ADDING GLOBAL FORBIDDING CONTEXT TO CONTEXT-FREE GRAMMARS

by

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ADDING GLOBAL FORBIDDING CONTEXT TO CONTEXT-FREE GRAMMARS

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ABSTRACT

A 1S grammar generalizes a context-free grammar in the following way: a production $A \to \alpha$ can be applied to a string uAv (to rewrite the designated occurrence of A) provided that all letters from u belong to a fixed alphabet X and all letters from v belong to a fixed alphabet Z (the alphabets X and Z are independent of the production). It is proved that a language is generated by a 1S grammar if and only if it is context-free: this solves an open problem from the theory of selective substitution grammars [KR2].

KEYWORDS: Selective substitution grammars, one sequential grammars, context-free languages.

CR CATEGORIES: F.4.2, F.4.3.

INTRODUCTION

The concept of a 1S grammar arose in a comparative study ([KR2]) of sequential and parallel ways of rewriting. This study was carried out within the theory of selective substitution grammars which provides a uniform framework for the grammatically oriented formal language theory (see, e.g., [R], [RW], [EMR], [KR1], [GKR1], [GKR2] and [K]). Within this framework 1S grammars form a generalization of Sequential rewriting (i.e. one symbol per step). The generalization consists of adding one specification of allowed left and right context sets of letters: an occurrence of a non-terminal in a sentential form cannot be rewritten if to its left or to its right a symbol appears which is not in the specified left or right, respectively, context set (a forbidden symbol).
Note that when no symbols are forbidden as left or right context, one gets the classical context-free way of rewriting.

The application of the context conditions is global: one set of rewritable non-terminals is specified together with left and right context sets which are independent of the productions to be used.

In a more local application of context conditions one specifies pairs of left and right context sets for various sets of non-terminals: if a letter from a given non-terminal set is to be rewritten, then its left-right context pair is taken into consideration.
It is shown in [KR2] that even the slightest local use of such context conditions increases the generative power of context-free grammars. However, it is left open in [KR2] whether 1S grammars are more powerful than context-free grammars. In the present paper we prove that a language is generated by a 1S grammar if and only if it is context-free.

It is worth noticing that the role of forbidding context in rewriting systems was investigated in depth within formal language theory (see, e.g., [vdW], [L], [M], [P], [RvS] and [C]). Although the notion of a 1S grammar came up within the theory of selective substitution grammars, it clearly forms a very natural and perhaps the most basic way of adding forbidding context to context-free grammars fitting very well into systematizing the above mentioned line of research. In view of this our main result seems to be rather surprising - until now all "non-trivial" ways of adding forbidding context to context-free grammars resulted in a (considerable) increase of generating power.

The paper is organized as follows.

After the introduction of some basic terminology in Section 1, in Section 2 a subclass of 1S grammars called special 1S grammars is introduced. It is shown that every 1S language is a homomorphic image of the language generated by a special 1S grammar. In Section 3 the successful derivations in special 1S grammars are considered and it is demonstrated that for those a "normal form" exists. Finally, by constructing (in Section 4) special 1S grammars in which the successful derivations in "normal form" cannot get "blocked", we are able to show that every language generated by a special 1S grammar is context-free. This implies that every 1S grammar generates a context-free language (Section 5).

1. BASIC DEFINITIONS

We assume the reader to be familiar with the basic concepts of formal language theory as presented, e.g., in [S].
As far as selective substitution grammars are concerned the paper is self-contained.

In order to fix our notation and terminology we recall some basic notions now.

Let Σ be an alphabet. For a word $w \in \Sigma^*$, $|w|$ denotes its length and alph(w) denotes the set of symbols occurring in w; Λ denotes the empty word.
For a language $L \subseteq \Sigma^*$, $\underline{alph}(L) = \bigcup_{w \in L} \underline{alph}(w)$.
For a binary relation $P \subseteq \Sigma \times \Sigma^*$, we write $\underline{rhs}(P) = \{w \in \Sigma^* : (A,w) \in P$, for some $A \in \Sigma^*\}$.
Two language generating devices are said to be equivalent, if the languages they generate are equal.
We often identify notationally a singleton set with its element.

Let Σ and Γ be two (finite) alphabets. A mapping h from Σ^* into the set of non-empty subsets of Γ^* is called a finite substitution from Σ into Γ, if, for all $a \in \Sigma$, h(a) is finite and, for all $a \in \Sigma$ and $w \in \Sigma^*$, $h(aw) = h(a)h(w)$. h is a homomorphism from Σ into Γ if, additionally, for all $a \in \Sigma$, h(a) is a singleton.
By FSUB(Σ,Γ) we denote the set of all finite substitutions from Σ into Γ and by HOM(Σ,Γ) the set of all homomorphisms from Σ into Γ is denoted.
For $L \subseteq \Sigma^*$ and $h \in$ FSUB(Σ,Γ), h(L) denotes the set $\{h(w) : w \in L\}$.

A context-free (abbreviated CF) grammar is specified in the form (Σ,P,S,Δ), where Σ is its total alphabet, $\Delta \subseteq \Sigma$ is its terminal alphabet, $S \in \Sigma\setminus\Delta$ is its axiom and $P \subseteq (\Sigma\setminus\Delta) \times \Sigma^*$ is its (finite) set of productions.

For a context-free grammar G, its direct derivation relation and its derivation relation are as usual denoted by $\underset{G}{\Rightarrow}$ and $\overset{*}{\underset{G}{\Rightarrow}}$, respectively, and if no confusion arises by $\Rightarrow$ and $\overset{*}{\Rightarrow}$, respectively.

In the general theory of rewriting systems it is often essential to provide productions also for terminal symbols. If one allows to rewrite also terminal symbols in a context-free grammar then one gets an EOS _system_ (see, e.g., [KR1]).

It is easily seen that $L(\mathrm{CF}) = L(\mathrm{EOS})$, where $L(\mathrm{CF})$ and $L(\mathrm{EOS})$ denote the families of languages generated by CF grammars and EOS systems, respectively.

In the sequel we will assume that an infinite alphabet of symbols is available. All symbols to be used are elements of the infinite alphabet $A \cup \bar{A}$, where $\bar{A} = \{\bar{a} : a \in A\}$, and A and $\bar{A}$ are disjoint. The bars appearing above symbols have a special meaning: they indicate that the original symbol is _activated_. A consists of non-activated symbols only and hence $\bar{A}$ consists of activated symbols only. For an arbitrary alphabet $\Sigma \subseteq A$, $\bar{\Sigma}$ denotes $\{\bar{a} : a \in \Sigma\}$.

Definition 1.1. (1). A CF _based_ 1S _grammar_ is a construct $(\Sigma,P,S,\Delta,X^*\bar{Y}Z^*)$, where (Σ,P,S,Δ) is a context-free grammar, $X,Z \subseteq \Sigma$ and $Y \subseteq \Sigma\backslash\Delta$.

(2). An EOS _based_ 1S _grammar_ is a construct $(\Sigma,P,S,\Delta,X^*\bar{Y}Z^*)$, where (Σ,P,S,Δ) is an EOS system and $X,Y,Z \subseteq \Sigma$. □

For a Q based 1S grammar $H = (\Sigma,P,S,\Delta,X^*\bar{Y}Z^*)$, where $Q \in \{\mathrm{CF},\mathrm{EOS}\}$, we refer to (Σ,P,S,Δ) as the _base of_ H, denoted by _base_(H).

Definition 1.2. Let $Q \in \{\mathrm{CF},\mathrm{EOS}\}$. Let $H = (\Sigma,P,S,\Delta,X^*\bar{Y}Z^*)$ be a Q based 1S grammar.

For $u,v \in \Sigma^*$, u _directly derives_ v (_in_ H), denoted by $u \underset{H}{\Rightarrow} v$ ($u \Rightarrow v$, if no confusion is possible) if there exist

(1) $u_1 \in X^*$, $A \in Y$ and $u_2 \in Z^*$, and

(2) $(A,w) \in P$, such that $u = u_1Au_2$ and $v = u_1wu_2$.

By $\overset{*}{\underset{H}{\Rightarrow}}$ ($\overset{*}{\Rightarrow}$) we denote the transitive and reflexive closure of $\underset{H}{\Rightarrow}$ ($\Rightarrow$, respectively).

The <u>language</u> <u>of</u> H, denoted by L(H), is defined by

$L(H) = \{w \in \Delta^* : S \overset{*}{\underset{H}{\Rightarrow}} w\}$. □

In [KR2] only EOS based 1S grammars are considered. In the present paper, however our constructions involve CF based 1S grammars only. This is justified by the following straightforward lemma (see also [KR1]).

<u>Lemma</u> 1.1. (1). For every CF based 1S grammar there exists (effectively) an equivalent EOS based 1S grammar.

(2). For every EOS based 1S grammar there exists (effectively) an equivalent CF based 1S grammar. □

<u>In the remainder of this paper we will consider CF based 1S grammars only and we will refer to them as</u> 1S <u>grammars</u>.

The family of languages generated by 1S grammars is denoted by $\mathcal{L}(1S)$.

Next we introduce some notions and notations concerning 1S grammars, that will be extensively used in the sequel.

Let $H = (\Sigma,P,S,\Delta,X^*\bar{Y}Z^*)$ be a 1S grammar.
Then $L_H = \Sigma\backslash Z$, $R_H = \Sigma\backslash X$ and $B_H = L_H \cap R_H$ are
the set of <u>left-blocking</u> <u>symbols</u> (<u>of</u> H),
the set of <u>right-blocking</u> <u>symbols</u> (<u>of</u> H), and
the set of <u>blocking</u> <u>symbols</u> (<u>of</u> H), respectively.
If a symbol is neither left-blocking, nor right-blocking (hence it is in $X \cap Z$) we will refer to it as <u>neutral</u> (<u>in</u> H).

<u>Example</u> 1.1. Let $H = (\Sigma,P,S,\Delta,X^*\bar{Y}Z^*)$ be defined in the following way.
$\Sigma = \{S,A,B,C,a,b,c\}$

$P = \{(S,ABC),(A,AA),(A,a),(A,c),(B,b),(C,CA),(C,BCC),(C,a),(C,c)\}$,

$\Delta = \{a,b,c\}$, $X = \{a,b,A\}$, $Y = \{S,A,B,C\}$ and $Z = \{c,b,C\}$.

Then $L_H = \{S,A,B,a\}$, $R_H = \{S,B,C,c\}$ and $B_H = \{S,B\}$.

Then $S \Rightarrow ABC$ and in ABC, B is the only symbol that can be rewritten. Since B is blocking it prohibits the rewriting of A and C.

$ABC \Rightarrow AbC \Rightarrow Abc \Rightarrow AAbc$.

In AAbc only the right-most A can be rewritten. Since A is left-blocking it prohibits the rewriting of symbols occurring to its left.

Analogously in ab^2CC (derived via $AbC \Rightarrow abC \Rightarrow abBCC \Rightarrow ab^2CC$) only the left-most occurrence of C can be rewritten, because C is right-blocking.

In AbCA (derived from AbC) no symbol can be rewritten because the right-most A "blocks" AbC and C "blocks" this A. □

Let $w = a_1 \ldots a_k$, for some $a_i \in \Sigma$ and $1 \leq i \leq k$, where $k \geq 0$.

Then $Locc_H(w) = \max(\{i : 1 \leq i \leq k \text{ and } a_i \in (\Sigma \backslash \Delta) \cap L_H\} \cup \{0\})$

and $Rocc_H(w) = \min(\{i : 1 \leq i \leq k \text{ and } a_i \in (\Sigma \backslash \Delta) \cap R_H\} \cup \{k+1\})$.

Hence $Locc_H(w)$ denotes the right-most occurrence of a left-blocking non-terminal symbol in w if any, otherwise $Locc_H(w) = 0$, and $Rocc_H(w)$ denotes the left-most occurrence of a right-blocking non-terminal symbol in w if any, otherwise $Rocc_H(w) = |w| + 1$.

Note that if $Rocc_H(w) < Locc_H(w)$, then w cannot be rewritten in H.

Whenever possible we will omit the subscript H from $Locc_H$ and $Rocc_H$.

<u>Example</u> 1.1. (Continued.)

$Locc_H(ABC) = 2 = Rocc_H(ABC)$,

$Locc_H(AbC) = 1$ and $Rocc_H(AbC) = 3$.

$Locc_H(AAbc) = 2$ and $Rocc_H(AAbc) = 5$.

$Locc_H(ab^2CC) = 0$ and $Rocc_H(ab^2CC) = 4$.

$Locc_H(AbCA) = 4$ and $Rocc_H(AbCA) = 3$ and AbCA is "blocked". □

Let $w \in \Sigma^*$. Then $\underline{\text{contr}}_H(w) = \{u : u \in \Delta^* \text{ and } w \underset{\underline{\text{base}}(H)}{\overset{*}{\Rightarrow}} u\}$.

Notice that $\underline{\text{contr}}_H(w) \supseteq \{u : u \in \Delta^* \text{ and } w \underset{H}{\overset{*}{\Rightarrow}} u\}$.

