$, the elements of $A^{\otimes(n+1)}$ up to application of t, we have the complex

$$C_*^{Cyc} : \ldots \xrightarrow{b} C_n^\lambda(A) \xrightarrow{b} C_{n-1}^\lambda(A) \xrightarrow{b} \ldots \xrightarrow{b} C_0^\lambda(A)$$

with homology $HC_n(A)$, called the cyclic homology of A.

These two homology groups are related by the long exact sequence

$$\ldots \longrightarrow HH_n(A) \xrightarrow{I} HC^n(A) \xrightarrow{S} HC^{n-2}(A) \xrightarrow{B} HH^{n-1}(A) \longrightarrow \ldots .$$

The map S is called Connes' Periodicity Operator.

Let A be an associative and unital k-algebra. The dual $A^* = \mathrm{Hom}(A, k)$ of A is also denoted $C^0(A)$. More generally we put $C^n(A) == \mathrm{Hom}(A^{\otimes(n+1)}, k)$

Let $C_\lambda^n(A)$ denote the cyclic n-cochains of A, that is, $f \in C^n(A)$ with

$$f(a_0, \ldots, a_n) = (-1)^n f(a_n, a_0, \ldots, a_{n-1}).$$

We then have the complex (C_λ^*, b^*) which has homology $H_\lambda^*(A)$, called the cyclic cohomology of A. This then gives a similar long exact sequence

$$\ldots \longrightarrow HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \xrightarrow{I} HH^{n+1}(A) \longrightarrow \ldots .$$

1.2 Poisson Algebras and Pseudodifferential Operators

A Poisson algebra is a commutative algebra A over k with a bracket that is a bilinear map

$$(a, b) \mapsto \{a, b\}$$

such that

(1) A is a Lie algebra with respect to the Poisson bracket,

(2) $\{f, gh\} = g\{f, h\} + h\{f, g\}$ for all $f, g, h \in A$.

One example of a Poisson algebra is the space of smooth functions on a symplectic manifold $(M.\omega)$, with bracket defined by

$$\{f, g\} = \omega^{-1}(df, dg).$$

A pseudodifferential operator is a generalization of the Fourier transform

$$(Fu)(\xi) = \hat{u}(\xi) = \int e^{-ix\dot{\xi}} u(x)dx$$

where $u \in C^\infty(\mathbb{R}^n)$ with

$$\sup_{x\in\mathbb{R}^n} \left| x^\alpha (\partial^\beta u)(x) \right| < \infty,$$

called the Schwartz space of functions on $\mathbb{R}^n$. A function $a \in C^\infty(X \times \mathbb{R}^n)$ is called a symbol of order m if for any compact $K \subset X$, and multi-indices α, β there exists a constant $C_{\alpha,\beta,K}$ such that

$$\left| \partial_\theta^\alpha \partial_x^\beta a(x, \theta) \right| \le C_{\alpha,\beta,K} \langle\theta\rangle^{m-|\alpha|}$$

where $x \in K$, $\theta \in \mathbb{R}^n$, and $\langle\theta\rangle = (1+|\theta|^2)^{\frac{1}{2}}$. The collection of these functions is denoted $S^m(X\times\mathbb{R}^n)$. Each symbol defines a pseudodifferential operator $A : C_0^\infty(X) \to C^\infty(X)$, defined by

$$Au(x) = \iint e^{i(x-y)\dot{\xi}} a(x, y, \xi)u(y)\, dy\, d\xi.$$

An operator $A : C_0^\infty(M) \to C^\infty(M)$ is called a pseudodifferential operator on a manifold, M, if for every chart (U, ϕ_U) of M, the operator, A_U defined by the diagram

$$\begin{array}{ccc} C_0^\infty(U) & \xrightarrow{A} & C^\infty(U) \\ \uparrow{\scriptstyle \phi_U^*} & & \uparrow{\scriptstyle \phi_U^*} \\ C_0^\infty(\phi(U)) & \xrightarrow{A_U} & C^\infty(\phi(U)) \end{array}$$

is a pseudodifferential operator. The algebra of symbols on the manifold M of degree m are denoted $\Psi^m(M)$. This is a filtered algebra, with

$$\Psi^{-\infty} = \bigcap_m \Psi^m(M)$$

the algebra of smoothing operators. This leads to a new algebra, $\mathcal{P} = \Psi^\infty/\Psi^{-\infty}$ with product

$$a \star b = ab + \{a, b\}/2i + \sum_{j=2}^{\infty} \phi_j(a, b)$$

where $\phi_j : \Psi^k/\Psi^{-\infty} \times \Psi^l/\Psi^{-\infty} \to \Psi^{k+l-i}/\Psi^{-\infty}$ is a bilinear map. This algebra is also endowed with a trace, [1]

$$R(a(x, y)) = \int_{S^*M} a_{-n}(x, y)\iota(\mathcal{R})\omega^n.$$

Chapter 2

Conceptual Foundation

The following is a summary of results from a unpublished paper[4] of I. Zakharevich.

2.1 Alexander Spanier Cohomology

Consider the sheaf of vector spaces $\mathcal{O}$ over a manifold M. Let $\mathcal{O}^{\boxtimes n}$ be the exterior tensor product of $\mathcal{O}$ with itself. The sheaf over M^n is given by

$$\Gamma\left(U_1 \times \ldots \times U_n, \mathcal{O}^{\boxtimes n}\right) = \Gamma(U_1, \mathcal{O}) \otimes \ldots \otimes \Gamma(U_n, \mathcal{O}).$$

Let $\mathrm{Alt}\mathcal{O}^{\boxtimes n}$ be the subsheaf of skewsymmetric sections. These sheaves form a complex, with differential given by exterior product by any fixed sheaf, denoted $\wedge 1$. The sheaves $\Lambda^k \mathcal{O} := \Delta^*(\mathrm{Alt}\mathcal{O}^{\boxtimes n})$ form a complex, whose sections over $U \subset M$ are skewsymmetric sections over the diagonal in M^k. This gives us a representation of the Alexander-Spanier complex, denoted $C^k_{AS}(\mathcal{O}) = \Lambda^{k+1}\mathcal{O}$. We also have another complex, $C^*_{HAS}(\mathcal{O}) = (\Gamma(M, \Delta^*(\mathcal{O}^{\boxtimes * + 1})), \Delta^* d)$, called the Hochschild Alexander-Spanier complex for $\mathcal{O}$, with differential

$$d(f_0 \boxtimes \cdots \boxtimes f_k) = \sum_{i=0}^{k+1} (-1)^{k+1-i} f_0 \otimes \cdots \boxtimes f_{i-1} \boxtimes 1 \boxtimes f_i \boxtimes \cdots \boxtimes f_k.$$

Furthermore, if $\mathcal{O}$ is soft, then the cohomology of these complexes are quasi-isomorphic to the cohomology of the manifold.

The sheaf of pseudodifferential symbols lives on the product of a punctured disk and the spherization of the cotangent bundle. So, that base space only has cohomology in degree 0 and 1, both of dimension 1. Particularly, a generator in degree 1 in the Alexander-Spanier cohomology is $\omega = f(z_1, z_2) = \log \frac{z_2}{z_1} = 1 \wedge \log z$. Since $\log z \notin C^1_{AS}$, then ω is the symplectic 2-form making $\Psi^\infty/\Psi^{-\infty}$ into a Poisson algebra.

2.2 A Pairing on Cycles

Given a trace on an associative algebra A, we have a map $i : CC_*(A) \longrightarrow CC^*(A)$

$$i_{f_0\otimes\ldots\otimes f_k}(g_0, \ldots, g_k) = \sum_{i=0}^{k} (-1)^{ki} \mathrm{Tr} f_0 g_i \ldots f_{k-i} g_k f_{k-i+1} g_0 \ldots f_k g_{i-1}$$

and the following diagram commutes:

$$\begin{array}{ccc} CC_k(A) & \xrightarrow{\wedge 1} & CC_{k+1}(A) \\ \big\downarrow i & & \big\downarrow i \\ CC^k(A) & \xrightarrow{b} & CC^{k+1}(A) \end{array}$$

where $\wedge 1(f_0 \otimes \ldots \otimes f_k) = \sum_{i=0}^{k+1} (-1)^{k+1-i} f_0 \otimes \ldots \otimes f_{i-1} \otimes 1 \otimes f_i \otimes \ldots \otimes f_k$. For $\mathcal{O}$ a sheaf of associative algebras over M, the pairing gives a map

$$C^*_{cAS}(\mathcal{O}) \longrightarrow CC^*(\Gamma(\mathcal{O}))$$

Similarly, the pairings

$$\langle f_1 \wedge \ldots \wedge f_k, g_1 \wedge \ldots \wedge g_k \rangle = \mathrm{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_k \\ \tau\in S_k}} f_{\sigma_1} g_{\tau_1} \cdots f_{\sigma_k} g_{\tau_k}$$

$$\langle f_0 \otimes \ldots \otimes f_k, g_0 \otimes \ldots \otimes g_k \rangle = \mathrm{Tr} f_0 0 g_0 \ldots f_k g_k$$

give the maps

$$C^*_{AS}(\mathcal{O}) \longrightarrow C^*_{\mathrm{Lie}}(\Gamma(\mathcal{O}))$$

$$C^*_{HAS}(\mathcal{O}) \longrightarrow HC^*(\Gamma(\mathcal{O}), \Gamma(\mathcal{O})^*)$$

Moreover, these maps induce maps on the cohomologies.

Now, let $*$ be an associative product on $\mathcal{A} \otimes_K K[[h]]$ with $\mathcal{A}[[h]]$ a $K[[h]]$-algebra. Let $\cdot_h$ an associative product on $\mathcal{A}$ for small h, and $\cdot = \cdot_0$ be commutative. Then, with product $*$, we have $fg - gf = O(h)$, and so, a product

$$\{f, g\} = \lim_{h \to 0} \frac{f \cdot_h g - g \cdot_h f}{h}.$$

which is a Poisson bracket. Furthermore, for $\mathcal{A}$ a Poisson algebra, we have a sheaf of Lie algebras $\mathcal{O}$ on Spec$\mathcal{A}$ from the Poisson bracket. We then, from the pairing, have a map

$$H^*_{AS}(X) \longrightarrow H^*_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{A}, \{,\})).$$

The goal now is to prove the following result:

Theorem 2.2.1. *Let $\mathcal{A}$ be a Poisson algebra with products $\cdot_h$ and common trace,* Tr*. For any $\alpha_n = f_0 \wedge \ldots \wedge f_n \in \Lambda^{n+1}\mathcal{A}$, we have a cochain $\alpha_h^{n,r} = S^r(\alpha_n) \in CC^{n+1+2r}(\mathcal{A}, \cdot_h)$. Restricting to acting on antisymmetric chains, we then have a cochain $\hat{\alpha}_h^{n,r} \in \Lambda^{n+2r+1}\mathcal{A}^*$ which can be written as*

$$\hat{\alpha}_h^{n,r} = \hat{\alpha}^{n,r} h^{n+r} + O\left(h^{n+r+1}\right)$$

with

$$\hat{\alpha}^{n,r} = \frac{n!\, r!}{(n+2r)!} 2^{-r} \sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} 2^{-2M} \binom{r+n-M}{r, n-2M, M} \hat{\alpha}^{n,r}_{(n-2M)}$$

and

$$\hat{\alpha}^{n,r}_{(M)}(g_0 \wedge \ldots g_{n+2r}) = \mathrm{Tr} \underset{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}}{\mathrm{Alt}} f_{\sigma_0} \cdot g_{\tau_0} \cdot \{f_{\sigma_1}, g_{\tau_1}\} \cdot \ldots \cdot \{f_{\sigma_M}, g_{\tau_M}\}$$
$$\cdot \left\{f_{\sigma_{M+1}}, f_{\sigma_{M+2}}\right\} \cdot \ldots \cdot \left\{f_{\sigma_{n-1}}, f_{\sigma_n}\right\} \cdot \left\{g_{\tau_{M+1}}, g_{\tau_{M+2}}\right\} \cdot \ldots \cdot \left\{g_{\tau_{n+2r-1}}, g_{\tau_{n+2r}}\right\}$$

Chapter 3

Preliminary Computations

3.1 Repeated Action of the Periodicity Operator

First, some technical results I will use later.

Lemma 3.1.1.

(a) $\displaystyle\sum_{l=c}^{d}(-1)^l = \frac{1}{2}\left((-1)^c + (-1)^d\right)$

(b) $\displaystyle\sum_{k=a}^{b}\sum_{l=k}^{d}(-1)^{k+l} = \frac{1}{2}(b-a+1) + \frac{1}{4}(-1)^d\left((-1)^a + (-1)^b\right)$

(c) $\displaystyle\sum_{0\leq k\leq l\leq 2m}(-1)^{k+l} = m+1$

Proof.

(a) Note that

$$\sum_{l=0}^{d}(-1)^l = \begin{cases} 1 & \text{if d is even} \\ 0 & \text{if d is odd} \end{cases} = \frac{1}{2}\left(1+(-1)^d\right)$$

So,

$$\begin{aligned}\sum_{l=c}^{d}(-1)^l &= \sum_{l=0}^{d}(-1)^l - \sum_{l=0}^{c-1}(-1)^l \\ &= \frac{1}{2}\left(1+(-1)^d\right) - \frac{1}{2}\left(1+(-1)^{c-1}\right) \\ &= \frac{1}{2}\left((-1)^d + (-1)^c\right)\end{aligned}$$

(b)

$$\begin{aligned}\sum_{k=a}^{b}\sum_{l=k}^{d}(-1)^{k+l} &= \sum_{k=a}^{b}(-1)^k\sum_{l=k}^{d}(-1)^l\\ &= \sum_{k=a}^{b}(-1)^k\frac{1}{2}\left((-1)^k+(-1)^d\right)\\ &= \sum_{k=a}^{b}\frac{1}{2}\left(1+(-1)^{d+k}\right)\\ &= \frac{1}{2}(b-a+1)+\frac{1}{2}(-1)^d\left(\frac{1}{2}\left((-1)^a+(-a)^b\right)\right)\\ &= \frac{1}{2}(b-a+1)+\frac{1}{4}(-1)^d\left((-1)^a+(-1)^b\right)\end{aligned}$$

(c)

$$\begin{aligned}\sum_{0\le k\le l\le 2m}(-1)^{k+l} &= \sum_{k=0}^{2m}\sum_{l=k}^{2m}(-1)^{k+l}\\ &= \frac{1}{2}(2m-0+1)+\frac{1}{4}(-1)^{2m}\left((-1)^0+(-1)^{2m}\right)\\ &= m+\frac{1}{2}+\frac{1}{4}(2)\\ &= m+1\end{aligned}$$

□

Proposition 3.1.2. *For Connes' periodicity operator, S, which can be expressed [2] as*

$$S=\frac{-1}{(n+1)(n+2)}\sum_{0\le a_1<a_2\le n+2}(-1)^{a_1+a_2}d_{a_2}d_{a_1}$$

we have that

$$S^r=\frac{n!\,r!}{(n+2r)!}(-1)^r\sum_{0\le a_1<\cdots<a_{2r}\le n+2r}(-1)^{\sum_{i=1}^{2r}a_i}d_{a_{2r}}\cdots d_{a_1}.$$

Proof. First, note that $S=\dfrac{n!}{(n+2)!}(-1)^1\sum_{0\le a_1<a_2\le n+2}(-1)^{a_1+a_2}d_{a_2}d_{a_1}$ is already of the claimed

form. Now, suppose S^{r-1} is of the above form. Then

$$\begin{aligned} S^r = SS^{r-1} = &\left(\frac{-1}{(n+2r-1)(n+2r)} \sum_{0 \le a_1 < a_2 \le n+2r} (-1)^{a_1+a_2} d_{a_2} d_{a_1} \right) \\ &\cdot \left(\frac{n!\,(r-1)!}{(n+2r-2)!} (-1)^{r-1} \sum_{0 \le b_1 < \cdots < b_{2(r-1)} \le n+2(r-1)} (-1)^{\sum_{i=1}^{2r-2} b_i} d_{b_{2(r-1)}} \cdots d_{b_1} \right) \\ &= \frac{n!\,(r-1)!}{(n+2r)!} (-1)^r \sum_{\substack{0 \le a_1 < a_2 \le n+2r \\ 0 \le b_1 < \cdots < b_{2(r-1)} \le n+2(r-1)}} (-1)^{a_1+a_2+\sum_{i=1}^{2r-2} b_i} d_{a_2} d_{a_1} d_{b_{2(r-1)}} \cdots d_{b_1} \end{aligned}$$

Note that S^r has

$$\begin{aligned} \binom{n+2r+1}{2}\binom{n+2r-1}{2r-2} &= \frac{(n+2r+1)!}{2!\,(n+2r-1)!} \frac{(n+2r-1)!}{(2r-2)!\,(n+1)!} \\ &= \frac{(n+2r+1)!}{(2r)!\,(n+1)!} \frac{(2r)!}{2!\,(2r-2)!} \\ &= \binom{n+2r+1}{2r}\binom{2r}{2} \end{aligned}$$

terms.

