Independent Partitions in Boolean Algebras

by

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B.A., University of Colorado at Boulder, 2004

A thesis submitted to the

Faculty of the Graduate School of the

University of Colorado in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

2012

This thesis entitled:
Independent Partitions in Boolean Algebras
written by R. M. Chestnut
has been approved for the Department of Mathematics

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Date

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Chestnut, R. M. (Ph.D., Mathematics)

Independent Partitions in Boolean Algebras

Thesis directed by Prof. J. Donald Monk

We introduce a natural generalization of the cardinal invariant independence on Boolean algebras, suggested by the proof of the Balcar-Franěk Theorem [1]. We develop some basic theory and generalize some known results regarding independence, including the Balcar-Franěk Theorem itself.

Dedication

To the Iron Fist.

Acknowledgements

Thanks to Don Monk, Keith Kearnes, Agnes Szendrei, Bart Kastermans, Graeme Forbes, my Parents, Kevin Selker, Betzi Sherman, and everyone who helped me deal with the necessary technology.

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Chapter 1

Introduction

In 1936, Hausdorff improved on a 1935 result by Kantorovich and Fichtenholz, who showed that $\wp(\omega)$ has 2^{2^ω} ultrafilters, by generalizing to $\wp(\kappa)$ for any cardinal κ [3]. The proof rests on the existence of an independent family of size 2^κ in the Boolean algebra $\wp(\kappa)$. A subset X of a Boolean algebra A is said to be independent if its members generate a free subalgebra of A, or equivalently if no monomial over X is 0. The question of whether this result can be generalized to any infinite complete Boolean algebra was probably first forlumated by Efimov in 1970 (the situation for finite Boolean algebras is too simple to be interesting, and it is easy to find large non-complete Boolean algebras without very many ultrafilters, e.g. $\mathrm{Finco}(\kappa)$, the set of finite and cofinite subsets of κ, has only κ many ultrafilters). This question was finally answered in the affirmative by Balcar and Franěk in 1982 [1], though prior to this partial solutions where porvided by Kesl'yakov, Koppelberg, Monk, and Blaszczyk. As in Hausdorff's proof, Balcar and Franěk guarantee a large amount of ultrafilters by exhibiting a large independent family. The study of independent families is interesting in its own right and is a natural part of the study of cardinal invariants on Boolean algebras, treated extensively in [10]. The large and small independence functions ind and i for a Boolean algebra A are defined as follows:

$$\mathrm{ind}(A) = \sup\{|X| : X \text{ is an independent subset of } A\}$$

$$\mathrm{i}(A) = \min\{|X| : X \text{ is a in infinite maximal independent subset of } A\}$$

These functions have been studied in some detail in [4],[6],[7],[8],[9], and [10]. From this perspective the Balcar-Franěk Theorem can be restated as "If A is complete and infinite then $\mathrm{ind}(A) = |A|$".

In the proof of their famous theorem, Balcar and Franěk introduce a more general notion of independence. If $X \subseteq A$, then $\{\{x, -x\} \,|x \in X\}$ forms a set of partitions of unity in A, and the condition that X be independent is equivalent to the condition that $\prod_{x \in F} f(x) \neq 0$ whenever F is a finite subset of X and $\forall x \in F \ (f(x) \in \{x, -x\})$. If we replace $\{\{x, -x\} \,|x \in X\}$ with a set P whose members are partitions of unity of arbitrary size, then we say analogously P is independent if $\prod_{p \in F} f(p) \neq 0$ whenever F is a finite subset of P and $\forall p \in P \ (f(p) \in p)$. If each member of P has size λ, we say P is λ-independent in A. If P is λ-independent and $P \cup \{q\}$ is not independent whenever q is a λ-sized partition of unity in A, we say P is maximal λ-independent. For any cardinal λ, the large and small λ-independence functions λ-ind and λ-i can now be defined in the natural way:

$$\lambda\text{-ind}(A) = \sup\{|P| : P \text{ is } \lambda\text{-independent in } A\}$$

$$\lambda\text{-i}(A) = \min\{|P| : P \text{ is infinite and maximal } \lambda\text{-independent in } A\}$$

It is natural to ask which of the known results pertaining to ind and i generalize to λ-ind and λ-i and under what conditions on λ, and this thesis provides some answers to these types of questions. In addition to some more basic results, we formulate and prove a generalized version of the Balcar-Franěk theorem itself and prove the equivalence of n-i on infinite algebras for all $n \in \omega$.

Chapter 2

Definitions and Notation

We adopt the set theoretical notation of [5] and notation for the arithmetic of Boolean algebras of [4]. For sets x and y and a cardinal κ, ${}^{y}x$ is the set of functions from y to x, $[x]^{\kappa}$ is the set of subsets of x of size κ, and $[x]^{<\kappa}$ is the set of subsets of x of size less than κ.

When it is clear from the context, 0 and 1 are understood to mean the additive and multiplicative identities of the Boolean algebra under discussion. When it is necessary to be explicit, subscripts will be used, e.g. 0_A is the additive identity in A. We will always use $+$ and $\cdot$ and $-$ for the Boolean operations, usually implying $\cdot$ by adjacency and omitting the symbol.

We will use the shorthand "BA" for "Boolean algebra". If A is a BA and $B \subseteq A$, $\langle B \rangle$ will denote the subalgebra of A generated by B, $\langle B \rangle^{\mathrm{id}}$ will denote the ideal of A generated by B, and if A is complete, $\langle B \rangle^{\mathrm{cm}}$ will denote the smallest complete subalgebra of A containing B, while $\langle A \rangle^{\mathrm{cm}}$ will generally denote the completion of A. In formulating products it will sometimes be convenient to use the convention $x^1 = x$, $x^0 = -x$ for x an element of a BA. The set of nonzero elements of a BA A will be denoted A^+. This notation may also be applied to a subset S of a BA that is not necessarily a subalgebra, so $S^+ = S \setminus \{0\}$. The notation $x \restriction y$ will be used in two different ways: If A is a BA and $a \in A$ then $A \restriction a$ is the BA $\{ab | b \in A\}$, with operations inherited from A, except that $1_{A\restriction a} = a$ and $(-b)_{A\restriction a} = (-b)_A \cdot a$. If f is a function and S is a subset of its domain, then $f \restriction S$ is the restriction of f to S. The meaning of $\restriction$ will always be clear from the context.

If A is a Boolean algebra and $X \subseteq \wp(A)$, we define X-mon, the set of monomials over X, by

$$X\text{-mon} = \left\{ \prod_{x \in F} f(x) \middle| F \in [X]^{<\omega},\ f : F \to \bigcup F, \text{ and } \forall x \in F\ (f(x) \in x) \right\}.$$

Usually each member of X will consist of partitions of unity, X will be indexed by some cardinal κ and each member of X by some cardinal λ, e.g. $X = \{p_\alpha | \alpha \in \kappa\}$ and $\forall \alpha \in \kappa\ (p_\alpha = \{x_{\alpha\beta} | \beta \in \lambda\})$, in which case

$$X\text{-mon} = \left\{ \prod_{\alpha \in F} x_{\alpha f(\alpha)} \middle| F \in [\kappa]^{<\omega} \text{ and } f : F \to \lambda \right\}.$$

If $Y \subseteq A$, let $X = \{\{y, -y\} | y \in Y\}$ and let Y-mon $= X$-mon. Thus Y-mon is the set of monomials over Y, and Y is independent if and only if $0 \notin Y$-mon. Accordingly, a subset X of $\wp(A)$ is independent if and only if $0 \notin X$-mon. We generalize the spectrum of maximal independent sets of a BA

$$\operatorname{spind}(A) = \{|X| : X \text{ is infinite and maximal independent in } A\}$$

in the natural way:

$$\lambda\text{-}\operatorname{spind}(A) = \{|P| : P \text{ is infinite and maximal } \lambda\text{-independent in } A\}.$$

Note that by definition $2\text{-}\operatorname{spind}(A) = \operatorname{spind}(A)$, $2\text{-}\mathrm{i}(A) = \mathrm{i}(A)$, and $2\text{-}\operatorname{ind}(A) = \operatorname{ind}(A)$.

Chapter 3

The Boolean Algebra Freely Generated by κ-many λ-partitions

It will be useful to define and prove some results regarding a canonical "almost free" algebra generated by an independent set of λ-partitions.

For λ and κ cardinals, let $X = \{x_{\alpha\beta} | \alpha \in \kappa,\ \beta \in \lambda\}$ be a set with $(\alpha, \beta) \neq (\alpha', \beta') \to x_{\alpha\beta} \neq x_{\alpha'\beta'}$ and define $\mathrm{Fr}_\lambda(\kappa) = \mathrm{Fr}(X)/I$, where

$$I = \begin{cases} \langle \{x_{\alpha\beta}x_{\alpha\gamma} | \alpha \in \kappa,\ \beta, \gamma \in \lambda, \text{ and } \beta \neq \gamma\} \rangle^{\mathrm{id}} & \lambda \geq \omega \\ \left\langle \{x_{\alpha\beta}x_{\alpha\gamma} | \alpha \in \kappa,\ \beta, \gamma \in \lambda, \text{ and } \beta \neq \gamma\} \cup \left\{ -\sum_{\beta \in \lambda} x_{\alpha\beta} | \alpha \in \kappa \right\} \right\rangle^{\mathrm{id}} & \lambda < \omega \end{cases}.$$

Let $\pi : \mathrm{Fr}(X) \to \mathrm{Fr}_\lambda(\kappa)$ be the natural homomorphism. For all $\alpha \in \kappa$ and $\beta \in \lambda$ let $y_{\alpha\beta} = \pi(x_{\alpha\beta})$, let $p_\alpha = \{y_{\alpha\beta} | \beta \in \lambda\}$ and let $P = \{p_\alpha | \alpha \in \kappa\}$. Henceforth P defined thusly will be called the canonical set of generating partitions of $\mathrm{Fr}_\lambda(\kappa)$.

Claim. *P is a λ-independent set in $\mathrm{Fr}_\lambda(\kappa)$ and $\bigcup P$ generates $\mathrm{Fr}_\lambda(\kappa)$.*

Proof. Clearly $\bigcup P$ generates $\mathrm{Fr}(X)/I$, as $\bigcup P = \pi[X]$. To see that each p_α is a partition of unity, fix α. For distinct β and γ in λ $x_{\alpha\beta}x_{\alpha\gamma} \in I \Rightarrow y_{\alpha\beta}y_{\alpha\gamma} = 0$, so p_α is pairwise disjoint. If $\lambda < \omega$, $-\sum_{\beta\in\lambda} x_{\alpha\beta} \in I \Rightarrow -\sum_{\beta\in\lambda} y_{\alpha\beta} = 0 \Rightarrow \sum_{\beta\in\lambda} y_{\alpha\beta} = 1$. If $\lambda \geq \omega$, suppose for contradiction that, for some nonzero a in $\mathrm{Fr}_\lambda(\kappa)$, $\forall\beta \in \lambda$ $(ax_{\alpha\beta} = 0)$. Fix such a, and fix $b \in \mathrm{Fr}(X) \setminus I$ such that $\pi(b) = a$. We can write b as a finite sum of monomials over X and $b \notin I \Rightarrow \exists m \in X\text{-mon} \setminus I$ such that $m \leq b$. Fix such m and fix $F \in [X]^{<\omega}$, $f : F \to 2$ such that $m = \prod_{x\in F} x^{f(x)}$. Because $m \notin I$, $\forall\delta \in \kappa$ $(|\{\beta \in \lambda | x_{\delta\beta} \in F \ \wedge\ f(x_{\delta\beta}) = 1\}|) \leq 1$. In particular, there is at most one $\beta \leq \lambda$ such that $x_{\alpha\beta} \in F$ and $f(x_{\alpha\beta}) = 1$. Fix such β if it exists, and otherwise take an arbitrary $\beta \in \lambda$ such that $x_{\alpha\beta} \notin F$, using finiteness of F. By assumption $ay_{\alpha\beta} = 0 \Rightarrow bx_{\alpha\beta} \in I \Rightarrow mx_{\alpha\beta} \in I \Rightarrow \exists G$ a finite subset of $\{x_{\delta\gamma}x_{\delta\varepsilon} | \delta \in \kappa,\ \gamma, \varepsilon \in \lambda \text{ and } \gamma \neq \varepsilon\}$ such that $mx_{\alpha\beta} \leq \sum G$. Let $h : \mathrm{Fr}(X) \to 2$ be a homomorphism with $\forall x \in X$

$$h(x) = \begin{cases} 1 & x \in F \ \wedge \ f(x) = 1 \\ 1 & x = x_{\alpha\beta} \\ 0 & \text{otherwise} \end{cases} .$$

Note that $\forall \delta \in \kappa$ there is at most one $\gamma \in \lambda$ such that $h(x_{\delta\gamma}) = 1 \Rightarrow h(\sum G) = 0$, but also $h(mx) = 1$, contradiction. So in fact $\forall a \in \mathrm{Fr}_\lambda(\kappa)^+ \ \exists \beta \in \lambda \ (ay_{\alpha\beta} \neq 0) \Rightarrow \sum p_\alpha$ exists and is 1.

To see that the p_α are independent, suppose for contradiction that $0 \in P$-mon. Then $\exists F \in [X]^{<\omega}$ and $f : F \to 2$ such that $\prod_{x \in F} x^{f(x)} \in I$, with $\forall \alpha \in \kappa \ (|\{\beta \in \lambda | x_{\alpha\beta} \in F \ \wedge \ f(x_{\alpha\beta}) = 1\}| \leq 1)$. Let $h : \mathrm{Fr}(X) \to 2$ be a homomorphism with $\forall x \in X$

$$f(x) = \begin{cases} 1 & x \in F \ \wedge \ f(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

so that, as above, for all $\alpha \in \kappa$ and distinct $\beta, \gamma \in \lambda$ $(h(x_{\alpha\beta}) = 0 \ \vee \ h(x_{\alpha\gamma}) = 0) \Rightarrow h(x_{\alpha\beta}x_{\alpha\gamma}) = 0$, while $h\left(\prod_{x \in F} x^{f(x)}\right) = 1$. But $\prod_{x \in F} x^{f(x)} \in I \Rightarrow \exists G \in [\{x_{\alpha\beta}x_{\alpha\gamma} | \alpha \in \kappa,\ \beta, \gamma \in \lambda \text{ and } \beta \neq \gamma\}]^{<\omega}$ such that $\prod_{x \in F} x^{f(x)} \leq \sum G$. Applying h to both sides of this inequality yields a contradiction. $\square$

Any element of $Fr(X)$ can be written as a finite sum of disjoint monomials over X. It would be nice to have a similar normal form for elements of $\mathrm{Fr}_\lambda(\kappa)$, using P as above, but unfortunately not every element of $\mathrm{Fr}_\lambda(\kappa)$ can be written as a finite sum of P-monomials; with notation as above, $-y_{00} \in \mathrm{Fr}_\omega(\omega)$ is an example. However, a nice normal form result can be obtained by slightly modifying our notion of a monomial.