In particular $\underline{\text{contr}}_H(S) = L(\underline{\text{base}}(H)) \supseteq L(H)$.

A <u>derivation</u> D (<u>in</u> H) is a sequence $D = (x_0, x_1, \ldots, x_n)$, where $n \geq 0$, $x_0 = S$ and $x_i \in \Sigma^*$ for $0 \leq i \leq n$, such that $x_i \underset{H}{\Rightarrow} x_{i+1}$, for $0 \leq i \leq n-1$. For $0 \leq i \leq n-1$ the pair (x_i, x_{i+1}) is called a (the $(i+1)$-th) <u>derivation step</u> (<u>in</u> D).

For $0 \leq i \leq n-1$, Prod(D,i) denotes the production used in the $(i+1)$-th derivation step in D and Rew(D,i) denotes the position of the symbol in x_i that is rewritten in that step. Thus if, for some $0 \leq i \leq n-1$, Prod $(D,i) = (A,\alpha)$ for some $(A,\alpha) \in P$ and Res $(D,i) = k$, for some $k \geq 1$, then $x_i = uAv$ and $x_{i+1} = u\alpha v$, where $u,v \in \Sigma^*$ and $|uA| = k$.

D is <u>successful</u>, if $x_n \in \Delta^*$.

For $0 \leq i \leq n$, the words x_i are said to <u>appear in</u> D.

Whenever a word appears in a derivation (in H) it is called a <u>sentential form</u> (<u>of</u> H)

A non-terminal sentential form x of H is <u>successful</u> if there exists a $w \in \Delta^*$ such that $x \underset{H}{\overset{*}{\Rightarrow}} w$.

Observe that if x is a successful non-terminal sentential form of H, then $x \in X^*YZ^*$ and hence $\text{Locc}_H(x) \leq \text{Rocc}_H(x)$.

If moreover $x = uAv$, for some $u,v \in \Sigma^*$ and $A \in \Sigma \backslash \Delta$, then $(\underline{\text{alph}}(u) \cap \Delta) \subseteq X$. and $(\underline{\text{alph}}(v) \cap \Delta) \subseteq Z$. Consequently, for all $w \in L(H)$, $w \in \Delta^* \cap (X^*\underline{\text{rhs}}(P)Z^*)$.

<u>Example</u> 1.1. (Continued).

Let D = (S,ABC,AbC,Abc,AAbc,Acbc,acbc). Then D is successful, (Abc,AAbc) is the fourth derivation step in D, Rew(D,3) = 1 and Prod(D,3) = (A,AA).

The sentential form AAbc is successful, but Aabc and AbCA are not successful sentential forms. □

2. SPECIAL 1S GRAMMARS

In this section we show that every language in $L(1S)$ is a homomorphic image of a language generated by a 1S grammar that satisfies certain conditions. To define this kind of 1S grammars we (re-)introduce the following notions.

First we consider context-free grammars (they are the underlying bases of 1S grammars).

Definition 2.1. Let $G = (\Sigma, P, S, \Delta)$ be a context-free grammar and let $A \in \Sigma$.

(1). A is reachable (in G), if there exist words $\alpha, \beta \in \Sigma^*$ such that $S \overset{*}{\Rightarrow} \alpha A \beta$.

(2). A is useful (in G), if either $A = S$ or there exists a word $\alpha \in \Delta^*$ such that $A \overset{*}{\Rightarrow} \alpha$.

(3). G is reduced if all elements of Σ are reachable and useful. □

The reader should note that this definition differs slightly from the usual definition of a reduced context-free grammar in that we consider the axiom always as useful, even if the generated language is empty.

Clearly in a 1S grammar all symbols appearing in a successful sentential form are reachable and useful.

Definition 2.2. Let $H = (\Sigma, P, S, \Delta, X^* \bar{Y} Z^*)$ be a 1S grammar and let $(A, \alpha) \in P$. (A, α) is safe (in H) if the following holds.

(1). If $\alpha \notin \Delta^*$, then, for all $\alpha_1, \alpha_2 \in \Sigma^*$ such that $\alpha = \alpha_1 \alpha_2$ either $\underline{\mathrm{alph}}(\alpha_1) \cap R_H = \emptyset$ or $\underline{\mathrm{alph}}(\alpha_2) \cap L_H = \emptyset$.

(2). If $\alpha = \alpha_1 a \alpha_2$, for some $a \in \Delta$ and $\alpha_1, \alpha_2 \in \Sigma^*$, then $a \in L_H$ implies that $\alpha_1 \in \Delta^*$, and $a \in R_H$ implies that $\alpha_2 \in \Delta^*$. □

Clearly, in a 1S grammar a production that is not safe is never applied in a successful derivation.

Definition 2.3. A 1S grammar $H = (\Sigma,P,S,\Delta,X^*\bar{Y}Z^*)$ is strongly reduced if the following holds.

(1). base(H) is reduced.

(2). $Y = \Sigma \backslash \Delta$.

(3). All productions in P are safe. □

The following definition describes the construction of a strongly reduced 1S grammar from a given 1S grammar.

Definition 2.4. Let $H = (\Sigma,P,S,\Delta,X^*\bar{Y}Z^*)$ be a 1S grammar. The strongly reduced version of H is the 1S grammar $(\Sigma_1,P_1,S_1,\Delta_1,X_1^*\bar{Y}_1Z_1^*)$ is constructed in the following way.

(1). $P' = P\backslash\{(A,w):(A,w) \in P$ and either $A \notin Y$ or (A,w) is not safe in $H\}$.

(2). Let G be the CF grammar (Σ,P',S,Δ).

Then $\Sigma_1 = \{A \in \Sigma : A$ is useful and reachable in $G\}$ and $\Delta_1 = \Delta \cap \Sigma_1$.

(3). $P_1 = \{(A,w):(A,w) \in P'$ and $A,w \in \Sigma_1^*\}$, $S_1 = S$, $X_1 = X \cap \Sigma_1$, $Y_1 = (Y \cap \Sigma_1) \cup \{S_1\}$ and $Z_1 = Z \cap \Sigma_1$. □

Clearly the procedure given by Definition 2.4 is effective. Note that the resulting 1S grammar may be quite "degenerated", e.g. of the form $(\{S\}, \emptyset, S, \emptyset, \bar{S})$.

The following lemma is an immediate consequence of the preceding definitions and so we state it without a proof.

Lemma 2.1. If H is a 1S grammar and H' is its strongly reduced version, then $L(H) = L(H')$ and H' is strongly reduced. □

Now we are ready to define one of the main notions of this paper and to prove an important intermediate result.

Definition 2.5. A 1S grammar $G = (\Sigma, P, S, \Delta, X^* \bar{Y} Z^*)$ is special if the following conditions are satisfied.

(C1) $L(G) \subseteq (\Delta \backslash B_G)^* B_G (\Delta \backslash B_G)^*$

(i.e., every word in $L(G)$ contains exactly one occurrence of a blocking symbol).

(C2) For all $\alpha \in \underline{rhs}(P)$, if $\underline{alph}(\alpha) \cap (B_G \cap \Delta) \neq \emptyset$, then $\alpha \in (B_G \cap \Delta)$

(i.e., blocking terminal symbols are never introduced together with another symbol).

(C3) $X \cap Z \subseteq \Delta$

(i.e., there are no neutral non-terminal symbols in G).

(C4) $S \notin \underline{alph}(\underline{rhs}(P))$ and, for all $A \in \Sigma \backslash (\Delta \cup \{S\})$, for all $v, w \in \underline{contr}_G(A)$, $\underline{alph}(v) = \underline{alph}(w)$

(i.e., S cannot be introduced during a derivation and all terminal words contributed by a non-terminal differing from S consist of the same symbols).

(C5) For all $\alpha_1 A \alpha_2 \in \underline{rhs}(P)$, such that $\alpha_1, \alpha_2 \in \Sigma^*$ and $A \in \Sigma \backslash \Delta$, if $\underline{alph}(\underline{contr}_G(A)) \cap B_G \neq \emptyset$, then $\underline{contr}_G(\alpha_1) \subseteq X^*$ and $\underline{contr}_G(\alpha_2) \subseteq Z^*$

(i.e., if a non-terminal symbol A can derive a blocking terminal symbol, then A is introduced with left and right context that contribute only terminal words which cannot block the rewriting of A or the rewriting of words derivable from A).

(C6) G is strongly reduced

(i.e., G has no a priori useless productions or symbols). □

Theorem 2.1. For every 1S grammar H there exist a special 1S grammar G and a homomorphism h, such that $h(L(G)) = L(H)$.

Proof. We will prove the theorem in five steps.

In the first step a 1S grammar G_1 is constructed from H and the homomorphism h is defined for which $h(L(G_1)) = L(H)$. G_1 will satisfy (C1) and (C2) from Definition 2.5.

In the next steps we will construct the 1S grammars G_2, G_3, G_4 and G from G_1, G_2, G_3 and G_4, respectively, in such a way that gradually all conditions from Definition 2.5 will be satisfied and the languages $L(G_i), i = 1,\ldots,4$ and $L(G)$ will be the same.

Then G will be special and $h(L(G)) = L(H)$.

Let $H = (\Sigma, P, S, \Delta, X^*\bar{Y}Z^*)$ be a 1S grammar.

(I) Let $G_1 = (\Sigma_1, P_1, S_1, \Delta_1, X_1^*\bar{Y}Z_1^*)$ be defined in the following way.

$\Sigma_1 = \Sigma \cup \Gamma \cup \Pi$, where $\Gamma = \{\hat{A} : A \in \Sigma\backslash\Delta\}$ and $\Pi = \{\langle w\rangle : w \in \Delta^*$ and $(A,w) \in P$, for some $A \in \Sigma\backslash\Delta\}$ are new mutually disjoint alphabets.

$P_1 = \{(A,w) : (A,w) \in P$ and if $|w| \geq 2$, then $\underline{\mathrm{alph}}(w) \cap (\Delta \cap B_H) = \emptyset\} \cup$
$\cup \{(\hat{A}, w_1\hat{B}w_2) : (A, w_1Bw_2) \in P$ and $B \in \Sigma\backslash\Delta\} \cup \{(\hat{A}, \langle w\rangle) : (A,w) \in P$ and $w \in \Delta^*\}$.

$S_1 = \hat{S}$.

$\Delta_1 = \Delta \cup \Pi$,

$X_1 = X \cup \{\hat{A} : A \in X \cap (\Sigma\backslash\Delta)\}$,

$Y_1 = Y \cup \{\hat{A} : A \in Y \cap (\Sigma\backslash\Delta)\}$ and

$Z_1 = Z \cup \{\hat{A} : A \in Z \cap (\Sigma\backslash\Delta)\}$.

Clearly, $L(G_1) \subseteq \Delta^*\Pi\Delta^*$. Moreover, since $\Pi \subseteq \Sigma_1\backslash(X_1 \cup Z_1)$, all elements from Π are blocking in G_1. In every successful derivation blocking terminals can only be introduced in the last derivation step. Hence this is done using a production of the form $(\hat{A}, \langle w\rangle)$, where $\langle w\rangle \in \Pi$.

This implies that $L(G_1) \subseteq (\Delta_1\backslash B_{G_1})^* B_{G_1} (\Delta_1\backslash B_{G_1})^*$ and so G_1 satisfies (C1).

From the construction of P_1 it follows that G_1 satisfies (C2).

The homomorphism $h \in HOM(\Delta_1, \Delta)$ is defined by $h(a) = a$, for all $a \in \Delta$, and $h(\langle w \rangle) = w$, for all $w \in \Delta^*$.

That $h(L(G_1)) = L(H)$ follows from the following observations.

Let $(S_1, x_1, \ldots, x_n)$ be a successful derivation in G_1. Then $(\psi(S_1), \psi(x_1), \ldots, \psi(x_n))$ is a successful derivation in H, where $\psi \in HOM(\Sigma_1, \Sigma)$ is defined by

$\psi(\hat{A}) = \psi(A) = A$, for all $A \in \Sigma_1 \backslash \Delta_1$ and

$\psi(a) = h(a)$, for all $a \in \Delta_1$.

Hence $h(L(G_1)) \subseteq L(H)$.

Let $D = (x_0 = S, x_1, \ldots, x_n)$ be a successful derivation in H. Then $x_{n-1} = uAv$ and $x_n = uwv$, for some $u,v,w \in \Delta^*$, $A \in \Sigma \backslash \Delta$ and $(A,w) \in P$. A derivation $\hat{D} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_n)$ in G_1 with $\hat{x}_0 = \hat{S}$, $\hat{x}_{n-1} = u\hat{A}v$ and $\hat{x}_n = u\langle w \rangle v$, can be defined in inductively by "capping all ancestors of $\hat{A}$". This can be illustrated as follows.