For ease of notation, let $b_{2r-1} = n+2r-1$. Using the identity $d_i d_j = d_{j+1} d_i$ for $i \le j$, each term of the product can be written as

$$d_{a_2} d_{a_1} d_{b_{2(r-1)}} \cdots d_{b_1} = d_{b_{2(r-1)+2}} \cdots d_{b_{l+1}+2} d_{a_2} d_{b_l+1} \cdots d_{b_{k+1}+1} d_{a_1} d_{b_k} \cdots d_{b_1}$$

for $b_l + 1 < a_2 < b_{l+1} + 2$, $b_k < a_1 < b_{k+1} + 1$, $a_1 < a_2$, with $0 \le k \le l \le 2r-2$. More concisely, each term is of the form

$$d_{c_{2r}} \cdots d_{c_1},\ 0 \le c_1 < \cdots < c_{2r} \le n+2r.$$

Fix $0 \le c_1 < \cdots < c_{2r} \le n+2r$ and $0 \le k \le l \le 2r-2$. If

$$b_i = \begin{cases} c_i & \text{if } 1 \le i \le k \\ c_{i+1} - 1 & \text{if } k+1 \le i \le l \\ c_{i+2} - 2 & \text{if } l+1 \le i \le 2(r-1) \end{cases} \qquad \text{and} \qquad \begin{aligned} a_1 &= c_{k+1} \\ a_2 &= c_{l+2} \end{aligned}$$

we have that

$$d_{b_{2(r-1)+2}} \cdots d_{b_{l+1}+2} d_{a_2} d_{b_l+1} \cdots d_{b_{k+1}+1} d_{a_1} d_{b_k} \cdots d_{b_1} = d_{c_{2r}} \cdots d_{c_1}.$$

Furthermore, note that there are $\binom{n+2r+1}{2r}$ ways to choose the $c_i's$ and $\binom{2r}{2}$ ways to choose k and l. So, there are the same number of terms in

$$\sum_{\substack{0\le a_1<a_2\le n+2r\\ 0\le b_1<\cdots<b_{2(r-1)}\le n+2(r-1)}} d_{a_2}d_{a_1}d_{b_{2(r-1)}}\cdots d_{b_1}$$

as in

$$\sum_{\substack{0\le c_1<\cdots<c_{2r}\le n+2r\\ 0\le k\le l\le 2r-2}} d_{c_{2r}}\cdots d_{c_1}$$

and every term in the second sum comes from a term in the first sum. Thus, we have

$$\begin{aligned}
S^r &= \frac{n!\,(r-1)!}{(n+2r)!}(-1)^r \sum_{\substack{0\le a_1<a_2\le n+2r\\ 0\le b_1<\cdots<b_{2(r-1)}\le n+2(r-1)}} (-1)^{a_1+a_2+\sum_{i=1}^{2r-2} b_i} d_{a_2}d_{a_1}d_{b_{2(r-1)}}\cdots d_{b_1}\\
&= \frac{n!\,(r-1)!}{(n+2r)!}(-1)^r \sum_{\substack{0\le c_1<\cdots<c_{2r}\le n+2r\\ 0\le k\le l\le 2r-2}} (-1)^{c_{k+1}+c_{l+2}+\sum_{i=1}^{k} c_i+\sum_{i=k+1}^{l}(c_{i+1}-1)+\sum_{i=l+1}^{2r-2}(c_{i+2}-2)} d_{c_{2r}}\cdots d_{c_1}\\
&= \frac{n!\,(r-1)!}{(n+2r)!}(-1)^r \sum_{0\le c_1<\cdots<c_{2r}\le n+2r}\ \sum_{0\le k\le l\le 2(r-1)} (-1)^{-(l-k)-2(2r-2-l)+\sum_{i=1}^{2r} c_i} d_{c_{2r}}\cdots d_{c_1}\\
&= \frac{n!\,(r-1)!}{(n+2r)!}(-1)^r \sum_{0\le c_1<\cdots<c_{2r}\le n+2r} (-1)^{\sum_{i=1}^{2r} c_i} \sum_{0\le k\le l\le 2(r-1)} (-1)^{l+k} d_{c_{2r}}\cdots d_{c_1}\\
&= \frac{n!\,(r-1)!}{(n+2r)!}(-1)^r \sum_{0\le c_1<\cdots<c_{2r}\le n+2r} (-1)^{\sum_{i=1}^{2r} c_i}\ (r-1)+1\ d_{c_{2r}}\cdots d_{c_1}\\
&= \frac{n!\,r!}{(n+2r)!}(-1)^r \sum_{0\le c_1<\cdots<c_{2r}\le n+2r} (-1)^{\sum_{i=1}^{2r} c_i} d_{c_{2r}}\cdots d_{c_1}.
\end{aligned}$$

□

3.2 Dual Action of Periodicity

Let A be an associative k-algebra with a trace, $\mathrm{Tr} : A \longrightarrow k$. Then for cyclic n-chains $f_0 \otimes \ldots \otimes f_n$, $g_0 \otimes \ldots \otimes g_n \in CC_n(A)$, we have the pairing

$$\langle (f_0,\ldots,f_n),(g_0,\ldots,g_n)\rangle = \sum_{i=0}^{n} (-1)^{ni}\mathrm{Tr} f_0 g_i \ldots f_{n-i} g_n f_{n-i+1} g_0 \ldots f_n g_{i-1}.$$

This defines a mapping, $\alpha \mapsto \langle \alpha, - \rangle$ from $CC_*(A)$ to $CC^*(A)$. We then have the degeneracy maps $s_i : CC_n(A) \longrightarrow CC_{n+1}(A)$ as the duals to the maps $d_i : CC^n(A) \longrightarrow CC^{n+1}(A)$, with respect to the pairing.

We will now compute the dual action on the periodicity operator with respect to the pairing. Denote $L_N(c_0, c_1, \ldots, c_k) = (a_0, a_1, \ldots, a_N) \in CC^N(A)$ where

$$a_i = \begin{cases} f_j & \text{if } \; i = c_j \\ 1 & \text{otherwise} \end{cases}$$

Lemma 3.2.1.

(a) *Let* $0 \leq i \leq N+1$ *and* $1 \leq c_1 < \cdots < c_n \leq N$. *Then*

$$d_i L_N(0, c_1, \ldots, c_n) = \begin{cases} L_{N+1}(0, c_1+1, \ldots, c_n+1) & \textit{if } \; 0 \leq i \leq c_1 - 1 \\ L_{N+1}(0, c_1, \ldots, c_j, c_{j+1}+1, \ldots, c_n+1) & \textit{if } \; c_j \leq i \leq c_{j+1} - 1 \\ L_{N+1}(0, c_1, \ldots, c_n) & \textit{if } \; c_n \leq i \leq N \\ L_{N+1}(N+1, c_1, \ldots, c_n) & \textit{if } \; i = N+1 \end{cases}$$

(b) *Let* $0 \leq i \leq N+1$ *and* $1 \leq c_1 < \cdots < c_n < M \leq N$. *Then*

$$d_i L_N(M, c_1, \ldots, c_n) = \begin{cases} L_{N+1}(M+1, c_1+1, \ldots, c_n+1) & \textit{if } \; 0 \leq i \leq c_1 - 1 \\ L_{N+1}(M+1, c_1, \ldots, c_j, c_{j+1}+1, \ldots, c_n+1) & \textit{if } \; c_j \leq i \leq c_{j+1} - 1 \\ L_{N+1}(M+1, c_1, \ldots, c_n) & \textit{if } \; c_n \leq i \leq M-1 \\ L_{N+1}(M, c_1, \ldots, c_n) & \textit{if } \; M \leq i \leq N+1 \end{cases}$$

Proof. The result follows immediately from the definition of d_i and $L_N(\ldots)$. $\square$

Lemma 3.2.2. *For* $0 \leq a_1 < \cdots < a_k \leq n+k-1$, *let* $1 \leq c_1 < \cdots < c_n \leq n+k$ *such that*

$$\{c_1, \ldots, c_n\} = \{1, 2, \ldots . n+k\} \setminus \{a_1+1, \ldots, a_k+1\}.$$

Then

$$d_{a_k} \cdots d_{a_1} L_n(0, 1, \ldots, n) = L_{n+k}(0, c_1, \ldots, c_n).$$

Proof. Let $k = 1$, and $0 \leq i \leq n$. Setting

$$\{c_1, \dots, c_n\} = \{1, 2, \dots . n+1\} \setminus \{i+1\} = \{1, \dots, i, i+2, \dots, n+1\},\ 1 \leq c_1 < \cdots < c_n \leq n+1$$

we have that

$$c_j = \begin{cases} j & \text{if } \ j \leq i \\ j+1 & \text{if } \ j > i \end{cases}$$

Thus,

$$d_i L_n(0, 1, \dots, n) = L_{n+1}(0, 1, \dots, i, i+2, \dots, n+1) = L_{n+1}(0, c_1, \dots, c_n).$$

Now, let $0 \leq a_1 < \cdots < a_{k+1} \leq n+k$. Set

$$\{e_1, \dots, e_n\} = \{1, 2, \dots, n+k\} \setminus \{a_1+1, \dots, a_k+1\},\ 1 \leq e_1 < \cdots < e_n \leq n+k$$

$$\{c_1, \dots, c_n\} = \{1, 2, \dots, n+k+1\} \setminus \{a_1+1, \dots, a_{k+1}+1\},\ 1 \leq c_1 < \cdots < c_n \leq n+k+1$$

Suppose $d_{a_k} \cdots d_{a_1} L_n(0, 1, \dots, n) = L_{n+k}(0, e_1, \dots, e_n)$.

Note that

$$\begin{aligned} \{c_1, \dots, c_n\} &= \{1, 2, \dots . n+k+1\} \setminus \{a_1+1, \dots, a_{k+1}+1\} \\ &= (\{1, 2, \dots, n+k\} \setminus \{a_1+1, \dots, a_{k+1}+1\}) \cup (\{n+k+1\} \setminus \{a_1+1, \dots, a_{k+1}+1\}) \\ &= (\{e_1, \dots, e_n\} \setminus \{a_{k+1}+1\}) \cup (\{n+k+1\} \setminus \{a_{k+1}+1\}) \\ &= \{e_1, \dots, e_n, n+k+1\} \setminus \{a_{k+1}+1\}. \end{aligned}$$

Since $0 \leq a_{k+1} \leq n+k$ and $a_{k+1} \neq a_i$ for $1 \leq i \leq k$, then either $a_{k+1} = n+k$, or

$$a_{k+1}+1 \in \{1, \dots, n+k\} \setminus \{a_1+1, \dots, a_k+1\} = \{e_1, \dots, e_n\}.$$

If $a_{k+1} = n+k$, then $c_i = e_i$ for $1 \leq i \leq n$, and so,

$$d_{a_{k+1}} \cdots d_{a_1} L_n(0, 1, \dots, n) = d_{n+k} L_{n+k}(0, e_1, \dots, e_n) = L_{n+k+1}(0, c_1, \dots, c_n).$$

Assume $a_{k+1}+1 = e_j$ for some $1 \leq j \leq n$. Then

$$c_i = \begin{cases} e_i & \text{if } \ i < j \\ e_{i+1} & \text{if } \ j \leq i < n \\ n+k+1 & \text{if } \ i = n \end{cases}$$

Also,

$$0 \le a_1 < \cdots < a_k < a_{k+1} < a_{k+1} + 1 = e_j < \cdots < e_n \le n + k.$$

We then have $a_k < j + k - 1$, and so, $1 \le a_1 + 1 < \cdots < a_k + 1 < j + k$. So,

$$\{e_1, \ldots, e_n\} = \{1, 2, \ldots, n + k\} \setminus \{a_1 + 1, \ldots, a_k + 1\} \supset \{j + k, \ldots, n + k\}.$$

This shows that $e_i = i + k$ for $j \le i \le n$. Thus,

$$c_i = \begin{cases} e_i & \text{if } \ i < j \\ i + k + 1 & \text{if } \ j \le i \le n \end{cases}$$

and

$$\begin{aligned} d_{a_{k+1}} \cdots d_{a_1} L_n(0, 1, \ldots, n) &= d_{a_{k+1}} L_{n+k}(0, e_1, \ldots, e_n) \\ &= d_{j+k-1} L_{n+k}(0, e_1, \ldots, e_{j-1}, j + k, \ldots, n + k) \\ &= L_{n+k+1}(0, e_1, \ldots, e_{j-1}, j + k + 1, \ldots, n + k + 1) \\ &= L_{n+k+1}(0, c_1, \ldots, c_n). \end{aligned}$$

Then, by induction, the result follows. $\square$

Lemma 3.2.3.

$$\begin{aligned} &\sum_{0 \le a_1 < \cdots < a_k \le n+k-1} (-1)^{\sum_{i=1}^{k} a_i} d_{a_k} \ldots d_{a_1} L_n(0, 1, \ldots, n) \\ &= \sum_{1 \le c_1 < \cdots < c_n \le n+k} (-1)^{\sum_{i=1}^{n} c_i + \binom{n+k+1}{2} - k} L_{n+k}(0, c_1, \ldots, c_n). \end{aligned}$$

Proof. Notice that both sides of the equation have $\binom{n+k}{k} = \binom{n+k}{n}$ terms. Also, for each fixed $0 \le a_1 < \cdots < a_k \le n + k - 1$, there are unique $1 \le c_1 < \cdots < c_n \le n + k$ such that

$\{a_1+1,\ldots,a_k+1\}\cup\{c_1,\ldots,c_n\}=\{1,\ldots,n+k\}$. Furthermore, from Lemma 3.2.2, we have that

$$
\begin{aligned}
(-1)^{\sum\limits_{i=1}^{k} a_i} d_{a_k}\ldots d_{a_1} L_n(0,1,\ldots,n) &= (-1)^{-\left(-\sum\limits_{i=1}^{n+k} i+\sum\limits_{i=1}^{n+k} i-\sum\limits_{i=1}^{k}(a_k+1)+\sum\limits_{i=1}^{k} 1\right)} L_{n+k}(0,c_1,\ldots,c_n)\\
&= (-1)^{-\binom{n+k+1}{2}+\sum\limits_{i=1}^{n} c_i+k} L_{n+k}(0,c_1,\ldots,c_n)\\
&= (-1)^{\sum\limits_{i=1}^{n} c_i+\binom{n+k+1}{2}-k} L_{n+k}(0,c_1,\ldots,c_n)
\end{aligned}
$$

Since both sides of the equation have the same number of terms, and every term on the left is equal to a term on the right, we have that

$$
\begin{aligned}
&\sum_{0\le a_1<\cdots<a_k\le n+k-1} (-1)^{\sum\limits_{i=1}^{k} a_i} d_{a_k}\ldots d_{a_1} L_n(0,1,\ldots,n)\\
&= \sum_{1\le c_1<\cdots<c_n\le n+k} (-1)^{\sum\limits_{i=1}^{n} c_i+\binom{n+k+1}{2}-k} L_{n+k}(0,c_1,\ldots,c_n).
\end{aligned}
$$

$\square$

Proposition 3.2.4.

(a)

$$S^r L_0(0) = \frac{r!}{(2r)!} L_{2r}(0) - \frac{r!}{(2r)!}\sum_{k=1}^{2r}(-1)^k L_{2r}(k)$$

(b) For $n\ge 1$,

$$
\begin{aligned}
S^r L_n(0,1,\ldots,n) &= \frac{n!\,r!}{(n+2r)!}\sum_{1\le c_1<\cdots<c_n\le n+2r}(-1)^{\sum\limits_{i=1}^{n} c_i+\binom{n+1}{2}} L_{n+2r}(0,c_1,\ldots,c_n)\\
&+\frac{n!\,r!}{(n+2r)!}\sum_{l=n+1}^{n+2r}\sum_{1\le c_1<\cdots<c_n\le l-1}(-1)^{\sum\limits_{i=1}^{n} c_i+\binom{n+2}{2}+l} L_{n+2r}(l,c_1,\ldots,c_n).
\end{aligned}
$$

Proof.