If $P = \{p_\alpha | \alpha \in \kappa\}$ is a set of λ-sized partitions of a BA and $\forall \alpha \in \kappa \ \ p_\alpha = \{x_{\alpha\beta} | \alpha \in \kappa,\ \beta \in \lambda\}$, define P-mon*, the augmented set of monomials over P, by P-mon*=

$$\left\{ \prod_{\alpha \in F} x_{\alpha f(\alpha)} \prod_{\alpha \in G} \left(- \sum_{\beta \in H_\alpha} x_{\alpha\beta} \right) \middle| F, G \in [\kappa]^{<\omega},\ F \cap G = \emptyset,\ f : F \to \lambda, \text{ and } \forall \alpha \in G\ H_\alpha \in [\lambda]^{<\omega} \right\},$$

or, equivalently,

$$P\text{-mon}^* = \left\{ \prod_{\alpha\in F} f(\alpha) \middle| F \in [\kappa]^{<\omega} \text{ and } f(\alpha) \in p_\alpha \cup \left\{ -\sum G \middle| G \in [p_\alpha]^{<\omega} \right\} \text{ for all } \alpha \in F \right\}.$$

Thus we allow, for each α in a finite subset of κ, either one member of p_α or the complement of the sum of finitely many members of p_α as a factor in our product.

Theorem 1. *Each element of* $\mathrm{Fr}_\lambda(\kappa)$ *can be written as a finite sum of disjoint members of* P-mon**, where* P *is the canonical set of generating partitions of* $\mathrm{Fr}_\lambda(\kappa)$*.*

Proof. Again using p_α, $x_{\alpha\beta}$, and $y_{\alpha\beta}$ as in the remarks following the definition of $\mathrm{Fr}_\lambda(\kappa)$, $\forall a \in \mathrm{Fr}_\lambda(\kappa)$ fix $b \in \mathrm{Fr}(X)$ such that $a = \pi(b)$ and M a finite pairwise disjoint subset of X-mon such that $b = \sum M$. It suffices now to show that $\forall m \in M \setminus I$ $(\pi(m) \in P\text{-mon}^*)$, because then M pairwise disjoint $\Rightarrow \pi[M \setminus I]$ pairwise disjoint and clearly $a = \sum \pi[M] = \sum \pi[M \setminus I]$.

For $m \in M \setminus I$ write m as a finite product of monomials over the $\pi^{-1}[p_\alpha]$, i.e.

$$m = \prod_{\alpha\in F} \prod_{\beta\in G_\alpha} x_{\alpha\beta}^{f_\alpha(\beta)},$$

where $F \in [\kappa]^{<\omega}$ and $\forall \alpha \in F$ $(G_\alpha \in [\lambda]^{<\omega})$ and $f_\alpha : G_\alpha \to 2$. For any $\alpha \in F$ there is at most one $\beta \in G_\alpha$ such that $f_\alpha(\beta) = 1$, because otherwise $m \in I$. Let $F' = \{\alpha \in F | 1 \in f_\alpha[G_\alpha]\}$ and define $g : F' \to \lambda$ by $g(\alpha) =$ the unique $\beta \in G_\alpha$ with $f_\alpha(\beta) = 1$. Note that $\forall \alpha \in F'$ $\forall \beta \in G_\alpha \setminus \{g(\alpha)\}$ $(x_{\alpha g(\alpha)} x_{\alpha\beta} \in I) \Rightarrow y_{\alpha g(\alpha)} \leq -y_{\alpha\beta}$, so

$$\prod_{\beta\in G_\alpha} y_{\alpha\beta}^{f_\alpha(\beta)} = y_{\alpha g(\alpha)} \cdot \left(\prod_{\beta\in G_\alpha\setminus\{g(\alpha)\}} -y_{\alpha\beta} \right) = y_{\alpha g(\alpha)}.$$

If $\alpha \in F \setminus F'$ then $\prod_{\beta\in G_\alpha} y_{\alpha\beta}^{f_\alpha(\beta)} = \prod_{\beta\in G_\alpha} -y_{\alpha\beta} = -\sum_{\beta\in G_\alpha} y_{\alpha\beta}$, so putting these together we have

$$\pi[m] = \prod_{\alpha\in F} \prod_{\beta\in G_\alpha} y_{\alpha\beta}^{f_\alpha(\beta)} = \prod_{\alpha\in F'} \prod_{\beta\in G_\alpha} y_{\alpha\beta}^{f_\alpha(\beta)} \prod_{\alpha\in F\setminus F'} \prod_{\beta\in G_\alpha} y_{\alpha\beta}^{f_\alpha(\beta)} = \prod_{\alpha\in F'} y_{\alpha g(\alpha)} \prod_{\alpha\in F\setminus F'} \left(-\sum_{\beta\in G_\alpha} y_{\alpha\beta} \right),$$

which is in P-mon*, as desired. $\square$

Recall that, for two subsets X and Y of a BA, X is dense in Y means $\forall y \in Y^+$ $\exists x \in X^+$ such that $x \leq y$.

Corollary 2. *If P is the canonical set of generating partitions for* $\mathrm{Fr}_\lambda(\kappa)$ *then* P-mon *is dense in* $\mathrm{Fr}_\lambda(\kappa)$*.*

Proof. By Theorem 1 it suffices to show P-mon is dense in P-mon*. For any nonzero $m \in P$-mon* write

$$m = \prod_{\alpha\in F} y_{\alpha f(\alpha)} \prod_{\alpha\in G} \left(-\sum_{\beta\in H_\alpha} y_{\alpha\beta} \right)$$

with F and G disjoint finite subsets of κ, $f : F \to \lambda$, and $\forall\alpha \in G$ $(H_\alpha \in [\lambda]^{<\omega})$. If λ is finite then, by the way I is defined in the definition of $\mathrm{Fr}_\lambda(\kappa)$, we have $-\sum p_\alpha = 0$. Thus $\forall\alpha \in G$ there is some $n_\alpha \in \lambda\setminus H_\alpha$, because otherwise $m \leq -\sum_{\beta\in H_\alpha} y_{\alpha\beta} = 0$. We extend f to $F\cup G$ by letting $f(\alpha) = n_\alpha$. If λ is infinite then each $\lambda \setminus H_\alpha$ is nonempty because H_α is finite, and we extend f to $F \cup G$ by setting $f(\alpha)$ to be an arbitrary member of $\lambda\setminus H_\alpha$. Either way $f(\alpha) \notin H_\alpha \Rightarrow y_{\alpha f(\alpha)} \leq -\sum_{\beta\in H_\alpha} y_{\alpha\beta}$, so

$$m = \prod_{\alpha\in F} y_{\alpha f(\alpha)} \prod_{\alpha\in G} \left(-\sum_{\beta\in H_\alpha} y_{\alpha\beta} \right) \geq \prod_{\alpha\in F\cup G} y_{\alpha f(\alpha)} \in P\text{-mon},$$

as desired. □

Theorem 3. *For any cardinals κ and λ*

$$\mathrm{Fr}_\lambda(\kappa) \cong \bigoplus_{\alpha\in\kappa} \mathrm{Finco}(\lambda),$$

the free product of κ many copies of $\mathrm{Finco}(\lambda)$*.*

Proof. By the characterization of free products [4, Proposition 11.4.], it suffices to exhibit a set $\{h_\alpha | \alpha \in \kappa\}$ of one-to-one homomorphisms from $\mathrm{Finco}(\lambda)$ into $\mathrm{Fr}_\lambda(\kappa)$ such that $\{h_\alpha[\mathrm{Finco}(\lambda)] | \alpha \in \kappa\}$ is an independent set of subalgebras who's union generates $\mathrm{Fr}_\lambda(\kappa)$. Again we use the canonical set of generating partitions of $\mathrm{Fr}_\lambda(\kappa)$ as defined above. For $\alpha \in \kappa$ define $h_\alpha : \mathrm{Finco}(\lambda) \to \mathrm{Fr}_\lambda(\kappa)$ by setting $h_\alpha(\{\beta\}) = y_{\alpha\beta}$ for all $\beta \in \lambda$ and extending h to a homomorphism, so that

$$h_\alpha(x) = \begin{cases} \sum_{\beta\in x} y_{\alpha\beta} & x \text{ is finite} \\ -\sum_{\beta\in\lambda\setminus x} y_{\alpha\beta} & \lambda \setminus x \text{ is finite} \end{cases}$$

Clearly $h_\alpha[\mathrm{Finco}(\lambda)] = \langle p_\alpha \rangle$, and if $x \in \mathrm{Finco}(\lambda)^+$, then $h_\alpha(x)$ is a nonempty sum of P-monomials $\Rightarrow h_\alpha(x) \neq 0$ by independence of P, showing h_α is one-to-one. If $F \in [\kappa]^{<\omega}$ and $\forall \alpha \in F$ $(x_\alpha \in \langle p_\alpha \rangle^+)$, then $\prod_{\alpha \in F} x_\alpha$ is a nonempty sum of P-monomials, so again by independence of P $\prod_{\alpha \in F} x_\alpha \neq 0$, showing $\{h_\alpha[\mathrm{Finco}(\lambda)] | \alpha \in \kappa\} = \{\langle p_\alpha \rangle \,| \alpha \in \kappa\}$ is an independent family of subalgebras of $\mathrm{Fr}_\lambda(\kappa)$. Finally, $\bigcup_{\alpha \in \kappa} h_\alpha[\mathrm{Finco}(\lambda)] = \bigcup P$, which generates $\mathrm{Fr}_\lambda(\kappa)$ by the claim following the definition of $\mathrm{Fr}_\lambda(\kappa)$. $\square$

Theorem 4. *A Boolean Algebra A has a λ-independent set of size κ if and only if A contains an isomorphic copy of* $\mathrm{Fr}_\lambda(\kappa)$

Proof. The "if" direction follows from the claim following the definition of $\mathrm{Fr}_\lambda(\kappa)$. For the other direction, suppose A is a BA and $Q = \{q_\alpha | \alpha \in \kappa\}$ is an independent set of partitions of unity in A. The proof of Theorem 3 can now be applied with Q in place of P to show $\langle \bigcup Q \rangle \cong \bigoplus_{\alpha \in \kappa} \mathrm{Finco}(\lambda) \cong \mathrm{Fr}_\lambda(\kappa)$, so $\langle \bigcup Q \rangle$ is the desired subalgebra of A. $\square$

Chapter 4

λ-independence

We prove some basic results regarding λ-ind.

Theorem 5. *If A is a BA and λ-ind$(A) = \kappa$ for cardinals $\lambda \geq 3$, $\kappa \geq 1$, then $\forall n \in \omega \cap \lambda$ (n-ind$(A) \geq \kappa$). In particular* ind$(A) \geq \kappa$.

Proof. Let $P = \{p_\alpha | \alpha \in \kappa\}$ be a λ-independent set in A and $\forall \alpha \in \kappa$ let $p_\alpha = \{x_{\alpha\beta} | \beta \in \lambda\}$. Fix $n \in \omega \cap \lambda$. We construct a κ-sized n-independent set in A. Thus if λ-ind(A) is not attained the argument can be applied to all $\kappa < \lambda$-ind(A) to prove n-ind$(A) \geq \lambda$-ind(A). For each $\alpha \in \kappa$ let

$$q_\alpha = \{x_{\alpha\beta} | \beta < n-1\} \cup \left\{ -\sum_{\beta < n-1} x_{\alpha\beta} \right\}.$$

Let $Q = \{q_\alpha | \alpha \in \kappa\}$. Clearly Q is a set of n-partitions, and P-mon is dense in Q-mon, so Q inherits independence from P. □

The above sum in the definition of q_α may not exist if $n \geq \omega$, so for a more general version we require that A have the necessary sums, and the proof is identical:

Theorem 6. *If A is a BA and λ-ind$(A) = \kappa$ for cardinals $\lambda \geq 3$, $\kappa \geq 1$, then $\forall \mu < \lambda$, if A is μ^+-complete, μ-*ind$(A) \geq \kappa$.

□

The following theorem shows that the completeness condition in Theorem 6 is necessary by showing that, for example, $\mathrm{Fr}_{\aleph_2}(\omega)$ has no $\aleph_1$-sized partition of unity. First, we make a definition and prove a convenient lemma. Recall that a BA A is compact if whenever $S \subseteq A$ and $\sum S = 1$ there is a finite subset S' of S such that $\sum S' = 1$ (This definition is not very interesting under the axiom of choice, because then every infinite BA has an infinite partition of unity, so compact just means finite). If λ is an infinite cardinal, in [2] Cichon defines A to be λ-compact if whenever $S \in [A]^{\leq\lambda}$ and $\sum S = 1$ there is a finite subset S' of S such that $\sum S' = 1$ (In fact he makes an equivalent definition using the dual notion of sets who's products are 0). We extend this definition by saying A is $<\lambda$-compact if whenever $S \in [A]^{<\lambda}$ and $\sum S = 1$ there is a finite subset S' of S such that $\sum S' = 1$. An easy example of a $<\lambda$-compact BA is $\mathrm{Finco}(\lambda)$.