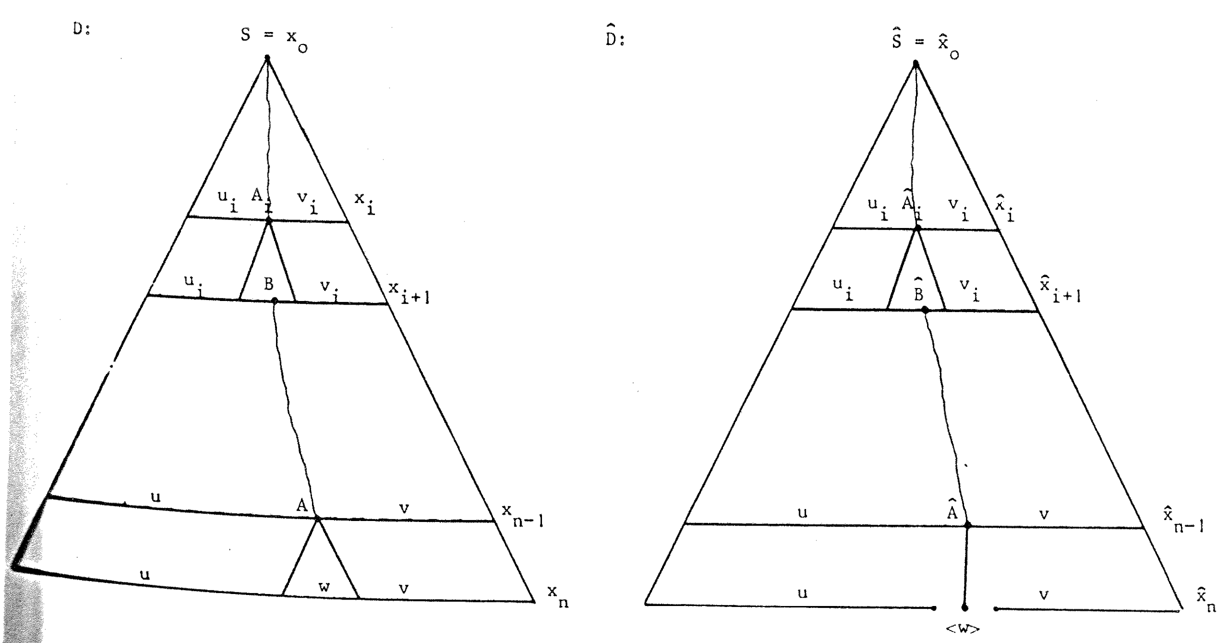


Figure 2.1.

Then $\hat{D}$ is a successful derivation in G_1 and $h(\hat{x}_n) = x_n$.

Hence $L(H) \subseteq h(L(G_1))$.

We conclude that $h(L(G_1)) = L(H)$ and G_1 satisfies (C1) and (C2).

(II) Let $G_2 = (\Sigma_2, P_2, S_2, \Delta_2, X_2^* \bar{Y}_2 Z_2^*)$ be defined in the following way.

$\Sigma_2 = \Sigma_1 \backslash ((X_1 \cap Z_1) \backslash \Delta_1) \cup \Gamma_X \cup \Gamma_Z$, where $\Gamma_X = \{[A,1] : A \in (X_1 \cap Z_1) \backslash \Delta_1\}$ and $\Gamma_Z = \{[A,2] : A \in (X_1 \cap Z_1) \backslash \Delta_1\}$ are two new mutually disjoint alphabets.

The homomorphism $\varphi \in HOM(\Sigma_2, \Sigma_1)$ is defined by

$\varphi(a) = a$, for all $a \in \Sigma_1 \cap \Sigma_2$, and

$\varphi([A,i]) = A$, for all $A \in (X_1 \cap Z_1) \backslash \Delta_1$ and $i \in \{1,2\}$.

$Q_2 = Q_1 \cup Q_2$, where $Q_1 = \{(A,\alpha) : A \in \Sigma_2$ and $(\varphi(A), \varphi(\alpha)) \in P_1\}$ and

$Q_2 = \{([A,1],[A,2]) : A \in (X_1 \cap Z_1) \backslash \Delta_1\} \cup \{([A,2],[A,1]) : A \in (X_1 \cap Z_1) \backslash \Delta_1\}$.

If $S_1 \in \Sigma_2$, then $S_2 = S_1$ else $S_2 = [S_1,1]$. $\Delta_2 = \Delta_1$.

$X_2 = (X_1 \cap \Sigma_2) \cup \Gamma_X$,

$Y_2 = (Y_1 \cap \Sigma_2) \cup \{[A,i] : A \in Y_1 \cap X_1 \cap Z_1$ and $i \in \{1,2\}\}$ and

$Z_2 = (Z_1 \cap \Sigma_2) \cup \Gamma_Z$.

From the construction of G_2 it follows that $X_2 \cap Z_2 \subseteq \Delta_2$. Hence G_2 has only left-blocking and right-blocking non-terminal symbols and satisfies (C3) of Definition 2.5.

G_2 satisfies (C2), because G_1 satisfies (C2), the blocking symbols of G_1 and G_2 are the same, Q_1 "corresponds" to P_1 and no productions violating (C2) have been included in Q_2.

If $L(G_1) = L(G_2)$, then G_2 satisfies (C1) because the blocking terminals in G_1 and G_2 are the same.

That $L(G_2) = L(G_1)$ can be seen as follows.

Let $D = (x_0 = S_1, x_1, \dots, x_n)$ be a successful derivation in G_2. Clearly, for every $i \in \{0,\dots,n-1\}$ such that $Prod(D,i) \in Q_1$, $\varphi(x_i)$ derives $\varphi(x_{i+1})$ in G_1. For every $i \in \{0,\dots,n-1\}$ such that $Prod(D,i) \in Q_2$, $\varphi(x_i) = \varphi(x_{i+1})$. Let $i_1,\dots,i_k$, $k \geq 1$, be - in ascending order - the elements of $\{i : i \in \{0,\ldots,n-1\}$ and $\mathrm{Prod}(D,i) \in Q_1\}$. Then $\varphi(x_{i_1}) = S_1$ and $(\varphi(x_{i_1}),\ldots,\varphi(x_{i_k}),x_n)$ is a derivation in G_1.

Hence $L(G_2) \subseteq L(G_1)$.

To show that $L(G_1) \subseteq L(G_2)$ we proceed as follows.

Let $D_1 = (x_0,x_1,\ldots,x_n)$ be a successful derivation in G_1. We construct a successful derivation $D_2 = (y_0,y_1,\ldots,y_m)$ in G_2 and a monotonic function δ from $\{0,\ldots,n\}$ to $\{0,\ldots,m\}$ such that for all $i \in \{0,\ldots,n\}$, $\varphi(y_{\delta(i)}) = x_i$ and $\mathrm{Locc}_{G_2}(y_{\delta(i)}) \le \mathrm{Rocc}_{G_2}(y_{\delta(i)})$.
We will define δ and D_2 inductively.

We set $\delta(0) = 0$ and $y_0 = S_2$. Hence $\varphi(y_{\delta(0)}) = x_0$ and $\mathrm{Locc}_{G_2}(y_{\delta(0)}) \le \mathrm{Rocc}_{G_2}(y_{\delta(0)})$.
Assume that δ and D_2 have been defined upto k and $y_{\delta(k)}$, for some $k \in \{0,\ldots,n-1\}$.

Let $\mathrm{Rew}(D_1,k) = p$, for some $1 \le p \le |x_k|$ and let $\mathrm{Prod}(D_1,k) = (A,\alpha)$, for some $(A,\alpha) \in P_1$. Hence $x_k = uAv$ and $x_{k+1} = u\alpha v$, for some $u,v \in \Sigma_1^*$ such that $|uA| = p$. Then $y_{\delta(k)} = u'A'v'$, with $\varphi(u') = u$, $\varphi(A') = A$ and $\varphi(v') = v$.
Let (A',α') be a production in P_2 such that $\varphi(\alpha') = \alpha$ and such that $\mathrm{Locc}_{G_2}(\alpha') \le \mathrm{Rocc}_{G_2}(\alpha')$. Since D_1 is a successful derivation in G_1, (A,α) is a safe production and hence a production (A',α') as described exists. Note that $A' \in Y_2$.

We distinguish the following cases.

(1). $u' \in X_2^*$ and $v' \in Z_2^*$.
In this case $y_{\delta(k)} = u'A'v' \underset{G_2}{\Rightarrow} u'\alpha'v'$. We set $y_{\delta(k)+1} = u'\alpha'v'$ and $\delta(k+1) = \delta(k) + 1$. Then $\varphi(y_{\delta(k+1)}) = \varphi(u'\alpha'v') = u\alpha v = x_{k+1}$ and $\mathrm{Locc}_{G_2}(y_{\delta(k+1)}) \le \mathrm{Rocc}_{G_2}(y_{\delta(k+1)})$.
$(y_0,\ldots,y_{\delta(k)}, y_{\delta(k+1)})$ is a derivation in G_2.

(2). $u' \notin X_2^*$ or $v' \notin Z_2^*$.
(2.1). $u' \notin X_2^*$.
Since $\varphi(u') = u$ and $u \in X_1^*$, u' contains at least one symbol from Γ_Z.

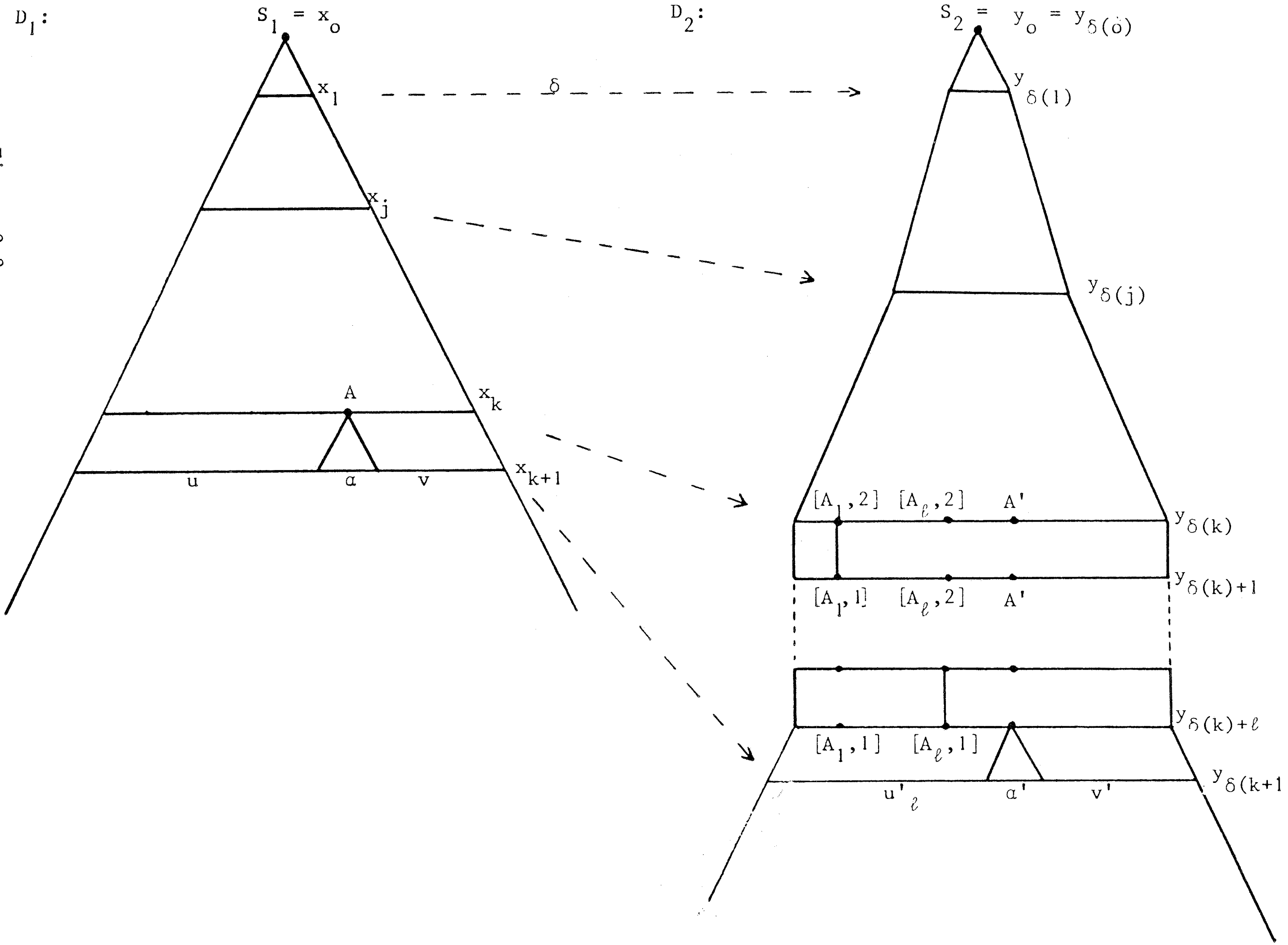

D1:
S1 = xo
x1
xj
A
xk
xk+1
u
α
v
δ
D2:
S2 = yo = yδ(o)
yδ(1)
yδ(j)
[A1,2] [Aℓ,2] A'
yδ(k)
[A1,1] [Aℓ,2] A'
yδ(k)+1
yδ(k)+ℓ
[A1,1] [Aℓ,1]
yδ(k+1)
u'ℓ
α'
v'

Figure 2.2

Since $\mathrm{Locc}_{G_2}(y_{\delta(k)}) \leq \mathrm{Rocc}_{G_2}(y_{\delta(k)})$, it follows that $A' \in Z_2$ and, since $\varphi(v') = v$ with $v \in Z_1^*$, $v' \in Z_2^*$. Moreover, $u = u_1[A_1,2]u_2[A_2,2]\ldots u_\ell[A_\ell,2]u_{\ell+1}$, for some $u_1 \in X_1^*$, $u_2,\ldots,u_{\ell+1} \in (\Delta_1 \cap (X_1 \cap Z_1))^*$ and $[A_i,2] \in \Gamma_Z$, for $1 \leq i \leq \ell$.