(a)

$$
\begin{aligned}
S^r L_0(0) &= \frac{r!}{(2r)!}(-1)^r \sum_{0\le a_1<\dots<a_{2r}\le 2r} (-1)^{\sum_{i=1}^{2r} a_i} d_{a_{2r}}\dots d_{a_1} L_0(0) \\
&= \frac{r!}{(2r)!}\sum_{k=0}^{2r}(-1)^{r+\binom{2r+1}{2}-k} d_{2r}\dots d_{k+1}d_{k-1}\dots d_0 L_0(0) \\
&= \frac{r!}{(2r)!}\sum_{k=0}^{2r}(-1)^{-k} d_{2r}\dots d_{k+1} L_k(0) \\
&= \frac{r!}{(2r)!}L_{2r}(0) + \frac{r!}{(2r)!}\sum_{k=0}^{2r-1}(-1)^k d_{2r}\dots d_{k+2}L_{k+1}(k+1) \\
&= \frac{r!}{(2r)!}L_{2r}(0) - \frac{r!}{(2r)!}\sum_{k=1}^{2r}(-1)^k L_{2r}(k)
\end{aligned}
$$

(b) From the preceding Lemmas and Proposition 3.1.2, we have that

$$
\begin{aligned}
\frac{(n+2r)!}{n!\,r!}(-1)^r S^r L_n(0,1,\dots,n) &= \sum_{0\le a_1<\dots<a_{2r}\le n+2r} (-1)^{\sum_{i=1}^{2r} a_i} d_{a_{2r}}\cdots d_{a_1} L_n(0,1,\dots,n) \\
&= \sum_{0\le a_1<\dots<a_{2r}\le n+2r-1} (-1)^{\sum_{i=1}^{2r} a_i} d_{a_{2r}}\cdots d_{a_1} L_n(0,1,\dots,n) \\
&+ \sum_{l=2}^{2r}\sum_{0\le a_1<\dots<a_{l-1}\le n+l-2} (-1)^{\sum_{i=1}^{l-1} a_i+\sum_{i=l}^{2r}(n+i)} d_{n+2r}\dots d_{n+l}d_{a_{l-1}}\dots d_{a_1} L_n(0,1,\dots,n) \\
&+ (-1)^{\sum_{i=1}^{2r}(n+i)} d_{n+2r}\dots d_{n+1} L_n(0,1,\dots,n) \\
&= \sum_{1\le c_1<\dots<c_n\le n+2r} (-1)^{\sum_{i=1}^{n} c_i+\binom{n+2r+1}{2}-2r} L_{n+k}(0,c_1,\dots,c_n) \\
&+ \sum_{l=2}^{2r}\sum_{1\le c_1<\dots<c_n\le n+l-1} (-1)^{\sum_{i=1}^{n} c_i+\binom{n+l}{2}-(l-1)+\sum_{i=l}^{2r}(n+i)} d_{n+2r}\dots d_{n+l}L_{n+l-1}(0,c_1,\dots,c_n) \\
&+ (-1)^{2rn+r(2r+1)} d_{n+2r}\dots d_{n+1} L_n(0,1,\dots,n) \\
&= \sum_{1\le c_1<\dots<c_n\le n+2r} (-1)^{\sum_{i=1}^{n} c_i+\frac{(n+2r+1)(n+2r)}{2}} L_{n+k}(0,c_1,\dots,c_n) \\
&+ \sum_{l=n+2}^{n+2r}\sum_{1\le c_1<\dots<c_n\le l-1} (-1)^{\sum_{i=1}^{n} c_i+\binom{l}{2}-(l-n-1)+\sum_{i=l}^{n+2r} i} d_{n+2r}\dots d_l L_{l-1}(0,c_1,\dots,c_n)
\end{aligned}
$$

$$
\begin{aligned}
&+ (-1)^r d_{n+2r} \dots d_{n+1} L_n(0,1,\dots,n) \\
&= \sum_{1\le c_1<\dots<c_n\le n+2r} (-1)^{\sum_{i=1}^n c_i + \binom{n+1}{2} + (n+1)r + rn + 2r^2} L_{n+k}(0,c_1,\dots,c_n) \\
&+ \sum_{l=n+2}^{n+2r} \sum_{1\le c_1<\dots<c_n\le l-1} (-1)^{\sum_{i=1}^n c_i + \binom{n+2r+1}{2} - (l-n-1)} d_{n+2r}\dots d_{l+1} L_l(l,c_1,\dots,c_n) \\
&+ \sum_{1\le c_1<\dots<c_n\le n} (-1)^{2\binom{n+2}{2}+r} d_{n+2r}\dots d_{n+2} L_{n+1}(n+1,c_1,\dots,c_n) \\
&= \sum_{1\le c_1<\dots<c_n\le n+2r} (-1)^{\sum_{i=1}^n c_i + \binom{n+1}{2} + r} L_{n+k}(0,c_1,\dots,c_n) \\
&+ \sum_{l=n+2}^{n+2r} \sum_{1\le c_1<\dots<c_n\le l-1} (-1)^{\binom{n+1}{2}+r+n+1-l} L_{n+3r}(l,c_1,\dots,c_n) \\
&+ \sum_{1\le c_1<\dots<c_n\le n} (-1)^{\binom{n+1}{2}+\binom{n+2}{2}+n+1+r} L_{n+2r}(n+1,c_1,\dots,c_n) \\
&= \sum_{1\le c_1<\dots<c_n\le n+2r} (-1)^{\sum_{i=1}^n c_i + \binom{n+1}{2} + r} L_{n+k}(0,c_1,\dots,c_n) \\
&+ \sum_{l=n+2}^{n+2r} \sum_{1\le c_1<\dots<c_n\le l-1} (-1)^{\binom{n+2}{2}+r-l} L_{n+2r}(l,c_1,\dots,c_n) \\
&+ \sum_{1\le c_1<\dots<c_n\le n} (-1)^{\sum_{i=1}^n c_i + \binom{n+1}{2}+n+1+r} L_{n+2r}(n+1,c_1,\dots,c_n) \\
&= \sum_{1\le c_1<\dots<c_n\le n+2r} (-1)^{\sum_{i=1}^n c_i + \binom{n+1}{2} + r} L_{n+k}(0,c_1,\dots,c_n) \\
&+ \sum_{l=n+1}^{n+2r} \sum_{1\le c_1<\dots<c_n\le l-1} (-1)^{\binom{n+2}{2}+r+l} L_{n+2r}(l,c_1,\dots,c_n)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
S^r L_n(0,1,\dots,n) = &\frac{n!\,r!}{(n+2r)!} \sum_{1\le c_1<\dots<c_n\le n+2r} (-1)^{\sum_{i=1}^n c_i + \binom{n+1}{2}} L_{n+2r}(0,c_1,\dots,c_n) \\
&+ \frac{n!\,r!}{(n+2r)!} \sum_{l=n+1}^{n+2r} \sum_{1\le c_1<\dots<c_n\le l-1} (-1)^{\sum_{i=1}^n c_i + \binom{n+2}{2}+l} L_{n+2r}(l,c_1,\dots,c_n).
\end{aligned}
$$

$\square$

Chapter 4

Periodicity in a Poisson Algebra

Define the notation

$$
\begin{aligned}
K_N(c_0,c_1,\dots,c_n) &= (-1)^{\sum_{i=0}^{n} c_i+\binom{n+1}{2}} g_{\tau_0}\cdot_h\cdots\cdot_h g_{\tau_{c_0-1}}\cdot_h \prod_{i=0}^{n-1}{}_{\cdot_h}\left(f_{\sigma_i}\cdot_h g_{\tau_{c_i}}\cdot_h\cdots\cdot_h g_{\tau_{c_{i+1}-1}}\right)\\
&\quad\cdot_h f_{\sigma_n}\cdot_h g_{\tau_{c_n}}\cdot_h\cdots\cdot_h g_{\tau_N}\\
\begin{bmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{bmatrix}_\gamma &= [f_{\alpha_1},g_{\alpha_2}]\cdot_h\cdots\cdot_h[f_{\alpha_1+\gamma-1},g_{\alpha_2+\gamma-1}]\cdot_h[f_{\alpha_1+\gamma},f_{\alpha_1+\gamma+1}]\cdot_h\cdots\cdot_h[f_{\beta_1-1},f_{\beta_1}]\\
&\quad\cdot_h[g_{\alpha_2+\gamma},g_{\alpha_2+\gamma+1}]\cdot_h\cdots\cdot_h[g_{\beta_2-1},g_{\beta_2}]\\
\begin{Bmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{Bmatrix}_\gamma &= f_{\alpha_1}\cdot_h g_{\alpha_2}\cdot_h\begin{bmatrix}\alpha_1+1 & \beta_1\\ \alpha_2+1 & \beta_2\end{bmatrix}_\gamma\\
R_{n,r} &= \langle S^r(f_0\wedge\dots\wedge f_n), g_0\wedge\dots\wedge g_{n+2r}\rangle_{\cdot_h}\\
m_{n,r} &= \frac{n!\,r!}{(n+2r)!}\\
T_{n,r} &= m_{n,r}\sum_{0\le c_0<\cdots<c_n\le n+2r} K_{n+2r}(c_0,c_1,\dots,c_n)\\
T^{(0)}_{n,r} &= m_{n,r}\sum_{1\le c_1<\cdots<c_n\le n+2r} K_{n+2r}(0,c_1,\dots,c_n)
\end{aligned}
$$

4.1 Computations in $\cdot_h$

Here I shall prove some general results in computing in the product. The end goal is to express every long product, as a product of commutators. This will mainly come from thwo main facts:

(1) $f \cdot_h g = g \cdot f + [f, g]$

(2)

$$\begin{aligned}
\operatorname*{Alt}_{\sigma \in S^2} f_{\sigma_0} \cdot_h f_{\sigma_1} &= 2^{-1}\mathrm{Alt}(f_{\sigma_0} \cdot_h f_{\sigma_1} + f_{\sigma_0} \cdot_h f_{\sigma_1}) \\
&= 2^{-1}\mathrm{Alt}(f_{\sigma_0} \cdot_h f_{\sigma_1} - f_{\sigma_1} \cdot_h f_{\sigma_0}) \\
&= 2^{-1}\mathrm{Alt}\,[f_{\sigma_0}, f_{\sigma_1}]
\end{aligned}$$

Also of note, which will e shown later, is that every product of the form $f_0 \cdot_h g_0 \cdot_h \dots \cdot_h f_N \cdot_h g_N$, will have a term with N commutators that does not cancel, none of which are nested. Furthermore, any term with nested commutators can be expressed as combinations of terms with more than N commutators. Because of this, it is okay to assume that commutators will commute with anything, with respect to the product $\cdot_h$.

Lemma 4.1.1 (Properties of $K_n(c_0, \dots, c_n)$)**.**

(1) $K_{2r}(0) = 2^{-r} \left\{ \begin{matrix} 0 & 0 \\ 0 & 2r \end{matrix} \right\}_0$

(2) $K_{2r}(1) = -2^{-r} \left\{ \begin{matrix} 0 & 0 \\ 0 & 2r \end{matrix} \right\}_0 + 2^{-r} \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix}_1$

(3) $K_N(c_0, \dots, c_n) =$

$$\begin{cases}
2^{-1}K_{N-2}(c_0 - 2, \dots, c_n - 2) \begin{bmatrix} n+1 & n \\ N-1 & N \end{bmatrix}_0 & \textit{if } \; c_0 \geq 2 \\
2^{-1}K_{N-2}(c_0, \dots, c_i, c_{i+1} - 2, \dots, c_n - 2) \begin{bmatrix} n+1 & n \\ N-1 & N \end{bmatrix}_0 & \textit{if } \; c_i < c_{i+1} - 2 \\
2^{-1}K_{N-2}(c_0, \dots, c_n) \begin{bmatrix} n+1 & n \\ N-1 & N \end{bmatrix}_0 & \textit{if } \; c_n \leq N - 2
\end{cases}$$

(4) $K_{2r}(c_0) + K_{2r}(c_0 + 1) = 2^{-r} \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix}_1$

(5)

$$\begin{aligned}
&K_{n+2r}(c_0, \dots, c_i, \dots, c_n) + K_{n+2r}(c_0, \dots, c_i + 1, \dots, c_n) \\
&\qquad = K_{n+2r-1}(c_0, \dots, c_{i-1}, c_{i+1} - 1, \dots, c_{n-1}) \begin{bmatrix} n & n \\ n+2r & n+2r \end{bmatrix}_1
\end{aligned}$$

(6) $K_1(0,1) = 2^{-2} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}_0 + \begin{Bmatrix} 0 & 1 \\ 0 & 1 \end{Bmatrix}_1$

(7) For $n \geq 2$,

$$\operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} K_{n+2r}(c_0,\dots,c_i,c_i+1,\dots,c_n)$$

$$= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} 2^{-2} K_{n+2r-2}(c_0,\dots,c_{i-1},c_{i+2}-2,\dots,c_n-2) \begin{bmatrix} n-1 & n \\ n+2r-1 & n+2r \end{bmatrix}_0$$

$$+ \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} K_{n+2r-1}(c_0,\dots,c_i,c_{i+2}-1,\dots,c_n-1) \begin{bmatrix} n & n \\ n+2r & n+2r \end{bmatrix}_1$$

(8) $K_3(0,2) = -2^{-3} \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}_0$

(9) For $n \geq 2$,

$$K_N(0,2,c_2,\dots,c_n) = -2^{-2} K_{N-2}(c_2-2,\dots,c_n-2) \begin{bmatrix} n-1 & n \\ N-1 & N \end{bmatrix}_0$$

Proof.

(1)

$$\operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} K_{2r}(0) = \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} f_{\sigma_0} \cdot_h g_{\tau_0} \cdot_h g_{\tau_1} \cdot_h \cdots \cdot_h g_{\tau_{2r}}$$

$$= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} f_{\sigma_0} \cdot_h g_{\tau_0} \cdot_h 2^{-1}\left[g_{\tau_1}, g_{\tau_2}\right] \cdot_h \cdots \cdot_h 2^{-1}\left[g_{\tau_{2r-1}}, g_{\tau_{2r}}\right]$$

$$= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} 2^{-r} \begin{Bmatrix} 0 & 0 \\ 0 & 2r \end{Bmatrix}_0$$

(2)

$$\operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} K_{2r}(1) = \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} -g_{\tau_0} \cdot_h f_{\sigma_0} \cdot_h g_{\tau_1} \cdot_h \cdots \cdot_h g_{\tau_{2r}}$$

$$= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} -\left(f_{\sigma_0} \cdot_h g_{\tau_0} + [g_{\tau_0}, f_{\sigma_0}]\right) \cdot_h 2^{-1}\left[g_{\tau_1}, g_{\tau_2}\right] \cdot_h \cdots \cdot_h 2^{-1}\left[g_{\tau_{2r-1}}, g_{\tau_{2r}}\right]$$

$$= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} -2^{-r} \begin{Bmatrix} 0 & 0 \\ 0 & 2r \end{Bmatrix}_0 + 2^{-r} \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix}_1$$

(3) Suppose $c_0 \geq 2$. Applying the permutation $(N\ N-1\ \ldots\ 0)$ twice to τ, we have have

$$\begin{aligned}
\operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{N+1}}} K_N(c_0,\ldots,c_n) &= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{N+1}}} (-1)^{\sum_{i=0}^{n} c_i + \binom{n+1}{2}} g_{\tau_0} \cdot_h g_{\tau_1} \cdot_h \cdots \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{N+1}}} (-1)^{2(N+1)}(-1)^{2(n+1)+\sum_{i=0}^{n}(c_i-2)+\binom{n+1}{2}} g_{\tau_{N-1}} \cdot_h g_{\tau_N} \cdot_h \cdots \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{N+1}}} 2^{-1} \left[g_{\tau_{N-1}}, g_{\tau_N}\right] \cdot_h K_{N-2}(c_0-2,\ldots,c_n-2) \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{N+1}}} 2^{-1} K_{N-2}(c_0-2,\ldots,c_n-2) \left[\begin{array}{c|c} n+1 & n \\ N-1 & N \end{array}\right]_0
\end{aligned}$$

The other two cases are nearly identical, except the applied permutations would be $(N\ N-1\ \ldots\ c_i)$ and $(N\ N-1\ \ldots\ c_n)$ for $c_i < c_{i+1}-2$ and $c_n \leq N-2$, respectively.