Lemma 7. *If B is a $<\lambda$-compact BA, then the free product $B \oplus \mathrm{Finco}(\lambda)$ is $<\lambda$-compact.*

Proof. Let $A = \mathrm{Finco}(\lambda)$. Following [4, 11.5] and the subsequent remarks, we view A and B as subalgebras of $A \oplus B$ and make use of the fact that $A \cap B = \{0, 1\}$ and $0 \notin A^+ \cdot B^+$. Suppose $S \subseteq B \oplus A$, $|S| = \kappa < \lambda$, and $\sum S = 1$. If $\kappa < \omega$ we are done, so assume $\kappa \geq \omega$. By [4, Proposition 11.4.(c)] $B \oplus A$ is generated by $A \cup B$, so each $x \in S$ is a finite sum of products of the form ab with $a \in A$ and $b \in B$. Without loss of generality each $x \in S$ is itself such a product, so for each $x \in S$ we can write $x = a_x b_x$ with $a_x \in A$ and $b_x \in B$. For each $\alpha \in \lambda$ let $S_\alpha = \{x \in S | \{\alpha\} \leq a_x\}$. For all $\alpha \in \lambda$, because $\{\alpha\}$ is an atom of A we have

$$\{\alpha\} = \{\alpha\} \cdot \sum S = \sum_{x \in S} (\{\alpha\} \cdot x) = \sum_{x \in S} (\{\alpha\} \cdot a_x \cdot b_x) = \sum_{x \in S_\alpha} (\{\alpha\} \cdot b_x).$$

This implies that $\sum_{x \in S_\alpha} b_x = 1$. In fact, otherwise there is some nonzero $c \in B$ such that $b_x c = 0$ for all $x \in S_\alpha$, hence

$$0 \neq \{\alpha\} \cdot c = \sum_{x \in S_\alpha} (\{\alpha\} \cdot b_x \cdot c) = 0,$$

contradiction. Thus for each $\alpha \in \lambda$ we can fix F_α a finite subset of S_α with $\sum_{x \in S_\alpha} b_x = 1$. Each F_α is a finite subset of S and $|[S]^{<\omega}| = |S| = \kappa < \lambda$, so there is some fixed $F \in [S]^{<\omega}$ and $R \in [\lambda]^{\geq\omega}$ such that $\forall \alpha \in R$ $(F_\alpha = F)$. For all $x \in F$ there are infinitely many α in R, all of which satisfy $\{\alpha\} \leq a_x$, and $a_x \in \mathrm{Finco}(\lambda)$, so a_x must be cofinite. Let $G = \sum_{x \in F} -a_x$, so G is finite. We claim $S' := F \cup \left(\bigcup_{\alpha \in G} F_\alpha\right)$ is the desired finite subset of S with $\sum S' = 1$. It suffices to show

$\forall\alpha \in \lambda\ \forall b \in B^+\ \exists x \in S'$ such that $x\{\alpha\}b \neq 0$. If $\alpha \in G$, take $x \in F_\alpha$ such that $b_x b \neq 0$ and if $\alpha \notin G$ take $x \in F$ such that $b_x b \neq 0$. In either case $\alpha \in a_x$, so $x\{\alpha\}b = a_x b_x \{\alpha\}b = \{\alpha\}(b_x b) \neq 0$, as desired. □

Corollary 8. *For any infinite cardinal λ and $n \in \omega$, $\bigoplus_{i\in n} \mathrm{Finco}(\lambda)$ is $<\lambda$-compact.*

Proof. We proceed by incuction on n. The base case is clear, and if $\bigoplus_{i\in n} \mathrm{Finco}(\lambda)$ is $<\lambda$-compact then $\bigoplus_{i\in n+1} \mathrm{Finco}(\lambda) = \left(\bigoplus_{i\in n} \mathrm{Finco}(\lambda)\right) \oplus \mathrm{Finco}(\lambda)$ is $<\lambda$-compact by Lemma 7. □

Theorem 9. *If κ, λ, and μ are infinite cardinals and $\kappa < \mu < \lambda$, then $\mathrm{Fr}_\lambda(\kappa)$ has no partition of unity of size μ.*

Proof. Suppose S is a μ-sized subset of $\mathrm{Fr}_\lambda(\kappa)$ and $\sum S = 1$. We show S is not pairwise disjoint. Let P be the canonical set of generating partitions for $\mathrm{Fr}_\lambda(\kappa)$. By Theorem 1, without loss of generality we may assume each $x \in S$ is a member of P-mon*. Note that in particular this means $0 \notin S$. For each $x \in S$ write

$$x = \prod_{\alpha\in F_x} x_\alpha,$$

where F_x is a finite subset of κ and each $x_\alpha \in \langle p_\alpha\rangle$. There are only κ many finite subsets of κ, so there is some fixed $F \in [\kappa]^{<\omega}$ and $R \in [S]^{\geq\omega}$ such that $\forall x \in R\ (F_x = F)$. Thus for $x \in R$ we have $x \in \left\langle\bigcup_{\alpha\in F} p_\alpha\right\rangle$. Note also that $\mathrm{Fr}_\lambda(\kappa) = \left\langle\bigcup_{\alpha\in F} p_\alpha\right\rangle \oplus \left\langle\bigcup_{\alpha\in\kappa\setminus F} p_\alpha\right\rangle$, and by assumption each $x \in S$ can be written as $a_x b_x$, with $a \in \left\langle\bigcup_{\alpha\in F} p_\alpha\right\rangle$ and $b_x \in \left\langle\bigcup_{\alpha\in\kappa\setminus F} p_\alpha\right\rangle$. If $x \in R$ then $b_x = 1$, so $x = a_x$.

$$1 = \sum S \leq \sum_{x\in S} a_x \in \left\langle \bigcup_{\alpha\in F} p_\alpha \right\rangle,$$

so by Corollary 8 there is a finite subset S' of S with $\sum_{x\in S'} a_x = 1$. Now we can take any $x \in R\setminus S'$ and use the fact that $\sum_{y\in S'} a_y = 1$ to find $y \in S'$ such that $a_y a_x \neq 0$. But $x \in R$ and $y \in S$, so $b_y \neq 0$ and thus $xy = a_x a_y b_y \neq 0$ by freeness of the product $\left\langle\bigcup_{\alpha\in F} p_\alpha\right\rangle \oplus \left\langle\bigcup_{\alpha\in\kappa\setminus F} p_\alpha\right\rangle$. This shows S is not pairwise disjoint, as desired. □

Theorem 10. *If A is a BA and $\mathrm{ind}(A) \geq \omega$, then n-$\mathrm{ind}(A) = \mathrm{ind}(A)$ for $2 \leq n < \omega$.*

Proof. That $\operatorname{ind}(A) \geq n\text{-}\operatorname{ind}(A)$ follows from Theorem 5. For the other direction, suppose $\operatorname{ind}(A) = \kappa$ and let X be an independent subset of A with $|X| = \kappa$. Fix $n \in \omega \setminus 2$. We construct a κ-sized n-independent set in A, handling the case when $\operatorname{ind}(A)$ is not attained as in Theorem 5. Let $Y = \{y_\alpha | \alpha \in \kappa\}$ be a partition of X into κ many subsets of size n. For each $\alpha \in \kappa$ let $y_\alpha = \{x_{\alpha i} | i < n\} \subset X$ and $\forall f \in {}^n 2$ let

$$z_{\alpha f} = \prod_{i<n} x_{\alpha i}^{f(i)}.$$

For each $\alpha \in \kappa$ let $p_\alpha = \{z_{\alpha f} | f : n \to 2\}$ and let $P = \{p_\alpha | \alpha \in \kappa\}$. We check that each p_α is a partition of unity:

$$\sum p_\alpha = \sum_{f:n\to 2} \prod_{i<n} x_{\alpha i}^{f(i)} = \prod_{i<n} (x_{\alpha i} + -x_{\alpha i}) = 1,$$

and, if $f(j) \neq g(j)$ for some $j \in n$,

$$z_{\alpha f} z_{\alpha g} = \left(\prod_{i<n} x_{\alpha i}^{f(i)}\right)\left(\prod_{i<n} x_{\alpha i}^{g(i)}\right) \leq x_{\alpha j}^{f(j)} x_{\alpha j}^{g(j)} = 0,$$

so p_α is a partition of unity. By disjointness of Y, P-mon $\subset X$-mon, so P inherits indpendence from X. $|P| = \kappa$ and $\forall \alpha \in \kappa$ $(|p_\alpha| = 2^n > n)$, so by Theorem 5 $n\text{-}\operatorname{ind}(A) \geq 2^n\text{-}\operatorname{ind}(A) \geq \kappa$. $\square$

Theorem 11. *If A is ω_1-complete and* $\operatorname{ind}(A) \geq \omega$*, then* $\omega\text{-}\operatorname{ind}(A) = \operatorname{ind}(A)$.

Proof. That $\omega\text{-}\operatorname{ind}(A) \leq \operatorname{ind}(A)$ follows from Theorem 5. To prove $\operatorname{ind}(A) \leq \omega\text{-}\operatorname{ind}(A)$, we use a κ-sized independent subset of A to construct a κ-sized ω-independent set in A. Let $X \subseteq A$ be independent with $|X| = \kappa$. Let $\{Y_\alpha | \alpha \in \kappa\}$ be a partition of X into κ many countably infinite sets. For each $\alpha \in \kappa$ let $Y_\alpha = \{x_{\alpha n} | n \in \omega\}$ and define a partition of unity $p_\alpha = \{y_{\alpha n} | n \in \omega\}$ as follows:

$$\forall n > 0 \text{ let } y_{\alpha n} = x_{\alpha n} \cdot \prod_{m<n} (-x_{\alpha m}) \text{ and let } y_{\alpha 0} = -\sum_{n=1}^{\omega} y_{\alpha n}.$$

It is clear from the definition that each p_α is a partition of unity. Note that $x_{\alpha 0} \leq y_{\alpha 0}$ and $p_\alpha \setminus \{x_{\alpha 0}\} \subseteq X$-mon, so $0 \notin p_\alpha \Rightarrow |p_\alpha| = \omega$. To see that the p_α are independent, we again use the fact that $x_{\alpha 0} \leq y_{\alpha 0}$ to note that $\forall F \in [\kappa]^{<\omega}$ $\forall f : F \to \omega$

$$\prod_{\alpha\in F} y_{\alpha f(\alpha)} \geq \prod_{\alpha \in F}\left(x_{\alpha f(\alpha)} \cdot \prod_{n<f(\alpha)} x_{\alpha n}\right) \in X\text{-mon} \Rightarrow \prod_{\alpha\in F} y_{\alpha f(\alpha)} \neq 0,$$

so $\{p_\alpha | \alpha \in \kappa\}$ is the desired ω-independent set. $\square$

It is perhaps worth noting that, in building λ-independent sets in BAs, completeness conditions are often necessary only make infinite pairwise disjoint sets into partitions of unity. If we replaced "partition of unity" with "pairwise disjoint set" in the definition of λ-independent, while we would have perhaps a less natural generalization, several proofs would be a bit simpler. For example, in Theorem 11 we could remove the completeness condition and replace $-\sum_{n=1}^{\omega} y_{\alpha n}$ with $x_{\alpha 0}$ in the proof, ending up with an independent κ-sized set of λ-sized pairwise disjoint sets. This result is proved later as Lemma 17.

Monk has shown that, for BAs A_0 and A_1, $\mathrm{ind}(A_0 \times A_1) = \max\{\mathrm{ind}(A_0), \mathrm{ind}(A_1)\}$ [7, Corollary 1.2]. In the proof, the implicit assumption that $A_0 \times A_1$ have an infinite independent set is essential, so we formulate the generalization accordingly.

Theorem 12. *For BAs A_0 and A_1 and λ a cardinal, if $(A_0 \times A_1)$ has an infinite λ-independent set then* $\lambda\text{-ind}(A_0 \times A_1) = \max\{\lambda\text{-ind}(A_0), \lambda\text{-ind}(A_1)\}$.

Proof. Let $A = A_0 \times A_1$. To show $\lambda\text{-ind}(A) \leq \max\{\lambda\text{-ind}(A_0), \lambda\text{-ind}(A_1)\}$, suppose P is an infinite λ-independent set in A. We show either A_0 or A_1 has a λ-independent set of size $|P|$.

Case 1. $\forall m \in P\text{-mon}\ (\pi_0(m) \neq 0)$

Then in particular each $x \in \bigcup P$ is also in P-mon $\Rightarrow \pi_0(x) \neq 0$, so $\forall p \in P\ (|\pi_0[p]| = \lambda)$. Clearly each $\pi_0[p]$ is a partition of unity, and $\{\pi_0[p] | p \in P\}$ is independent by assumption so $\{\pi_0[p] | p \in P\}$ is the desired λ-independent set in A_0.

Case 2. $\exists m \in P\text{-mon}\ (\pi_0(m) = 0)$

Fix $F \in [P]^{<\omega}$ and $f : F \to \bigcup F$ such that $\forall p \in F\ (f(p) \in p)$ and $m = \prod_{p \in F} f(p)$. I claim $\{\pi_1[p] | p \in P \setminus F\}$ is the desired λ-independent set in A_1. It suffices to show that $\{\pi_1[p] | p \in P \setminus F\}$ is independent, because then it follows as in Case 1 that $\pi_1[p]$ is a λ-partition whenever $p \in P \setminus F$. Suppose for contradiction that $n \in (P \setminus F)$ and $\pi_1(n) = 0$. Then $nm \in P$-mon and $\pi_0(nm) = \pi_1(nm) = 0 \Rightarrow nm = 0$, contradicting the independence of P. Thus $\{\pi_1[p] | p \in P \setminus F\}$ is the desired λ-independent set in A_1, finishing this direction of the proof.

To show $\lambda\text{-ind}(A) \geq \max\{\lambda\text{-ind}(A_0),\ \lambda\text{-ind}(A_1)\}$, suppose P is a λ-independent set in A_i for $i \in 2$. We show A has a λ-independent set of size $|P|$. By symmetry assume $i = 0$. For all $p \in P$ let $p = \{x_\alpha | \alpha \in \lambda\}$ and define $p' \in A$ by $p' = \{(x_0, 1)\} \cup \{(x_\alpha, 0) | \alpha \in \lambda \setminus 1\}$. Clearly p' inherits pairwise disjointness from p, and $\sum p' = (\sum p, 1) = (1, 1) = 1_A$, so $\{p' | p \in P\}$ is a set of λ-partitions of A. Let $P' = \{p' | p \in P\}$. $|P'| = |P|$, and $0_{A_0} \notin P\text{-mon} = \pi_0[P'\text{-mon}] \Rightarrow 0_A \notin P'\text{-mon}$, showing that P' is the desired λ-independent set in A. □

An easy induction yields the following:

Corollary 13. *If $\{A_i | i \in I\}$ is a finite set of atomless BAs and λ is a cardinal, then*

$$\lambda\text{-ind}\left(\prod_{i \in I} A_i\right) = \max\{\lambda\text{-ind}(A_i) | i \in I\}$$

□

The following theorem concerns the moderate product of a set $\{A_i | i \in I\}$ of BAs over a subalgebra B of $\wp(I)$, denoted $\prod_{i \in I}^{B} A_i$. The moderate product is defined as follows:

Definition. *If $\{A_i | i \in I\}$ is a set of BAs and $B \leq \wp(I)$, then*

$$\prod_{i \in I}^{B} A_i = \left\{ f \in \prod_{i \in I} A_i \middle| \{i \in I | f(i) \notin \{0, 1\}\} \text{ is finite and } \{i \in I | f(i) = 1\} \in B \right\}.$$

Thus the moderate product generalizes the weak product in that if $B = \{0, 1\}$ then $\prod_{i \in I}^{B} A_i = \prod_{i \in I}^{\mathrm{W}} A_i$. In general the moderate product is inbetween the weak and full product, in the sense that it may contain more members of the full product.

