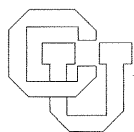


A Combinatorial Analysis of k -pseudo Trees

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CU-CS-509-91 January 1991



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ACKNOWLEDGMENTS SECTION.**

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Abstract

A k -pseudo tree is a family of sets such that each member U contains at least one element not contained in any member V incomparable to U , and the intersection of $k + 1$ incomparable members is empty. We show that the maximum cardinality of a k -pseudo tree consisting of subsets of an n -element set is $(k + 1)n - (k + 1)k/2$.

Definitions: Let \mathcal{F} be a family of subsets of an n -set X . We will consider \mathcal{F} partially ordered by the subset relation. Thus, $\{U_1, \dots, U_n\}$ is a *chain* if the U_i are linearly ordered by containment and it is an *antichain* if no U_i is contained in any U_j with $j \neq i$. U and V are *comparable* if either U is contained in V or V is contained in U , else they are *incomparable*.

The *center* of a member U of \mathcal{F} is the set of all elements of U not contained in any member of \mathcal{F} incomparable to U .

\mathcal{F} is a *pseudo tree* if the center of everyone of its members is non-empty and it contains all the singletons and X .

\mathcal{F} satisfies the *k -intersection property* if every antichain in \mathcal{F} of size greater than k has empty intersection.

\mathcal{F} is a *k -pseudo tree* if \mathcal{F} is a pseudo tree and satisfies the k -intersection property.

Further information on families of sets and partial orders can be found in [1], [2] and [4]. [3] discusses commonly studied intersection properties of structures.

Observation 1 *For any family \mathcal{F} on X containing X and the singletons, if U is X or a singleton, the center of U is itself.*

Observation 2 *\mathcal{F} is a 1-pseudo tree if and only if it is a tree, that is, if and only if any two of its members are either disjoint or comparable and it contains the singletons (the leaves) and X (the root).*

Observation 3 *In a pseudo tree, the centers of the members of an antichain must be mutually disjoint and also disjoint from the intersection.*

Therefore, an antichain of size k with non-empty intersection contains at least $k + 1$ elements in its union.

Observation 4 *A pseudo tree on an n -set is an $n - 1$ -pseudo tree.*

This implies that if \mathcal{F} is a k -pseudo tree on an n -set we may assume that $k < n$.

Theorem 5 *The maximum cardinality of a k -pseudo tree \mathcal{F} on an n -set X is $(k + 1)n - (k + 1)k/2$.*

Proof of the lower bound:

Example: Let $X = \{1, 2, \dots, n\}$ with $n > k$.

$$\mathcal{F} = \{\{i\} \mid i \leq n\} \cup \{\{1, \dots, j\} \cup \{j + l\} \mid 1 \leq j, 1 \leq l \leq k, j + l \leq n\}$$

Note that the second component of \mathcal{F} consists of initial segments of X with one of the next k singletons adjoined.

The cardinality of \mathcal{F} is $n + kn - (k + (k - 1) + \dots + 1) = (k + 1)n - (k + 1)k/2$.

For any set U in \mathcal{F} observe that if V is in \mathcal{F} and V contains the greatest element of U , then either $V \subset U$ or $U \subset V$. Therefore the center of U contains U 's greatest member and we can conclude that \mathcal{F} satisfies the second condition on k -pseudo trees.

To show that \mathcal{F} satisfies the k -intersection property suppose that $\{U_1, \dots, U_l\}$ is an antichain. Let x_i be the greatest member of U_i . Since x_i is in the center of U_i , observation 3 implies that the x_i are distinct. Without loss of generality assume that $x_1 \leq x_2 \leq \dots \leq x_l$. Then l must be less than or equal to k , for if u and v are the greatest members of U and V respectively, and $v - u > k$ then $U \subset V$ or $U \cap V$ is empty.