(4)

$$\begin{aligned}
&\operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} K_{2r}(c_0) + K_{2r}(c_0+1) \\
&\quad= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} g_{\tau_0} \cdot_h \cdots g_{\tau_{c_0-1}} \cdot_h \left((-1)^{c_0} f_{\sigma_0} \cdot_h g_{\tau_{c_0}} + (-1)^{c_0+1} g_{\tau_{c_0}} \cdot_h f_{\sigma_0}\right) \cdot_h g_{\tau_{c_0+1}} \cdots g_{\tau_{2r}} \\
&\quad= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} (-1)^{c_0} g_{\tau_0} \cdot_h \cdots g_{\tau_{c_0-1}} \cdot_h \left[f_{\sigma_0}, g_{\tau_{c_0}}\right] \cdot_h g_{\tau_{c_0+1}} \cdots g_{\tau_{2r}} \\
&\quad= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} \left[f_{\sigma_0}, g_{\tau_0}\right] \cdot_h g_{\tau_1} \cdot_h \cdots \cdot_h g_{\tau_{2r}} \\
&\quad= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} \left[f_{\sigma_0}, g_{\tau_0}\right] \cdot_h 2^{-1} \left[g_{\tau_1}, g_{\tau_2}\right] \cdot_h \cdots \cdot_h 2^{-}1 \left[g_{\tau_{2r-1}}, g_{\tau_{2r}}\right] \\
&\quad= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_1\\ \tau\in S_{2r+1}}} 2^{-r} \left[\begin{array}{c|c} 0 & 0 \\ 0 & 2r \end{array}\right]_1
\end{aligned}$$

(5) Applying the permutations $(n\ n-1\ \ldots\ i)$ to σ and $(n+2r\ n+2r-1\ \ldots\ c_i)$ to τ, we have

$$\operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} K_{n+2r}(c_0,\ldots,c_i,\ldots,c_n) + K_{2r}(c_0,\ldots,c_i+1,\ldots,c_n)$$

$$= \operatorname{Tr} \underset{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}}{\operatorname{Alt}} (-1)^{\sum_{j=0}^{n} c_j + \binom{n+1}{2}} \cdots \cdot_h f_{\sigma_0} \cdot_h g_{\tau_{c_0}} \cdot_h \cdots \cdot_h \left(f_{\sigma_i} \cdot_h g_{\tau_{c_i}} - g_{\tau_{c_i}} \cdot_h f_{\sigma_i} \right)$$

$$\cdot_h \cdots \cdot_h f_{\sigma_n} \cdot_h g_{\tau_{c_n}} \cdot_h \cdots$$

$$= \operatorname{Tr} \underset{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}}{\operatorname{Alt}} (-1)^{\sum_{j=0}^{n} c_j + \binom{n+1}{2}} \cdots \cdot_h f_{\sigma_0} \cdot_h g_{\tau_{c_0}} \cdot_h \cdots \cdot_h \left[f_{\sigma_i}, g_{\tau_{c_i}} \right]$$

$$\cdot_h \cdots \cdot_h f_{\sigma_n} \cdot_h g_{\tau_{c_n}} \cdot_h \cdots$$

$$= \operatorname{Tr} \underset{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}}{\operatorname{Alt}} (-1)^{\sum_{j=0}^{n} c_j + \binom{n+1}{2}} (-1)^{(n-i)+(n+2r-c_i)} \cdots \cdot_h f_{\sigma_0} \cdot_h g_{\tau_{c_0}} \cdot_h \cdots \cdot_h f_{\sigma_{i-1}} \cdot_h g_{\tau_{c_{i-1}}}$$

$$\cdot_h \cdots \cdot_h f_{\sigma_i} \cdot_h g_{\tau_{c_{i+1}-1}} \cdot_h \cdots \cdot_h f_{\sigma_{n-1}} \cdot_h g_{\tau_{c_n-1}} \cdot_h \cdots \cdot_h \left[f_{\sigma_n}, g_{\tau_{n+2r}} \right]$$

$$= \operatorname{Tr} \underset{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}}{\operatorname{Alt}} (-1)^{\sum_{j=0}^{i-1} c_j + \sum_{j=i+1}^{n} (c_j-1) + \binom{n}{2}} \cdots \cdot_h f_{\sigma_0} \cdot_h g_{\tau_{c_0}} \cdot_h \cdots \cdot_h f_{\sigma_{i-1}} \cdot_h g_{\tau_{c_{i-1}}}$$

$$\cdot_h \cdots \cdot_h f_{\sigma_i} \cdot_h g_{\tau_{c_{i+1}-1}} \cdot_h \cdots \cdot_h f_{\sigma_{n-1}} \cdot_h g_{\tau_{c_n-1}} \cdot_h \cdots \cdot_h \left[f_{\sigma_n}, g_{\tau_{n+2r}} \right]$$

$$= \operatorname{Tr} \underset{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}}{\operatorname{Alt}} K_{n+2r-1}(c_0 \ldots, c_{i-1}, c_{i+1}-1, \ldots, c_n - 1) \left[\begin{array}{r|l} n & n \\ n{+}2r & n{+}2r \end{array} \right]_1$$

(6)

$$\operatorname{Tr} \underset{\substack{\sigma\in S_2\\ \tau\in S_2}}{\operatorname{Alt}} K_1(0,1) = \operatorname{Tr} \underset{\substack{\sigma\in S_2\\ \tau\in S_2}}{\operatorname{Alt}} (-1)^2 f_{\sigma_0} \cdot_h g_{\tau_0} \cdot_h f_{\sigma_1} \cdot_h g_{\tau_1}$$

$$= \operatorname{Tr} \underset{\substack{\sigma\in S_2\\ \tau\in S_2}}{\operatorname{Alt}} f_{\sigma_0} \cdot_h g_{\tau_0} \cdot_h \left(g_{\tau_1} \cdot_h f_{\sigma_1} + [f_{\sigma_1}, g_{\tau_1}] \right)$$

$$= \operatorname{Tr} \underset{\substack{\sigma\in S_2\\ \tau\in S_2}}{\operatorname{Alt}} f_{\sigma_0} \cdot_h 2^{-1} [g_{\tau_0}, g_{\tau_1}] \cdot_h f_{\sigma_1} + f_{\sigma_0} \cdot_h g_{\tau_0} \cdot_h [f_{\sigma_1}, g_{\tau_1}]$$

$$= \operatorname{Tr} \underset{\substack{\sigma\in S_2\\ \tau\in S_2}}{\operatorname{Alt}} 2^{-1} f_{\sigma_0} \cdot_h f_{\sigma_1} \cdot_h [g_{\tau_0}, g_{\tau_1}] + \left\{ \begin{array}{c|c} 0 & 1 \\ 0 & 1 \end{array} \right\}_1$$

$$= \operatorname{Tr} \underset{\substack{\sigma\in S_2\\ \tau\in S_2}}{\operatorname{Alt}} 2^{-2} [f_{\sigma_0}, f_{\sigma_1}] \cdot_h [g_{\tau_0}, g_{\tau_1}] + \left\{ \begin{array}{c|c} 0 & 1 \\ 0 & 1 \end{array} \right\}_1$$

$$= \operatorname{Tr} \underset{\substack{\sigma\in S_2\\ \tau\in S_2}}{\operatorname{Alt}} 2^{-2} \left[\begin{array}{c|c} 0 & 1 \\ 0 & 1 \end{array} \right]_0 + \left\{ \begin{array}{c|c} 0 & 1 \\ 0 & 1 \end{array} \right\}_1$$

(7)

$$\begin{aligned}
&\operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} K_{n+2r}(c_0,\dots,c_i,c_i+1,\dots,c_n)\\
&= \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} (-1)^{\sum\limits_{j=0}^{i-1} c_j+2c_i+1+\sum\limits_{j=i+2}^{n} c_j+\binom{n+1}{2}} g_{\tau_0}\cdot_h\cdots\cdot_h f_{\sigma_0}\cdot_h g_{\tau_{c_0}}\cdot_h\cdots\\
&\quad \cdot_h f_{\sigma_i}\cdot_h g_{\tau_{c_i}}\cdot_h f_{\sigma_{i+1}}\cdot_h g_{\tau_{c_i+1}}\cdot_h\cdots\cdot_h f_{\sigma_n}\cdot_g g_{\tau_{c_n}}\cdot_h\cdots\cdot_h g_{\tau_{n+2r}}\\
&= \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} (-1)^{1+\sum\limits_{j=0}^{i-1} c_j+\sum\limits_{j=i+2}^{n} c_j+\binom{n+1}{2}} g_{\tau_0}\cdot_h\cdots\cdot_h f_{\sigma_0}\cdot_h g_{\tau_{c_0}}\cdot_h\cdots\\
&\quad \cdot_h f_{\sigma_i}\cdot_h g_{\tau_{c_i}}\cdot_h \left(g_{\tau_{c_i+1}}\cdot_h f_{\sigma_{i+1}}+\left[f_{\sigma_{i+1}},g_{\tau_{c_i+1}}\right]\right)\cdot_h\cdots\cdot_h f_{\sigma_n}\cdot_g g_{\tau_{c_n}}\cdot_h\cdots\cdot_h g_{\tau_{n+2r}}\\
&= \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} 2^{-2}(-1)^{1+\sum\limits_{j=0}^{i-1} c_j+\sum\limits_{j=i+2}^{n} c_j+(2n-1)+\binom{n-1}{2}} g_{\tau_0}\cdot_h\cdots\cdot_h f_{\sigma_0}\cdot_h g_{\tau_{c_0}}\cdot_h\cdots\\
&\quad \cdot_h \left[f_{\sigma_i},f_{\sigma_{i+1}}\right]\cdot_h\left[g_{\tau_{c_i}},g_{\tau_{c_i+1}}\right]\cdot_h\cdots\cdot_h f_{\sigma_n}\cdot_g g_{\tau_{c_n}}\cdot_h\cdots\cdot_h g_{\tau_{n+2r}}\\
&+ \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} (-1)^{1+\sum\limits_{j=0}^{i-1} c_j+\sum\limits_{j=i+2}^{n} c_j+n+\binom{n}{2}} g_{\tau_0}\cdot_h\cdots\cdot_h f_{\sigma_0}\cdot_h g_{\tau_{c_0}}\cdot_h\cdots\\
&\quad \cdot_h f_{\sigma_i}\cdot_h g_{\tau_{c_i}}\cdot_h\left[f_{\sigma_{i+1}},g_{\tau_{c_i+1}}\right]\cdot_h\cdots\cdot_h f_{\sigma_n}\cdot_g g_{\tau_{c_n}}\cdot_h\cdots\cdot_h g_{\tau_{n+2r}}\\
&= \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} 2^{-2}(-1)^{\sum\limits_{j=0}^{i-1} c_j+\sum\limits_{j=i+2}^{n}(c_j-2)+2(n-i-1)+\binom{n-1}{2}}(-1)^{2(n-i)+2(n+2r-c_i)}\\
&\quad g_{\tau_0}\cdot_h\cdots\cdot_h f_{\sigma_0}\cdot_h g_{\tau_{c_0}}\cdot_h\cdots f_{\sigma_{i-1}}\cdot_h g_{\tau_{c_{i-1}}}\cdot_h\cdots\cdot_h f_{\sigma_i}\cdot_h g_{\tau_{c_{i+2}-2}}\cdot_h\cdots\\
&\quad \cdot_h f_{\sigma_{n-2}}\cdot_g g_{\tau_{c_n-2}}\cdot_h\cdots\cdot_h g_{\tau_{n+2r}}\cdot_h\left[f_{\sigma_{n-1}},f_{\sigma_n}\right]\cdot_h\left[g_{\tau_{n+2r-1}},g_{\tau_{n+2r}}\right]\\
&+ \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} (-1)^{n+1-c_i+\sum\limits_{j=0}^{i} c_j+\sum\limits_{j=i+2}^{n}(c_j-1)+(n-i-1)+\binom{n}{2}}(-1)^{(n-i-1)+(n+2r-c_i-1)}\\
&\quad g_{\tau_0}\cdot_h\cdots\cdot_h f_{\sigma_0}\cdot_h g_{\tau_{c_0}}\cdot_h\cdots\cdot_h f_{\sigma_i}\cdot_h g_{\tau_{c_i}}\cdot_h\cdots\\
&\quad \cdot_h f_{\sigma_{i+1}}\cdot_h g_{\tau_{c_{i+2}-1}}\cdot_h\cdots\cdot_h f_{\sigma_{n-1}}\cdot_g g_{\tau_{c_n-1}}\cdot_h\cdots\cdot_h g_{\tau_{n+2r-1}}\cdot_h\left[f_{\sigma_n},g_{\tau_{n+2r}}\right]\\
&= \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} 2^{-2}K_{n+2r-2}(c_0,\dots,c_{i-1},c_{i+2}-2,\dots,c_n-2)\left[\begin{array}{r|l} n-1 & n\\ n+2r-1 & n+2r\end{array}\right]_0\\
&+ \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} K_{n+2r-1}(c_0,\dots,c_i,c_{i+2}-1,\dots,c_n-1)\left[\begin{array}{r|l} n & n\\ n+2r & n+2r\end{array}\right]_1
\end{aligned}$$

(8)

$$\begin{aligned}
\operatorname*{Tr\,Alt}_{\substack{\sigma\in S_2\\ \tau\in S_4}} K_3(0,2) &= \operatorname*{Tr\,Alt}_{\substack{\sigma\in S_2\\ \tau\in S_4}} (-1)^3 f_{\sigma_0}\cdot_h g_{\tau_0}\cdot_h g_{\tau_1}\cdot_h f_{\sigma_1}\cdot_h g_{\tau_2}\cdot_h g_{\tau_3}\\
&= \operatorname*{Tr\,Alt}_{\substack{\sigma\in S_2\\ \tau\in S_4}} - f_{\sigma_0}\cdot_h 2^{-1}\left[g_{\tau_0}, g_{\tau_1}\right]\cdot_h f_{\sigma_1}\cdot_h 2^{-1}\left[g_{\tau_2}, g_{\tau_3}\right]\\
&= \operatorname*{Tr\,Alt}_{\substack{\sigma\in S_2\\ \tau\in S_4}} - 2^{-2} f_{\sigma_0}\cdot_h f_{\sigma_1}\cdot_h \left[g_{\tau_0}, g_{\tau_1}\right]\cdot_h \left[g_{\tau_2}, g_{\tau_3}\right]\\
&= \operatorname*{Tr\,Alt}_{\substack{\sigma\in S_2\\ \tau\in S_4}} - 2^{-3}\left[f_{\sigma_0}, f_{\sigma_1}\right]\cdot_h \left[g_{\tau_0}, g_{\tau_1}\right]\cdot_h \left[g_{\tau_2}, g_{\tau_3}\right]\\
&= \operatorname*{Tr\,Alt}_{\substack{\sigma\in S_2\\ \tau\in S_4}} - 2^{-3}\left[\begin{array}{c|c} 0 & 1\\ 0 & 3\end{array}\right]_0
\end{aligned}$$

(9)

$$\begin{aligned}
&\operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} K_{n+2r}(c_0,\dots,c_i,c_i+2,\dots,c_n)\\
&= \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} (-1)^{2(c_i+1)+\sum_{j=0}^{i-1} c_j+\sum_{j=i+2}^{n} c_j+\binom{n+1}{2}} g_{\tau_0}\cdot_h\cdots\cdot_h f_{\sigma_0}\cdot_h g_{\tau_{c_0}}\cdot_h\cdots f_{\sigma_i}\cdot_h g_{\tau_{c_i}}\\
&\qquad \cdot_h g_{\tau_{c_i+1}}\cdot_h f_{\sigma_{i+1}}\cdot_h g_{\tau_{c_i+2}}\cdot_h\cdots\cdot_h f_{\sigma_{i+2}}\cdot_h g_{\tau_{c_{i+2}}}\cdot_h\cdots\cdot_h f_{\sigma_n}\cdot_h g_{\tau_{c_n}}\cdot_h\cdots\cdot_h g_{\tau_{n+2r}}\\
&= \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} 2^{-2}(-1)^{2(n-i-1)+\sum_{j=0}^{i-1} c_j+\sum_{j=i+2}^{n}(c_j-2)+(2n-1)+\binom{n-1}{2}} g_{\tau_0}\cdot_h\cdots\cdot_h f_{\sigma_0}\cdot_h g_{\tau_{c_0}}\\
&\qquad \cdot_h\cdots\left[f_{\sigma_i}, f_{\sigma_{i+1}}\right]\cdot_h\left[g_{\tau_{c_i}}, g_{\tau_{c_i+1}}\right]\cdot_h g_{\tau_{c_i+2}}\cdot_h\cdots\cdot_h f_{\sigma_{i+2}}\cdot_h g_{\tau_{c_{i+2}}}\cdot_h\cdots\\
&\qquad \cdot_h f_{\sigma_n}\cdot_h g_{\tau_{c_n}}\cdot_h\cdots\cdot_h g_{\tau_{n+2r}}\\
&= \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} -2^{-2}(-1)^{\sum_{j=0}^{i-1} c_j+\sum_{j=i+2}^{n}(c_j-2)+\binom{n-1}{2}}(-1)^{2(n-i)+2(n+2r-c_i)} g_{\tau_0}\cdot_h\cdots\\
&\qquad \cdot_h f_{\sigma_0}\cdot_h g_{\tau_{c_0}}\cdot_h\cdots\cdot_h f_{\sigma_{i-1}}\cdot_h g_{\tau_{c_{i-1}}}\cdot_h\cdots\cdot_h f_{\sigma_i}\cdot_h g_{\tau_{c_{i+2}-2}}\cdot_h\cdots\\
&\qquad \cdot_h f_{\sigma_{n-2}}\cdot_h g_{\tau_{c_n-2}}\cdot_h\cdots\cdot_h g_{\tau_{n+2r-2}}\cdot_h\left[f_{\sigma_{n-1}}, f_{\sigma_n}\right]\cdot_h\left[g_{\tau_{n+2r-1}}, g_{\tau_{n+2r}}\right]\\
&= \operatorname{Tr}\operatorname*{Alt}_{\substack{\sigma\in S_{n+1}\\ \tau\in S_{n+2r+1}}} -2^{-2}K_{n+2r-2}(c_0,\dots,c_{i-1},c_{i+2}-2,\dots,c_n-2)\left[\begin{array}{c|c} n-1 & n\\ n+2r-1 & n+2r\end{array}\right]_0
\end{aligned}$$

$\square$

Lemma 4.1.2.