Theorem 14. *If $\{A_i\}_i \in I$ is an infinite set of atomless BAs, $B \leq \wp(I)$, and λ is an infinite cardinal,*

$$\lambda\text{-ind}\left(\prod_{i \in I}^{B} A_i\right) \leq \sup\left(\{\lambda\text{-ind}(A_i) | i \in I\} \cup \{\lambda\text{-ind}(B)\}\right)$$

Proof. Let $A = \prod_{i \in I}^{B} A_i$. Suppose P is a set of λ-partitions of A and $|P| = \kappa$, where κ is greater than $\sup\left(\{\lambda\text{-ind}(A_i) | i \in I\} \cup \{\lambda\text{-ind}(B)\}\right)$. We show P is not independent. Let $P = \{p_\alpha | \alpha \in \kappa\}$

and $\forall\alpha \in \kappa$ let $p_\alpha = \{x_{\alpha\beta} | \beta \in \lambda\}$. For each $\alpha \in \kappa$ and $\beta \in \lambda$ let $S_{\alpha\beta} = \{i \in I | x_{\alpha\beta}(i) \notin \{0,1\}\}$ and define $y_{\alpha\beta} \in A$ as follows:

$$y_{\alpha\beta}(i) = \begin{cases} 1 & i \in S_{\alpha\beta} \setminus \bigcup_{\gamma<\beta} S_{\alpha\gamma} \\ 0 & i \in S_{\alpha\beta} \cap \bigcup_{\gamma<\beta} S_{\alpha\gamma} \\ x_{\alpha\beta}(i) & i \in I \setminus S_{\alpha\beta} \end{cases}.$$

Note that $S_{\alpha\beta}$ is finite so $y_{\alpha\beta}$ differs from $x_{\alpha\beta}$ on finitely many indeces, and hence $y_{\alpha\beta} \in B \leq A$. For each $\alpha \in \kappa$ let $q_\alpha = \{y_{\alpha\beta} | \beta \in \lambda\}$ and let $Q = \{q_\alpha | \alpha \in \kappa\}$. I claim Q is a set of partitions of unity in B. To see that each q_α is pairwise disjoint, fix $\alpha \in \kappa$ and suppose $\beta, \gamma \in \lambda,\ \gamma < \beta$. We partition I into four subsets:

Case 1. $i \in S_{\alpha\beta} \cap S_{\alpha\gamma}$

Then $y_{\alpha\beta}(i) = 0 \Rightarrow y_{\alpha\beta}(i)y_{\alpha\gamma}(i) = 0$.

Case 2. $i \in S_{\alpha\beta} \setminus S_{\alpha\gamma}$.

Then $x_{\alpha\beta}(i) \neq 0$, $x_{\alpha\gamma}(i) \in \{0,1\}$, and p_α is pairwise disjoint $\Rightarrow x_{\alpha\gamma}(i) = 0 \Rightarrow y_{\alpha\gamma}(i) = 0 \Rightarrow y_{\alpha\beta}(i)y_{\alpha\gamma}(i) = 0$.

Case 3. $i \in I \setminus S_{\alpha\beta}$ *and* $i \in S_{\alpha\gamma}$.

As in case 2, but with the roles of β and γ switched, $y_{\alpha\beta}(i) = 0 \Rightarrow y_{\alpha\beta}(i)y_{\alpha\gamma}(i) = 0$.

Case 4. $i \in I \setminus (S_{\alpha\beta} \cup S_{\alpha\gamma})$.

Then $y_{\alpha\beta}(i) = x_{\alpha\beta}(i)$ and $y_{\alpha\gamma}(i) = x_{\alpha\gamma}(i) \Rightarrow y_{\alpha\beta}(i)y_{\alpha\gamma}(i) = x_{\alpha\beta}(i)x_{\alpha\gamma}(i) = 0$. In any case $y_{\alpha\beta}(i)y_{\alpha\gamma}(i) = 0$, so $y_{\alpha\beta}y_{\alpha\beta} = 0$, showing q_α is pairwise disjoint. To see that q_α is a partition of unity, we partiton I into two subsets:

Case 1. $i \in \bigcup_{\beta\in\lambda} S_{\alpha\beta}$

Let $\varepsilon = \min\{\beta \in \lambda | i \in S_{\alpha\beta}\}$ so that $y_{\alpha\varepsilon}(i) = 1 \Rightarrow \sum_{\beta\in\lambda} y_{\alpha\beta}(i) = 1$.

Case 2. $i \in I \setminus \bigcup_{\beta\in\lambda} S_{\alpha\beta}$

Then $\sum_{\beta\in\lambda} y_{\alpha\beta}(i) = \sum_{\beta\in\lambda} x_{\alpha\beta}(i) = 1$. In any case $(\sum q_\alpha)(i) = 1$, so $\sum q_\alpha = 1$ as desired.

Our goal is to produce a monomial $m \in P$-mon such that $|\{i \in I | m(i) \neq 0\}| < \omega$. If $y_{\alpha\beta} = 0$ for some $\alpha \in \kappa$ and $\beta \in \lambda$, then $\forall i \in I \setminus S_{\alpha\beta}$ $(x_{\alpha\beta}(i) = y_{\alpha\beta}(i) = 0)$, so $m = x_{\alpha\beta}$ is the desired monomial. In this case let $F = \{\alpha\}$. Otherwise Q is a set of λ-partitions of B and $|Q| = \kappa > \lambda\text{-ind}(B)$, so Q is not independent. Fix $F \in [\kappa]^{<\omega}$ and $f : F \to \lambda$ such that $\prod_{\alpha\in F} y_{\alpha f(\alpha)} = 0$ and let $m = \prod_{\alpha\in F} x_{\alpha f(\alpha)}$. Note that $m \in P$-mon, $|\bigcup_{\alpha\in F} S_{\alpha f(\alpha)}| < \omega$, and $\forall i \in I \setminus \bigcup_{\alpha\in F} S_{\alpha f(\alpha)}$ $\forall \alpha \in F$ $(x_{\alpha f(\alpha)}(i) = y_{\alpha f(\alpha)}(i) \Rightarrow m(i) = \prod_{\alpha\in F} y_{\alpha f(\alpha)}(i) = 0)$, so m is the desired monomial. In either case fix such m and F and let $S = \{i \in I | m(i) \neq 0\}$. If $x_{\alpha\beta}[S] = \{0\}$ for some $\alpha \in \kappa \setminus F$ and $\beta \in \lambda$, then $0 = m x_{\alpha\beta} \in P$-mon so P is not independent and we are done. Thus we may assume $\forall \alpha \in \kappa \setminus F$ $\forall \beta \in \lambda$ $\exists i \in S$ such that $x_{\alpha\beta}(i) \neq 0$. For each $\alpha \in \kappa \setminus F$ let $p'_\alpha = \{x_{\alpha\beta} \restriction S | \beta \in \lambda\}$ and let $P' = \{p'_\alpha | \alpha \in \kappa \setminus F\}$. Note that P' is a set of λ-partitions of $\prod_{i\in S} A_i$. By Corollary 13 $\lambda\text{-ind}\left(\prod_{i\in S} A_i\right) < \kappa = |P'|$, so there is some $n' \in P'$-mon such that $n' = 0$. Fix $G \in [\kappa \setminus F]^{<\omega}$ and $g : G \to \lambda$ such that $n' = \prod_{\alpha\in G} \left(x_{\alpha g(\alpha)} \restriction S\right)$ and let $n = \prod_{\alpha\in G} x_{\alpha g(\alpha)}$. Note that $n(i) = n'(i) = 0$ whenever $i \in S$ and that $m(i) = 0$ whenever $i \in I \setminus S$, so $mn = 0$. $F \cap G = \emptyset$ so $mn \in P$-mon, showing P is not independent. $\square$

For $\lambda > \omega$, $\text{Fr}(\kappa)$ is an example of an arbitrarily large BA without even one λ-partition; [4, Corollary 9.18] states that every free algebra satisfies the countable chain condition, and thus has no uncountable pairwise disjoint set. The following theorem shows that, for $\lambda > \omega$, there are BA's with arbitrarily large independence and arbitrarily large pairwise disjoint sets that still have no λ-independent sets of size larger than 1.

Theorem 15. *For κ, λ uncountable cardinals, let $X = \{x_\alpha | \alpha \in \kappa\}$, let $Y = \{y_\alpha | \alpha \in \lambda\}$, and let $A = \text{Fr}\,(X \cup Y)\,/I$, where $I = \langle\{x_\alpha x_\beta | \alpha \neq \beta \ \wedge \ \alpha, \ \beta \in \kappa\}\rangle^{id}$. Then no two uncountable partitions of A are independent.*

Proof. For $a \in \text{Fr}(X \cup Y)$, we use the shorthand $[a]$ for $\{x \in \text{Fr}(X \cup Y) | x \Delta a \in I\} \in A$. First we show that $\forall \alpha \in \kappa$ the function $f : \text{Fr}(Y) \to A \restriction [x_\alpha]$ defined by $f(a) = [x_\alpha a]$ is an isomorphism. It is easy to see f is a homomorphism. Because Y-mon is dense in $\text{Fr}(Y)$, to show injectivity it suffices to show $f(m) \neq 0$ whenever $m \in Y\text{-mon}^+$. For any $m \in Y\text{-mon}^+$ take g a homomorphism from $\mathrm{Fr}(X \cup Y)$ to 2 with $g(m) = g(x_\alpha) = 1$ and $\forall \beta \in \kappa \setminus \{\alpha\}\ (g(x_\beta) = 0)$. Now $g(x_\alpha m) = 1$ and $\forall a \in I\ (g(a) = 0)$, so $f(m) = [x_\alpha m] \neq 0$ as desired. Each element of $\mathrm{Fr}(X \cup Y)$ is a finite sum of monomials in $(X \cup Y)$-mon, so to show surjectivity is suffices to show $\forall m \in X \cup Y\text{-mon}\ [x_\alpha m] \in f[\mathrm{Fr}(Y)]$. To see this, note that if $\beta \in \kappa \setminus \{\alpha\}$ then $[x_\alpha x_\beta] = 0$, so $[x_\alpha \cdot -x_\beta] = [x_\alpha]$. It follows that $\forall m \in X\text{-mon}\ ([x_\alpha m] \in \{0, [x_\alpha]\})$, so $\forall m \in X \cup Y\text{-mon}$, writing $m = m_X m_Y$ with $m_X \in X\text{-mon}$ and $m_Y \in Y\text{-mon}$, we have $[x_\alpha m] = [x_\alpha m_X m_Y] \in \{0, [x_\alpha m_Y]\} \subseteq f[\mathrm{Fr}(Y)]$ as desired.

Now suppose for contradiction that P and Q are two independent uncountable partitions of A. For each $p \in \mathrm{Fr}(X \cup Y)$ with $[p] \in P$, fix $M_p \in [(X \cup Y)\text{-mon}]^{<\omega}$ such that $p = \sum M_p$. For each $m \in (X \cup Y)$-mon fix $F_m \in [X \cup Y]^{<\omega}$ and $f_m : F_m \to 2$ such that $m = \prod_{x \in F_m} x^{f_m(x)}$.

Case 1. $\exists p \in \mathrm{Fr}(X \cup Y)$ *with* $[p] \in P$ *such that* $\forall m \in M_p\ (1 \in f_m[X \cap F_m])$.

Fix such p and $\forall m \in M_p$ take $\alpha_m \in \kappa$ such that $f_m(x_{\alpha_m}) = 1$. Note that $[p] \leq \sum_{m \in M_p} [x_{\alpha_m}]$. By independence of P and Q we have $\forall [q] \in Q\ ([pq] \neq 0) \Rightarrow \forall [q] \in Q\ \exists m \in M_p$ such that $[pqx_{\alpha_m}] \neq 0$. Q is uncountable, so we can take $Q' \subseteq Q$ such that Q' is uncountable and $m \in M_p$ such that $\forall [q] \in Q'([pqx_{\alpha_m}] \neq 0)$. Now $\{[pqx_{\alpha_m}] \mid [q] \in Q'\}$ is an uncountable partition of $A \restriction [x_{\alpha_m}] \cong \mathrm{Fr}(Y)$, contradiction.

Case 2. $\forall p \in \mathrm{Fr}(X \cup Y)$ *with* $[p] \in P\ \exists m \in M_p$ *such that* $1 \notin f_m[X \cap F_m]$.

For each such p fix such m and call it m_p. For any pair $p,\ p' \in \mathrm{Fr}(X \cup Y)$ with $[p],\ [p'] \in P$ and $[p] \neq [p']$, write $m_p = m_{pX} m_{pY}$ and $m_{p'} = m_{p'X} m_{p'Y}$ with $m_{pX},\ m_{p'X} \in X$-mon and $m_{pY},\ m_{p'Y} \in Y$-mon and fix $\alpha \in \kappa$ with $x_\alpha \notin F_{m_p} \cup F_{m_{p'}}$. Note that, by choice of m_p and $m_{p'}$,

$$m_{pX} = \prod_{x \in F_{m_{pX}}} -x \quad \text{and} \quad m_{p'X} = \prod_{x \in F_{m_{p'X}}} -x.$$

As in the proof of the surjectivity of f above, it follows that $[x_\alpha m_{pX}] = \left[x_\alpha m_{p'X}\right] = [x_\alpha] \Rightarrow \left[x_\alpha m_{pY} m_{p'Y}\right] = \left[x_\alpha m_p m_{p'}\right] \leq [pp'] = 0 \Rightarrow x_\alpha m_{pY} m_{p'Y} \in I$. As in the proof of the injectivity of f above, the right choice of a homomorphism from $\mathrm{Fr}(X \cup Y)$ to 2 gives a contradiction unless

$m_{pY}m_{p'Y} = 0$. By independence of Y, this implies $\exists\beta \in \lambda$ such that $(y_\beta \in F_{m_p} \cap F_{m_{p'}} \wedge f_{m_p}(y_\beta) \neq f_{m_{p'}}(y_\beta)) \Rightarrow$

$$\prod_{y \in Y \cap F_{m_p}} y^{f_{m_p}(y)} \prod_{y \in Y \cap F_{m_{p'}}} y^{f_{m_{p'}}(y)} = 0.$$

Thus

$$\left\{ \prod_{y \in Y \cap F_{m_p}} y^{f_{m_p}(y)} \middle| [p] \in P \right\}$$

is an uncountable pairwise disjoint subset of $\mathrm{Fr}(Y)$, contradiction.

□

4.1 A Generalized Version of the Balcar-Franěk Theorem

We point out some necessary concessions in attempting a full generalization of the Balcar-Franěk Theorem to partitions of arbitrary size. The Balcar-Franěk Theorem states that any infinite complete BA A has an independent subset of size $|A|$. We would like to find necessary and sufficient conditions for a complete BA A to have a λ-independent set of size κ for cardinals λ and κ. The obvious requirement that A have at least one partition of unity of size λ is not enough, as shown by taking $\lambda > \omega$, $\kappa > 2^\lambda$, and setting $A = 2^\lambda \times \langle \mathrm{Fr}(\kappa)\rangle^{\mathrm{cm}}$. A is a product of complete algebras and hence complete, $|A| \geq \kappa = \kappa^\lambda$, and 2^λ provides a partition of unity of size λ, but A has no λ-independent set of size κ.