Proof of the upper bound: By identifying some of the sets in \mathcal{F} with their centers in a one-to-one way, and then identifying the remaining sets in \mathcal{F} with elements of X in a k -to-one way, we first establish a coarse upper bound of $(k + 2)n - 1$. To reduce this to the bound given in the theorem, we will define a notion of deficiency for arbitrary centers of fixed size. The deficiency measures the minimum reduction of the size of \mathcal{F} relative to the coarse bound due to such a center.

Observation 6 *Let A be any subset of X . The set of members of \mathcal{F} with center A forms a chain.*

By definition, if U intersects the center of V , then U must be comparable to V .

Lemma 7 *If U is contained in V and U intersects the center of V , then the center of U is contained in the center of V .*

Proof of Lemma 7: If x is any element not in the center of V , there is a set W in \mathcal{F} incomparable to V with $x \in W$. W does not intersect the center of V , so W must be incomparable to U . Therefore x is not in the center of U . \square

Corollary 8 *\mathcal{C} , the set of centers of members of \mathcal{F} , is a tree. That is, any two centers are either comparable or disjoint.*

Proof of Corollary 8: If A and B are centers of U and V respectively, and A and B have non-empty intersection, U and V must be comparable. The result now follows by Lemma 7. \square

Lemma 9 *If \mathcal{G} is a tree on X , then the cardinality of \mathcal{G} is at most $2n - 1$.*

Proof of Lemma 9: Consider the members of \mathcal{G} smallest sets first. That is, write $\mathcal{G} = \{U_1, \dots, U_l\}$ where U_i is maximal in $\mathcal{G}_i = \{U_1, \dots, U_i\}$ and the singletons are listed first. Let \mathcal{M}_i be the set of maximal members of \mathcal{G}_i . Suppose $i \geq n$. Then \mathcal{M}_i is a partition of X . Since the sets in \mathcal{G} are distinct and U_{i+1} is maximal in \mathcal{G}_{i+1} , \mathcal{M}_{i+1} is strictly coarser than \mathcal{M}_i . We obtain $l - n \leq n - 1$, thus $l \leq 2n - 1$. \square

If A is in \mathcal{C} , and U is the smallest set in \mathcal{F} with center A (see Observation 6), identify U with A . This identification (which is one-to-one) accounts for $|\mathcal{C}| \leq 2n - 1$ many sets in \mathcal{F} .

For each set not yet accounted for, we will select one of its element not in its center in such a way that if the same element is associated with l many sets in \mathcal{F} , then these sets form an antichain.

Lemma 10 *Let V_1, \dots, V_r be a decreasing chain of sets in \mathcal{F} such that for any j , V_{j+1} is maximal below V_j with respect to the sets in \mathcal{F} intersecting the center of V_j . If $W \in \mathcal{F}$ intersects the center of V_r and is not properly contained in V_r , then either the center of W contains the center of V_r or W properly contains V_1 .*

Proof of Lemma 10: By Lemma 7, the centers of the V_j form a decreasing chain. Suppose that W does not contain V_1 . Since W intersects the center of

every V_j , W is comparable to every V_j . In particular, it must be contained in V_1 . Let j be the largest index such that $W \subset V_j$. If $j \neq r$ we have $V_j \supset W \supset V_{j+1}$, so by the maximality assumption, $W = V_j$ or $W = V_{j+1}$, and the result follows. \square

Lemma 11 *Let U be a set in \mathcal{F} not identified with its center. Let V be the greatest set in \mathcal{F} contained in U such that the center of V is the same as the center of U . Then there is a $x \in U \setminus V$ contained in the center of every member W of \mathcal{F} properly contained in U with $x \in W$.*