(1) $$\operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{\theta_1-\alpha_1+1}\\ \tau\in S_{\theta_2-\alpha_2+1}}} \left[\begin{array}{c|c}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{array}\right]_\gamma \left[\begin{array}{c|c}\beta_1{+}1 & \theta_1\\ \beta_2{+}1 & \theta_2\end{array}\right]_\lambda = \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{\theta_1-\alpha_1+1}\\ \tau\in S_{\theta_2-\alpha_2+1}}} \left[\begin{array}{c|c}\alpha_1 & \theta_1\\ \alpha_2 & \theta_2\end{array}\right]_{\gamma+\lambda}$$

(2) $$\operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{\theta_1-\alpha_1+1}\\ \tau\in S_{\theta_2-\alpha_2+1}}} \left\{\begin{array}{c|c}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{array}\right\}_\gamma \left[\begin{array}{c|c}\beta_1{+}1 & \theta_1\\ \beta_2{+}1 & \theta_2\end{array}\right]_\lambda = \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{\theta_1-\alpha_1+1}\\ \tau\in S_{\theta_2-\alpha_2+1}}} \left\{\begin{array}{c|c}\alpha_1 & \theta_1\\ \alpha_2 & \theta_2\end{array}\right\}_{\gamma+\lambda}$$

Proof.

(1) Notice that both sides of the equation are products of commutators. Each side have

$$(\beta_1-\alpha_1+1-\gamma)+(\theta_1-\beta_1-\lambda)=\theta_1-\alpha_1+1-\gamma-\lambda$$

commutators of the form $[f_i, f_{i+1}]$,

$$(\beta_2-\alpha_2+1-\gamma)+(\theta_2-\beta_2-\lambda)=\theta_2-\alpha_2+1-\gamma-\lambda$$

commutators of the form $[g_i, g_{i+1}]$, and $\gamma+\lambda$ commutators of the form $[f_i, g_i]$. So, the two sides only differ by the order of the commutators. But, since commutators commute up to a higher-order term in h, the two sides must be equal.

(2)

$$\begin{aligned}
\operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{\theta_1-\alpha_1+1}\\ \tau\in S_{\theta_2-\alpha_2+1}}} \left\{\begin{array}{c|c}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{array}\right\}_\gamma \left[\begin{array}{c|c}\beta_1{+}1 & \theta_1\\ \beta_2{+}1 & \theta_2\end{array}\right]_\lambda &= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{\theta_1-\alpha_1+1}\\ \tau\in S_{\theta_2-\alpha_2+1}}} f_{\alpha_1} \cdot_h g_{\alpha_2} \cdot_h \left[\begin{array}{c|c}\alpha_1{+}1 & \beta_1\\ \alpha_2{+}1 & \beta_2\end{array}\right]_\gamma \left[\begin{array}{c|c}\beta_1{+}1 & \theta_1\\ \beta_2{+}1 & \theta_2\end{array}\right]_\lambda \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{\theta_1-\alpha_1+1}\\ \tau\in S_{\theta_2-\alpha_2+1}}} f_{\alpha_1} \cdot_h g_{\alpha_2} \cdot_h \left[\begin{array}{c|c}\alpha_1{+}1 & \theta_1\\ \alpha_2{+}1 & \theta_2\end{array}\right]_{\gamma+\lambda} \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma\in S_{\theta_1-\alpha_1+1}\\ \tau\in S_{\theta_2-\alpha_2+1}}} \left\{\begin{array}{c|c}\alpha_1 & \theta_1\\ \alpha_2 & \theta_2\end{array}\right\}_{\gamma+\lambda}
\end{aligned}$$

$\square$

4.2 Recurrence Relation Computations for Sums over $\cdot_h$

Lemma 4.2.1 (Properties of $T_{n,r}$ and $T^{(0)}_{n,r}$)**.**

(1) For $n \geq 2$,

$$T_{n,0} = 2^{-2} T_{n-2,0} \left[\begin{array}{c|c} n-1 & n \\ n-1 & n \end{array}\right]_0 + T_{n-1,0} \left[\begin{array}{c|c} n & n \\ n & n \end{array}\right]_1$$

(2)

$$T_{n,0} = \sum_{M=1}^{\lfloor \frac{n+1}{2} \rfloor} 2^{-2(M+1)} \binom{n-M}{n+1-2M} \left[\begin{array}{c|c} 0 & n \\ 0 & n \end{array}\right]_{n+1-2M} + \sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} 2^{-2M} \binom{n-M}{n-2M} \left\{\begin{array}{c|c} 0 & n \\ 0 & n \end{array}\right\}_{n-2M}$$

(3) $T_{0,r} = m_{0,r} 2^{-r} \left(\left\{\begin{array}{c|c} 0 & 0 \\ 0 & 2r \end{array}\right\}_0 + r \left[\begin{array}{c|c} 0 & 0 \\ 0 & 2r \end{array}\right]_1 \right)$

(4) $T_{1,r} = m_{1,r} 2^{-r} \left(2^{-2}(r+1) \left[\begin{array}{c|c} 0 & 1 \\ 0 & 2r+1 \end{array}\right]_0 + \binom{r+1}{2} \left[\begin{array}{c|c} 0 & 1 \\ 0 & 2r+1 \end{array}\right]_2 + (r+1) \left\{\begin{array}{c|c} 0 & 1 \\ 0 & 2r+1 \end{array}\right\}_1 \right)$

(5) For $n \geq 2$ and $r \geq 1$,

$$\begin{aligned} T_{n,r} = & \frac{m_{n,r}}{m_{n-1,r}} T_{n-1,r} \left[\begin{array}{c|c} n & n \\ n+2r & n+2r \end{array}\right]_1 + 2^{-2} \frac{m_{n,r}}{m_{n-2,r}} T_{n-2,r} \left[\begin{array}{c|c} n-1 & n \\ n+2r-1 & n+2r \end{array}\right]_0 \\ & - 2^{-1} \frac{m_{n,r}}{m_{n,r-1}} T_{n,r-1} \left[\begin{array}{c|c} n+1 & n \\ n+2r-1 & n+2r \end{array}\right]_0 \end{aligned}$$

(6)

$$\begin{aligned} T_{n,r} = & m_{n,r} 2^{-r} \sum_{M=0}^{\lfloor \frac{n+1}{2} \rfloor} 2^{-2(M+1)} \binom{r+M}{M} \binom{r+n-M}{n+1-2M} \left[\begin{array}{c|c} 0 & n \\ 0 & n+2r \end{array}\right]_{n+1-2M} \\ & + m_{n,r} 2^{-r} \sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} 2^{-2M} \binom{r+n-M}{r, n-2M, M} \left\{\begin{array}{c|c} 0 & n \\ 0 & n+2r \end{array}\right\}_{n-2M} \end{aligned}$$

(7)

$$\begin{aligned} T^{(0)}_{n,0} = & \sum_{M=1}^{\lfloor \frac{n+1}{2} \rfloor} 2^{-2M} \binom{n-M}{0, n+1-2M, M-1} \left[\begin{array}{c|c} 0 & n \\ 0 & n+2r \end{array}\right]_{n+1-2M} \\ & + \sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} 2^{-2M} \binom{n-M}{0, n-2M, M} \left\{\begin{array}{c|c} 0 & n \\ 0 & n+2r \end{array}\right\}_{n-2M} \end{aligned}$$

(8) $T^{(0)}_{1,r} = m_{1,r}2^{-r}\left(2^{-2}\binom{r}{r,0,0}\begin{bmatrix}0 & 1\\ 0 & 1+2r\end{bmatrix}_0 + \binom{r+1}{r,1,0}\begin{Bmatrix}0 & 1\\ 0 & 1+2r\end{Bmatrix}_1\right)$

(9) $T^{(0)}_{2,r} = m_{2,r}2^{-r}\left(2^{-2}\binom{r+1}{r,1,0}\begin{bmatrix}0 & 2\\ 0 & 2+2r\end{bmatrix}_1 + 2^{-2}\binom{r+1}{r,0,1}\begin{Bmatrix}0 & 2\\ 0 & 2+2r\end{Bmatrix}_0 + \binom{r+2}{r,2,0}\begin{Bmatrix}0 & 2\\ 0 & 2+2r\end{Bmatrix}_2\right)$

(10) *For* $n \geq 3$ *and* $r \geq 1$,

$$\begin{aligned}T^{(0)}_{n,r} &= 2^{-2}\frac{m_{n,r}}{m_{n-2,r}}T^{(0)}_{n-2,r}\begin{bmatrix}n-1 & n\\ n+2r-1 & n+2r\end{bmatrix}_0 + \frac{m_{n,r}}{m_{n-1,r}}T^{(0)}_{n-1,r}\begin{bmatrix}n & n\\ n+2r & n+2r\end{bmatrix}_1\\ &+ 2^{-1}\frac{m_{n,r}}{m_{n,r-1}}T^{(0)}_{n,r-1}\begin{bmatrix}n+1 & n\\ n+2r-1 & n+2r\end{bmatrix}_0\end{aligned}$$

(11)

$$\begin{aligned}T^{(0)}_{n,r} &= m_{n,r}2^{-r}\sum_{M=1}^{\lfloor\frac{n+1}{2}\rfloor}2^{-2M}\binom{r+n-M}{r,n+1-2M,M-1}\begin{bmatrix}0 & n\\ 0 & n+2r\end{bmatrix}_{n+1-2M}\\ &+ m_{n,r}2^{-r}\sum_{M=0}^{\lfloor\frac{n}{2}\rfloor}2^{-2M}\binom{r+n-M}{r,n-2M,M}\begin{Bmatrix}0 & n\\ 0 & n+2r\end{Bmatrix}_{n-2M}\end{aligned}$$

Proof.

(1)

$$\begin{aligned}T_{n,0} &= m_{n,0}K_n(0,1,\ldots,n)\\ &= 1\cdot\left(2^{-2}K_{n-2}(0,1,\ldots,n-2)\begin{bmatrix}n-1 & n\\ n-1 & n\end{bmatrix}_0 + K_{n-1}(0,1,\ldots,n-1)\begin{bmatrix}n & n\\ n & n\end{bmatrix}_1\right)\\ &= 2^{-2}T_{n-2,0}\begin{bmatrix}n-1 & n\\ n-1 & n\end{bmatrix}_0 + T_{n-1,0}\begin{bmatrix}n & n\\ n & n\end{bmatrix}_1\end{aligned}$$

(2) Proceeding by induction, we have

$$\begin{aligned}T_{0,0} &= K_0(0) = f_0 \cdot_h g_0 = \begin{Bmatrix}0 & 0\\ 0 & 0\end{Bmatrix}_0\\ T_{1,0} &= K_1(0,1) = 2^{-2}\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}_0 + \begin{Bmatrix}0 & 1\\ 0 & 1\end{Bmatrix}_1\end{aligned}$$

Now, suppose for some $N \geq 2$,

$$T_{n,0} = \sum_{M=1}^{\lfloor \frac{n+1}{2} \rfloor} 2^{-2M} \binom{n-M}{n+1-2M} \left[\begin{array}{c|c} 0 & n \\ 0 & n \end{array}\right]_{n+1-2M} + \sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} 2^{-2M} \binom{n-M}{n-2M} \left\{\begin{array}{c|c} 0 & n \\ 0 & n \end{array}\right\}_{n-2M}$$

for all $n < N$. Then

$$T_{N,0} = 2^{-2} T_{N-2} \left[\begin{array}{c|c} N-1 & N \\ N-1 & N \end{array}\right]_0 + T_{N-1} \left[\begin{array}{c|c} N & N \\ N & N \end{array}\right]_1$$

$$\begin{aligned}
&= 2^{-2} \sum_{M=1}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2M} \binom{N-2-M}{N-1-2M} \left[\begin{array}{c|c} 0 & N-2 \\ 0 & N-2 \end{array}\right]_{N-1-2M} \left[\begin{array}{c|c} N-1 & N \\ N-1 & N \end{array}\right]_0 \\
&+ 2^{-2} \sum_{M=0}^{\lfloor \frac{N-2}{2} \rfloor} 2^{-2M} \binom{N-2-M}{N-2-2M} \left\{\begin{array}{c|c} 0 & N-2 \\ 0 & N-2 \end{array}\right\}_{N-2-2M} \left[\begin{array}{c|c} N-1 & N \\ N-1 & N \end{array}\right]_0 \\
&+ \sum_{M=1}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{N-1-M}{N-2M} \left[\begin{array}{c|c} 0 & N-1 \\ 0 & N-1 \end{array}\right]_{N-2M} \left[\begin{array}{c|c} N & N \\ N & N \end{array}\right]_1 \\
&+ \sum_{M=0}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2M} \binom{N-1-M}{N-1-2M} \left\{\begin{array}{c|c} 0 & N-1 \\ 0 & N-1 \end{array}\right\}_{N-1-2M} \left[\begin{array}{c|c} N & N \\ N & N \end{array}\right]_1
\end{aligned}$$

$$\begin{aligned}
&= \sum_{M=1}^{\lfloor \frac{N+1}{2} \rfloor -1} 2^{-2(M+1)} \binom{N-2-M}{N-1-2M} \left[\begin{array}{c|c} 0 & N \\ 0 & N \end{array}\right]_{N-1-2M} \\
&+ \sum_{M=1}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{N-1-M}{N-2M} \left[\begin{array}{c|c} 0 & N \\ 0 & N \end{array}\right]_{N-2M+1} \\
&+ \sum_{M=0}^{\lfloor \frac{N}{2} \rfloor -1} 2^{-2(M+1)} \binom{N-2-M}{N-2-2M} \left\{\begin{array}{c|c} 0 & N \\ 0, & N \end{array}\right\}_{N-2-2M} \\
&+ \sum_{M=0}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2M} \binom{N-1-M}{N-1-2M} \left\{\begin{array}{c|c} 0 & N \\ 0 & N \end{array}\right\}_{N-2M}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{M=2}^{\lfloor \frac{N+1}{2} \rfloor} 2^{-2M} \binom{N-1-M}{N+1-2M} \left[\begin{array}{c|c} 0 & N \\ 0 & N \end{array}\right]_{N+1-2M} \\
&+ \sum_{M=1}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{N-1-M}{N-2M} \left[\begin{array}{c|c} 0 & N \\ 0 & N \end{array}\right]_{N+1-2M}
\end{aligned}$$

$$+\sum_{M=1}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{N-1-M}{N-2M} \left\{ {}^{0}_{0} \middle| {}^{N}_{N} \right\}_{N-2M}$$
$$+\sum_{M=0}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2M} \binom{N-1-M}{N-1-2M} \left\{ {}^{0}_{0} \middle| {}^{N}_{N} \right\}_{N-2M}$$

$$= 2^{-2} \binom{N-2}{N-2} \left[{}^{0}_{0} \middle| {}^{N}_{N} \right]_{N-1} + \binom{N-1}{N-1} \left\{ {}^{0}_{0} \middle| {}^{N}_{N} \right\}_{N}$$
$$+\sum_{M=2}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \left(\binom{N-1-M}{N+1-2M} + \binom{N-1-M}{N-2M} \right) \left[{}^{0}_{0} \middle| {}^{N}_{N} \right]_{N+1-2M}$$
$$+\sum_{M=1}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2M} \left(\binom{N-1-M}{N-2M} + \binom{N-1-M}{N-1-2M} \right) \left\{ {}^{0}_{0} \middle| {}^{N}_{N} \right\}_{N-2M}$$
$$+\frac{1}{2}\left(1+(-1)^{N+1}\right) 2^{-(N+1)} \binom{N-1-\frac{N+1}{2}}{0} \left[{}^{0}_{0} \middle| {}^{N}_{N} \right]_{0}$$
$$+\frac{1}{2}\left(1+(-1)^{N}\right) 2^{-N} \binom{N-1-\frac{N}{2}}{0} \left\{ {}^{0}_{0} \middle| {}^{N}_{N} \right\}_{0}$$

$$= 2^{-2} \binom{N-1}{N-1} \left[{}^{0}_{0} \middle| {}^{N}_{N} \right]_{N-1} + \binom{N}{N} \left\{ {}^{0}_{0} \middle| {}^{N}_{N} \right\}_{N}$$
$$+\sum_{M=2}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{N-M}{N+1-2M} \left[{}^{0}_{0} \middle| {}^{N}_{N} \right]_{N+1-2M} + \sum_{M=1}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2M} \binom{N-M}{N-2M} \left\{ {}^{0}_{0} \middle| {}^{N}_{N} \right\}_{N-2M}$$
$$+\frac{1}{2}\left(1+(-1)^{N+1}\right) 2^{N+1} \binom{N-\frac{N+1}{2}}{0} \left[{}^{0}_{0} \middle| {}^{N}_{N} \right]_{0} + \frac{1}{2}\left(1+(-1)^{N}\right) 2^{N} \binom{N-\frac{N}{2}}{0} \left\{ {}^{0}_{0} \middle| {}^{N}_{N} \right\}_{0}$$

$$= \sum_{M=1}^{\lfloor \frac{N+1}{2} \rfloor} 2^{-2M} \binom{N-M}{N+1-2M} \left[{}^{0}_{0} \middle| {}^{N}_{N} \right]_{N+1-2M} + \sum_{M=0}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{N-M}{N-2M} \left\{ {}^{0}_{0} \middle| {}^{N}_{N} \right\}_{N-2M}$$

Then, by induction, the result follows.