Proof. Let $\pi_0 : A \to 2^\lambda$ and $\pi_1 : A \to \langle \mathrm{Fr}(\kappa)\rangle^{\mathrm{cm}}$ be the projection maps. Suppose for contradiction P is a κ-sized λ-independent set in A. $\mathrm{Fr}(\kappa)$ has no uncountable pairwise disjoint subset and is dense in $\langle \mathrm{Fr}(\kappa)\rangle^{\mathrm{cm}} \Rightarrow \langle \mathrm{Fr}(\kappa)\rangle^{\mathrm{cm}}$ has no uncountable pairwise disjoint subset. Thus $|\{x \in p | \pi_1(x) \neq 0\}| < \lambda$, so $|\{x \in p | \pi_1(x) = 0\}| = \lambda$ for all $p \in P$. For each $p \in P$ let $p' = \{x \in p | \pi_1(x) = 0\}$. The set p' does not contain 0 for any $p \in P$ so $0 \notin \pi_0[p']$. The p' are pairwise disjoint so the $\pi_0(p')$ are as well, which means $\{\pi_0[p'] | p \in P\}$ is an independent set of λ-sized pairwise disjoint subsets of 2^λ. But then $|\langle \pi_0[\bigcup X'] \rangle| \geq \kappa > 2^\lambda$, contradiction. □

In this example 2^λ provides the desired λ-sized pairwise disjoint set and $\langle \mathrm{Fr}(\kappa)\rangle^{\mathrm{cm}}$ provides the cardinality, without either providing a large λ-independent set. We may try to remedy this by requiring that A be atomless, but then taking $\lambda > \omega$ and $\kappa > 2^\lambda$, $A = \langle \mathrm{Fr}_\lambda(\omega) \rangle^{\mathrm{cm}} \times \langle \mathrm{Fr}(\kappa) \rangle^{\mathrm{cm}}$ provides a similar counterexample without atoms.

Proof. Each factor of A is complete and atomless so A is as well. In forming the completion of $\mathrm{Fr}_\lambda(\omega)$ there are at most $2^{|\,\mathrm{Fr}_\lambda(\omega)|} = 2^\lambda$ sums and products that must be added to the algebra, so $|\,\langle \mathrm{Fr}_\lambda(\omega) \rangle^{\mathrm{cm}}\,| \leq 2^\lambda < \kappa$ and $|A| = |\,\langle \mathrm{Fr}(\kappa) \rangle^{\mathrm{cm}}\,| \geq \kappa$. Suppose for contradiction that P is a κ-sized λ-independent set in A. As above $\mathrm{Fr}(\kappa)$ has no uncountable pairwise disjoint sets, so there is a κ-sized independent set of λ-sized pairwise disjoint sets in $\langle \mathrm{Fr}_\lambda(\omega) \rangle^{\mathrm{cm}}$. These pairwise disjoint sets generate a subalgebra of size $\geq \kappa$, contradiction. $\square$

To motivate the correct set of conditions on A, we must generalize the notion of atomlessness. In Balcar and Franěk's proof, the complete algebra A is written as a product $A_0 \times A_1$, there A_0 is atomic and A_1 is atomless. The cases $|A_0| = |A|$ and $|A_1| = |A|$ are then treated separately. If $|A_0| = |A|$ then $|A_0|$ must be isomorphic to an infinite powerset algebra, and Hausdorff's 1936 result [3] provides the desired large intependent set. The bulk of the proof is the case $|A_1| = |A|$, and uses heavily the atomlessness of A_1. In formulating the generalization we restrict our attention to atomless BA's, but treat powerset algebras along the way in Corollary 21. The statement "A is atomless" is equivalent to the statement "$\forall a \in A\ \exists$ a 2-partition of $A \restriction a$", which generalizes naturally to the condition "$\forall a \in A\ \exists$ a λ-partition of $A \restriction a$". This, along with completeness, is enough to guarantee the desired $|A|$-sized independent set of λ-partitions of A.

Theorem 16. *If A is a complete BA and $\lambda \leq |A|$, then A has a λ-independent set of size $|A|$ iff $\exists B \leq A$ such that $|B| = |A|$ and $B \restriction b$ has a λ-partition for all $b \in B^+$.*

Before embarking on the proof, we prove four lemmas.

Lemma 17. *If a BA A has an independent subset of size $\kappa \geq \omega$ then there is a κ-sized independent set P of infinite pairwise disjoint subsets of A.*

Proof. Let $X = \{a_\alpha | \alpha \in \kappa\}$ be an independent subset of A. Take $Q = \{q_\alpha | \alpha \in \kappa\} \subseteq [\kappa]^\omega$ a partition of κ into κ-many ω-sized subsets and $\forall \alpha \in \kappa$ write $q_\alpha = \{\beta_{\alpha n} | n \in \omega\}$. For each $\alpha \in \kappa$ let

$$p_\alpha = \left\{ a_{\beta_{\alpha n}} \cdot - \sum_{m<n} a_{\beta_{\alpha m}} | n \in \omega \right\}$$

and let $P = \{p_\alpha | \alpha \in \kappa\}$. Then $\forall \alpha \in \kappa$

$$\forall n, m \in \omega \quad m < n \to \left(a_{\beta_{\alpha n}} \cdot - \sum_{k<n} a_{\beta_{\alpha k}} \right) \cdot \left(a_{\beta_{\alpha m}} \cdot - \sum_{k<m} a_{\beta_{\alpha k}} \right) \leq -a_{\beta_{\alpha m}} \cdot a_{\beta_{\alpha m}} = 0,$$

so p_α is pairwise disjoint. By pairwise disjointness of Q we have P-mon $\subseteq X$-mon, so P inherits independence from X. $\square$

Lemma 18. *Let, for $i \in I$ and $\lambda > 2$, U_i be an infinite independent set of λ-partitions of a BA A_i. Then $\prod_{i \in I} A_i$ has a λ-independent set of size $\prod_{i \in I} |U_i|$.*

This is a direct generalization of [4, Corollary 13.10], which states that if U_i is an independent subset of A_i for each $i \in I$ then $\prod_{i \in I} A_i$ has an independent set of size $\prod_{i \in I} |U_i|$. The proof is almost identical, and makes use of [4, Lemma 13.9], which states that for a family $(X_i)_{i \in I}$ of infinite sets, $\prod_{i \in I} X_i$ has a finitely distinguished subset of size $|\prod_{i \in I} X_i|$. A subset F of $\prod_{i \in I} X_i$ is finitely distinguished if, for each finite subset $\{f_1, ..., f_n\}$ of F with $f_1, ..., f_n$ pairwise distinct, there is some $i \in I$ such that $f_1(i), ..., f_n(i)$ are pairwise distinct.

Proof. By [4, Lemma 13.9], let U be a finitely distinguished subset of $\prod_{i \in I} U_i$ with $|U| = |\prod_{i \in I} U_i|$. For each $f \in U$ and $i \in I$ enumerate $f(i) = \{a_{i\alpha} | \alpha \in \lambda\}$ in a one-to-one fashion. For each $\alpha \in \lambda$ define $f_\alpha \in \prod_{i \in I} A_i$ by $f_\alpha(i) = a_{i\alpha}$ and let $P_f = \{f_\alpha | \alpha \in \lambda\}$. We show that $P := \{P_f | f \in U\}$ is the desired λ-independent set. For all $f \in U$ and $i \in I$, $\{f_\alpha(i) | \alpha \in \lambda\} \in U_i \Rightarrow \sum_{\alpha \in \lambda} f_\alpha(i) = 1_{A_i} \Rightarrow \sum_{\alpha \in \lambda} f_\alpha = 1$. Moreover, if $\alpha \neq \beta$ then $\forall i \in I$ $(f_\alpha(i) \cdot f_\beta(i) = 0_{A_i}) \Rightarrow f_\alpha \cdot f_\beta = 0$, so the P_f are partitions of unity. To see that P is independent, $\forall F \in [U]^{<\omega}$ $\forall \delta : F \to \bigcup \{P_f | f \in F\}$ with $\delta(f) \in P_f$ we show $\prod_{f \in F} \delta(f) \neq 0$. Take $i \in I$ such that $f(i) \neq g(i)$ for any distinct $f, g \in F$. Now by independence of $\pi_i[F]$ we have $\left(\prod_{f \in F} \delta(f)\right)(i) \neq 0_{A_i} \Rightarrow \prod_{f \in F} \delta(f) \neq 0$, as desired. $\square$

Lemma 19. *If a BA A is λ^+-complete and there is a κ-sized independent set of λ-sized pairwise disjoint subsets of A, then A has a κ-sized λ-independent set.*

Proof. Suppose $P = \{p_\alpha | \alpha \in \kappa\}$ is an independent set of pairwise disjoint sets and $\forall \alpha \in \kappa\ p_\alpha = \{x_{\alpha\beta} | \beta \in \lambda\}$. For each $\alpha \in \kappa$ let

$$q_\alpha = (p_\alpha \setminus \{x_{\alpha 0}\}) \cup \left\{ - \sum_{\beta \in \lambda \setminus \{0\}} x_{\alpha\beta} \right\}$$

and let $Q = \{q_\alpha | \alpha \in \kappa\}$. Thus we use λ^+-independence to enlarge each $x_{\alpha 0}$ as necessary to make sure $\sum q_\alpha = 1$. Clearly Q is the desired λ-independent set. $\square$

Lemma 20. *For any infinite cardinals κ and λ, if $\prod_{\alpha \in \lambda} \mathrm{Fr}(\kappa) \leq B$ for a complete BA B then B has a κ-sized λ-independent set.*

Proof. By Lemma 17 let $Q = \{q_\alpha | \alpha \in \kappa\}$ be an independent set of ω-sized pairwise disjoint sets of $\mathrm{Fr}(\kappa)$ and $\forall \alpha \in \kappa$ write $q_\alpha = \{b_{\alpha n} | n \in \omega\}$ with $n \neq m \to b_{\alpha n} b_{\alpha m} = 0$. The following construction requires a λ-sized subset U of $\wp(\lambda)$ with $\forall F \in [U]^{<\omega}$ $(|\bigcap F| = \lambda)$ and $\forall F \in [U]^{\geq\omega}$ $(\bigcap F = \emptyset)$. In other words U generates a regular ultrafilter on $\wp(\lambda)$. To that end, take $f : \wp([\lambda]^{<\omega}) \to \wp(\lambda)$ an isomorphism of BA's, let $U' = \{\{S \in [\lambda]^{<\omega} | S \supseteq R\} | R \in [\lambda]^{<\omega}\}$, and let $U = f[U']$, as in the construction in [**] of a regular ultrafilter on $\wp(\lambda)$. Enumerate $U = \{U_\alpha | \alpha \in \lambda\}$. For each $\alpha \in \kappa$ let $p_\alpha = \{a_{\alpha\beta} | \beta \in \lambda\}$ where, for each $\beta \in \lambda$, $a_{\alpha\beta} \in \prod_{\alpha \in \lambda} \mathrm{Fr}(\kappa)$ is defined inductively by

$$\forall \gamma \in \lambda \quad a_{\alpha\beta}(\gamma) = \begin{cases} b_{\alpha m} & \gamma \in U_\beta \ \wedge\ m = \min\{n \in \omega | \forall \delta < \beta \quad a_{\alpha\delta}(\gamma) \neq b_{\alpha n}\} \\ 0 & \gamma \notin U_\beta \end{cases}.$$

Note that such m will always exist when $\gamma \in U_\beta$, because in that case $\gamma \in \bigcap \{U_\delta | \delta < \beta \ \wedge\ \gamma \in U_\delta\} \Rightarrow |\{U_\delta | \delta < \beta \ \wedge\ \gamma \in U_\delta\}| < \omega \Rightarrow |\{n \in \omega | \exists \delta < \beta\ a_{\alpha\delta}(\gamma) = b_{\alpha n}\}| < \omega$. It remains only to show that $P := \{p_\alpha | \alpha \in \kappa\}$ is an independent set of λ-sized pairwise disjoint subsets of B, because then by Lemma 19 B has a κ-sized λ-independent set. To prove pairwise disjointness, note that if $\alpha \in \kappa$ and β and δ are distict members of λ then, $\forall \gamma \in \lambda$,

$$[a_{\alpha\beta}(\gamma) = 0 \ \vee\ a_{\alpha\delta}(\gamma) = 0 \ \vee\ \exists m, n \in \omega\ \ (a_{\alpha\beta}(\gamma) = b_{\alpha n} \neq b_{\alpha m} = a_{\alpha\delta}(\gamma))]\text{, so } a_{\alpha\beta}(\gamma) \cdot a_{\alpha\delta}(\gamma) = 0.$$

Thus $a_{\alpha\beta} \cdot a_{\alpha\delta} = 0$, showing p_α is pairwise disjoint.

To prove independence, $\forall F \in [\kappa]^{<\omega} \quad \forall f : F \to \lambda$ let $G = f[F]$ and fix $\gamma \in \bigcap_{\beta\in G} U_\beta$, so that $a_{\alpha f(\alpha)} \neq 0$ whenever $\alpha \in F$. By independence of Q $\left(\prod_{\alpha\in F} a_{\alpha f(\alpha)}\right)(\gamma) \neq 0$ so $\prod_{\alpha\in F} a_{\alpha f(\alpha)} \neq 0$, showing P is independent. □

The following corollary was probably known to Balcar and Franěk in 1982 and is proved directly in Monk's upcoming book.

Corollary 21. *For any cardinals κ and λ, if $\lambda \leq \kappa$ then λ-ind$(\wp(\kappa)) = 2^\kappa$*

Proof. Clearly $\lambda\text{-ind}(\wp(\kappa)) \leq 2^\kappa$, as $|\wp(\kappa)| = 2^\kappa$. For the other direction, by Theorem 6 it suffices to show $\kappa\text{-ind}(\wp(\kappa)) \geq 2^\kappa$. Let $P = \{x_\alpha | \alpha \in \kappa\}$ be a partition of κ into κ many subsets of size κ.

$$\wp(\kappa) \cong \prod_{\alpha\in\kappa} \wp(\kappa) \restriction x_\alpha \cong \prod_{\alpha\in\kappa} \wp(\kappa).$$

By Hausdorff's 1936 result [3], $\text{ind}(\wp(\kappa)) = \kappa$, so $\wp(\kappa)$ has a κ-sized κ-independent set by Lemma 20. □

We are now equipped to prove the main theorem. Like the proof of the Balcar-Franěk theorem, this proof relies on [4, Lemma 13.12], which allows us to write a complete BA A as $\prod_{i\in I} A_i$ where each A_i is homogeneous with respect to a fixed finite list of order preserving cardinal functions on A. Recall that A is homogeneous with respect to the cardinal function f if $f(A \restriction a) = f(A)$ for all $a \in A^+$, and f is order preserving on A if $f(A \restriction b) \leq f(A \restriction a)$ whenever $a, b \in A$ with $b \leq a$. We apply the lemma for the single cardinal function ind, which is easily seen to be order preserving on any BA.

Proof. For the forward direction, if A has a λ-independent set X of size $|A|$, then $B := \langle \bigcup X \rangle \leq A$ is the desired subalgebra; $\forall a \in B^+$ take $b \in X$-mon such that $b \leq a$ and $F \in [X]^{<\omega}$ such that $b \in \langle \bigcup F \rangle$. Fixing $p \in X \setminus F$, for any $x \in p \quad xb \in X$-mon, so by independence of X $xb \neq 0$ and thus $xa \neq 0$. p is a partition of unity, so $\{xa | x \in p\}$ is the desired λ-partition of $B \restriction a$.