Then we define u_j', $1 \leq j \leq \ell$ as follows

$u_j' = u_1[A_1,1]\, u_2\ldots u_j[A_j,1]\, u_{j+1}[A_{j+1},2]\, u_{j+2}\ldots u_\ell[A_\ell,2]u_{\ell+1}$.

Let $y_{\delta(k)+j} = u_j'A'v'$, for $1 \leq j \leq \ell$.

Then $y_{\delta(k)+j-1} \underset{G_2}{\Rightarrow} y_{\delta(k)+j}$, for $1 \leq j \leq \ell$. Moreover, $y_{\delta(k)+\ell} = u_\ell'A'v'$ and $y_{\delta(k)+\ell} \underset{G_2}{\Rightarrow} u_\ell'\alpha'v'$, where $\varphi(u_\ell') = u$.

We set $\delta(k+1) = \delta(k) + \ell + 1$ and $y_{\delta(k+1)} = u_\ell'\alpha'v'$. Hence $\varphi(y_{\delta(k+1)}) = x_{k+1})$ and $\mathrm{Locc}_{G_2}(y_{\delta(k+1)}) \leq \mathrm{Rocc}_{G_2}(y_{\delta(k+1)}).(y_0,\ldots,y_{\delta(k)}, y_{\delta(k)+1},\ldots,y_{\delta(k)+\ell},y_{\delta(k+1)})$ is a derivation in G_2. This reasoning is illustrated in Figure 2.2.

(2.ii) The case that $v' \notin Z_2^*$ is symmetric to (2.i) and can be treated analogously.

Now, let $m = \delta(n)$. Then $\varphi(y_m) = x_n$ and $(y_0,y_1,\ldots,y_m)$ is a successful derivation in G_2.

Hence $L(G_1) \subseteq L(G_2)$.

This implies that $L(G_2) = L(G_1)$ and that G_1 satisfies (C1).

We conclude that G_2 satisfies (C1),(C2) and (C3), and $L(G_2) = L(G_1)$.

(III). Let $G_3 = (\Sigma_3,P_3,S_3,\Delta_3,X_3^*\bar{Y}_3Z_3^*)$ be defined in the following way. $\Sigma_3 = \{[A,T] : A \in \Sigma_2\backslash\Delta_2 \text{ and } T \subseteq \Delta_2\} \cup \{S_3\} \cup \Delta_3$, where $\Delta_3 = \Delta_2$ and S_3 is a new symbol.

The homomorphism $\rho \in \mathrm{HOM}(\Sigma_3\backslash\{S_3\},\Sigma_2)$ is defined by

$\rho(a) = a$, for all $a \in \Delta_3$, and

$\rho([A,T]) = A$, for all $A \in \Sigma_2\backslash\Delta_2$ and $T \subseteq \Delta_2$.

The finite substitution $\tau \in \mathrm{FSUB}(\Sigma_3\backslash\{S_3\},\Delta_2)$ is defined by

$\tau(a) = a$, for all $a \in \Delta_3$, and

$\tau([A,T]) = T$, for all $A \in \Sigma_2\backslash\Delta_2$ and $T \subseteq \Delta_2$.

$P_3 = \{(S_3,[S_2,T]) : T \subseteq \Delta_2\} \cup \{(A,w) : A \in \Sigma_3\setminus\Delta_3,\ w \in \Sigma_3^*,\ (\rho(A),\rho(w)) \in P_2$ and $\tau(A) = \underline{alph}(\tau(w))\}$.

$X_3 = \rho^{-1}(X_2)$,

$Y_3 = \rho^{-1}(Y_2) \cup \{S_3\}$ and

$Z_3 = \rho^{-1}(Z_2)$.

Note that, for $[A,T]$ with $A \in \Sigma_2\setminus\Delta_2$ and $T \subseteq \Delta_2$, T "promises" what terminal symbols will be contributed by $[A,T]$.

Firstly we prove that $L(G_3) = L(G_2)$.

Let $(S_3,x_1,\ldots,x_n)$ be a successful derivation in G_3. Then $\rho(x_1) = S_2$ and $\rho(x_n) = x_n$. From the construction of G_3 it follows immediately that $(\rho(x_1),\ldots,\rho(x_n))$ is a derivation in G_2.

Thus $L(G_3) \subseteq L(G_2)$.

To prove the converse inclusion we proceed as follows.

Let $D = (x_0,x_1,\ldots,x_n)$ be a successful derivation in G_2.

Then we set $\hat{x}_n = x_n$. Hence $\rho(\hat{x}_n) = x_n$.

Assume that for some $k \in \{1,\ldots,n\}$, $\hat{x}_k,\ldots,\hat{x}_n$ have been defined in such a way that $\rho(\hat{x}_j) = x_j, k \leq j \leq n$, and $\hat{x}_j \underset{G_3}{\Rightarrow} \hat{x}_{j+1}$, $k \leq j \leq n-1$.

Let $\mathrm{Prod}(D,k-1) = (A,\alpha)$ for some $(A,\alpha) \in P_2$. Hence $\hat{x}_k = \hat{u}\hat{\alpha}\hat{v}$, for some $\hat{u},\hat{\alpha},\hat{v} \in \Sigma_3^*$ such that $\rho(\hat{\alpha}) = \alpha$ and $\mathrm{Rew}(D,k-1)=|\hat{u}| + 1$. Let $T = \underline{alph}(\tau(\hat{\alpha}))$.

Then $([A,T],\hat{\alpha}) \in P_3$.

Let $\hat{x}_{k-1} = \hat{u}[A,T]\hat{v}$. Hence $\rho(\hat{x}_{k-1}) = x_{k-1}$ and $x_{k-1} \underset{G_3}{\Rightarrow} x_k$.

Then $(\hat{S}_3,\hat{x}_0,\hat{x}_1,\ldots,\hat{x}_n)$ is a derivation in G_3.

Thus $L(G_3) \supseteq L(G_2)$

Together with the above we have $L(G_3) = L(G_2)$.

From the construction of P_3 it follows that $S_3 \notin \underline{alph}(\underline{rhs}(P_3))$.

Moreover, clearly for every $B \in \Sigma_3$, where $B = [A,T]$, for some $A \in \Sigma_2\setminus\Delta_2$ and $T \subseteq \Delta_2$, $\underline{contr}_{G_3}(B) \subseteq \{w \in T^* : \underline{alph}(w) = T\}$.

Hence G_3 satisfies (C4).

That G_2 also satisfies (C1), (C2) and (C3) can be shown using the following observations.

(1). $X_3 \cap \Delta_3 = X_2 \cap \Delta_2$, $Z_3 \cap \Delta_3 = Z_2 \cap \Delta_2$, $L(G_2) = L(G_3)$ and G_2 satisfies (C1). Hence (C1) holds for G_3.

(2). From the construction of P_3 from P_2 and arguments similar to the above, it follows that G_3 satisfies (C2).

(3). $X_3 \cap Z_3 = \rho^{-1}(X_2) \cap \rho^{-1}(Z_2) = \rho^{-1}(X_2 \cap Z_2) \subseteq \Delta_2$ and hence (C3) holds for G_3.

We conclude that G_3 satisfies (C1), (C2), (C3) and (C4), and $L(G_3) = L(G_2)$.

(IV) Let $G_4 = (\Sigma_4, P_4, S_4, \Delta_4, X_4^* Y_4 Z_4^*)$ be defined in the following way. $\Sigma_4 = \Sigma_3$, $S_4 = S_3$, $\Delta_4 = \Delta_3$, $X_4 = X_3$, $Y_4 = Y_3$, $Z_4 = Z_3$ and $P_4 = P_3 \setminus \{(C, \alpha_1 A \alpha_2) : A \in \theta,\ \underline{\mathrm{contr}}_{G_3}(\alpha_1) \nsubseteq X_3^*$ or $\underline{\mathrm{contr}}_{G_3}(\alpha_2) \nsubseteq Z_3^*\}$, where $\theta = \{A : A \in \Sigma_3 \setminus (\Delta_3 \cup \{S_3\})$ and $\underline{\mathrm{contr}}_{G_3}(A) \subseteq \Delta_3^* B_{G_3} \Delta_3^*\}$.

Since G_4 results from G_3 by removing productions from P_3 we have $L(G_4) \subseteq L(G_3)$.

We will prove the converse inclusion by showing that no production from $P_3 \setminus P_4$ can be used in a successful derivation in G_3.

Let x and y be sentential forms of G_3 such that $x \underset{G_3}{\Rightarrow} y$ using a production from $P_3 \setminus P_4$. Hence $x = uCv$ and $y = u\alpha_1 A \alpha_2 v$, for some $u, v \in \Sigma_3^*$, $C, A \in \theta$ and $(C, \alpha_1 A \alpha_2) \in P_3 \setminus P_4$.

Then every $w \in \underline{\mathrm{contr}}_{G_3}(y)$ is of the form $u_1 \alpha_1' \beta_1 b \beta_2 \alpha_2' v_1$, where $u_1 \in \underline{\mathrm{contr}}_{G_3}(u)$, $\alpha_i' \in \underline{\mathrm{contr}}_{G_3}(\alpha_i)$, $i = 1,2$, $\beta_1 b \beta_2 \in \underline{\mathrm{contr}}_{G_3}(A)$, where b is a blocking terminal and $v_1 \in \underline{\mathrm{contr}}_{G_3}(v)$.

Observe that, since G_3 satisfies (C4), for all $A \in \Sigma_3 \setminus \Delta_3$ and $T \in \Delta_3$, whenever $\alpha \notin T^*$, for some $\alpha \in \underline{\mathrm{contr}}_{G_3}(A)$, then, for all $\beta \in \underline{\mathrm{contr}}_{G_3}(A)$, $\beta \notin T^*$.

Hence $\alpha_1' \notin X_3^*$ or $\alpha_2' \notin Z_3^*$ and so w cannot be in $L(G_3)$ which is a subset of $(X_3 \cap \Delta_3)^* B_{G_3} (Z_3 \cap \Delta_3)^*$.

Thus no derivation in G_3 that uses a production from $P_3 \setminus P_4$ is successful.

Hence $L(G_3) \subseteq L(G_4)$.

G_3 satisfies (C1) through (C4) and G_4 results from G_3 by removing the productions that violate (C5) - hence G_4 satisfies (C1) through (C5).

We conclude that G_4 satisfies (C1), (C2), (C3), (C4) and (C5), and $L(G_4) = L(G_3)$.

(V) Let G be the strongly reduced version of G_4. Then G and the homomorphism h (defined in (I)) satisfy the statement of the theorem because, by Lemma 2.1, $L(G) = L(G_4)$ and hence $L(G) = L(G_1)$, and $L(G_1) = h(L(G))$. G is strongly reduced.

G_4 satisfies (C1) through (C5) and since G results from G_4 by removing useless symbols and productions, G also satisfies (C1) through (C5).

We conclude that G is special and $h(L(G)) = L(H)$. □

3. CENTRAL DERIVATIONS IN SPECIAL 1S GRAMMARS

In this section we concentrate on (successful) derivations in special 1S grammars. In particular we will show that a "normal form" (a fixed strategy) for successful derivations exists.

We need the following terminology and notation.

Let $G = (\Sigma,P,S,\Delta,X^*\bar{Y}Z^*)$ be a special 1S grammar.

If $A \in \Sigma\backslash\Delta$ is such that $\underline{\mathrm{alph}}(\underline{\mathrm{contr}}_G(A)) \cap B_G \neq \emptyset$ then we call A a *central symbol* (*of* G). By C(G) we denote the set of central symbols of G.

Let u be a sentential form of G. Then u contains at most one occurrence of a central symbol. If u contains exactly one occurrence of a central symbol, then we say that u is a *central sentential form* (*of* G).

If u is a central sentential form of G then we denote by $\underline{\mathrm{cent}}_G(u)$ the unique central symbol in u and by $\mathrm{Occ}_G(u)$ its position in u. Hence if $u = u_1Au_2$, for some $u_1,u_2 \in (\Sigma\backslash C(G))^*$ and $A \in C(G)$, then $\underline{\mathrm{cent}}_G(u) = A$ and $\mathrm{Occ}_G(u) = |u_1A|$.