(3)

$$T_{0,r} = m_{0,r} \sum_{i=0}^{2r} K_{2r}(i)$$
$$= m_{0,r} \left(K_{2r}(2r) + \sum_{i=0}^{r-1} \left(K_{2r}(2i) + K_{2r}(2i+1) \right) \right)$$

$$
\begin{aligned}
&= m_{0,r}\left(2^{-r}K_0(0)\left[\begin{array}{r|l}1 & 0\\ 1 & 2r\end{array}\right]_0 + \sum_{i=0}^{r-1} 2^{-i}\left(K_{2(r-i)}(0)+K_{2(r-i)}(1)\right)\left[\begin{array}{r|l}1 & 0\\ 2(r-i)+1 & 2r\end{array}\right]_0\right)\\
&= m_{0,r}\left(2^{-r}\left\{\begin{array}{r|l}0 & 0\\ 0 & 2r\end{array}\right\}_0 + \sum_{i=0}^{r-1} 2^{-r}\left(\left\{\begin{array}{r|l}0 & 0\\ 0 & 2(r-i)\end{array}\right\}_0 - \left\{\begin{array}{r|l}0 & 0\\ 0 & 2(r-i)\end{array}\right\}_0 + \left[\begin{array}{r|l}0 & 0\\ 0 & 2(r-i)\end{array}\right]_1\right)\left[\begin{array}{r|l}1 & 0\\ 2(r-i)+1 & 2r\end{array}\right]_0\right)\\
&= m_{0,r}2^{-r}\left(\left\{\begin{array}{r|l}0 & 0\\ 0 & 2r\end{array}\right\}_0 + r\left[\begin{array}{r|l}0 & 0\\ 0 & 2r\end{array}\right]_1\right)
\end{aligned}
$$

(4)

$$
\begin{aligned}
T_{1,r} &= m_{1,r}\sum_{0\le i<j\le 1+2r} K_{1+2r}(i,j)\\
&= m_{1,r}\left(K_{1+2r}(2r,1+2r) + \sum_{i=0}^{r-1}\left(\sum_{j=2i+1}^{1+2r} K_{1+2r}(2i,j) + \sum_{j=2i+2}^{1+2r} K_{1+2r}(2i+1,j)\right)\right)\\
&= m_{1,r}K_{1,2r}(2r,1+2r)\\
&\quad + m_{1,r}\sum_{i=0}^{r-1}\left(K_{1+2r}(2i,2i+1) + \sum_{j=2i+2}^{2r+1} K_{2r}(j-1)\left[\begin{array}{r|l}1 & 1\\ 2r+1 & 2r+1\end{array}\right]_1\right)\\
&= m_{1,r}\sum_{i=0}^{r} K_{1+2r}(2i,2i+1)\\
&\quad + m_{1,r}\sum_{i=0}^{r-1-1}\sum_{j=0}^{r-i}\left(K_{2r}(2(i+j)+1) + K_{2r}(2(i+j)+2)\right)\left[\begin{array}{r|l}1 & 1\\ 2r+1 & 2r+1\end{array}\right]_1\\
&= m_{1,r}\sum_{i=0}^{r} 2^{-i}K_{1+2(r-i)}(0,1)\left[\begin{array}{r|l}2 & 1\\ 2(r-i)+2 & 2r+1\end{array}\right]_0\\
&\quad + m_{1,r}\sum_{i=0}^{r-1}\sum_{j=0}^{r-i-1}\left(K_{2r}(2(i+j)+1) + K_{2r}(2(i+j)+2)\right)\left[\begin{array}{r|l}1 & 1\\ 2r+1 & 2r+1\end{array}\right]_1\\
&= m_{1,r}\sum_{i=0}^{r} 2^{-r}K_1(0,1)\left[\begin{array}{r|l}2 & 1\\ 2 & 2r+1\end{array}\right]_0 + m_{1,r}\sum_{i=0}^{r-1}\sum_{j=0}^{r-i-1} 2^{-r}\left[\begin{array}{r|l}0 & 1\\ 0 & 2r+1\end{array}\right]_2\\
&= m_{1,r}2^{-r}\left((r+1)\left(2^{-2}\left[\begin{array}{r|l}0 & 1\\ 0 & 1\end{array}\right]_0 + \left\{\begin{array}{r|l}0 & 1\\ 0 & 1\end{array}\right\}_1\right)\left[\begin{array}{r|l}2 & 1\\ 2 & 2r+1\end{array}\right]_0 + \binom{r+1}{2}\left[\begin{array}{r|l}0 & 1\\ 0 & 2r+1\end{array}\right]_2\right)\\
&= m_{1,r}2^{-r}\left(2^{-2}(r+1)\left[\begin{array}{r|l}0 & 1\\ 0 & 2r+1\end{array}\right]_0 + \binom{r+1}{2}\left[\begin{array}{r|l}0 & 1\\ 0 & 2r+1\end{array}\right]_2 + (r+1)\left\{\begin{array}{r|l}0 & 1\\ 0 & 2r+1\end{array}\right\}_1\right)
\end{aligned}
$$

(5)

$$T_{n,r} = m_{n,r} \sum_{0 \leq c_0 < \cdots < c_n \leq n+2r} K_{n+2r}(c_0, \ldots, c_n)$$

$$= m_{n,r} \sum_{1 \leq c_1 < \cdots < c_n \leq n+2r} K_{n+2r}(0, c_1, \ldots, c_n) + m_{n,r} \sum_{2 \leq c_1 < \cdots < c_n \leq n+2r} K_{n+2r}(1, c_1, \ldots, c_n)$$
$$+ m_{n,r} \sum_{2 \leq c_0 < \cdots < c_n \leq n+2r} K_{n+2r}(c_0, \ldots, c_n)$$

$$= m_{n,r} \sum_{2 \leq c_2 < \cdots < c_n \leq n+2r} K_{n+2r}(0, 1, c_2 \ldots, c_n)$$
$$+ m_{n,r} \sum_{2 \leq c_1 < \cdots < c_n \leq n+2r} (K_{n+2r}(0, c_1, \ldots, c_n) + K_{n+2r}(1, c_1, \ldots, c_n))$$
$$+ m_{n,r} 2^{-1} \sum_{2 \leq c_0 < \cdots < c_n \leq n+2r} K_{n+2r-2}(c_0 - 2, \ldots, c_n - 2) \left[\begin{array}{r|l} n+1 & n \\ n+2r-1 & n+2r \end{array}\right]_0$$

$$= m_{n,r} 2^{-2} \sum_{2 \leq c_2 < \cdots < c_n \leq n+2r} K_{n+2r-2}(c_2 - 2, \ldots, c_n - 2) \left[\begin{array}{r|l} n-1 & n \\ n+2r-1 & n+2r \end{array}\right]_0$$
$$+ m_{n,r} \sum_{2 \leq c_2 < \cdots < c_n \leq n+2r} K_{n+2r-1}(0, c_2 - 1, \ldots, c_n - 1) \left[\begin{array}{r|l} n & n \\ n+2r & n+2r \end{array}\right]_1$$
$$+ m_{n,r} \sum_{2 \leq c_1 < \cdots < c_n \leq n+2r} K_{n+2r-1}(c_1 - 1, \ldots, c_n - 1) \left[\begin{array}{r|l} n & n \\ n+2r & n+2r \end{array}\right]_1$$
$$+ m_{n,r} 2^{-1} \sum_{0 \leq c_0 < \cdots < c_n \leq n+2(r-1)} K_{n+2r-2}(c_0, \ldots, c_n) \left[\begin{array}{r|l} n+1 & n \\ n+2r-1 & n+2r \end{array}\right]_0$$

$$= m_{n,r} 2^{-2} \sum_{0 \leq c_0 < \cdots < c_{n-2} \leq n+2r-2} K_{n+2r-2}(c_0, \ldots, c_{n-2}) \left[\begin{array}{r|l} n-1 & n \\ n+2r-1 & n+2r \end{array}\right]_0$$
$$+ m_{n,r} \sum_{1 \leq c_1 < \cdots < c_{n-1} \leq n+2r-1} K_{n+2r-1}(0, c_1, \ldots, c_{n-1}) \left[\begin{array}{r|l} n & n \\ n+2r & n+2r \end{array}\right]_1$$
$$+ m_{n,r} \sum_{1 \leq c_0 < \cdots < c_{n-1} \leq n+2r-1} K_{n+2r-1}(c_0, c, 1, \ldots, c_{n-1}) \left[\begin{array}{r|l} n & n \\ n+2r & n+2r \end{array}\right]_1$$
$$+ \frac{m_{n,r}}{m_{n,r-1}} 2^{-1} T_{n,r-1} \left[\begin{array}{r|l} n+1 & n \\ n+2r-1 & n+2r \end{array}\right]_0$$

$$
\begin{aligned}
&= \frac{m_{n,r}}{m_{n-2,r}} 2^{-2} T_{n-2,r} \left[\begin{array}{r|l} n-1 & n \\ n+2r-1 & n+2r \end{array}\right]_0 \\
&+ m_{n,r} \sum_{0 \le c_0 < \cdots < c_{n-1} \le n+2r-1} K_{n+2r-1}(c_0, c_1, \ldots, c_{n-1}) \left[\begin{array}{r|l} n & n \\ n+2r & n+2r \end{array}\right]_1 \\
&+ \frac{m_{n,r}}{m_{n,r-1}} 2^{-1} T_{n,r-1} \left[\begin{array}{r|l} n+1 & n \\ n+2r-1 & n+2r \end{array}\right]_0
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{m_{n,r}}{m_{n-2,r}} 2^{-2} T_{n-2,r} \left[\begin{array}{r|l} n-1 & n \\ n+2r-1 & n+2r \end{array}\right]_0 + \frac{m_{n,r}}{m_{n-1,r}} T_{n-1,r} \left[\begin{array}{r|l} n & n \\ n+2r & n+2r \end{array}\right]_1 \\
&+ \frac{m_{n,r}}{m_{n,r-1}} 2^{-1} T_{n,r-1} \left[\begin{array}{r|l} n+1 & n \\ n+2r-1 & n+2r \end{array}\right]_0
\end{aligned}
$$

(6) Suppose for some $N \ge 2$ and $R \ge 1$. we have for $1 \le n < N$ and $r < R$ that

$$
\begin{aligned}
T_{n,r} = m_{n,r} 2^{-r} &\sum_{M=0}^{\lfloor \frac{n+1}{2} \rfloor} 2^{-2M} \binom{r+M}{M} \binom{r+n-M}{n+1-2M} \left[\begin{array}{r|l} 0 & n \\ 0 & n+2r \end{array}\right]_{n+1-2M} \\
&+ m_{n,r} 2^{-r} \sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} 2^{-2M} \binom{r+n-M}{r, n-2M, M} \left\{\begin{array}{r|l} 0 & n \\ 0 & n+2r \end{array}\right\}_{n-2M}.
\end{aligned}
$$

Then

$$
\begin{aligned}
T_{N,R} &= \frac{m_{N,R}}{m_{N-2,R}} 2^{-2} T_{N-2,R} \left[\begin{array}{r|l} N-1 & N \\ N+2R-1 & N+2R \end{array}\right]_0 + \frac{m_{N,R}}{m_{N-1,R}} T_{N-1,R} \left[\begin{array}{r|l} N & N \\ N+2R & N+2R \end{array}\right]_1 \\
&+ \frac{m_{N,R}}{m_{N,R-1}} 2^{-1} T_{N,R-1} \left[\begin{array}{r|l} N+1 & N \\ N+2R-1 & N+2R \end{array}\right]_0
\end{aligned}
$$

$$
\begin{aligned}
&= m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2(M+1)} \binom{R+M}{M} \binom{R+N-2-M}{N-1-2M} \left[\begin{array}{r|l} 0 & N \\ 0 & N+2R \end{array}\right]_{N-1-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N-2}{2} \rfloor} 2^{-2(M+1)} \binom{R+N-2-M}{R, N-2-2M, M} \left\{\begin{array}{r|l} 0 & N \\ 0 & N+2R \end{array}\right\}_{N-2-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-2M} \left[\begin{array}{r|l} 0 & N \\ 0 & N+2R \end{array}\right]_{N+1-2M}
\end{aligned}
$$

$$
\begin{aligned}
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2M} \binom{R+N-1-M}{R, N-1-2M, M} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N+1}{2} \rfloor} 2^{-2M} \binom{R-1+M}{M} \binom{R+N-1-M}{N+1-2M} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N+1-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{R+N-1-M}{R-1, N-2M, M} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N-2M}
\end{aligned}
$$

$$
\begin{aligned}
&= m_{N,R} 2^{-R} \sum_{M=1}^{\lfloor \frac{N+1}{2} \rfloor} 2^{-2M} \binom{R+M-1}{M-1} \binom{R+N-1-M}{N+1-2M} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N+1-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N+1}{2} \rfloor} 2^{-2M} \binom{R-1+M}{M} \binom{R+N-1-M}{N+1-2M} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N+1-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-2M} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N+1-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=1}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{R-1+M}{M-1} \binom{R+N-1-M}{N-2M} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{R-1+M}{M} \binom{R+N-1-M}{N-2M} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-1-2M} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N-2M}
\end{aligned}
$$

$$
\begin{aligned}
&= m_{N,R} 2^{-R} \binom{R-1}{0} \binom{R+N-1}{N+1} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N+1} \\
&+ m_{N,R} 2^{-R} \sum_{M=1}^{\lfloor \frac{N+1}{2} \rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N+1-2M} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N+1-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-2M} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N+1-2M} \\
&+ m_{N,R} 2^{-R} \binom{R-1}{0} \binom{R+N-1}{N} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N} \\
&+ m_{N,R} 2^{-R} \sum_{M=1}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-2M} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N-2M}
\end{aligned}
$$

$$+\, m_{N,R} 2^{-R} \sum_{M=0}^{\left\lfloor \frac{N-1}{2} \right\rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-1-2M} \left\{ \begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right\}_{N-2M}$$

$$\begin{aligned}
&= m_{N,R} 2^{-R} \binom{R}{0} \binom{R+N-1}{N+1} \left[\begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right]_{N+1} \\
&+ m_{N,R} 2^{-R} \frac{1}{2} \left(1 + (-1)^{N+1}\right) 2^{-(N+1)} \binom{R + \frac{N+1}{2}}{\frac{N+1}{2}} \binom{R+N-1-\frac{N+1}{2}}{0} \left[\begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right]_{0} \\
&+ m_{N,R} 2^{-R} \sum_{M=1}^{\left\lfloor \frac{N}{2} \right\rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N+1-2M} \left[\begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right]_{N+1-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\left\lfloor \frac{N}{2} \right\rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-2M} \left[\begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right]_{N+1-2M} \\
&+ m_{N,R} 2^{-R} \binom{R}{0} \binom{R+N-1}{N} \left\{ \begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right\}_{N} \\
&+ m_{N,R} 2^{-R} \frac{1}{2} \left(1 + (-1)^{N}\right) 2^{-N} \binom{R + \frac{N}{2}}{\frac{N}{2}} \binom{R+N-1-\frac{N}{2}}{0} \left[\begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right]_{1} \\
&+ m_{N,R} 2^{-R} \sum_{M=1}^{\left\lfloor \frac{N-1}{2} \right\rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-2M} \left\{ \begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right\}_{N-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\left\lfloor \frac{N-1}{2} \right\rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-1-2M} \left\{ \begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right\}_{N-2M}
\end{aligned}$$