Proof of Lemma 11: To find the desired x , let W_1 be maximal below U among members of \mathcal{F} below U intersecting $U \setminus V$. Since the singletons are in \mathcal{F} , such sets exist. Since W_1 is incomparable to V , the center of W_1 is contained in $U \setminus V$. Let W_1, \dots, W_r be a decreasing chain starting with W_1 satisfying the conditions given in Lemma 10 with W_r minimal in \mathcal{F} . Suppose Y is properly contained in U and intersects the center of W_r . By minimality of W_r , Y is not properly contained in W_r . By the maximality condition on W_1 , Y does not properly contain W_1 . Hence, by Lemma 10, the center of Y contains the center of W_r . It follows that any x in the center of W_r satisfies the desired conditions (note that W_r must be a singleton). \square

For any U in \mathcal{F} not identified with its center, we select any one of the x satisfying the conclusion of Lemma 11. By the lemma, if x is selected for l many distinct sets, these sets must form an antichain, so, by the k -intersection property, no x is selected more than k times. Since we have now accounted for all the sets in \mathcal{F} , we obtain the primary upper bound of $|\mathcal{F}| \leq b_p = 2n - 1 + kn = (k + 2)n - 1$.

Definition: To reduce b_p we associate with each $x \in X$ a positive integer $d(x)$, the *deficiency* due to x , in such a way that $|\mathcal{F}|$ is at least the sum of all these deficiencies less than b_p . Let $s(x)$ be the number of times x is selected for sets in \mathcal{F} . Clearly, we can let $d(x)$ be any number less than or equal to $k - s(x)$. Let l be the maximum size of any antichain in \mathcal{F} with x in its intersection. If x is selected exactly l times, we may set the deficiency of x to be as great as $k + 1 - s(x)$. This is due to a forced reduction in the size of the tree of centers (Lemma 12 below). Since in either case the given value is at least $k + 1 - l$ we define $d(x) = k + 1 - l$.

Lemma 12 *Let X' be the set of all $x \in X$ selected exactly l times, where l is the maximum size of any antichain containing x in its intersection. Then $|\mathcal{C}| \leq 2n - 1 - |X'|$.*

Proof of Lemma 12: For each x in X' , let C_x be the least set in \mathcal{C} containing both x and the centers of the (antichain of) sets for which x was

selected. Since the center of X is X , such a set exists. We will show that C_x has at least $r + 2$ many children (i.e. members of \mathcal{C} maximal below C_x), where r is the number of y 's in X' for which $C_y = C_x$. An analysis of the proof of the bound on the size of trees then shows that $|\mathcal{C}|$ is reduced by at least r for this C_x , and the result follows.

Suppose that $x \in X'$, and C_1, \dots, C_l are the centers of the sets U_1, \dots, U_l in \mathcal{F} for which x was selected. First we observe that $l \geq 2$: Otherwise, the sets in the pseudo tree containing x form a chain, which implies that x is in the center of every one of these sets, so that x is never selected. Since the maximum size of an antichain containing x is at least 1, it follows that x is not in X' , contrary to assumption.

Lemma 12.1 *If $C \in \mathcal{C}$ contains x and intersects at least one of the C_i then $C \supset C_x$.*

Proof of Lemma 12.1: Without loss of generality, assume that C intersects C_1 . Remember that the C_i are disjoint, and x is not contained in any of the C_i . Let W be any set in \mathcal{F} with center C . Since x is in the center of W , all of the U_i are comparable to W . If W were contained in U_1 , by Lemma 7, C would be contained in C_1 , which is not the case. Hence W contains U_1 . Since U_1 is incomparable to any of the other U_i , W also contains all the other U_i . Since all the U_i contain x which is in the center of W , again by Lemma 7, all the C_i are contained in C . The result now follows by definition of C_x . \square

Lemma 12.2 *If $C \in \mathcal{C}$ contains all the C_i , then $C \supset C_x$.*

Proof of Lemma 12.2: Again, let W be a set in \mathcal{F} with center C . W contains all the U_i , for else (by comparability), its center would be contained in one of the C_i , but there are at least two of these, both contained in C . Suppose that x is not in C . Let V be a set in \mathcal{F} incomparable to W containing x . Since C is disjoint from V (otherwise it wouldn't be the center of W), V is incomparable to all the U_i , so V together with the U_i form an antichain with non-empty intersection containing x , contradicting the maximality assumption on l . Hence x must be in C . \square

The last two lemmas show that C_x has at least three children: one containing x , and at least two partitioning $\{C_1, \dots, C_l\}$, since $l \geq 2$. It remains to consider what happens when $C_x = C_y$ for $y \neq x$. Let V_1, \dots, V_l be the sets for which y was selected.