Whenever G is clear from the context we omit the subscript G.

Since G is special all non-terminals of G are either in L_G or in R_G (or in both). Hence to rewrite a successful sentential form u one has at most two options:

Either one rewrites the right-most left-blocking non-terminal in u (if any) or one rewrites the left-most right-blocking non-terminal in u (if any).

Notice that since either $\mathrm{Locc}(u) \geq 1$ or $\mathrm{Rocc}(u) \leq |u|$, at least one of these situations occurs and that it may be that $\mathrm{Locc}(u) = \mathrm{Rocc}(u)$ (a blocking symbol).

We will describe a strategy, how to make a choice between Locc(u) and Rocc(u), in the case one has two options. This strategy depends on the central symbol in the current sentential form.

Observe that whenever u is a non-central sentential form of G, then there exists no $v \in \Sigma^*$ such that $u \underset{G}{\Rightarrow} v$ (u is *blocked*).

Definition 3.1. (1). Let u be a central sentential form of G. Then the designated occurrence in u, denoted $Des_G(u)$, is defined by

$$Des_G(u) = \begin{cases} Locc_G(u), & \text{if } Rocc_G(u) > |u|, \\ Locc_G(u), & \text{if } 1 \le Locc_G(u) \le Rocc_G(u) \le Occ_G(u) \le |u|, \\ Rocc_G(u), & \text{if } Locc_G(u) < 1, \\ Rocc_G(u), & \text{if } 1 \le Occ_G(u) \le Locc_G(u) \le Rocc_G(u) \le |u|. \end{cases}$$

(2). Let $D = (x_0, x_1, \ldots, x_n)$, $n \ge 0$, be a derivation in G. For $0 \le i \le n-1$, the derivation step (x_i, x_{i+1}) is central if $Rew(D,i) = Des_G(x_i)$. D is a central derivation (in G) if, for all $0 \le i \le n-1$, (x_i, x_{i+1}) is central. □

We will write Des(u) rather than $Des_G(u)$, whenever G is clear from the context.

Hence in a central derivation step the designated occurrence is rewitten, i.e.

(1). the right-most left-blocking non-terminal if

(1.i) no right-blocking non-terminals are present, or

(1.ii) the central symbol is right-blocking;

(2). the left-most right-blocking non-terminal if

(2.i) no left-blocking non-terminals are present, or

(2.ii) the central symbol is left-blocking.

Observe that if the sentential form contains a blocking non-terminal, then the right-most left-blocking and left-most right-blocking non-terminals are the same.

That the central derivations are derivations in a "normal form" and that one can consider central derivations only in special 1S grammars is proved as follows.

Theorem 3.1. Let G be a special 1S grammar. For every $w \in L(G)$ there exists a central derivation of w in G.

Proof. Let $G = (\Sigma, P, S, \Delta, X^*\bar{Y}Z^*)$. For a derivation D in G, $v(D)$ denotes the number of non-central derivation steps in D.

Let $w \in L(G)$ and let D_w be a derivation of w such that $v(D_w) = \min\{v(D) : D$ is a derivation of w in $G\}$. We will prove by contradiction that $v(D_w) = 0$. Hence, we assume that $v(D_w) > 0$.

Let $D_w = (x_0, x_1, \ldots, x_n)$, where $x_i \in \Sigma^*$ for $1 \le i \le n$, $x_0 = S$ and $x_n = w$. Let $i_0 \in \{1, \ldots, n-1\}$ be the smallest i such that the derivation step (x_i, x_{i+1}) in D is non-central.

If $\underline{\text{cent}}(x_{i_0}) \in B_G$, then $\text{cent}(x_{i_0})$ is rewritten in (x_{i_0}, x_{i_0+1}). This would imply that (x_{i_0}, x_{i_0+1}) is central. A contradiction. Hence $\underline{\text{cent}}(x_{i_0}) \in X \cup Z$ and since G is special, this implies that $\underline{\text{cent}}(x_{i_0}) \in (X \setminus Z) \cup (Z \setminus X)$.

Let us assume that $\underline{\text{cent}}(x_{i_0}) \in X \setminus Z$ (the other case is symmetric and can be dealt with analogously). Since $\underline{\text{cent}}(x_{i_0}) \in L_G \setminus R_G$ and (x_{i_0}, x_{i_0+1}) is non-central, it follows that $\text{Rocc}(x_{i_0}) \le |x_{i_0}|$, $\text{Locc}(x_{i_0}) \ne \text{Rocc}(x_{i_0})$ and $\text{Rew}(D, i_0) = \text{Locc}(x_{i_0})$. Let $r = \text{Rocc}(x_{i_0})$ and $k = |x_{i_0}|$; hence $1 < r \le k$.

Let $\alpha_0 = a_1 \ldots a_{r-1}$ and $\beta_0 = a_r \ldots a_k$, where $a_i \in \Sigma$, for $1 \le i \le k$, and such that $x_{i_0} = \alpha_0 \beta_0$.

Hence $D_w = (x_0, x_1, \ldots, x_{i_0-1}, \alpha_0\beta_0, \alpha_1\beta_1, \ldots, \alpha_q\beta_q)$, where $q = n - i_0$ and, for $0 \le i \le q$, $\alpha_i\beta_i = x_{i_0+i}$ and for $0 \le i \le q-1$:

(1) if $\text{Rew}(D_w, i_0+i) > |\alpha_i|$, then $\alpha_i = \alpha_{i+1}$ and $\beta_i \underset{G}{\Rightarrow} \beta_{i+1}$, and

(2) if $\text{Rew}(D_w, i_0+i) \le |\alpha_i|$, then $\alpha_i \underset{G}{\Rightarrow} \alpha_{i+1}$ and $\beta_i = \beta_{i+1}$.

Let $I = \{i : 0 \le i \le q-1$ and $\text{Rew}(D_w, i_0+i) > |\alpha_i|\}$ and $J = \{i : 0 \le i \le q-1$ and $\text{Rew}(D_w, i_0+i) \le |\alpha_i|\}$. Note that $I \cap J = \emptyset$.

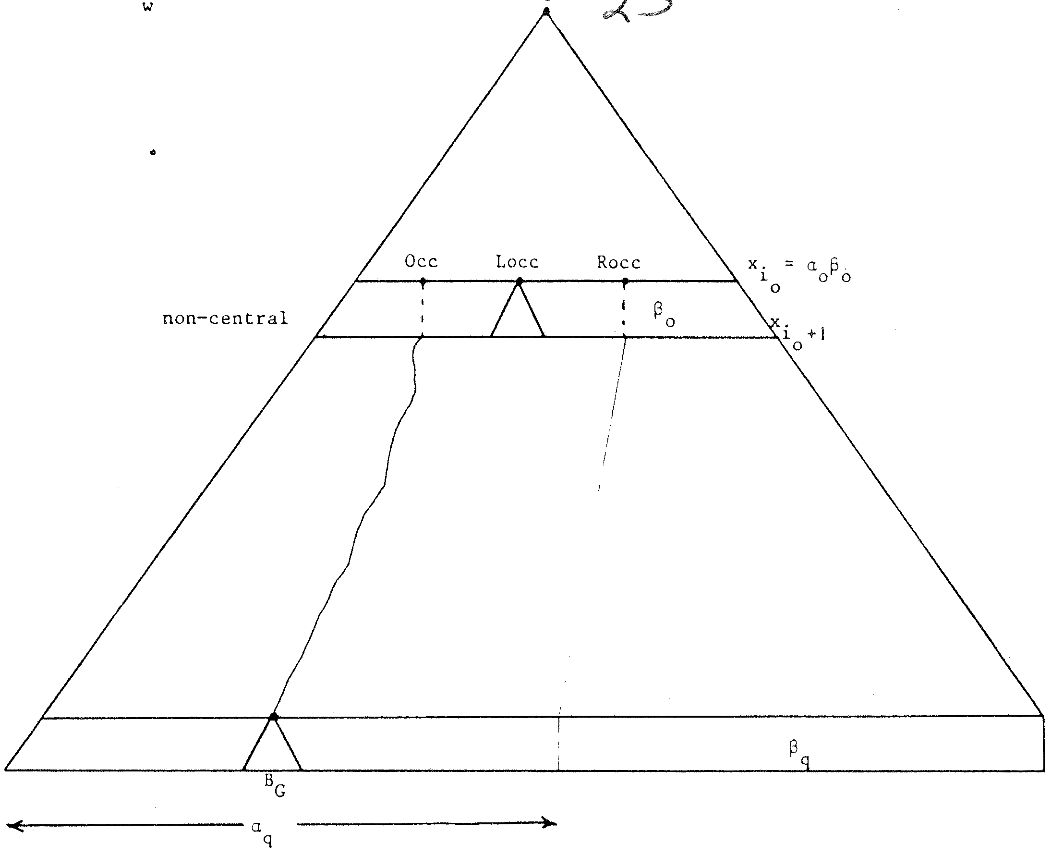
Let $i_1, \ldots, i_p$ be the elements of I in ascending order and let $j_1, \ldots, j_s$ be the elements of J in ascending order. Hence $p, s \ge 1$ and $p+s = q$, $\alpha_{j_1} = \alpha_0$, $\beta_{i_1} = \beta_0$, $\alpha_{j_i} \Rightarrow \alpha_{j_i+1}$, for $1 \le i \le s-1$, $\alpha_{j_s} \Rightarrow \alpha_q$ and $\beta_{i_j} \Rightarrow \beta_{i_{j+1}}$, for $0 \le j \le p-1$ and $\beta_{i_p} \Rightarrow \beta_q$.

We set $D' = (x_0, x_1, \ldots, x_{i_0-1}, \alpha_0\beta_{i_1}, \ldots, \alpha_0\beta_{i_p}, \alpha_0\beta_q, \alpha_{j_2}\beta_q, \ldots, \alpha_{j_s}\beta_q, \alpha_q\beta_q\}$

We claim that D' is a derivation of w in G and $v(D') \le v(D_w) - 1$.

The situation can be illustrated as follows.

D_w:



D':

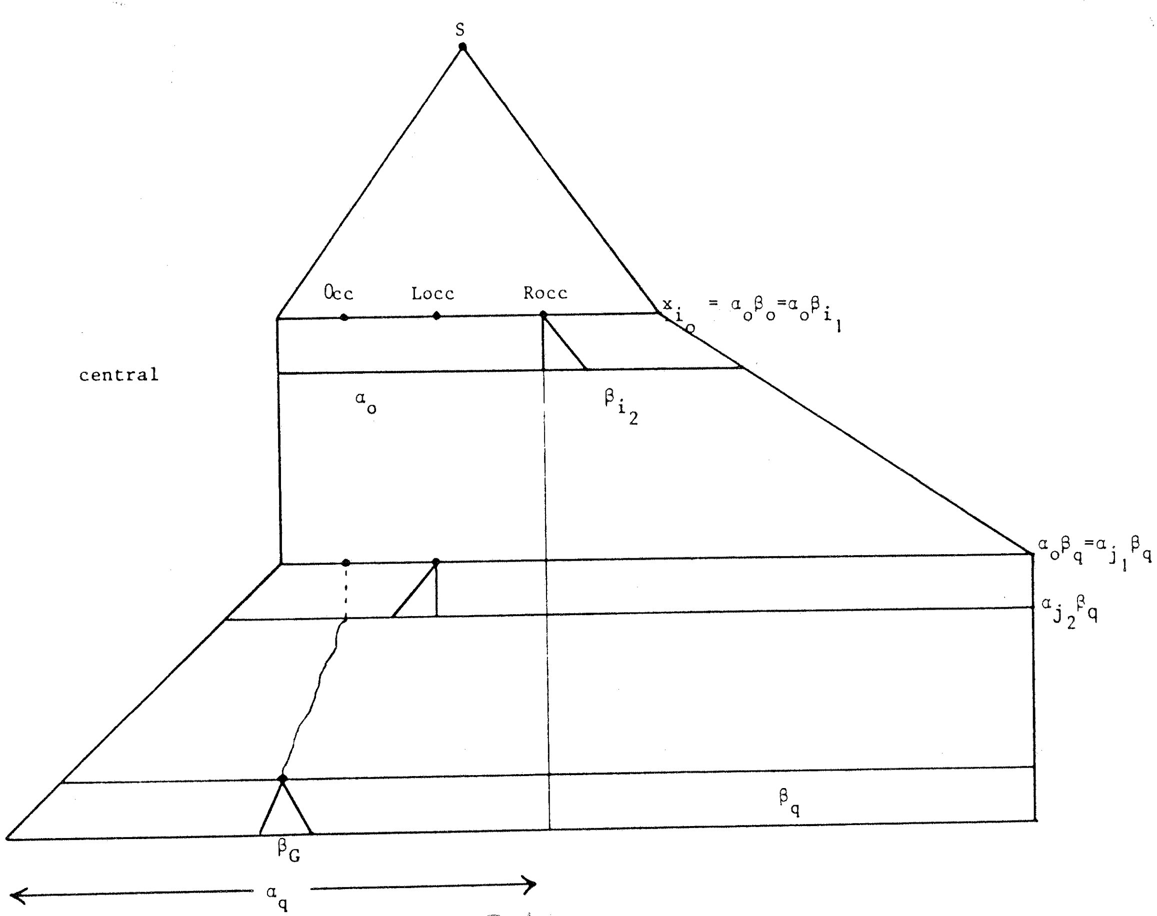


Figure 3.1.