$$\begin{aligned}
&= m_{N,R} 2^{-R} \frac{1}{2} \left(1 + (-1)^{N+1}\right) 2^{-(N+1)} \binom{R + \frac{N+1}{2}}{\frac{N+1}{2}} \binom{R+N-\frac{N+1}{2}}{0} \left[\begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right]_{0} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\left\lfloor \frac{N}{2} \right\rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N+1-2M} \left[\begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right]_{N+1-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\left\lfloor \frac{N}{2} \right\rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-2M} \left[\begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right]_{N+1-2M} \\
&+ m_{N,R} 2^{-R} \frac{1}{2} \left(1 + (-1)^{N}\right) 2^{-N} \binom{R + \frac{N}{2}}{\frac{N}{2}} \binom{R+N-\frac{N}{2}}{0} \left[\begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right]_{1} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\left\lfloor \frac{N-1}{2} \right\rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-2M} \left\{ \begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right\}_{N-2M} \\
&+ m_{N,R} 2^{-R} \sum_{M=0}^{\left\lfloor \frac{N-1}{2} \right\rfloor} 2^{-2M} \binom{R+M}{M} \binom{R+N-1-M}{N-1-2M} \left\{ \begin{array}{c|l} 0 & N \\ 0 & N+2R \end{array} \right\}_{N-2M}
\end{aligned}$$

$$= m_{N,R}2^{-R}\frac{1}{2}\left(1+(-1)^{N+1}\right)2^{-(N+1)}\binom{R+\frac{N+1}{2}}{\frac{N+1}{2}}\binom{R+N-\frac{N+1}{2}}{0}\left[\begin{matrix}0\\0\end{matrix}\middle|\begin{matrix}N\\N+2R\end{matrix}\right]_0$$
$$+ m_{N,R}2^{-R}\sum_{M=0}^{\lfloor\frac{N}{2}\rfloor}2^{-2M}\binom{R+M}{M}\binom{R+N-M}{N+1-2M}\left[\begin{matrix}0\\0\end{matrix}\middle|\begin{matrix}N\\N+2R\end{matrix}\right]_{N+1-2M}$$
$$+ m_{N,R}2^{-R}\frac{1}{2}\left(1+(-1)^{N}\right)2^{-N}\binom{R+\frac{N}{2}}{\frac{N}{2}}\binom{R+N-\frac{N}{2}}{0}\left[\begin{matrix}0\\0\end{matrix}\middle|\begin{matrix}N\\N+2R\end{matrix}\right]_1$$
$$+ m_{N,R}2^{-R}\sum_{M=0}^{\lfloor\frac{N-1}{2}\rfloor}2^{-2M}\binom{R+M}{M}\binom{R+N-M}{N-2M}\left\{\begin{matrix}0\\0\end{matrix}\middle|\begin{matrix}N\\N+2R\end{matrix}\right\}_{N-2M}$$

$$= m_{N,R}2^{-R}\sum_{M=0}^{\lfloor\frac{N+1}{2}\rfloor}2^{-2M}\binom{R+M}{M}\binom{R+N-M}{N+1-2M}\left[\begin{matrix}0\\0\end{matrix}\middle|\begin{matrix}N\\N+2R\end{matrix}\right]_{N+1-2M}$$
$$+ m_{N,R}2^{-R}\sum_{M=0}^{\lfloor\frac{N}{2}\rfloor}2^{-2M}\binom{R+M}{M}\binom{R+N-M}{N-2M}\left\{\begin{matrix}0\\0\end{matrix}\middle|\begin{matrix}N\\N+2R\end{matrix}\right\}_{N-2M}$$

Then, by induction, the result follows.

(7)

$$T_{n,0}^{(0)} = m_{n,0}\sum_{1\leq c_1<\cdots<c_n\leq n}K_n(0,c_1,\ldots,c_n)$$
$$= m_{n,0}K_n(0,1,\ldots,n)$$
$$= T_{n,0}$$
$$= \sum_{M=1}^{\lfloor\frac{n+1}{2}\rfloor}2^{-2M}\binom{n-M}{n+1-2M}\left[\begin{matrix}0\\0\end{matrix}\middle|\begin{matrix}n\\n\end{matrix}\right]_{n+1-2M} + \sum_{M=0}^{\lfloor\frac{n}{2}\rfloor}2^{-2M}\binom{n-M}{n-2M}\left[\begin{matrix}0\\0\end{matrix}\middle|\begin{matrix}n\\n\end{matrix}\right]_{n-2M}$$
$$= \sum_{M=1}^{\lfloor\frac{n+1}{2}\rfloor}2^{-2M}\binom{n-M}{0,n+1-2M,M-1}\left[\begin{matrix}0\\0\end{matrix}\middle|\begin{matrix}n\\n\end{matrix}\right]_{n+1-2M} + \sum_{M=0}^{\lfloor\frac{n}{2}\rfloor}2^{-2M}\binom{n-M}{0,n-2M,M}\left\{\begin{matrix}0\\0\end{matrix}\middle|\begin{matrix}n\\n\end{matrix}\right\}_{n-2M}$$

(8)

$$T_{1,r}^{(0)} = m_{1,r}\sum_{i=1}^{1+2r}K_{1+2r}(0,i)$$

$$
\begin{aligned}
&= m_{1,r}\left(K_{1+2r}(0,1)+\sum_{i=1}^{r}\left(K_{1+2r}(0,2i)+K_{1+2r}(0,2i+1)\right)\right)\\
&= m_{1,r}\left(2^{-r}K_{1}(0,1)\left[\begin{array}{c|l}2&1\\2&1+2r\end{array}\right]_0+\sum_{i=1}^{r}K_{2r}(0)\left[\begin{array}{r|l}1&1\\1+2r&1+2r\end{array}\right]_1\right)\\
&= m_{1,r}\left(2^{-r}\left(2^{-2}\left[\begin{array}{c|c}0&1\\0&1\end{array}\right]_0+\left\{\begin{array}{c|c}0&1\\0&1\end{array}\right\}_1\right)\left[\begin{array}{c|l}2&1\\2&1+2r\end{array}\right]_0+r2^{-r}K_0(0)\left[\begin{array}{c|l}1&0\\1&2r\end{array}\right]_0\left[\begin{array}{r|l}1&1\\1+2r&1+2r\end{array}\right]_1\right)\\
&= m_{1,r}2^{-r}\left(2^{-2}\left[\begin{array}{c|l}0&1\\0&1+2r\end{array}\right]_0+(r+1)\left\{\begin{array}{c|l}0&1\\0&1+2r\end{array}\right\}_1\right)\\
&= m_{1,r}2^{-r}\left(2^{-2}\binom{r}{r,0,0}\left[\begin{array}{c|l}0&1\\0&1+2r\end{array}\right]_0+\binom{r+1}{r,1,0}\left\{\begin{array}{c|l}0&1\\0&1+2r\end{array}\right\}_1\right)
\end{aligned}
$$

(9)

$$
T^{(0)}_{2,r} = m_{2,r}\sum_{1\le i<j\le 2+2r}K_{2+2r}(0,i,j)
$$

$$
\begin{aligned}
&= m_{2,r}K_{2+2r}(0,1+2r,2+2r)\\
&\quad + m_{2,r}\sum_{i=0}^{r-1}\left(\sum_{j=2i+2}^{2r+2}K_{2r+2}(0,2i+1,j)+\sum_{j=2i+3}^{2r+2}K_{2r+2}(0,2i+2,j)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= m_{2,r}K_{2+2r}(0,1+2r,2+2r)\\
&\quad + m_{2,r}\sum_{i=0}^{r-1}\left(K_{2r+2}(0,2i+1,2i+2)+\sum_{j=2i+3}^{2r+2}K_{2r+1}(0,j-1)\left[\begin{array}{r|l}2&2\\2r+2&2r+2\end{array}\right]_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= m_{2,r}\sum_{i=0}^{r}K_{2+2r}(0,2i+1,2i+2)\\
&\quad + m_{2,r}\sum_{i=0}^{r-1}\sum_{j=2i+2}^{2r+1}K_{2r+1}(0,j)\left[\begin{array}{r|l}2&2\\2r+2&2r+2\end{array}\right]_1
\end{aligned}
$$

$$
\begin{aligned}
&= m_{2,r}\sum_{i=0}^{r}2^{-i}2^{-(r-i)}K_{2}(0,1,2)\left[\begin{array}{c|l}3&2\\3&2(r-i)+2\end{array}\right]_0\left[\begin{array}{r|l}3&2\\2(r-i)+3&2+2r\end{array}\right]_0\\
&\quad + m_{2,r}\sum_{i=0}^{r-1}\sum_{j=0}^{r-i-1}\left(K_{2r+1}(0,2j+2i+2)+K_{2r+1}(0,2j+2i+3)\right)\left[\begin{array}{r|l}2&2\\2r+2&2r+2\end{array}\right]_1
\end{aligned}
$$

$$
\begin{aligned}
&= m_{2,r} 2^{-r}(r+1)\left(2^{-2}\left[\begin{array}{c|c} 0 & 2 \\ 0 & 2 \end{array}\right]_1 + 2^{-2}\left\{\begin{array}{c|c} 0 & 2 \\ 0 & 2 \end{array}\right\}_0 + \left\{\begin{array}{c|c} 0 & 2 \\ 0 & 2 \end{array}\right\}_2\right)\left[\begin{array}{c|c} 3 & 2 \\ 3 & 2{+}2r \end{array}\right]_0 \\
&\quad + m_{2,r}\sum_{i=0}^{r-1}\sum_{j=0}^{r-i-1} K_{2r}(0)\left[\begin{array}{r|l} 1 & 1 \\ 2r{+}1 & 2r{+}1 \end{array}\right]_1\left[\begin{array}{r|l} 2 & 2 \\ 2r{+}2 & 2r{+}2 \end{array}\right]_1
\end{aligned}
$$

$$
\begin{aligned}
&= m_{2,r} 2^{-r}\left(2^{-2}(r+1)\left[\begin{array}{c|l} 0 & 2 \\ 0 & 2{+}2r \end{array}\right]_1 + 2^{-2}(r+1)\left\{\begin{array}{c|l} 0 & 2 \\ 0 & 2{+}2r \end{array}\right\}_0 + (r+1)\left\{\begin{array}{c|l} 0 & 2 \\ 0 & 2{+}2r \end{array}\right\}_2\right) \\
&\quad + m_{2,r} 2^{-r}\binom{r+1}{2}\left\{\begin{array}{c|l} 0 & 0 \\ 0 & 2r \end{array}\right\}_0\left[\begin{array}{r|l} 1 & 2 \\ 2r{+}1 & 2r{+}2 \end{array}\right]_2
\end{aligned}
$$

$$
= m_{2,r} 2^{-r}\left(2^{-2}(r+1)\left[\begin{array}{c|l} 0 & 2 \\ 0 & 2{+}2r \end{array}\right]_1 + 2^{-2}(r+1)\left\{\begin{array}{c|l} 0 & 2 \\ 0 & 2{+}2r \end{array}\right\}_0 + \binom{r+2}{2}\left\{\begin{array}{c|l} 0 & 2 \\ 0 & 2{+}2r \end{array}\right\}_2\right)
$$

$$
= m_{2,r} 2^{-r}\left(2^{-2}\binom{r+1}{r,1,0}\left[\begin{array}{c|l} 0 & 2 \\ 0 & 2{+}2r \end{array}\right]_1 + 2^{-2}\binom{r+1}{r,0,1}\left\{\begin{array}{c|l} 0 & 2 \\ 0 & 2{+}2r \end{array}\right\}_0 + \binom{r+2}{r,2,0}\left\{\begin{array}{c|l} 0 & 2 \\ 0 & 2{+}2r \end{array}\right\}_2\right)
$$

(10)

$$
T^{(0)}_{n,r} = m_{n,r}\sum_{1\le c_1<\cdots<c_n\le n+2r} K_{n+2r}(0,c_1,\ldots,c_n)
$$

$$
\begin{aligned}
&= m_{n,r}\sum_{2\le c_2<\cdots<c_n\le n+2r} K_{n+2r}(0,1,c_2,\ldots,c_n) + m_{n,r}\sum_{3\le c_2<\cdots<c_n\le n+2r} K_{n+2r}(0,2,c_2,\ldots,c_n) \\
&\quad + m_{n,r}\sum_{3\le c_1<\cdots<c_n\le n+2r} K_{n+2r}(0,c_1,\ldots,c_n)
\end{aligned}
$$

$$
\begin{aligned}
&= m_{n,r}\sum_{2\le c_2<\cdots<c_n\le n+2r} 2^{-2}K_{n+2r-2}(c_2-2,\ldots,c_n-2)\left[\begin{array}{r|l} n{-}1 & n \\ n{+}2r{-}1 & n{+}2r \end{array}\right]_0 \\
&\quad + m_{n,r}\sum_{2\le c_2<\cdots<c_n\le n+2r} K_{n+2r-1}(0,c_2-1,\ldots,c_n-1)\left[\begin{array}{r|l} n & n \\ n{+}2r & n{+}2r \end{array}\right]_1 \\
&\quad - m_{n.r}\sum_{3\le c_2<\cdots<c_n\le n+2r} 2^{-2}K_{n+2r-2}(c_2-2,\ldots,c_n-2)\left[\begin{array}{r|l} n{-}1 & n \\ n{+}2r{-}1 & n{+}2r \end{array}\right]_0
\end{aligned}
$$

$$+ m_{n,r} \sum_{3\le c_1<\cdots<c_n\le n+2r} 2^{-1} K_{n+2r-2}(0, c_1-2, \ldots, c_n-2) \left[\begin{array}{c|c} n+1 & n \\ n+2r-1 & n+2r \end{array}\right]_0$$

$$\begin{aligned}
&= 2^{-2} m_{n,r} \sum_{0\le e_0<\cdots<e_{n-2}\le n+2r-2} K_{n+2r-2}(e_0, \ldots, e_{n-2}) \left[\begin{array}{c|c} n-1 & n \\ n+2r-1 & n+2r \end{array}\right]_0 \\
&+ m_{n,r} \sum_{1\le e_1<\cdots<e_{n-1}\le n+2r-1} K_{n+2r-1}(0, e_1, \ldots, e_{n-1}) \left[\begin{array}{c|c} n & n \\ n+2r & n+2r \end{array}\right]_1 \\
&- 2^{-2} m_{n.r} \sum_{1\le e_0<\cdots<e_{n-2}\le n+2r-2} K_{n+2r-2}(e_0, \ldots, e_{n-2}) \left[\begin{array}{c|c} n-1 & n \\ n+2r-1 & n+2r \end{array}\right]_0 \\
&+ 2^{-1} m_{n,r} \sum_{1\le e_1<\cdots<e_n\le n+2r-2} K_{n+2r-2}(0, e_1, \ldots, e_n) \left[\begin{array}{c|c} n+1 & n \\ n+2r-1 & n+2r \end{array}\right]_0
\end{aligned}$$

$$\begin{aligned}
&= 2^{-2} m_{n,r} \sum_{1\le e_1<\cdots<e_{n-2}\le n+2r-2} K_{n+2r-2}(0, e_1, \ldots, e_{n-2}) \left[\begin{array}{c|c} n-1 & n \\ n+2r-1 & n+2r \end{array}\right]_0 \\
&+ \frac{m_{n,r}}{m_{n-1,r}} T^{(0)}_{n-1,r} \left[\begin{array}{c|c} n & n \\ n+2r & n+2r \end{array}\right]_1 + 2^{-1} \frac{m_{n,r}}{m_{n,r-1}} T^{(0)}_{n,r-1} \left[\begin{array}{c|c} n+1 & n \\ n+2r-1 & n+2r \end{array}\right]_0
\end{aligned}$$

$$\begin{aligned}
&= 2^{-2} \frac{m_{n,r}}{m_{n-2,r}} T^{(0)}_{n-2,r} \left[\begin{array}{c|c} n-1 & n \\ n+2r-1 & n+2r \end{array}\right]_0 + \frac{m_{n,r}}{m_{n-1,r}} T^{(0)}_{n-1,r} \left[\begin{array}{c|c} n & n \\ n+2r & n+2r \end{array}\right]_1 \\
&+ 2^{-1} \frac{m_{n,r}}{m_{n,r-1}} T^{(0)}_{n,r-1} \left[\begin{array}{c|c} n+1 & n \\ n+2r-1 & n+2r \end{array}\right]_0
\end{aligned}$$

(11) Suppose for some $N \ge 3$, and $R \ge 1$ we have for $1 \le n < N$ and $0 \le r < R$ that

$$\begin{aligned}
T^{(0)}_{n,r} &= m_{n,r} 2^{-r} \sum_{M=1}^{\lfloor \frac{n+1}{2} \rfloor} 2^{-2M} \binom{r+n-M}{r, n+1-2M, M-1} \left[\begin{array}{c|c} 0 & n \\ 0 & n+2r \end{array}\right]_{n+1-2M} \\
&+ m_{n,r} 2^{-r} \sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} 2^{-2M} \binom{r+n-M}{r, n-2M, M} \left\{\begin{array}{c|c} 0 & n \\ 0 & n+2r \end{array}\right\}_{n-2M}
\end{aligned}$$