Lemma 12.3 *If C is the center of W in \mathcal{F} and C contains both x and y , then C contains $C_x = C_y$.*

Proof of Lemma 12.3: First suppose that for some i , y is not contained in U_i . Then U_i is contained in W , and by Lemma 7, C_i is contained in C . Lemma 12.1 now implies that C contains C_x . Suppose then that y is contained in U_i . By the k -intersection property, U_i must be comparable to at least one of the V_j . Suppose $U_i \subset V_j$ (the other case is symmetric). By Lemma 11, y is in C_i , so by Lemma 12.1, C , which also contains x , contains C_x . \square

Let x_1, \dots, x_r be all the x such that $C_x = C$. By the above lemmas, C must have at least one child for each x_i , none of which may intersect any of the centers of sets for which one of the x_i was selected. In addition C must have at least two children for, say, the centers of the sets for which x_1 was selected. This adds up to at least $r + 2$ many children, as desired. \square

Let $d(m, l)$ be the minimum possible value of the sum of the deficiencies of elements in the center of a member U of a k -pseudo tree where m is the cardinality of the center, and l is the maximum size of any antichain in the pseudo tree with U contained in its intersection. We may assume that $1 \leq l \leq k$.

Lemma 13

$$\begin{aligned} d(m, l) &\geq m - 1 \\ &+ (k + 1 - l) + (k + 1 - l - 1) + \dots + (k + 1 - l - \min(m - 1, k - l)) \end{aligned}$$

In particular, this shows that the total deficiency of X , the center of which is X , is at least $d(n, 1) = n - 1 + (k(k + 1))/2$, so $|\mathcal{F}| \leq b_p - d(n, 1) = (k + 1)n - k(k + 1)/2$ thus completing the proof of the theorem.

Proof of Lemma 13: If $l = k$ the lemma asserts $d(m, k) \geq (m - 1) + (k - k + 1) = m$. Since for any x , $d(x) \geq 1$, this part of the lemma is true, so we may assume $l < k$.

The remainder of the proof proceeds by induction on the first argument. Consider first $m = 1$. Let U be a set in a k -pseudo tree with center $C = \{x\}$. Suppose the maximum size of an antichain containing U in its intersection is l . By Lemma 7, every member of the pseudo tree contained in U which contains x , has x as its center. Therefore, if U_1, \dots, U_r is an antichain with x in its intersection, each U_i contains U so that $r \leq l$. It follows that $d(x) = k + 1 - l$. By generality, $d(1, l) \geq k + 1 - l$, which is as desired.

Let U be a set in a k -pseudo tree with center C of cardinality $m > 1$. We may assume that U is the least set in the pseudo tree with this center, for if C is contained in the intersection of an antichain of size greater than one, so is every set in the pseudo tree with center C (if it contains one of the members of the antichain, then by Lemma 7 its center must also contain the center of that member).

Every set in the pseudo tree contained in U intersecting C has its center strictly contained in C . Since the singletons are in the pseudo tree and $m > 1$, C has at least two children in the tree of centers. For each child of C , the greatest set in the pseudo tree with this child as its center is contained in U . Let \mathcal{U} consist of these sets, one for each child of C .