For the reverse direction, suppose $C \leq A$, $|C| = |A| = \kappa$, and $C \restriction c$ has a λ-partition for all $c \in C^+$. If $\lambda < \omega$, this simply means C is atomless, so $\kappa \geq \omega$. By the Balcar-Franěk Theorem A

has an independent subset of size κ. By Theorem 11 A has an ω-independent set of size κ, and now by Theorem 6 A has a λ-independent set of size κ.

If $\lambda \geq \omega$, let $B = \langle C \rangle^{\mathrm{cm}}$, the external completion of C, as opposed to the completion within A. Note that $|B| \geq \kappa$ and C is dense in B so $B \restriction b$ has a λ-partition for all $b \in B^+$. We show that B has a κ-sized λ-independent set.

By [4, Lemma 13.12] write $B_1 = \prod_{i \in I} B_i$ where each B_i is homogeneous with respect to independence and $\forall i \in I\ \exists b_i \in B^+$ such that $B_i = B \restriction b_i$. Each B_i is complete so by the Balcar-Franěk Theorem $\mathrm{ind}(B_i) = |B_i|$, and $B_i = B \restriction b_i$ so B_i has a λ-partition of unity X_i. Write $X_i = \{x_{i\alpha} | \alpha \in \lambda\}$ and $B_i = \prod_{\alpha \in \lambda} B_i \restriction x_{i\alpha}$. For all $\alpha \in \lambda$, by homogeneity $\mathrm{ind}(B_i \restriction x_{i\alpha}) = \mathrm{ind}(B_i) = |B_i|$, so B_i contains an isomorphic copy of

$$\prod_{\alpha \in \lambda} \mathrm{Fr}(|B_i|).$$

Thus by Lemma 20 B_i has a $|B_i|$-sized λ-independent set, and by Lemma 18 it follows that B has a κ-sized λ-independent set P. We construct from P a λ-sized independent set of pairwise disjoint sets in A. Using density of C in B, $\forall m \in P$-mon take $c_m \in C^+$ such that $c_m \leq m$. Using completeness of A, $\forall p \in P\ \forall b \in p$ let $a_b = \sum \{c_m | m \in P\text{-mon} \ \wedge\ m \leq b\}$. For each $p \in P$ let $p' = \{a_b | b \in p\}$ and let $P' = \{p' | p \in P\}$. Clearly $|P'| = \kappa$, and we claim P' is as desired. To see that each p' is pairwise disjoint and of cardinality λ, note that $\forall a_b, a_d \in p'$, if $a_b \cdot a_d \neq 0$ then $\exists m, n \in P$-mon such that $m \leq b$, $n \leq d$, and $c_m \cdot c_n \neq 0$. It follows that $m \cdot n \neq 0 \Rightarrow b \cdot d \neq 0 \Rightarrow b = d \Rightarrow a_b = a_d$, showing p' is pairwise disjoint. Clearly $0 \notin p'$, so $|p'| = |p| = \lambda$. To see that P' is independent, $\forall m' \in P$-mon take $F' \in [\bigcup P']^{<\omega}$ such that $m' = \prod F'$. Let $F = \{b \in B | a_b \in F'\}$ and note that $\prod F \in P$-mon, so $\prod F \neq 0$. For all $b \in F$

$$\prod F \leq b \Rightarrow c_{\prod F} \leq b \Rightarrow c_{\prod F} \leq a_b \Rightarrow c_{\prod F} \leq \prod F' \Rightarrow \prod F' \neq 0,$$

proving P' is independent. Thus A has a κ-sized independent set of λ-sized pairwise disjoint sets, and by Lemma 19 A has a κ-sized λ-independent set. $\square$

Chapter 5

λ-independence

We begin with an easy generalization of [9, Lemma 1.2], which states that $\mathrm{spind}(A) \subseteq \mathrm{spind}(A \times B)$, and an application to powerset algebras.

Theorem 22. *If A_0 and A_1 are Boolean algebras and A_0 has an maximal λ-independent set of size κ then $A_0 \times A_1$ has a maximal independent set of size κ. Thus* $\lambda\text{-i}(A_0 \times A_1) \leq \min\{\lambda\text{-i}(A_0), \lambda\text{-i}(A_1)\}$.

Proof. Let $A = A_0 \times A_1$. Suppose P is a maximal λ-independent set in A_0 and $|P| = \kappa$. Let $P = \{p_\alpha | \alpha \in \kappa\}$ and $\forall \alpha \in \kappa$ let $p_\alpha = \{x_{\alpha\beta} | \beta \in \lambda\}$. We build a set of λ-partitions in the product from the $x_{\alpha\beta}$. For each $\alpha \in \kappa$ let $y_{\alpha 0} = (x_{\alpha 0}, 1) \in A$, and for $\beta \in \lambda \setminus \{0\}$ let $y_{\alpha\beta} = (x_{\alpha\beta}, 0)$. Let $q_\alpha = \{y_{\alpha\beta} | \beta \in \lambda\}$ and let $Q = \{q_\alpha | \alpha \in \kappa\}$. Clearly each q_α is a partition of unity. To see that Q is independent, note that $\forall m \in Q\text{-mon}$ $(\pi_0(m) \in P\text{-mon} \Rightarrow \pi_0(m) \neq 0 \Rightarrow m \neq 0)$. To see that Q is maximal, suppose r is a λ-partition of A and let $r' = \pi_0[r]$. If $0 \in r'$, then fix $z \in r$ such that $\pi_0(z) = 0$. By definition $\pi_1(y_{01}) = 0$, so $0_A = zy_{01} \in (Q \cup \{r\})\text{-mon}$. If $0 \notin r'$, then r' is a λ-partition of A_0, so by maximality of P there are $m \in P\text{-mon}$ and $z \in r$ such that $\pi_0(z)m = 0$. Fix such z and m and write $m = \prod_{\alpha \in F} x_{\alpha f(\alpha)}$ for some $F \in [\kappa]^{<\omega}$ and $f : F \to \lambda$. Fix $\gamma \in \kappa \setminus F$ and define $n \in Q\text{-mon}$ by $n = y_{\gamma 1} \prod_{\alpha \in F} x_{\alpha f(\alpha)}$. The inclusion of $y_{\gamma 1}$ ensures that $\pi_1(zn) = 0$, and

$$\pi_0(zn) \leq \pi_0(z \prod_{\alpha \in F} y_{\alpha f(\alpha)}) = \pi_0(z) \prod_{\alpha \in F} x_{\alpha f(\alpha)} = \pi_0(z)m = 0,$$

so $0_A = zn \in (Q \cup \{r\})\text{-mon}$. In either case $0 \in (Q \cup \{r\})\text{-mon} \Rightarrow (Q \cup \{r\})$ is not independent, so Q is the desired maximal λ-independent set. $\square$

Corollary 23. *If $\kappa \geq \omega$ and $\lambda \geq 2$ then λ-$\mathrm{spind}(\wp(\omega)) \subseteq \lambda$-$\mathrm{spind}(\wp(\kappa))$. Thus λ-$\mathrm{i}(\wp(\kappa)) \leq \lambda$-$\mathrm{i}(\wp(\omega))$.*

Proof. If $\mu \in \lambda$-$\mathrm{spind}(\wp(\omega))$, let P be a maximal λ-independent set in $\wp(\omega)$ of size μ. Write $\wp(\kappa)$ as $\wp(\kappa) \upharpoonright \omega \times \wp(\kappa) \upharpoonright (\kappa \setminus \omega) \cong \wp(\omega) \times \wp(\kappa \setminus \omega)$. By Theorem 22 $\wp(\kappa)$ has a maximal λ-independent set of size $|P|$. □

From this it is easy to see that [8, Proposition 36], which states that $\mathrm{i}\,(\wp\,(\omega)) \geq \omega_1$, does not generalize to $\mathrm{i}\,(\wp\,(\kappa)) \geq \kappa^+$ for all cardinals κ; Choosing $\kappa \geq 2^\omega$ provides a counterexample in ZFC, as then $\mathrm{i}(\wp(\kappa)) \leq \mathrm{i}(\wp(\omega)) \leq 2^\omega < \kappa^+$, and κ can be forced down by introducing smaller maximal independent sets of $\wp\,(\omega)$.

However, if we note that each independent set in $\wp\,(\omega)$ maps to an independent set in $\wp\,(\omega)\,/\mathrm{fin}$ and instead generalize to $\wp\,(\kappa)\,/<\kappa$, the proof goes through, and in fact the same technique can be used to prove the a slightly more general result (Recall that $\wp\,(\kappa)\,/<\kappa = \wp\,(\kappa)\,/I$, where $I = \{x \in \wp(\kappa) : |x| < \kappa\}$).

Theorem 24. *If κ is an infinite cardinal and $\lambda < \kappa$, then λ-$\mathrm{i}\,(\wp\,(\kappa)\,/{<}\kappa) > \kappa$.*

Proof. Suppose P is an independent set of λ-partitions of κ, $|P| = \kappa$, and $\forall m \in P$-mon $|m| = \kappa$. Write P-mon $= \{m_\alpha | \alpha \in \kappa\}$ without redundancy, and $\forall \alpha \in \kappa\ \forall \beta \in \lambda$ recursively choose $x_{\alpha\beta} \in m_\alpha \setminus \{x_{\gamma\delta} | \gamma < \alpha\ \vee\ (\gamma = \alpha\ \wedge\ \delta < \beta)\}$ (This is possible because $|\{x_{\gamma\delta} | \gamma < \alpha\ \vee\ (\gamma = \alpha\ \wedge\ \delta < \beta)\}| \leq |\alpha|\lambda + |\beta| < \kappa$, while $|m_\alpha| = \kappa$). Now $\forall \beta \in \lambda \setminus \{0\}$ let $y_\beta = \{x_{\alpha\beta} | \alpha \in \kappa\}$, and let

$$y_0 = \kappa \setminus \bigcup_{\beta \in \lambda \setminus \{0\}} y_\beta \supseteq \{x_{\alpha 0} | \alpha \in \kappa\}\,.$$

Thus each y_β contains an element of each member of P-mon and $y_\beta \cap y_\delta = \emptyset$ for distinct β and δ. Let $q = \{y_\beta | \beta \in \lambda\}$. Clearly q is pairwise disjoint, and the definition of y_0 ensures that q is a partition of unity. Note that, for all $\alpha \in \kappa$, $|\{\gamma \in \kappa | a_\gamma \subset a_\alpha\}| = \kappa \Rightarrow\ |\{\gamma \geq \alpha | a_\gamma \subset a_\alpha\}| = \kappa \Rightarrow \forall \beta \in \lambda\ (|z_\beta \cap a_\alpha| = \kappa)$, so not only is $P \cup \{q\}$ independent in $\wp\,(\kappa)$, but also $\{f\,[p]\,| p \in P\} \cup f\,[q]$ is independent in $\wp\,(\kappa)\,/{<}\kappa$, where $f : \wp\,(\kappa) \to \wp\,(\kappa)\,/{<}\kappa$ is the natural homomorphism. Thus no κ-sized λ-independent subset of $(\wp\,(\kappa))\,/{<}\kappa$ is maximal, showing λ-$\mathrm{i}\,(\wp\,(\kappa)\,/{<}\kappa) > \kappa$. □

5.1 λ-i for Weak Products

Monk and Mckenzie [6, Theorem 4] have shown that, for I an infinite set and $\langle A_i : i \in I\rangle$ a system of atomless BAs,

$$\operatorname{spind}\left(\prod_{i\in I}^{\mathrm{W}} A_i\right) = \{\omega\} \cup \bigcup_{i\in I} \operatorname{spind}(A_i).$$

Most of the results leading to this can be readily generalized to maximal λ-independent sets for any cardinal λ, with the notable exception of the "not easy" direction of [6, Theorem 2], which shows that if A_0 and A_1 are atomless BAs and $A_0 \times A_1$ has a maximal independent set of size κ then either A_0 or A_1 has a maximal independent set of size κ. I suspect that the generalized version, the converse of Theorem 22, holds, but have not found a proof. The following are the generalized versions of the remaining pertinent results.

Theorem 25. *If $A = \prod_{i\in i}^{\mathrm{W}} A_i$ and for some $i \in I$ A_i has a maximal λ-independent set of size κ then A has a maximal λ-independent set of size κ.*

Proof. Let $P = \{p_\alpha | \alpha \in \kappa\}$ be a λ-independent subset of $\wp(A_i)$ for some fixed $i \in I$. Write each p_α as $\{x_{\alpha\beta} | \beta \in \lambda\}$, without redundancy. For each $\alpha \in \kappa$ and $\beta \in \lambda$ define $y_{\alpha\beta} \in A$ by $\forall j \in I$

$$y_{\alpha\beta}(j) = \begin{cases} x_{\alpha\beta} & j = i \\ 1 & j \neq i \ \wedge \ \beta = 0 \\ 0 & j \neq i \ \wedge \ \beta \neq 0 \end{cases}.$$

Let $q_\alpha = \{y_{\alpha\beta} | \beta \in \lambda\}$ and let $Q = \{q_\alpha | \alpha \in \kappa\}$. Clearly Q is a set of λ-partitions, and $\forall m \in Q\text{-mon}$ $\pi_i(m) \in P\text{-mon} \Rightarrow m \neq 0$, so Q is independent. To see that Q is maximal, suppose that r is a λ-partition of A. Either $\exists z \in r$ such that $\pi_i(z) = 0$ or $\pi_i[r]$ is a λ-partition of A_i, in which case by maximality of P in A_i $\exists z \in r$ $\exists m \in P\text{-mon}$ such that $\pi_i(zm) = 0$. In either case fix $z \in r$ and $m \in P\text{-mon}$ such that $\pi_i(zm) = 0$ and fix $F \in [\kappa]^{<\omega}$, $f : F \to \lambda$ such that $m = \prod_{\alpha\in F} x_{\alpha f(\alpha)}$. Let $n = \prod_{\alpha\in F} y_{\alpha f(\alpha)}$ and fix any $\gamma \in \kappa \setminus F$, so that $y_{\gamma 1}mz \in (Q \cup \{r\})\text{-mon}$. For all $j \in I \setminus \{i\}$ we have $\pi_j(y_{\gamma 1}mz) \leq \pi_j(y_{\gamma 1}) = 0$ and $\pi_i(y_{\gamma 1}mz) \leq \pi_i(mz) = 0$, so $y_{\gamma 1}mz = 0_A$, showing $Q \cup \{r\}$ is not independent. $\square$

Theorem 26. *If $\langle A_\alpha | \alpha \in \kappa \rangle$ is a system of Boolean algebras, $B = \prod^{\mathrm{W}}_{\alpha\in\kappa} A_\alpha$, λ is any cardinal, and A_α has an infinite λ-independent set for infinitely many α, then B has a countably infinite maximal λ-independent set.*