Then we have

$$\begin{aligned}
T^{(0)}_{N,R} &= 2^{-2} \frac{m_{N,R}}{m_{N-2,R}} T^{(0)}_{N-2,R} \left[\begin{array}{c|c} N-1 & N \\ N-1+2R & N+2R \end{array}\right]_0 + \frac{m_{N,R}}{m_{N-1,R}} T^{(0)}_{N-1,R} \left[\begin{array}{c|c} N & N \\ N+2R & N+2R \end{array}\right]_1 \\
&+ 2^{-1} \frac{m_{N,R}}{m_{N,R-1}} T^{(0)}_{N,R-1} \left[\begin{array}{c|c} N+1 & N \\ N-1+2R & N+2R \end{array}\right]_0
\end{aligned}$$

$$
\begin{aligned}
&= m_{N,R}2^{-R}\sum_{M=1}^{\lfloor \frac{N-1}{2}\rfloor} 2^{-2(M+1)}\binom{R+N-2-M}{R,N-1-2M,M-1}\left[\begin{array}{c|c}0 & N-2\\ 0 & N-2+2R\end{array}\right]_{N-1-2M}\left[\begin{array}{c|c}N-1 & N\\ N-1+2R & N+2R\end{array}\right]_{0}\\
&+ m_{N,R}2^{-R}\sum_{M=0}^{\lfloor \frac{N-2}{2}\rfloor} 2^{-2(M+1)}\binom{R+N-2-M}{R,N-2-2M,M}\left\{\begin{array}{c|c}0 & N-2\\ 0 & N-2+2R\end{array}\right\}_{N-2-2M}\left[\begin{array}{c|c}N-1 & N\\ N-1+2R & N+2R\end{array}\right]_{0}\\
&+ m_{N,R}2^{-R}\sum_{M=1}^{\lfloor \frac{N}{2}\rfloor} 2^{-2M}\binom{R+N-1-M}{R,N-2M,M-1}\left[\begin{array}{c|c}0 & N-1\\ 0 & N-1+2R\end{array}\right]_{N-2M}\left[\begin{array}{c|c}N & N\\ N+2R & N+2R\end{array}\right]_{1}\\
&+ m_{N,R}2^{-R}\sum_{M=0}^{\lfloor \frac{N-1}{2}\rfloor} 2^{-2M}\binom{R+N-1-M}{R,N-1-2M,M}\left\{\begin{array}{c|c}0 & N-1\\ 0 & N-1+2R\end{array}\right\}_{N-1-2M}\left[\begin{array}{c|c}N & N\\ N+2R & N+2R\end{array}\right]_{1}\\
&+ m_{N,R}2^{-R}\sum_{M=1}^{\lfloor \frac{N+1}{2}\rfloor} 2^{-2M}\binom{R+N-1-M}{R-1,N+1-2M,M-1}\left[\begin{array}{c|c}0 & N\\ 0 & N-2+2R\end{array}\right]_{N+1-2M}\left[\begin{array}{c|c}N+1 & N\\ N-1+2R & N+2R\end{array}\right]_{0}\\
&+ m_{N,R}2^{-R}\sum_{M=0}^{\lfloor \frac{N}{2}\rfloor} 2^{-2M}\binom{R+N-1-M}{R-1,N-2M,M}\left\{\begin{array}{c|c}0 & N\\ 0 & N-2+2R\end{array}\right\}_{N-2M}\left[\begin{array}{c|c}N+1 & N\\ N-1+2R & N+2R\end{array}\right]_{0}
\end{aligned}
$$

$$
\begin{aligned}
&= m_{N,R}2^{-R}\sum_{M=2}^{\lfloor \frac{N+1}{2}\rfloor} 2^{-2M}\binom{R+N-1-M}{R,N+1-2M,M-2}\left[\begin{array}{c|c}0 & N\\ 0 & N+2R\end{array}\right]_{N+1-2M}\\
&+ m_{N,R}2^{-R}\sum_{M=1}^{\lfloor \frac{N}{2}\rfloor} 2^{-2M}\binom{R+N-1-M}{R,N-2M,M-1}\left\{\begin{array}{c|c}0 & N\\ 0 & N+2R\end{array}\right\}_{N-2M}\\
&+ m_{N,R}2^{-R}\sum_{M=1}^{\lfloor \frac{N}{2}\rfloor} 2^{-2M}\binom{R+N-1-M}{R,N-2M,M-1}\left[\begin{array}{c|c}0 & N\\ 0 & N+2R\end{array}\right]_{N+1-2M}\\
&+ m_{N,R}2^{-R}\sum_{M=0}^{\lfloor \frac{N-1}{2}\rfloor} 2^{-2M}\binom{R+N-1-M}{R,N-1-2M,M}\left\{\begin{array}{c|c}0 & N\\ 0 & N+2R\end{array}\right\}_{N-2M}\\
&+ m_{N,R}2^{-R}\sum_{M=1}^{\lfloor \frac{N+1}{2}\rfloor} 2^{-2M}\binom{R+N-1-M}{R-1,N+1-2M,M-1}\left[\begin{array}{c|c}0 & N\\ 0 & N+2R\end{array}\right]_{N+1-2M}\\
&+ m_{N,R}2^{-R}\sum_{M=0}^{\lfloor \frac{N}{2}\rfloor} 2^{-2M}\binom{R+N-1-M}{R-1,N-2M,M}\left\{\begin{array}{c|c}0 & N\\ 0 & N+2R\end{array}\right\}_{N-2M}
\end{aligned}
$$

$$
= m_{N,R}2^{-R}\sum_{M=2}^{\lfloor \frac{N}{2}\rfloor} 2^{-2M}\binom{R+N-M}{R,N+1-2M,M-1}\left[\begin{array}{c|c}0 & N\\ 0 & N+2R\end{array}\right]_{N+1-2M}
$$

$$+ m_{N,R} 2^{-R} \sum_{M=1}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2M} \binom{R+N-M}{R, N-2M, M} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N-2M}$$

$$+ m_{N,R} 2^{-R} \frac{1}{2} \left(1 + (-1)^{N+1}\right) 2^{-(N+1)} \left(\binom{R + \frac{N+1}{2} - 2}{R, 0, \frac{N+1}{2} - 2} + \binom{R + \frac{N+1}{2} - 2}{R-1, 0, \frac{N+1}{2} - 1} \right) \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{0}$$

$$+ m_{N,R} 2^{-R} \frac{1}{2} \left(1 + (-1)^{N}\right) 2^{-N} \left(\binom{R + \frac{N}{2} - 1}{R, 0, \frac{N}{2} - 1} + \binom{R + \frac{N}{2} - 1}{R-1, 0, \frac{N}{2}} \right) \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{0}$$

$$+ m_{N,R} 2^{-R} 2^{-2} \left(\binom{R+N-2}{R, N-2, 0} + \binom{R+N-2}{R-1, N-1, 0} \right) \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N-1}$$

$$+ m_{N,R} 2^{-R} \left(\binom{R+N-1}{R, N-1, 0} + \binom{R+N-1}{R-1, N, 0} \right) \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N}$$

$$= m_{N,R} 2^{-R} \sum_{M=2}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{R+N-M}{R, N+1-2M, M-1} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N+1-2M}$$

$$+ m_{N,R} 2^{-R} \sum_{M=1}^{\lfloor \frac{N-1}{2} \rfloor} 2^{-2M} \binom{R+N-M}{R, N-2M, M} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N-2M}$$

$$+ m_{N,R} 2^{-R} \frac{1}{2} \left(1 + (-1)^{N+1}\right) 2^{-(N+1)} \binom{R + \frac{N+1}{2} - 1}{R, 0, \frac{N+1}{2} - 1} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{0}$$

$$+ m_{N,R} 2^{-R} \frac{1}{2} \left(1 + (-1)^{N}\right) 2^{-N} \binom{R + \frac{N}{2}}{R, 0, \frac{N}{2}} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{0}$$

$$+ m_{N,R} 2^{-R} 2^{-2} \binom{R+N-1}{R, N-1, 0} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N-1}$$

$$+ m_{N,R} 2^{-R} \binom{R+N}{R, N, 0} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N}$$

$$= m_{N,R} 2^{-R} \sum_{M=1}^{\lfloor \frac{N+1}{2} \rfloor} 2^{-2M} \binom{R+N-M}{R, N+1-2M, M-1} \left[\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right]_{N+1-2M}$$

$$+ m_{N,R} 2^{-R} \sum_{M=0}^{\lfloor \frac{N}{2} \rfloor} 2^{-2M} \binom{R+N-M}{R, N-2M, M} \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} N \\ N+2R \end{matrix} \right\}_{N-2M}$$

Then, by induction, the result follows.

$\square$

Now that we have computed how the sums of the $\cdot_h$ products behave, we can now compute the action of the periodicity operator in the pairing.

Proposition 4.2.2.

(1)

$$R_{0,r} = \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_1 \\ \tau \in S_{2r+1}}} 2^{-r} m_{0,r} \left(-r \left[\begin{array}{c|c} 0 & 0 \\ 0 & 2r \end{array}\right]_1 + \left\{\begin{array}{c|c} 0 & 0 \\ 0 & 2r \end{array}\right\}_0 \right)$$

(2)

$$\begin{aligned} R_{n,r} = \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}} m_{n,r} 2^{-r} \Bigg[& - \binom{r+n}{n+1} \left[\begin{array}{c|c} 0 & n \\ 0 & n+2r \end{array}\right]_{n+1} \\ & + \sum_{M=1}^{\lfloor \frac{n+1}{2} \rfloor} 2^{-2(M+1)} \left(8 \binom{r+n-M}{r, n-2M, M} - \binom{r+M}{M} \binom{r+n-M}{n+1-2M} \right) \left[\begin{array}{c|c} 0 & n \\ 0 & n+2r \end{array}\right]_{n+1-2M} \\ & + \sum_{M=0}^{\lfloor \frac{n}{2} \rfloor} 2^{-2M} \binom{r+n-M}{r, n-2M, M} \left\{\begin{array}{c|c} 0 & n \\ 0 & n+2r \end{array}\right\}_{n-2M} \Bigg] \end{aligned}$$

Proof. (1)

$$\begin{aligned} R_{0,r} &= \langle S^r (f_0), g_0 \wedge \ldots \wedge g_{2r} \rangle \\ &= m_{0,r} \langle L_{2r}(0), g_0 \wedge \ldots \wedge g_{2r} \rangle - m_{0,r} \sum_{i=1}^{2r} (-1)^i \langle L_{2r}(i), g_0 \wedge \ldots \wedge g_{2r} \rangle \\ &= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_1 \\ \tau \in S_{2r+1}}} m_{0,r} \left(K_{2r}(0) - \sum_{i=1}^{2r} K_{2r}(i) \right) \\ &= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_1 \\ \tau \in S_{2r+1}}} \left(2 m_{0,r} K_{2r}(0) - T_{0,r} \right) \\ &= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_1 \\ \tau \in S_{2r+1}}} 2^{-r} m_{0,r} \left(2 \left\{\begin{array}{c|c} 0 & 0 \\ 0 & 2r \end{array}\right\}_0 - \left(\left\{\begin{array}{c|c} 0 & 0 \\ 0 & 2r \end{array}\right\}_0 + r \left[\begin{array}{c|c} 0 & 0 \\ 0 & 2r \end{array}\right]_1 \right) \right) \\ &= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_1 \\ \tau \in S_{2r+1}}} 2^{-r} m_{0,r} \left(-r \left[\begin{array}{c|c} 0 & 0 \\ 0 & 2r \end{array}\right]_1 + \left\{\begin{array}{c|c} 0 & 0 \\ 0 & 2r \end{array}\right\}_0 \right) \end{aligned}$$

(2)

$$
\begin{aligned}
R_{n,r} &= \langle S^r(f_0 \wedge \ldots \wedge f_n), g_0 \wedge \ldots \wedge g_{n+2r} \rangle \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}} m_{n,r} \sum_{1 \le c_1 < \cdots < c_n \le n+2r} K_{n+2r}(0, c_1, \ldots, c_n) \\
&\quad + \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}} m_{n,r} \sum_{i=n+1}^{n+2r} \sum_{1 \le c_1 < \cdots < c_n \le i-1} (-1)^{i+\sum_{j=1}^n c_j + \binom{n+2}{2}} g_{\tau_0} \cdot_h \cdots \\
&\qquad \cdot_h f_{\sigma_1} \cdot_h g_{\tau_{c_1}} \cdot_h \cdots \cdot_h f_{\sigma_n} \cdot_h g_{\tau_{c_n}} \cdot_h \cdots \cdot_h f_{\sigma_0} \cdot_h g_{\tau_i} \cdot_h \cdots \cdot_h g_{\tau_{n+2r}} \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}} T^{(0)}_{n,r} \\
&\quad + \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}} m_{n,r} \sum_{i=n+1}^{n+2r} \sum_{1 \le c_1 < \cdots < c_n \le i-1} (-1)^{i+\sum\limits_{j=1}^n c_j + (n+1) + \binom{n+1}{2}} (-1)^n g_{\tau_0} \cdot_h \cdots \\
&\qquad \cdot_h f_{\sigma_0} \cdot_h g_{\tau_{c_1}} \cdot_h \cdots \cdot_h f_{\sigma_{n-1}} \cdot_h g_{\tau_{c_n}} \cdot_h \cdots \cdot_h f_{\sigma_n} \cdot_h g_{\tau_i} \cdot_h \cdots \cdot_h g_{\tau_{n+2r}} \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}} T^{(0)}_{n,r} - m_{n,r} \sum_{1 \le c_1 < \cdots < c_n < i \le n+2r} K_{n+2r}(c_1, \ldots, c_n, i) \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}} T^{(0)}_{n,r} - \left(T_{n,r} - T^{(0)}_{n,r} \right) \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}} 2T^{(0)}_{n,r} - T_{n,r} \\
&= \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}} m_{n,r} 2^{-r} \Bigg[-\binom{r+n}{n+1} \begin{bmatrix} 0 \big| n \\ 0 \big| n+2r \end{bmatrix}_{n+1} \\
&\quad + \sum_{M=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} 2^{-2(M+1)} \left(8 \binom{r+n-M}{r, n-2M, M} - \binom{r+M}{M} \binom{r+n-M}{n+1-2M} \right) \begin{bmatrix} 0 \big| n \\ 0 \big| n+2r \end{bmatrix}_{n+1-2M} \\
&\quad + \sum_{M=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 2^{-2M} \binom{r+n-M}{r, n-2M, M} \begin{Bmatrix} 0 \big| n \\ 0 \big| n+2r \end{Bmatrix}_{n-2M} \Bigg]
\end{aligned}
$$

$\square$

Any term containing $\begin{bmatrix} 0 \big| n \\ 0 \big| n+2r \end{bmatrix}_{\bullet}$ will have $n+1$ commutators, and so, will be of $O(h^{n+1})$. So, from here, the theorem follows.

Theorem 4.2.3. *Let A be a Poisson algebra with products $\cdot_h$ and common trace,* Tr*. For any*

$\alpha_n = f_0 \wedge \ldots \wedge f_n \in \Lambda^{n+1}\mathcal{A}$, we have a cochain $\alpha^h_{n,r} = S^r(\alpha_n) \in CC^{n+1+2r}(\mathcal{A}, \cdot_h)$. Restricting to acting on antisymmetric chains, we then have a cochain $\hat{\alpha}^h_{n,r} \in \Lambda^{n+2r+1}\mathcal{A}^*$ which can be written as

$$\hat{\alpha}^h_{n,r} = \hat{\alpha}_{n,r} h^{n+r} + O\left(h^{n+r+1}\right)$$

with

$$\hat{\alpha}_{n,r} = \frac{n!\,r!}{(n+2r)!} 2^{-r} \sum_{M=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 2^{-2M} \binom{r+n-M}{r, n-2M, M} \hat{\alpha}^{(n-2M)}_{n,r} \in \Lambda^{n+2r+1}\mathcal{A}^*$$

and

$$\begin{aligned} \hat{\alpha}^{(M)}_{n,r}\left(g_0 \wedge \ldots g_{n+2r}\right) = \operatorname{Tr} \operatorname*{Alt}_{\substack{\sigma \in S_{n+1} \\ \tau \in S_{n+2r+1}}} & f_{\sigma_0} \cdot g_{\tau_0} \cdot \{f_{\sigma_1}, g_{\tau_1}\} \cdot \ldots \cdot \{f_{\sigma_M}, g_{\tau_M}\} \\ & \cdot \left\{f_{\sigma_{M+1}}, f_{\sigma_{M+2}}\right\} \cdot \ldots \cdot \left\{f_{\sigma_{n-1}}, f_{\sigma_n}\right\} \cdot \left\{g_{\tau_{M+1}}, g_{\tau_{M+2}}\right\} \cdot \ldots \cdot \left\{g_{\tau_{n+2r-1}}, g_{\tau_{n+2r}}\right\} \end{aligned}$$

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