First we recall that $\alpha_q\beta_q = w$.

Next we consider D_w.

$Rocc(x_{i_0}) = Rocc(\alpha_0\beta_0) = |\alpha_0| + 1$. Hence $\alpha_0 \in X^*$. This implies that, for $0 \le j \le p-1$, $\alpha_0\beta_{i_j} \Rightarrow \alpha_0\beta_{i_{j+1}}$ and $\alpha_0\beta_{i_p} \Rightarrow \alpha_0\beta_q$. Recall furthermore that $Occ(\alpha_0\beta_0) \le Locc(\alpha_0\beta_0) \le |\alpha_0|$. Since G is special this implies that $\underline{contr}_G(\beta_0) \in Z^*$ and in particular that $\beta_q \in Z^*$. Hence, for $0 \le i \le s-1$, $\alpha_{j_i}\beta_q \Rightarrow \alpha_{j_{i+1}}\beta_q$ and $\alpha_{j_s}\beta_q \Rightarrow \alpha_q\beta_q$.

This proves that D' is a derivation of w in G.

It remains to show that $v(D') \le v(D_w)-1$.

(1) Consider the step $(\alpha_0\beta_{i_1},\alpha_0\beta_{i_2})$ in D'. Since $\alpha_0\beta_{i_1} = \alpha_0\beta_0$ and $Rocc(\alpha_0\beta_0) \ge |\alpha_0| + 1$, all non-terminals in β_0 are right-blocking. Hence it is the first non-terminal symbol of β_0 that is rewritten in this step. Since $Occ(\alpha_0\beta_0) \le Locc(\alpha_0\beta_0) \le |\alpha_0|$, it follows that this step in D' is central.

(2) Consider a step $(\alpha_0\beta_{i_j},\alpha_0\beta_{i_{j+1}})$ in D', for some $2 \le j \le p$ where $\beta_{i_{p+1}} = \beta_q$, and assume that this step is non-central. We will show that the step $(\alpha_{i_j}\beta_{i_j}, \alpha_{i_j+1}\beta_{i_j+1})$ in D_w is also non-central.
Since $\underline{cent}(\alpha_0\beta_{i_j}) \in L_G$, it follows that $Occ(\alpha_0\beta_{i_j}) \le |\alpha_0|$. Since the rewriting takes place in β_{i_j} and the step under consideration is non-central, it follows that $Locc(\alpha_0\beta_{i_j}) > |\alpha_0|$, $Rocc(\alpha_0\beta_{i_j}) \le |\alpha_0\beta_{i_j}|$ and the right-most left-blocking symbol is rewritten. Hence $Locc(\alpha_{i_j}\beta_{i_j}) > |\alpha_{i_j}|$, $Rocc(\alpha_{i_j}\beta_{i_j}) \le |\alpha_{i_j}\beta_{i_j}|$ and in the step $(\alpha_{i_j}\beta_{i_j},\alpha_{i_j+1}\beta_{i_j+1})$ in D_w the right-most left-blocking symbol is rewritten.
This implies that this step is not central in D_w.

(3) Consider a step $(\alpha_{j_i}\beta_q,\alpha_{j_{i+1}}\beta_q)$ in D', for some $1 \le i \le s$, where $\alpha_{j_{s+1}} = \alpha_q$ and assume that this step is non-central. We will show that the step $(\alpha_{j_i}\beta_{j_i},\alpha_{j_{i+1}}\beta_{j_{i+1}})$ in D_w is also non-central.
Since $\beta_q \in \Delta^*$ it follows that if $\underline{cent}(\alpha_{j_i}\beta_q) \in L_G$ then $Rocc(\alpha_{j_i}\beta_q) \le |\alpha_{j_i}|$ and the right-most left-blocking symbol in α_{j_i} is rewritten, and if

$\underline{cent}(\alpha_{j_i}\beta_q) \in R_G$ then $Locc(\alpha_{j_i}\beta_q) \geq 1$ and the left-most right-blocking symbol in α_{j_i} is rewritten in $(\alpha_{j_i}\beta_q, \alpha_{j_{i+1}}\beta_q)$.

Then $(\alpha_{j_i}\beta_{j_i}, \alpha_{j_{i+1}}\beta_{j_{i+1}})$ is non-central, because $Rew(D_w, j_i) = Rew(D', i_0+p+i)$.

(1), (2) and (3) together imply that the number of non-central steps in the first (i_0+1) derivation steps of D_w and D' are 1 and 0 respectively; in the last $n-i_0-1$ steps each non-central step in D' corresponds to a distinct non-central step in D_w. Hence $v(D') \leq v(D_w)-1$, which contradicts the minimality of $v(D_w)$.

Thus for every $w \in L(G)$ there exists a central derivation of w in $L(G)$ which proves the theorem. $\square$

4. SPECIAL 1S GRAMMARS GENERATE CONTEXT-FREE LANGUAGES

In this section we consider an arbitrary but fixed, special 1S grammar $G = (\Sigma,P,S,\Delta,X^*\bar{Y}Z^*)$ and its central derivations. We will show that only a "finite memory" is needed to prevent unsuccessfulness (blocking) of central derivations. By building in this finite memory an equivalent special 1S grammar is constructed from G, in which all central derivations are successful. Then it can be shown that L(G) is context-free.

First we identify the situations in which a central derivation gets blocked.

Let $(A,\alpha) \in P$, for some $A \in \Sigma \setminus \Delta$ and $\alpha \in \Sigma^*$.
(A,α) is a *left-blocking production* if $\mathrm{alph}(\alpha) \cap (\Delta\setminus Z) \neq \emptyset$,
(A,α) is a *right-blocking production* if $\mathrm{alph}(\alpha) \cap (\Delta\setminus X) \neq \emptyset$ and
(A,α) is a *blocking production* if it is both left- and right-blocking.

Observe that whenever a left-(right-) blocking production is used to rewrite a sentential from, then the non-terminals that occur to the left (right) of the rewritten occurrence cannot be rewritten anymore.
Hence in order not to block a central derivation, whenever a left-(right-) blocking production is applied to a sentential form w, the left (right) description of w should be terminal. We define two functions on the set of central sentential forms of G, that describe the left and the right context of the designated occurrences in sentential forms. These functions $\underline{\ell d}$ and $\underline{rd}$ respectively have as their codomain the set $\{\underline{t},\underline{nt}\}$, where $\underline{t}$ stands for terminal and $\underline{nt}$ for non-terminal.

Let w be a central sentential form of G. Let u, $v \in \Sigma^*$ and $A \in \Sigma \setminus \Delta$ be such that $w = uAv$ and $\mathrm{Des}(w) = |uA|$. Then

$$\underline{\ell d}(w) = \begin{cases} \underline{t} & \text{if } u \in \Delta^*, \\ \underline{nt} & \text{otherwise}, \end{cases}$$

$$\underline{rd}(w) = \begin{cases} \underline{t} & \text{if } v \in \Delta^*, \\ \underline{nt} & \text{otherwise}. \end{cases}$$

Let $D = (x_0, x_1, \ldots, x_n)$ be a central derivation in G. For $0 \leq k \leq n-1$, we call (x_k, x_{k+1}) careful if

whenever $\mathrm{Prod}(D,k)$ is left-blocking, then $\underline{\ell d}(x_k) = \underline{t}$ and

whenever $\mathrm{Prod}(D,k)$ is right-blocking, then $\underline{rd}(x_k) = \underline{t}$.

D is careful if, for all $0 \leq k \leq n-1$, (x_k, x_{k+1}) is careful.

Lemma 4.1. Let $D = (x_0, x_1, \ldots, x_n)$ be a central derivation. If $x_n \in \Delta^*$, then D is careful.

Proof. Obvious. □

Hence if a central derivation is not careful it will never "lead to success".

On the other hand all central derivations that get blocked (in a non-trivial way), are not careful. This is shown in the following lemma.

Lemma 4.2. Let $D = (x_0, x_1, \ldots, x_n)$, $n \geq 0$, be a central derivation in G, such that $x_n \notin \Delta^*$. If there exists no y such that $(x_0, x_1, \ldots, x_n, y)$ is a central derivation then either

(1) $n = 0$ and $P = \emptyset$, or

(2) $n \geq 1$ and D is not careful.

Proof. (1) $n = 0$.

In this case $D = (S)$. G is strongly reduced. Hence, if $P \neq \emptyset$, then $P \cap (S \times \Sigma^*) \neq \emptyset$. Since $S \in Y$, this implies that $S \underset{G}{\Rightarrow} \alpha$, for some $\alpha \in \Sigma^*$ and $(S, \alpha) \in P$. Then, obviously, (S, α) is a central derivation. This is a contradiction and so $P = \emptyset$.

(2) $n \geq 1$.

Hence $P \neq \emptyset$ and for all non-terminal symbols productions are available. We distinguish the following two cases.

(2.i) There exists no y such that $x_n \underset{G}{\Rightarrow} y$.

Consider the central derivation step (x_{n-1},x_n) in D. Let $x_{n-1} = xAz$, with $|xA| = \text{Des}\ (x_{n-1})$, $x_n = x\alpha z$, for some $x \in X^*$, $z \in Z^*$ and $(A,\alpha) \in P$. (A,α) is a safe production. Hence if $\alpha \notin \Delta^*$, then $\alpha \in X^*YZ^*$ and so $x_n \in X^*YZ^*$. This implies that there exists a $y \in \Sigma^*$ such that $x_n \underset{G}{\Rightarrow} y$. This contradicts our assumption.
Hence $\alpha \in \Delta^*$. If $A \notin C(G)$, then either $\text{Occ}(x_{n-1}) \leq |x|$ or $\text{Occ}(x_{n-1}) > |xA|$.
If $\text{Occ}(x_{n-1}) \leq |x|$, then $\alpha \in Z^*$(condition (C5) from Definition 2.5). Moreover, $x = uBv$, for some $u \in x^*$, $B \in \Sigma\backslash\Delta$ and $v \in (X \cap \Delta)^*$. Since $\text{Occ}(x_{n-1}) \leq |uB|$, $v \in (Z \cap X \cap \Delta)^*$. Hence $x_n = x\alpha z = uBv\alpha z$, with $u \in x^*$ and $v\alpha z \in Z^*$. This implies that there exists a $y \in \Sigma^*$ such that $x_n \underset{G}{\Rightarrow} y$. This contradicts our assumption.
Similarly one can prove that the case $\text{Occ}(x_{n-1}) > |xA|$ leads to a contradiction.
Hence $A \in C(G)$. This implies that $\alpha \in (B_G \cap \Delta)$ and (A,α) is a blocking production.
Since $x_n \notin \Delta^*$, either $\underline{\ell d}(x_{n-1}) = \underline{nt}$ or $\underline{rd}(x_{n-1}) = \underline{nt}$. Hence (x_{n-1},x_n) is not a careful derivation step and so D is not careful.

(2.ii) There exists a y such that $x_n \underset{G}{\Rightarrow} y$.
Hence x_n is a central sentential form. Since x_n can not be rewritten in the central way, x_n contains a left-blocking non-terminal and a right-blocking non-terminal, which are not the same. (Otherwise there is only one way of rewriting x, which then is automatically the central way.) Hence $1 \leq \text{Locc}(x_n) < \text{Rocc}(x_n) \leq |x_n|$ and either
(I) $\text{Des}\ (x_n) = \text{Locc}(x_n)$ and $\underline{\text{cent}}(x_n) \in R_G$ or
(II) $\text{Des}\ (x_n) = \text{Rocc}(x_n)$ and $\underline{\text{cent}}(x_n) \in L_G$.
We consider case(I).
In this case $x_n = \alpha A\beta B\gamma$, with $\alpha,\gamma \in \Sigma^*$, $\beta \in \Delta^*$ and $A,B \in \Sigma\backslash\Delta$, and $\text{Locc}(x_n) = |\alpha A|$ and $\text{Rocc}(x_n) = |\alpha A\beta B|$. Since x_n can be rewritten, but not in the central way, $\alpha A\beta \in X^*$ and $\gamma \in Z^*$. Moreover, $B \in Z\backslash X$ and so $\beta \in X^*\backslash Z^*$, i.e. $\beta = \beta_1 b\beta_2$, for some $\beta_1 \in \Delta^*, \beta_2 \in (\Delta\cap Z)^*$ and $b \in \Delta\backslash Z$. Observe, that since all productions are safe, this occurrence of b is introduced using a production of the form (A',ubv) where $u \in \Delta^*$ and $v \in (\Sigma\backslash\{b\})^*$.