Proof. Without loss of generality $\forall \alpha \in \omega$ let $\{p_{\alpha n} | n \in \omega\}$ be a countably infinite independent set of λ-partitions of A_α. For each $\alpha \in \omega$ and $n \in \omega$ write $p_{\alpha n} = \{x_{\alpha n \beta} | \beta \in \lambda\}$, without redundancy. We define a set $Q = \{q_n | n \in \omega\}$ of λ-partitions in B. For each $n \in \omega$ let $q_n = \{y_{n\beta} | \beta \in \lambda\}$, where the $y_{n\beta} \in B$ are defined as follows:

$$y_{n\beta}(\alpha) = \begin{cases} x_{\alpha n \beta} & \alpha < n \\ 0 & \alpha = n,\ \beta = 0 \\ x_{\alpha n 0} + x_{\alpha n 1} & \alpha = n,\ \beta = 1 \\ x_{\alpha n \beta} & \alpha = n,\ \beta > 1 \\ 1 & \alpha > n,\ \beta = 0 \\ 0 & \alpha > n,\ \beta > 0 \end{cases}$$

To see that Q is independent, it suffices to show that if $k \in \omega$ and $f : k \to \lambda$ then $\prod_{n<k} y_{nf(n)} \neq 0$. If $\forall n \in k\ (f(n) = 0)$ then $\prod_{n<k} y_{nf(n)}(k) = 1$. Otherwise let $m \in k$ be minimal such that $f(m) \neq 0$ and note that

$$\prod_{n<k} y_{nf(n)}(m) = \begin{cases} \prod_{n<m} 1\,(x_{mm0} + x_{mm1}) \prod_{m<n<k} x_{mnf(n)} & f(m) = 1 \\ \prod_{n<m} 1\left(x_{mmf(m)}\right) \prod_{m<n<k} x_{mnf(n)} & f(m) \neq 1 \end{cases}$$

Either way

$$\prod_{n<k} y_{n\delta(n)}(m) \geq x_{mm\delta(m)} \prod_{m<n<k} x_{mn\delta(n)} \neq 0$$

by independence of $\{p_{mn} | n \in \omega\}$. To see that Q is maximal, suppose $r = \{z_\alpha | \alpha \in \lambda\}$ is a partition of unity in B. If α and β are distinct members of λ and $\{\gamma \in \kappa | z_\alpha(\gamma) \neq 0\}$ is infinite, then $\{\gamma \in \kappa | z_\beta(\gamma) \neq 0\}$ is finite because otherwise $\{\gamma \in \kappa | z_\alpha(\gamma) \neq 1\}$ and $\{\gamma \in \kappa | z_\beta(\gamma) \neq 1\}$ are both finite $\Rightarrow$ $z_\alpha z_\beta \neq 0$, contradicting pairwise disjointness of r. Without loss of generality

$\{\alpha \in \kappa | z_0(\alpha) \neq 0\}$ is finite (if not we could use z_1 in place of z_0). Fix $m \in \omega$ such that $m > \max\{n \in \omega | z_0(n) \neq 0\}$. We show $z_0\left(\prod_{n<m} y_{n0}\right) y_{m1} = 0$ by partitioning κ into three sets.

Case 1. $\alpha < m$

Then $\prod_{n<m} y_{n0}(\alpha) = 0$.

Case 2. $\alpha \in \omega \setminus m$

Then $z_0(\alpha) = 0$.

Case 3. $\alpha \in \kappa \setminus \omega$

Then $y_{m1}(\alpha) = 0$.

In any case $z_0\left(\prod_{n<m} y_{n0}\right) y_{m1}(\alpha) = 0$, so $z_0\left(\prod_{n<m} y_{n0}\right) y_{m1} = 0$. Thus $Q \cup \{r\}$ is not independent, and Q is the desired set of partitions. □

Corollary 27. *If $\{A_\alpha | \alpha \in \kappa\}$ is an infinite set of atomless Boolean algebras, $B = \prod^{\mathrm{W}}_{\alpha\in\kappa} A_\alpha$, and $n \in \omega$, then B has a countably infinite maximal n-independent set.*

Proof. By Theorem 10 each A_α has an infinite n-independent set, so Theorem 26 applies. □

This does not suffice for a full characteriziation of λ-spind for weak products, but at least we can conclude the following:

Theorem 28. *If $\{A_\alpha | \alpha \in \kappa\}$ is an infinite set of Boolean algebras, λ is any cardinal, and A_α has an infinite independent set of λ-partitions for infinitely many α, then*

$$\lambda\text{-spind}\left(\prod^{\mathrm{W}}_{\alpha\in\kappa} A_\alpha\right) \subseteq \{\omega\} \cup \bigcup_{\alpha\in\kappa} \lambda\text{-spind}(A_\alpha).$$

5.2 n-i for Finite n

Results like Theorems 5,6,10, and 11 are not as easy to come by for λ-i, as the preservation of maximality when constructing an independent set of λ-partitions using an independent set of μ-partitions presents a bit of a challenge. As an example, consider $\lambda = 2$, $\mu = 3$. If A is a BA and

P is a κ-sized 3-independent set in A, we may construct an independent set of size κ as in Theorem 5: Let $P = \{p_\alpha | \alpha \in \kappa\}$ and $\forall \alpha \in \kappa$ let $p_\alpha = \{x_{\alpha 0}, x_{\alpha 1}, x_{\alpha 2}\}$. Let $X = \{x_{\alpha 0} | \alpha \in \kappa\}$. Then P-mon is dense in X-mon so X inherits independence from P, but X is not necessarily maximal. There may even be a 3-partition that is independent over $\langle X \rangle$ without being independent over $\langle \bigcup P \rangle$. The problem becomes even more difficult when μ and λ are infinite, but at least in the finite case the following holds.

Theorem 29. *If A is a BA and A has an infinite independent set, then n-$\mathrm{i}(A) = \mathrm{i}(A)$ for all $n \in \omega$.*

We break the bulk of the proof into two lemmas.

Lemma 30. *If a BA A has a maximal κ-sized n-independent set with $\kappa \geq \omega$, then A has a maximal κ-sized n^2-independent set.*

Proof. If P is a κ-sized maximal n-independent set in A, partition P into two κ-sized sets Q and R. Let $Q = \{q_\alpha | \alpha \in \kappa\}$ and $R = \{r_\alpha | \alpha \in \kappa\}$. For each $\alpha \in \kappa$, let $q_\alpha = \{x_{\alpha i} | i \in n\}$ and let $r_\alpha = \{y_{\alpha i} | i \in n\}$. Let $s_\alpha = \{x_{\alpha i} y_{\alpha j} | (i,j) \in n \times n\}$ and let $S = \{s_\alpha | \alpha \in \kappa\}$. We show S is the desired κ-sized maximal n^2-independent set.

First, $\forall \alpha \in \kappa\ \forall (i,j) \in n \times n$, by disjointness of Q and R we have $x_{\alpha i} y_{\alpha j} \in P$-mon $\Rightarrow x_{\alpha i} y_{\alpha j} \neq 0$. If (i,j) and $(k,l) \in n \times n$ and $(i,j) \neq (k,l)$, by symmetry assume $i \neq k$, and we have

$$(x_{\alpha i} y_{\alpha j})(x_{\alpha k} y_{\alpha l}) \leq x_{\alpha i} x_{\alpha k} = 0,$$

showing that $|s_\alpha| = n^2$ and that s_α is pairwise disjoint. To see that s_α is a partition of unity,

$$\sum s_\alpha = \sum_{(i,j) \in n \times n} x_{\alpha i} y_{\alpha j} = \left(\sum_{i \in n} x_{\alpha i} \right) \left(\sum_{j \in n} y_{\alpha j} \right) = 1 \cdot 1 = 1.$$

By disjointness of Q and R we have S-mon $\subseteq P$-mon $\Rightarrow S$ inherits independence from P. To see that S is maximal, suppose r is any n^2-partition of A. Let $r = \{z_i | i \in n^2\}$ and let

$$r' = \{z_i | i \in n-1\} \cup \left\{ \sum_{n-1 \leq j < n^2} z_j \right\}.$$

Clearly r' is an n-partition, so $P \cup \{r'\}$ is not independent by maximality of P. For any i with $n-1 \leq i < n^2$ we have

$$z_i \leq \sum_{n-1 \leq j < n^2} z_j,$$

so $P \cup \{r\}$ is also not independent. Fix $i \in n^2$ and $m \in P$-mon such that $z_i m = 0$. Using $P = R \cup Q$, write

$$m = \prod_{\alpha \in F} x_{\alpha f(\alpha)} \prod_{\alpha \in G} y_{\alpha g(\alpha)}$$

with $F, G \in [\kappa]^{<\omega}$, $f : F \to n$, and $g : G \to n$. Arbitrarily extend f and g to functions from $F \cup G$ to n and let

$$n = \prod_{\alpha \in F \cup G} x_{\alpha f(\alpha)} y_{\alpha g(\alpha)}.$$

Thus $n \in S$-mon and $n \leq m \Rightarrow z_i n = 0$, so $S \cup \{r\}$ is not independent, showing S is maximal. $\square$

Lemma 31. *If a BA B has a κ-sized maximal n-independent set for $3 \leq n < \omega$ and $\kappa \geq \omega$, then B has a κ-sized $(n-1)$-independent set.*

Proof. Let X be a κ-sized maximal n-independent set in B. In case $\kappa = \omega$, write $X = \{r_i | i \in \omega\}$ where $r_i = \{z_{ij} | j \in n\}$ without redundancy. Define a function $f : \bigcup X \to \mathrm{Intalg}[0,1)$ by

$$f(z_{ij}) = x_{ij} := \bigcup_{k \in n^i} \left[\frac{kn+j}{n^{i+1}}, \frac{kn+j+1}{n^{i+1}}\right).$$

For each $m \in \omega$ define a subaglebra A_m of Intalg$[0,1)$ by $A_m = \left\langle \bigcup_{i \in m} f[r_i] \right\rangle$ and let $A = \bigcup_{m \in \omega} A_m$.

Claim: The A_m are atomic with atoms $\left\{ \left[\frac{k}{n^m}, \frac{k+1}{n^m}\right) \middle| k \in n^m \right\}$, and each atom of A_m is $\prod_{i \in m} x_{i\delta(i)}$ for some $\delta : m \to n$. We prove by induction on m that $\forall m \in \omega$

$$\forall \delta : m \to n \quad \prod_{i \in m} x_{i\delta(i)} = \left[\frac{\sum_{i \in m} \delta(i)\, n^{m-1-i}}{n^m}, \frac{\sum_{i \in m} \delta(i)\, n^{m-1-i} + 1}{n^m}\right) \quad (*_m).$$

Because each $k \in n^m$ has a unique representation of the form $k = \sum_{i \in m} \delta(i)\, n^{m-1-i}$ for some $\delta : m \to n$ (this is the n-ary representation of k) and from the definition of the x_{ij} it is clear that $\forall a \in A_m$ $(L(a) \geq \frac{1}{n^m})$, where L is Lebesgue measure, $\forall m$ $(*_m)$ will be sufficient to prove the claim. $A_0 = \{\emptyset, [0,1)\} \Rightarrow [0,1)$ is the only atom of A_0, and the only function δ from 0 to n is $\delta = \emptyset$, for which $\sum_{i\in 0}\delta(i)\,n^{0-1-i}=0$, so $(*_0)$ holds. Given $(*_m)$, $\forall\delta : m+1\to n$

$$\prod_{i\in m+1} x_{i\delta(i)} = \prod_{i\in m} x_{i\delta(i)}\cdot x_{m\delta(m)} =$$

$$\left[\frac{\sum_{i\in m}\delta(i)\,n^{m-1-i}}{n^m},\frac{\sum_{i\in m}\delta(i)\,n^{m-1-i}+1}{n^m}\right)\cap\bigcup_{k\in n^m}\left[\frac{kn+\delta(m)}{n^{m+1}},\frac{kn+\delta(m)+1}{n^{m+1}}\right)=$$

$$\bigcup_{k\in n^m}\left(\left[\frac{k}{n^m}+\frac{\delta(m)}{n^{m+1}},\frac{k}{n^m}+\frac{\delta(m)+1}{n^{m+1}}\right)\cap\left[\frac{\sum_{i\in m}\delta(i)\,n^{m-1-i}}{n^m},\frac{\sum_{i\in m}\delta(i)\,n^{m-1-i}+1}{n^m}\right)\right).$$

The above intersection is nonempty exactly when $k=\sum_{i\in m}\delta(i)\,n^{m-1-i}$, so $\prod_{i\in m+1}x_{i\delta(i)}=$

$$\left[\frac{\sum_{i\in m}\delta(i)\,n^{m-1-i}}{n^m}+\frac{\delta(m)}{n^{m+1}},\frac{\sum_{i\in m}\delta(i)\,n^{m-1-i}}{n^m}+\frac{\delta(m)+1}{n^{m+1}}\right)\bigcap$$

$$\left[\frac{\sum_{i\in m}\delta(i)\,n^{m-1-i}}{n^m},\frac{\sum_{i\in m}\delta(i)\,n^{m-1-i}+1}{n^m}\right)$$

$$=\left[\frac{\sum_{i\in m}\delta(i)\,n^{m-1-i}}{n^m}+\frac{\delta(m)}{n^{m+1}},\frac{\sum_{i\in m}\delta(i)\,n^{m-1-i}}{n^m}+\frac{\delta(m)+1}{n^{m+1}}\right)$$

$$=\left[\frac{\sum_{i\in m}\delta(i)\,n^{m-i}+\delta(m)}{n^{m+1}},\frac{\sum_{i\in m}\delta(i)\,n^{m-i}+\delta(m)+1}{n^{m+1}}\right)$$

$$=\left[\frac{\sum_{i\in m+1}\delta(i)\,n^{(m+1)-1-i}}{n^{m+1}},\frac{\sum_{i\in m+1}\delta(i)\,n^{(m+1)-1-i}+1}{n^{m+1}}\right),$$

which is $(*_{m+1})$, finishing the proof by induction and proving the claim.

In particular this shows $\{\{x_{ij}|j\in n\}\,|i\in\omega\}$ is an independent set of partitions in A. Using this and the independence or the r_i, we see that $\forall F\in[\bigcup X]^{<\omega}\ \ \forall\varepsilon : F\to 2$

$$\prod_{z\in F} z^{\varepsilon(z)}=0\leftrightarrow(\exists r\in X\ \ \exists z,y\in r\ \ \varepsilon(z)=\varepsilon(y)=1)\vee(\exists r\in X\ \ r\subset F\wedge\varepsilon[r]=\{0\})\leftrightarrow\prod_{z\in F}f(z)^{\varepsilon(z)}=0,$$

so by Sikorski's extension criterion [4, Proposition 5.6] we can extend f to an isomorphism from $\langle\bigcup X\rangle$ to $\langle f[\bigcup X]\rangle=A$. The bulk of the proof now takes place inside A.

