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PROPAGATING MODES ON A BURIED LEAKY COAXIAL CABLE

by

Steven W. Plate, David D. Chang and Edward F. Kuester

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Department of Electrical Engineering
University of Colorado

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Characteristics of the discrete propagating modes on a buried leaky coaxial cable are investigated in this report. Provided the transfer inductance of the braid is small, we found three distinct propagating modes for most of the cable depths. The first mode has a field structure closely associated with the TEM-mode of a coaxial line and referred to as a bifilar mode because of its almost equal but opposite current on the two conductors. The second mode has a field structure located mainly outside the cable in the earth region and is referred to as a monofilar mode. The third mode which exists only for certain cable

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) depths has a field more spread out over the air-earth interface and is referred to as a surface-attached mode. Unlike the case of an elevated wire, no degeneracy of modes is observed for parameters of practical interest.

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1. Introduction

The problem of wave propagation along a buried, insultated wire has long been associated with long distance communication at very low and extremely low frequencies (VLF and ELF).[1] Operating at long wavelengths, the wire is usually very close to the earth surface, say within one or two skin-depths so that the interaction between the wire and the air-interface plays an important role in determining the propagation constant of the current waves supported by the wire structure.[2] More recently, buried wires operating at the higher end of the radio spectrum, 100 MHz for instance, have also found important applications as wave guiding structures in the design of groundwave radar detection systems either for vehicle monitoring or perimeter surveillance. These wires again have to be placed close to the earth surface in order to avoid unnecessary loss into the surrounding earth as well as to enhance area coverage. In many practical applications, leaky coaxial cables, either in the form of braided wires or periodic slots in the outer sheath, are used to allow electromagnetic waves to leak out continuously from the coaxial region into the surrounding medium. It is apparent that the characteristics of the propagating modes supported by such a waveguiding system would have to be influenced by the air-earth interface. The purpose of this report is to investigate such an influence.

A related problem concerning an elevated wire structure above earth surface has been studied extensively.[3,4,9,12] It is found that a bare or a dielectrically-coated wire usually supports two distinct modes. One of the two modes has a field structure more concentrated between the wire and its image and approaches the conventional transmission-line mode in the limiting case of a perfectly-conducting earth. Such a mode is referred to as structure-attached. The second mode, however, is more spead out over the air-earth interface, much like a groundwave field guided along the direction of the wire. This mode is then referred to as surface-attached. Although the two generally exhibit very different properties, both propagation constants are very close the wavenumber in air, and degeneracy of the two can occur when the wire parameters are properly chosen[12]. The result of our investigation indicates that such a phenomenon does not exist in the case of a buried wire, however.

2. The Modal Equation

Consider an infinitely long thin cable of exterior radius a buried in the earth at a depth h parallel to the z-axis. The earth,region 1, $(x > 0)$ is assumed to be nonmagnetic, having the refractive index $n_1 = (\varepsilon_{r1} + i\sigma_1/\omega\varepsilon_o)^{\frac{1}{2}}$ and region 2, the refractive index n_2. In this problem, region 2 usually represents free space which means that n_2 reduces to unity; however, n_2 is left arbitrary for generality. All field quantities are assumed to vary as $\exp(i\alpha k_o z - i\omega t)$ where α is the yet undetermined, complex propagation constant of a discrete mode. The geometry of this problem is illustrated in Figure 1. We further assume that the cable is thin compared to the depth at which it is buried $(a \ll h)$