Observe that \mathcal{U} has at least two maximal members U_1 and U_2 , respectively: If V is maximal in \mathcal{U} , and C' is the center of V , let x be any element in $C \setminus C'$. Let V' be a set maximal with respect to the family of sets in the pseudo tree incomparable to V containing x . Since x is not in the center of V , such sets exist. Since $C' \subset C$ is not contained in V' and x is in C , V' is contained in U , so its center is contained in C , and by maximality, V' is in \mathcal{U} . V' is also maximal in \mathcal{U} .

Let c_1 and c_2 be the cardinalities of the centers of U_1 and U_2 respectively. Let \mathcal{U}_3 be \mathcal{U} excluding U_1 and U_2 . Let U_3 be any maximal set in \mathcal{U}_3 and c_3 the cardinality of its center. Let \mathcal{U}_4 be \mathcal{U}_3 excluding U_3 and let U_4 be any maximal set in \mathcal{U}_4 and c_4 the cardinality of its center. Continue in this fashion obtaining U_i until $\max(s_i, l) \geq k$ where $s_i = c_1 + \dots + c_i$. Let r be the index of the last set thus obtained. For $r < i \leq t$, where $t = |\mathcal{U}|$, let c_{r+1}, \dots, c_t be the cardinalities of the centers of the remaining members of \mathcal{U} .

Lemma 13.1 *For $2 < i \leq r$, the maximal size of any antichain with U_i in its intersection is at most $\max(s_{i-1}, l)$. For $i = 1$ or $i = 2$, this quantity is at most l .*

Proof of Lemma 13.1: Let V_1, \dots, V_j be an antichain in the pseudo tree with U_i in its intersection. All the members of the antichain are comparable to U , since they intersect U 's center. Suppose V_h (say) contains U . Then, all the other members of the antichain contain U (else it would not be an antichain). In this case, $j \leq l$. So, suppose that all the V_h are contained in U . Since they intersect C , the centers of the V_h are (strictly) contained in C . In particular, each one is contained in or equal to at least one set in \mathcal{U} . By the maximality condition on U_1 and U_2 , this is impossible for $i = 1$ or $i = 2$ unless $j = 1$. Suppose then that $i > 2$. Since U_i is maximal in \mathcal{U}_i , unless $j = 1$, the V_h are not contained in any member of \mathcal{U}_i . Let C_h be the center of V_h . C_h must be contained in C . Let U' be the member of \mathcal{U} the center of which contains C_h . Then, since U' contains $V_h \supset U_i$, U' must be $U_{i'}$ for $i' < i$. Since h was arbitrary, the union of the centers of the V_h must be contained in the union of the centers of the $U_{i'}$ with $i' < i$, which has cardinality s_{i-1} . Since the centers of the V_h are disjoint, $j \leq s_{i-1}$. The result now follows. \square

For $2 < i \leq r$, let $l_i = \max(s_{i-1}, l)$. Let $l_1 = l_2 = l$. By choice of r , for all $i \leq r$, $l_i < k$. We can now bound the total deficiency d of the center of U from below:

$$d \geq d(c_1, l) + d(c_2, l) + d(c_3, l_3) + \dots + d(c_r, l_r) + d(c_{r+1}, k) + \dots + d(c_t, k)$$

The summands of the form $d(c_i, k)$ are equal to c_i . Using the induction hypothesis:

$$\begin{aligned} d \geq & m - r \\ & + (k + 1 - l) + (k + 1 - l - 1) + \dots + (k + 1 - l - r_1) \\ & + 1 + (k - l) + (k - l) + \dots + (k + 1 - l - r_2) \\ & \vdots \\ & + 1 + (k - l_r) + (k - l_r) + \dots + (k + 1 - l_r - r_r) \end{aligned}$$

where $r_i = \min(c_i - 1, k - l_i)$.