Let $j = \max\{i : 0 \le i \le n-1,\ Rew(D,i) \le |\alpha A\beta_1 b|\}$. Hence $x_j = \alpha A\beta_3 A'\delta$ and $Prod(D,j) = (A', ubv)$, as above, where $\beta_3 u = \beta_1$ and $\delta \in \Sigma^*$.

Hence $Prod(D,j)$ is left-blocking and $\underline{\ell d}(x_j) = \underline{nt}$.

This implies that (x_j, x_{j+1}) is not careful.

Case II can be treated analogously. □

Consequently, central derivations can be prolonged in the central way as long as the designated occurrences "know" about their context and "act" accordingly.

We will show that only a finite memory is needed for this knowledge.

We need the following notions.

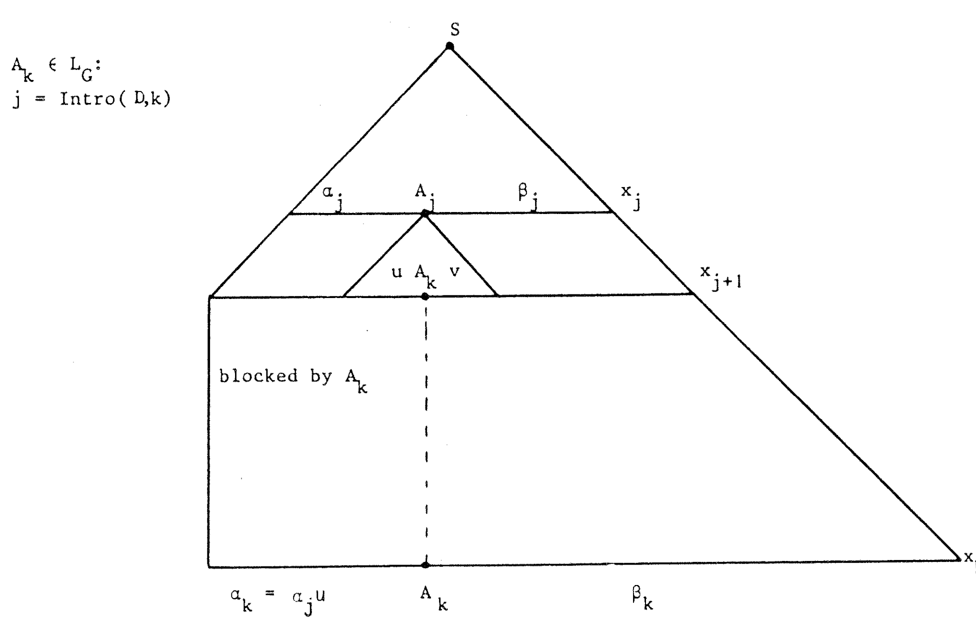
Let $D = (x_0, x_1, \ldots, x_n)$, $n \ge 0$, be a central derivation in G, such that x_n is a central sentential form. Let, for $1 \le k \le n$, $x_k = \alpha_k A_k \beta_k$, for some $\alpha_k, \beta_k \in \Sigma^*$ and $A_k \in \Sigma \setminus \Delta$, where $|\alpha_k A_k| = \underline{Des}(x_k)$.

Then, for $1 \le k \le n$,

$$Intro(D,k) = \begin{cases} \max\{j : 0 \le j \le k-1,\ |\alpha_j| \le |\alpha_k|\} & \text{if } A_k \in L_G, \\ \max\{j : 0 \le j \le k-1,\ |\beta_j| \le |\beta_k|\} & \text{if } A_k \in R_G. \end{cases}$$

Observe that $Intro(D,k)$ describes in which derivation step in D the designated occurrence in x_k has been introduced.

This can be illustrated as follows.



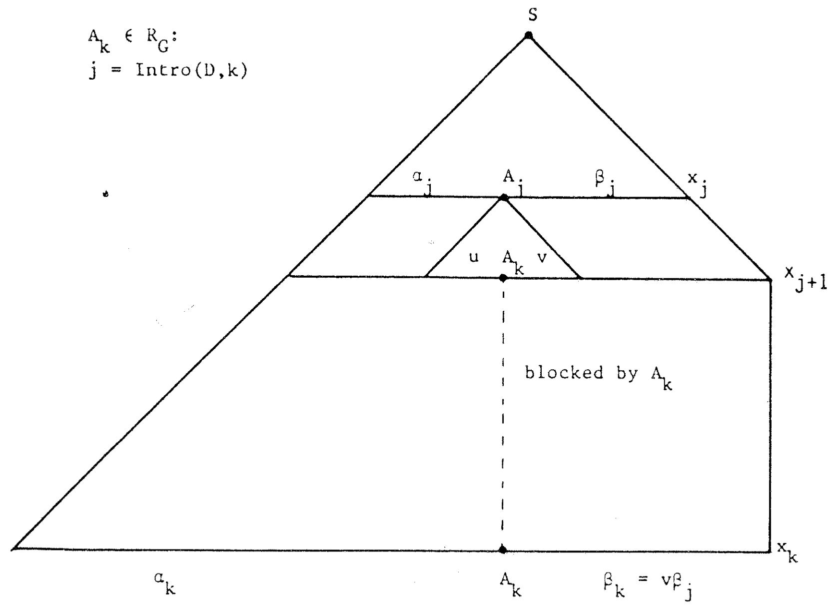


Figure 4.1.

Let k and j be such that $j = \underline{Intro}(D,k)$, $0 \le j < k \le n$. Let $u, v \in \Sigma^*$ be such that $Prod(D,j) = (A_j, uA_kv)$ and $\alpha_k = \alpha_j u$ if $A_k \in L_G$ and $\beta_k = v\beta_j$ if $A_k \in R_G$.

Then $\underline{prefintro}(D,k) = u$ and $\underline{postintro}(D,k) = v$.

Hence $\underline{prefintro}(D,k)$ and $\underline{postintro}(D,k)$ describe with what left and right, respectively, context the designated occurrence in x_k has been introduced.

If $Des(x_k) < Occ(x_k)$, for some $0 \le k \le n$, then $\underline{rd}(x_k) = \underline{nt}$ and, since G is special, the step (x_k, x_{k+1}) will be careful (Condition (C5) of Definition 2.5).

Analogously, if $Des(x_k) > Occ(x_k)$, then $\underline{\ell d}(x_k) = \underline{nt}$ and (x_k, x_{k+1}) is careful.

The following lemma shows that for the remaining situations it suffices to know about the introduction of $Des(x_k)$.

Lemma 4.3. There exist functions F_L and F_R, such that for every central derivation $D = (x_0, x_1, \ldots, x_n)$ in G, for every $i,j \in \{0,\ldots,n-1\}$ such that $i = Intro(D,j)$, $x_j = \alpha A \beta$, for some $\alpha,\beta \in \Sigma^*$ and $A \in \Sigma \setminus \Delta$, where $Des(x_j) = |\alpha A|$,

$\underline{\ell d}(x_j) = F_L(\underline{\ell d}(x_i), \underline{\text{prefintro}}(D,j),A)$ if $\text{Des}(x_j) \leq \text{Occ}(x_j)$, and

$\underline{rd}(x_j) = F_R(\underline{rd}(x_i), \underline{\text{postintro}}(D,j),A)$ if $\text{Des}(x_j) \geq \text{Occ}(x_j)$.

Proof. Let F_L and F_R be defined in the following way.

For $x \in \{\underline{t},\underline{nt}\}$, $w \in \Sigma^*$ and $E \in \Sigma\backslash\Delta$,

$$F_L(x,w,E) = \begin{cases} \underline{t} \text{ if } E \in R_G\backslash L_G, \\ \underline{t} \text{ if } E \in L_G,\ x = \underline{t} \text{ and } w \in \Delta^*, \\ \underline{nt} \text{ otherwise,} \end{cases}$$

and

$$F_R(x,w,E) = \begin{cases} \underline{t} \text{ if } E \in L_G\backslash R_G, \\ \underline{t} \text{ if } E \in R_G,\ x = \underline{t} \text{ and } w \in \Delta^*, \\ \underline{nt} \text{ otherwise.} \end{cases}$$

Let D,i,j and x_j be as in the statement of the lemma and let $\text{Prod}(D,i) = (A',uAv)$, for some $u,v \in \Sigma^*$ and $A' \in \Sigma\backslash\Delta$, such that $u = \underline{\text{prefintro}}(D,j)$ and $v = \underline{\text{postintro}}(D,j)$. Hence $x_i = \gamma A'\delta$ and $x_{i+1} = \gamma uAv\delta$ for some $\gamma,\delta \in \Sigma^*$.

Moreover

if $A \in L_G\backslash R_G$, then $\gamma u = \alpha$

if $A \in R_G\backslash L_G$, then $v\delta = \beta$, and

if $A \in L_G \cap R_G$, then $j = i+1$, $\gamma u = \alpha$ and $v\delta = \beta$.

If $\text{Des}(x_j) \leq \text{Occ}(x_j)$, then

if $A \in R_G\backslash L_G$, then $\text{Locc}(x_j) = 0$ and so $\alpha \in \Delta^*$.

If $\text{Des}(x_j) \geq \text{Occ}(x_j)$, then

if $A \in L_G\backslash R_G$, then $\text{Rocc}(x_j) = |x_j|+1$ and so $\beta \in \Delta^*$.

Hence if $\text{Des}(x_j) \leq \text{Occ}(x_j)$, then

$$\underline{\ell d}(x_j) = \begin{cases} \underline{t} \text{ if } A \in R_G\backslash L_G, \\ \underline{t} \text{ if } A \in L_G,\ \gamma \in \Delta^* \text{ and } u \in \Delta^*, \\ \underline{nt} \text{ otherwise,} \end{cases}$$

hence $\underline{\ell d}(x_j) = F_L(\underline{\ell d}(x_i), u, A)$.

Similarly, it follows that if $\text{Des}(x_j) \geq \text{Occ}(x_j)$, then $\underline{rd}(x_j) = F_R(\underline{rd}(x_i),v,A)$. □

From G we construct the 1S grammar $G_1 = (\Sigma_1, P_1, S_1, \Delta_1, X_1^* \; \bar{Y}_1 \; Z_1^*)$ in the following way.

$\Sigma_1 = \Delta \cup [\cdot,\Gamma] \cup [\cdot,\Gamma,\cdot] \cup [\Gamma,\cdot]$. Here $[\cdot,\Gamma], [\cdot,\Gamma,\cdot]$ and $[\Gamma,\cdot]$ are mutually disjoint alphabets also disjoint with Σ, defined as follows. For $x,y \in \{\underline{t},\underline{nt}\}$, $[x,\Gamma] = \{[x,A] : A \in (\Sigma\setminus\Delta) \setminus C(G)\}$, $[x,\Gamma,y] = \{[x,A,y] : A \in C(G)\}$ and $[\Gamma,y] = \{[A,y] : A \in (\Sigma\setminus\Delta) \setminus C(G)\}$ are mutually disjoint and $[\cdot,\Gamma] = [\underline{t},\Gamma] \cup [\underline{nt},\Gamma]$,

$$[\cdot,\Gamma,\cdot] = \bigcup_{x,y \in \{\underline{t},\underline{nt}\}} [x,\Gamma,y] \text{ and}$$

$[\Gamma,\cdot] = [\Gamma,\underline{t}] \cup [\Gamma,\underline{nt}]$.

$S_1 = [\underline{t},S,\underline{t}]$ and $\Delta_1 = \Delta$.

P_1 is defined in the following way. For $\pi = (A,\alpha) \in P$ with $A \notin C(G)$ and $\alpha = \alpha_1 A_1 \alpha_2 \ldots \alpha_n A_n \alpha_{n+1}$, for some $n \geq 0$, $\alpha_1,\ldots,\alpha_{n+1} \in \Delta^*$ and $A_1,\ldots,A_n \in \Sigma\setminus\Delta$, we define $[x,\pi]$ and $[\pi,x]$, $x \in \{\underline{t},\underline{nt}\}$, as follows.

$[x,\pi] = \emptyset$ if $x = \underline{nt}$ and $\alpha_1 \ldots \alpha_{n+1} \notin Z^*$,

$[\pi,x] = \emptyset$ if $x = \underline{nt}$ and $\alpha_1 \ldots \alpha_{n+1} \notin X^*$,

otherwise $[x,\pi] = \{([x,A], \alpha_1 [x_1,A_1] \alpha_2 \ldots \alpha_n [x_n,A_n] \alpha_{n+1})$: for $1 \leq i \leq n$,
$x_i = F_L(x, \alpha_1 A_1 \ldots \alpha_{i-1} A_{i-1} \alpha_i, A_i)\}$

and

$[\pi,x] = \{([A,x], \alpha_1 [A_1,x_1] \alpha_2 \ldots \alpha_n [A_n,x_n] \alpha_{n+1})$: for $1 \leq i \leq n$, $x_i =$
$F_R(x, \alpha_{i+1} A_{i+1} \ldots \alpha_n A_n \alpha_{n+1}, A_n)\}$.