For each $i\in\omega$ let $R_i=\{r\in[0,1)|r$ is and endpoint of some interval in $A_i\}=\left\{\frac{k}{n^i}\middle|k<n^i\right\}$ and let $R=\bigcup_{i\in\omega}R_i$. We inductively define $S_i\in[R]^{<\omega}$, $g_i : R\to\omega$, $\left\{h_i^j|j\in n\right\}\subset{}^R(R\cup\{1\})$, and $q_i\subset A$ so that $Q:=\{q_i|i\in\omega\}$ is an independent set of $(n-1)$-partitions of A with the property that Q-mon is dense in A.

First, let $S_0=\{0\}$. If S_i has been defined, $\forall r\in R$ let $r_i^+=\min(((r,1)\cap S_i)\cup\{1\})$ and let $g_i(r)\in\omega$ be minimal such that $R_{g_i(r)}\cap\left(r,r_i^+\right)\neq\emptyset$. Let $h_i^0(r)=r$ and let $h_i^{n-1}(r)=r_i^+$. Note that if $r \in S_i$ then $h_i^{n-1}(r) \in S_i \cup \{1\}$, and no elements of S_i are inbetween r and $h_i^{n-1}(r)$. We now use g_i to define an increasing sequence of real numbers inbetween r and $h_i^{n-1}(r)$ using elements of R_l for the smallest possible indeces l. Let $h_i^1(r) = \min\left(R_{g_i(r)} \cap \left(r, r_i^+\right)\right)$, and for $0 < j < n-2$ let $h_i^{j+1}(r) = h_i^1(h_i^j(r))$. If S_i, g_i, and $\left\{h_i^j | j \in n\right\}$ have been defined, let $q_i = \{y_{ij} | j \in n-1\}$ where $\forall j \in n-1$

$$y_{ij} = \bigcup_{r \in S_i} \left[h_i^j(r), h_i^{j+1}(r)\right).$$

Given S_i, g_i, $\left\{h_i^j | j \in n\right\}$, and q_i, let $S_{i+1} = \left\{h_i^j(r) | r \in S_i,\ j \in n-1\right\}$.

For all $i \in \omega$ and $r \in S_i$, we make some usefull observations regarding the above definitions. First note that h_i^0 is the identity function, so $S_i \subseteq S_{i+1}$. We prove by induction that $1 \leq j < n-1 \rightarrow (r < h_i^j(r) < r_i^+) \wedge (h_i^j(r) < h_i^{j+1}(r))$. Clearly $r < h_i^j(r)$, and $g(i)$ is defined to be just large enough so that the minimality of $h_i^1(r)$ guarantees $h_i^1(r) < r_i^+$. Now assume that $j < n-2$ and $r < h_i^j(r) < r_i^+$. Then $r_i^+ \leq (h_i^j(r))_i^+$ and $r_i^+ \in R_i \Rightarrow (h_i^j(r))_i^+ \leq r_i^+$, so $(h_i^j(r))_i^+ = r_i^+$. Thus $h_i^j(r) < h_i^1(h_i^j(r)) < (h_i^j(r))_i^+ = r_i^+$. If $j < n-2$ then $h_i^j(r) < h_i^1(h_i^j(r)) = h_i^{j+1}(r)$, and if $j = n-2$ then $h_i^j(r) < (h_i^j(r))_i^+ = r_i^+ = h_i^{j+1}(r)$, finishing the induction. It follows that $r < h_i^1(r) < ... < h_i^{n-1}(r) = r_i^+$. Finally, by definition of S_{i+1}, $\forall j \in n-1$ we have $h_i^{j+1}(r) = (h_i^j(r))_{i+1}^+ = h_{i+1}^{n-1}(h_i^j(r))$. We are now equiped to prove Q is an independent set of partitions of unity. For all $i \in \omega$,

$$\bigcup_{j \in n-1} y_{ij} = \bigcup_{j \in n-1} \bigcup_{r \in S_i} \left[h_i^j(r), h_i^{j+1}(r)\right) = \bigcup_{r \in S_i} \bigcup_{j \in n-1} \left[h_i^j(r), h_i^{j+1}(r)\right) = \bigcup_{r \in S_i} \left[r, h_i^{n-1}(r)\right) = [0,1)$$

and

$$\forall j,k \in n-1 \quad j \neq k \rightarrow y_{ij} \cap y_{ik} = \bigcup_{r \in S_i} \left(\left[h_i^j(r), h_i^{j+1}(r)\right) \cap \left[h_i^k(r), h_i^{k+1}(r)\right)\right) = \emptyset,$$

so q_i is a partition of unity. To see that Q is independent, we prove the stronger statement, also useful in proving Q-mon is dense in A, that $\forall m \in \omega$

$$\forall \delta : m \rightarrow n-1 \quad \exists r \in S_m \text{ such that } \prod_{i \in m} y_{i\delta(i)} = \left[r, h_m^{n-1}(r)\right) \quad (**_m)$$

by induction on m. If $m = 0$, any product over m is $1_A = [0,1) = [0, h_0^{n-1}(0))$, so $r = 0$ works for this case. Given $**_m$, $\forall\delta : m+1 \to n-1$ fix $r \in S_m$ such that $\prod_{i\in m} y_{i,\delta(i)} = [r, h_m^{n-1}(r))$ so that

$$\prod_{i\in m+1} y_{i,\delta(i)} = y_{m\delta(m)} \cap \left[r, h_m^{n-1}(r)\right) = \bigcup_{s\in S_m} \left(\left[h_m^{\delta(m)}(s), h_m^{\delta(m)+1}(s)\right) \cap \left[r, h_m^{n-1}(r)\right)\right).$$

The above intersection is nonempty if and only if $r = s$, in which case it is $\left[h_m^{\delta(m)}(r), h_m^{\delta(m)+1}(r)\right)$, so

$$\prod_{i\in m+1} y_{i,\delta(i)} = \left[h_m^{\delta(m)}(r), h_m^{\delta(m)+1}(r)\right) = \left[h_m^{\delta(m)}(r), h_{m+1}^{n-1}\left(h_m^{\delta(m)}(r)\right)\right)$$

and $h_m^{\delta(m)}(r) \in S_{m+1}$, as desired. Let $S = \bigcup_{i\in\omega} S_i$. For any $r \in S$ and any $m \in \omega$, $[r, h_m^{n-1}(r)) \neq \emptyset$, so by $**_m$ Q is independent.

To see that Q-mon is dense in A, it now suffices to show $R = S$, because then for any interval $[r, s) \in A$ we can take $m \in \omega$ such that $r, s \in S_m$ and note that

$$\sum_{\delta:m\to n-1} \prod_{i\in m} y_{i\delta(i)} = \prod_{i\in m} \sum_{j\in n-1} y_{ij} = 1 \Rightarrow$$

$\exists\delta : m \to n-1$ such that $[r, s) \cap \prod_{i\in m} y_{i\delta(i)} \neq \emptyset$. Fixing such δ, by $**_m$ $\exists t \in S_m$ such that $\prod_{i\in m} y_{i\delta(i)} = \left[t, h_m^{n-1}(t)\right)$. For such t we have $\left(t, h_m^{n-1}(t)\right) \cap S_m = \emptyset \Rightarrow r, s \notin \left(t, h_m^{n-1}(t)\right) \Rightarrow \prod_{i\in m} y_{i\delta(i)} \subseteq [r, s)$, showing Q-mon dense in A.

We show $R = S$. It is clear from the definition of S_i that $S \subseteq R$. For the other inclusion, suppose for contradiction that $R \setminus S \neq \emptyset$. Fix i minimal such that $R_i \setminus S \neq \emptyset$ and k minimal such that $\frac{k}{n^i} \notin S$. S does not contain 0 so $k \neq 0$, and by minimality we can fix $l \in \omega$ such that $\frac{k-1}{n^i} \in S_l$. Let $r = \max\left(S_l \cap [0, \frac{k}{n^i})\right)$ and note that $\forall s \in S_l \cap (r, 1)$ $\left(s > \frac{k}{n^i} \notin S_l\right) \Rightarrow g_l(r) \leq i$ by minimality of $g_l(r)$. The same argument shows that $\forall m > l$, if $(r, \frac{k}{n^i}) = \emptyset$ then $g_m(r) \leq i$.

Case 1. *$\exists m > l$ such that $(r, \frac{k}{n^i}) \cap S_m \neq \emptyset$.*

Fix miminal such m. Note that $r \in S_{m-1}$ and $h_{m-1}^1(r)$ is the smallest element of $S_m \cap (r, 1) \Rightarrow h_{m-1}^1(r) < \frac{k}{n^i}$. Thus $h_{m-1}^1(r) \in R_{g_{m-1}(r)}$ and $r > \frac{k-1}{n^i} \Rightarrow R_i \cap (r, \frac{k}{n^i}) = \emptyset \Rightarrow g_{m-1}(r) > i$. But by the above also $g_{m-1}(r) \leq i$, contradiction.

Case 2. $\forall m > l$ $\left((r, \frac{k}{n^i}) \cap S_m = \emptyset\right)$.

Then $g_m(r) \leq i$ for all $m > l$. But $\forall m > l$ $\big(r \in S_m \Rightarrow h^1_m(r) = r^+_{m+1} \in S_{m+1} \Rightarrow h^1_{m+1}(r) < h^1_m(r)\big)$. By minimality of $h^1_m(r)$ in $R_{g_m(r)} \cap (r,1)$, it follows that $h^1_{m+1}(r) \notin R_{g_m(r)}$. The R_i are increasing, so $R_{g_m(r)} \subset R_{g_{m+1}(r)} \Rightarrow g_m(r) < g_{m+1}(r)$, which means $\{g_m(r)|m > l\}$ is an infinite set of natural numbers bounded by i, contradiction.

Thus $S = R$, and hence Q-mon is dense in A. If we abuse notation a bit and let $f[X] = \{f[p]|p \in X\}$ and let $f^{-1}[Q] = \big\{f^{-1}[q]|q \in Q\big\}$, then because $f[X]$-mon $\subseteq A$, Q-mon is dense in $f[X]$-mon, and it follows that $f^{-1}[Q]$-mon is dense in X-mon. Now that we have a countable set of $(n-1)$-partitions of B whose monomials are dense in X-mon and in who's monomials X-mon is dense, we do the same for the uncountable case and then finish the proof for both cases together.

If $\kappa \geq \omega$, write $X = \{p_\alpha|\alpha \in \kappa\}$ where $p_\alpha = \{x_i|i \in n\}$ for all $\alpha \in \kappa$, without redundancy. Partition κ into κ many subsets of size ω, say $\kappa = \bigcup_{\beta\in\kappa} S_\beta$, and $\forall\beta \in \kappa$ let $X_\beta = \{p_\alpha|\alpha \in S_\beta\}$. Using the result obtained in the case $\kappa = \omega$, $\forall\beta \in \kappa$ take Y_β an ω-sized independent set of $(n-1)$-partitions of B such that Y_β-mon is dense in X_β-mon and vice-versa. For each $a \in Y$-mon write $a = a_{\beta_1}a_{\beta_2}...a_{\beta_k}$ where the β_i are distinct and each $a_{\beta_i} \in Y_{\beta_i}$-mon. For each $i \leq k$ take $b_{\beta_i} \in X_{\beta_i}$-mon such that $b_{\beta_i} \leq a_{\beta_i}$. $a_{\beta_1}a_{\beta_2}...a_{\beta_k} \geq b_{\beta_1}b_{\beta_2}...b_{\beta_k} > 0$ by independence of X, showing $\bigcup_{\beta\in\kappa} Y_\beta$ is independent. Let Y be an extension of $\bigcup_{\beta\in\kappa} Y_\beta$ to a maximal independent set of $(n-1)$-partitions. I claim $|Y| = \kappa$, and thus Y is the desired maximal independent set. If not, then $|Y| > \kappa$ and we can fix $p, q \in Y \setminus \bigcup_{\beta\in\kappa} Y_\beta$. Let $p = \{z_i|i \in n-1\}$ and take $b \in q$. Let $r = \{z_0 b, z_0(-b), z_1, z_2, \ldots, z_{n-2}\}$ so r is a partition of unity and $\bigcup_{\beta\in\kappa} Y_\beta \cup \{p, q\}$ is independent $\Rightarrow \bigcup_{\beta\in\kappa} Y_\beta \cup \{r\}$ is independent. But r is an n-partition and X is maximal, so $\exists a \in X$-mon $\exists z \in r$ such that $az = 0$. As above, this time using the density of the Y_β-mon in the X_β-mon, we can find $a' \in \bigcup_{\beta\in\kappa} Y_\beta$-mon such that $a' < a \Rightarrow a'z = 0$, contradiction. So $|Y| = \kappa$ (in fact $|Y \setminus \bigcup_{\beta\in\kappa} Y_\beta| < 2$). □

proof of Theorem 29. For P a maximal n-independent set in A with $n > 2$, repeated application of Lemma 31 yields a maximal 2-independent set of size $|P|$. For P a maximal 2-independent set, repeated application of Lemma 30 yields a maximal 2^{2^k}-independent set of size $|P|$ for arbitrarily large $k \in \omega$. Having reached $k > \log_2(\log_2 n)$, repeated application of Lemma 31 now yields a maximal n-independent set of size $|P|$. $\square$

The case $\kappa = \omega$ in Lemma 31 is admittedly a bit messy. The following is an alternate, less constructive but shorter proof. I have included the original above because it shows the relationship between the n-partitions and the $(n-1)$-partitions in a way that is visually presentable; for small values of n, it is feasable to draw the x_{ij} for the first several i and illustrate how the y_{ij} are built from these. Many of the messy-to-prove claims in the proof then become readily apparent.

Proof. Suppose X is a countably infinite n-independent set in A. Let f be an isomorphism from $\langle \bigcup X \rangle$ onto $\mathrm{Fr}_n(\omega)$. Both $\mathrm{Fr}_n(\omega)$ and $\mathrm{Fr}_{(n-1)}(\omega)$ are countable and atomless, so by [4, Corollary 5.16](Any two countably infinite atomless BA's are isomorphic) there is an isomorphism $g : \mathrm{Fr}_n(\omega) \to \mathrm{Fr}_{(n-1)}(\omega)$. Let $h = f^{-1} \circ g^{-1}$, let $P = \{p_\alpha | \alpha \in \omega\}$ be the canonical set of generating partitions for $\mathrm{Fr}_{(n-1)}(\omega)$, and let $Y = \{h[p_\alpha] | \alpha \in \omega\}$. Because h is an isomorphism and P is a countably infinite n-independent set, so is Y. For the proof of Lemma 31 it is also necessary that Y-mon be dense in X-mon and vice-versa. To see this, note that by Corollary 2 P-mon is dense in $\mathrm{Fr}_{(n-1)}(\omega) \Rightarrow Y$-mon is dense in $\langle \bigcup X \rangle \supseteq X$-mon. The proof of Corollary 2 can be applied to $\langle \bigcup X \rangle$ as well to show X-mon is dense in $\langle \bigcup X \rangle$, so a symmetric argument shows X-mon dense in Y-mon. $\square$

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