The leading 1's together with the term $m - r$ sum up to $m - 1$. It remains to show that the sum of the first $\min(m - 1, k - l) + 1$ many remaining summands dominate the series $(k + 1 - l) + (k + 1 - l - 1) + \dots + (k + 1 - l - \min(m - 1, k - l))$ consisting of the descending sequence of consecutive integers starting at $k + 1 - l$ with $\min(m, k - l + 1)$ many terms. Here are the remaining summands again:

$$\begin{aligned} & (k + 1 - l) + (k + 1 - l - 1) + \dots + (k + 1 - l - r_1) \\ & + (k - l) + (k - l) + \dots + (k + 1 - l - r_2) \\ & \vdots \\ & + (k - l_r) + (k - l_r) + \dots + (k + 1 - l_r - r_r) \end{aligned}$$

This is greater than or equal to:

$$\begin{aligned} (1) \quad & (k + 1 - l) + (k - l - 1) + \dots + (k - l - r_1) \\ (2) \quad & + (k - l) + (k - l - 1) + \dots + (k - l - r_2) \\ & \vdots \\ (r) \quad & + (k - l_r) + (k - l_r - 1) + \dots + (k - l_r - r_r) \end{aligned}$$

Note that all the terms are at least 0.

Let $r \geq i \geq 1$. By definition, $l_i \leq l + s_{i-1} + 1$. Thus, if $s_i \leq k - l$

$$c_i - 1 = s_i - s_{i-1} - 1 \leq k - l - s_{i-1} - 1 \leq k - l_i.$$

This inequality implies that $r_i = c_i - 1$. Since $s_i - l_i \leq c_i$, this in turn implies:

$$\begin{aligned} k - l_{i+1} &= k - \max(l, s_i) \\ &\geq k - \max(l_i, s_i) \\ &= k - l_i - \max(s_i - l_i, 0) \\ &\geq k - l_i - c_i \\ &= k - l_i - r_i - 1 \end{aligned}$$

That is, the first term on the $i + 1$ 'th line is at least as great as the last term on line i less one.

This shows that if $s_i \leq k - l$, the terms of this sum up to and including the last term of the $(i + 1)$ 'th line, dominate consecutive non-negative decreasing integers starting with $k + 1 - l$ ('excess' terms dominate 0's).

The number of terms on the (i) 'th line is $\min(c_i, k - l_i + 1)$. Suppose that $c_i - 1 > k - l_i$. If $s_{i-1} < l$, $l_i = l$, so $c_i > k - l + 1$, and line (i) has $k - l + 1$ many terms. If $i = 1$, since the first term on line (1) is $k + 1 - l$, the induction is complete. Else, the first term is $k - l$ (since $l_i = l$), so we simply add in the first term of line (1), and again we are done. If $s_{i-1} \geq l$, $l_i = s_{i-1}$, so $s_i = c_i + s_{i-1} > k - l_i + 1 + s_{i-1} = k + 1$ hence $i = r$. We can subsume this case in the next one.

Let h be the least i such that $s_i > k - l$ or r if no such i exists. In either case, the terms in the first h lines dominate term by term the descending series of consecutive integers starting with $(k + 1 - l)$ and for $i < h$, the number of terms on line (i) is $r_i + 1 = c_i$. We have already dealt with the case where $c_i > k - l_i + 1$ with $i < r$, so we may assume this is not the case except possibly for $i = h = r$. The number of terms in lines (1) through (h) is therefore $c_1 + \dots + c_{h-1} + r_h + 1 = s_{h-1} + \min(c_h, k - l_h + 1)$. If $h \neq r$, the last term is c_h , so we get at least $s_h \geq k - l + 1$ many terms the sum of which dominates $(k + 1 - l) + \dots + 1$. If $h = r$ and $c_h \leq k - l_h + 1$ we get s_r many such terms. By construction, either $s_r \geq k \geq k - l + 1$, or $s_r = m$ (which is the case where $m < k - l + 1$), and we are done. Otherwise, by the previous paragraph, we may assume that $l_h = s_{h-1}$, so we get $s_{h-1} + k - s_{h-1} + 1 = k + 1 \geq k - l + 1$ many terms. \square

The proof of the theorem is now complete.

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