For $\pi = (A,\alpha) \in P$, with $A \in C(G)$ and $\alpha \in \Delta \cap B_G$, we define $[x,\pi,y]$, $x,y, \in \{\underline{t},\underline{nt}\}$, as follows.

$[x,\pi,y] = \emptyset$ if $x = \underline{nt}$ or $y = \underline{nt}$, and

$[\underline{t},\pi,\underline{t}] = \{([\underline{t},A,\underline{t}],\alpha)\}$.

For $\pi = (A,\alpha C\beta) \in P$, with $A,C \in C(G)$, $\alpha = \alpha_1 A_1 \alpha_2 \ldots \alpha_n A_n \alpha_{n+1}$ and $\beta = \beta_1 B_1 \beta_2 \ldots \beta_m B_m \beta_{m+1}$, for some $n,m \geq 0$, $\alpha_1,\ldots,\alpha_{n+1},\beta_1,\ldots,\beta_{m+1} \in \Delta^*$, $A_1,\ldots,A_n,B_1,\ldots,B_m \in (\Sigma\setminus\Delta) \setminus C(G)$, we define $[x,\pi,y]$, $x,y \in \{\underline{t},\underline{nt}\}$ as follows.

$[x,\pi,y] = \emptyset$ if $x = \underline{nt}$ and $\alpha_1 \cdots \alpha_{n+1} \notin Z^*$,

$[x,\pi,y] = \emptyset$ if $y = \underline{nt}$ and $\beta_1 \cdots \beta_{m+1} \notin X^*$,

otherwise $[x,\pi,y] = \{([x,A,y], \alpha_1 [x_1,A_1] \alpha_2 \cdots \alpha_n [x_n,A_n] \alpha_{n+1} [x_{n+1},C,y_0]$

$\beta_1 [B_1,y_1] \beta_2 \cdots \beta_m [B_m,y_m] \beta_{m+1})$: for $1 \le i \le n$, $x_i = F_L(x,\alpha_1 A_1 \cdots \alpha_{i-1} A_{i-1} \alpha_i, A_i)$,

$x_{n+1} = F_L(x,\alpha,C), y_0 = F_R(y,\beta,C)$, for $1 \le i \le m$, $y_i = F_R(y,\beta_{i+1} B_{i+1} \cdots \beta_m B_m \beta_{m+1}, B_i)\}$.

Now we set
$$P_1 = \bigcup_{x \in \{\underline{t},\underline{nt}\}} \Big(\Big(\bigcup_{\substack{(A,\alpha) \in P \\ A \notin C(G)}} ([x,(A,\alpha)] \cup [(A,\alpha),x]) \Big)$$
$$\cup \Big(\bigcup_{y \in \{\underline{t},\underline{nt}\}} \; \bigcup_{\substack{(A,\alpha) \in P \\ A \in C(G)}} [x,(A,\alpha),y] \Big).$$

Let $\mu \in HOM(\Sigma_1,\Sigma)$ be such that

$\mu(a) = a$, for all $a \in \Delta$,

$\mu([x,A]) = \mu([A,x]) = A$, for all $A \in (\Sigma \backslash \Delta) \backslash C(G)$ and $x \in \{\underline{t},\underline{nt}\}$,

$\mu([x,A,y]) = A$, for all $A \in C(G)$ and $x,y \in \{\underline{t},\underline{nt}\}$.

$X_1 = \mu^{-1}(X)$, $Y_1 = \mu^{-1}(Y)$ and $Z_1 = \mu^{-1}(Z)$.

Since G is special, G_1 satisfies (C1) through (C5) from Definition 2.5, as can easily be checked.

Let G_2 be the strongly reduced version of G_1. Then G_2 satisfies all conditions from Definition 2.5 (cf. step (V) in the proof of Theorem 2.1) and hence G_2 is special.

__Lemma__ 4.4. $L(G) \subseteq L(G_2)$.

__Proof__. Let $D = (x_0,x_1,\ldots,x_n)$ be a successful central derivation in G. We show that there exist $y_0,y_1,\ldots,y_{n-1} \in \Sigma_1^*$ such that $y_i \in \mu^{-1}(x_i)$, for $0 \le i \le n-1$ and $D' = (y_0,y_1,\ldots,y_{n-1},x_n)$ is a successful derivation in G_1. Since G_2 is the strongly reduced version of G_1, D' is also a successful derivation in G_2.

Let, for $0 \le i \le n-1$, $x_i = u_i A_i v_i$, for some $u_i, v_i \in \Sigma^*$ and $A_i \in \Sigma\backslash\Delta$, such that $Rew(D,i) = |u_i A_i|$; $x_n = u_{n-1} a v_{n-1} \in \Delta^*$ with $a \in B_G$.

Let $y_0 = [\underline{t}, S, \underline{t}] = S_1$.

Assume that for some $k \in \{0, \ldots, n-1\}$, $y_0, \ldots, y_k$ have been defined in such a way that $D_k' = (y_0, \ldots, y_k)$ is a derivation in G_1, and, for $0 \le i \le k$, $Des(D_k', i) = Rew(D,i)$, $y_i = u_i' A_i' v_i'$, with $\mu(u_i') = u_i$, $\mu(A_i') = A_i$ and $\mu(v_i') = v_i$ and

$A_i' = [x, A_i]$ if $Rew(D,i) < Occ(x_i)$ with $x = \underline{\ell d}(x_i)$,

$A_i' = [A_i, x]$ if $Rew(D,i) > Occ(x_i)$ with $x = \underline{rd}(x_i)$, and

$A_i' = [x, A_i, y]$ if $A_i = \underline{cent}(x_i)$, with $x = \underline{\ell d}(x_i)$ and $y = \underline{rd}(x_i)$.

Consider the step (x_k, x_{k+1}). Let $Prod(D,k) = (A_k, \alpha)$, for some $\alpha \in \Sigma^*$.

If $k < n-1$, then we proceed as follows.

(x_k, x_{k+1}) is a careful derivation step (Lemma 4.1.). Hence if $\underline{\ell d}(x_k) = \underline{nt}$, then $\underline{alph}(\alpha) \cap (\Delta\backslash Z) = \emptyset$ and if $\underline{rd}\ (x_k) = \underline{nt}$, then $\underline{alph}(\alpha) \cap (\Delta\backslash X) = \emptyset$. This implies that P_1 contains a unique production (A_k', α') such that $\mu(A_k') = A_k$ and $\mu(\alpha') = \alpha$. Then we set $y_{k+1} = u_k' \alpha' v_k'$. Clearly $\mu(y_k) = x_k$ and $y_k \underset{G_1}{\Rightarrow} y_{k+1}$ with $Des(D_{k+1}', k) = Rew(D,k)$.

Let u_{k+1}', v_{k+1}' and $A_{k+1}' \in \Sigma^*$ be such that $\mu(u_{k+1}') = u_{k+1}$, $\mu(v_{k+1}') = v_{k+1}$, $\mu(A_{k+1}') = A_{k+1}$ and $y_{k+1} = u'_{k+1} A'_{k+1} v'_{k+1}$.

Let $j = Intro(D,k+1)$ and $Prod(D,j) = (A_j', \beta A_{k+1} \gamma)$, with $\underline{prefintro}(D,k+1) = \beta$ and $\underline{postintro}(D,k+1) = \gamma$.

Since (x_j, x_{j+1}) is careful, $Prod(D_k', j) = (A_j', \beta' A_{k+1}' \gamma')$ with $\mu(A_j') = A_j$, $\mu(\beta') = \beta$, $\mu(A_{k+1}') = A_{k+1}$ and $\mu(\gamma') = \gamma$, is unique. From the construction of G_1 it follows that

$A_{k+1}' = [\underline{\ell d}(x_{k+1}), A_{k+1}]$ if $Rew(D,k+1) < Occ(x_{k+1})$,

$A_{k+1}' = [A_{k+1}, \underline{rd}(x_{k+1})]$ if $Rew(D,k+1) > Occ(x_{k+1})$ and

$A_{k+1}' = [\underline{\ell d}(x_{k+1}), A_{k+1}, \underline{rd}(x_{k+1})]$ if $A_{k+1} = \underline{cent}(x_{k+1})$.

If $k = n-1$, then $y_{n-1} = u_{n-1}[\underline{t}, A_{n-1}, \underline{t}]v_{n-1}$, and $\alpha = a$. $([\underline{t}, A_{n-1}, \underline{t}], a) \in P_1$ and so $y_{n-1} \underset{G_1}{\Rightarrow} x_n$. Hence $(y_0, \ldots, y_{n-1}, x_n)$ is a successful derivation in G_1. □

Lemma 4.5. Let $D = (x_0, x_1, \ldots, x_n)$ be a central derivation in G_2. Then $D_\mu = (\mu(x_0), \mu(x_1), \ldots, \mu(x_n))$ is a careful central derivation in G.

Proof. Immediate. □

Theorem 4.1. $L(G) = L(G_2)$.

Proof. Immediate from Lemma 4.4. and Lemma 4.5. □

The following lemma shows that unless $P_2 = \emptyset$, no central derivations in G_2 can get unsuccessfully blocked.

Lemma 4.6. Let $D = (x_0, x_1, \ldots, x_n)$ be a central derivation in G_2 such that there exists no $y \in \Sigma_2^*$ for which $(x_0, x_1, \ldots, x_n, y)$ is a central derivation in G_2.
Then either $n = 0$ and $P_2 = \emptyset$ or $x_n \in \Delta_2^*$.

Proof. (1). $n = 0$.
As in case (1) in the proof of Lemma 4.2 it follows that $P_2 = \emptyset$.

(2). $n \geq 1$. Hence $P_2 \neq \emptyset$ and productions for all non-terminal symbols of G_2 are available. From Lemma 4.5 it follows that $(\mu(x_0), \mu(x_1), \ldots, \mu(x_n))$ is a careful central derivation in G. From Lemma 4.2 it follows that if $\mu(x_n) \notin \Delta^*$, then there exists a $v \in \Sigma^*$ such that $(\mu(x_0), \mu(x_1), \ldots, \mu(x_n), v)$ is a central derivation in G. Hence $x_n = \alpha A \beta$, for some $\alpha \in X^*, \beta \in Z^*$ and $A \in \Sigma_2 \setminus \Delta_2$, such that $\mathrm{Des}(x_n) = |\alpha A|$. Then there exists a word $y \in \Sigma_2^*$ such that (x_n, y) is a central derivation step in G_2. This is a contradiction. Hence $\mu(x_n) \in \Delta^*$ which implies that $\mu(x_n) = x_n \in \Delta_2^*$. □

Lemma 4.7. $L(G_2) = L(\underline{\mathrm{base}}(G_2))$.

Proof. For every 1S grammar H, $L(H) \subseteq L(\underline{base}(H))$. Hence $L(G_2) \subseteq L(\underline{base}(G_2))$. It thus remains to show that $L(\underline{base}(G_2)) \subseteq L(G_2)$.

Consider the context-free grammar $\underline{base}(G_2)$. It is well known that one can adopt for context-free grammars any strategy to select in a sentential form a non-terminal to rewrite. As long as this strategy does not lead to failures, the language will not be changed. From Lemma 4.6 it follows that in $\underline{base}(G_2)$ the central derivation strategy of picking out non-terminals does not fail and if $P_2 = \emptyset$, then $L(G_2) = \emptyset = L(\underline{base}(G_2))$.

Hence $L(\underline{base}(G_2)) \subseteq L(G_2)$. □

Theorem 4.2. $L(G) \in \mathcal{L}(CF)$.

Proof. By Theorem 4.1., $L(G) = L(G_2)$ and Lemma 4.7 implies that $L(G_2) \in \mathcal{L}(CF)$. Hence $L(G) \in \mathcal{L}(CF)$. □

5. THE MAIN RESULT AND CONCLUSIONS

Now we can prove that all 1S grammars generate context-free languages only.

Theorem 5.1. $L(1S) = L(CF)$.

Proof. That $L(CF) \subseteq L(1S)$ is clear.

From Theorem 2.1 and Theorem 4.2 it follows that for every 1S grammar H there exists a CF grammar G_H and a homomorphism h such that $L(H) = h(L(G_H))$. Since $L(CF)$ is closed under homomorphisms, $L(H) \in L(CF)$.

Thus, $L(1S) \subseteq L(CF)$. □

Corollary 5.1. Every EOS based 1S grammar generates a context-free language. □

It is important to note at this point that all our constructions (in particular in the proof of Theorem 2.1) have been effective. Given a homomorphism h and a context-free grammar H_1 one can also effectively construct a context-free grammar H_2 such that $L(H_2) = h(L(H_1))$. As a consequence we have the following.

Theorem 5.2. Given a (EOS based) 1S grammar G it is decidable, whether or not

(1). $L(G)$ is empty,

(2). $L(G)$ is finite. □

Another consequence is the following.

Theorem 5.3. For every (EOS based) 1S grammar there exists an equivalent (EOS based) 1S grammar without erasing productions ($\Lambda \notin \underline{rhs}(P)$, where P is its set of productions). □

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CONTEXT TO CONTEXT-FREE GRAMMARS

by

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