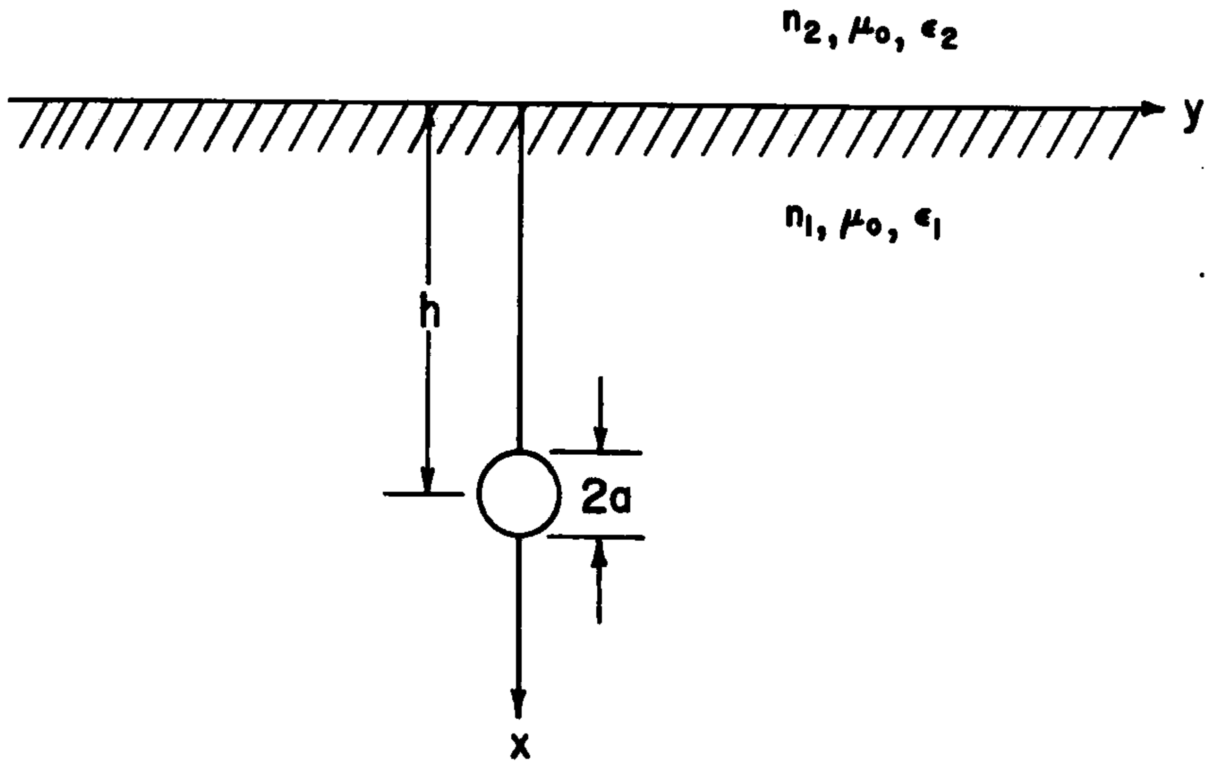

Figure 1. Geometry of the problem

and to the wavelength in region 1 ($k_o a|n_1| \ll 1$ where $k_o = \omega(\mu_o \varepsilon_o)^{\frac{1}{2}}$). This implies that the current on the cable will be virtually only z-directed and will be ϕ-independent. The corresponding modal equation for propagating modes supported by the wire structure is then found by assuming a current I on the cable, calculating the average z-directed E-field on the exterior surface of the cable due to the current I, and then enforcing the boundary condition $E_z = IZ(\alpha)$ where $Z(\alpha)$ is the surface impedance defined at the surface of the cable. All of the dependency on the internal structure of the cable is contained in the surface impedance $Z(\alpha)$. The resulting modal equation is given in [3] and [4]:

$$M(\alpha) = M_o(\alpha) + 4\omega\varepsilon_o Z(\alpha)/k_o^2 = 0 \tag{1}$$

where

$$M_o(\alpha) = (\zeta_1^2/n_1^2)[H_o^{(1)}(\zeta_1 A) - H_o^{(1)}(\zeta_1 H)] + P(\zeta;H) - \alpha^2 Q(\alpha;H) \tag{2}$$

$$P(\alpha;H) = \frac{2}{i\pi} \int_{\infty}^{\infty} \frac{\exp(-u_1 H)}{u_1 + u_2} \, d\lambda \tag{3}$$

$$Q(\alpha;H) = \frac{2}{i\pi} \int_{\infty}^{\infty} \frac{\exp(-u_1 H)}{n_2^2 u_1 + n_1^2 u_2} \, d\lambda \tag{4}$$

$$u_1 = (\lambda^2 - \zeta_1^2)^{\frac{1}{2}} \ ; \quad -\pi/2 \le \arg u_1 < \pi/2 \tag{5a}$$

$$u_2 = (\lambda^2 - \zeta_2^2)^{\frac{1}{2}} \ ; \quad -\pi/2 \le \arg u_2 < \pi/2 \tag{5b}$$

$$\zeta_1 = (n_1^2 - \alpha^2)^{\frac{1}{2}} \ ; \quad 0 \le \arg \zeta_1 < \pi \tag{6a}$$

$$\zeta_2 = (n_2^2 - \alpha^2)^{\frac{1}{2}} \ ; \quad 0 \le \arg \zeta_2 < \pi \tag{6b}$$

$$A = k_o a \ ; \quad H = 2k_o h$$

Here, $H_o^{(1)}(x)$ is the Hankel function of the first kind and of order zero. We note that the arguments $\zeta_{1,2}$ are chosen so that the fields found in the derivation of (1) are bounded at infinity in the transverse direction. The arguments of $u_{1,2}$ are chosen so that all of the integrals converge everywhere in the complex λ-plane. While equation (1) is applicable to a class of thin-wire structures[1],[7] we are specifically interested in the application to a leaky coaxial line composed of a center conductor of radius c and refractive index n_w; a dielectric insulator around the conductor of index n_b; a thin uniform metal braided sheath of radius b and transfer impedance Z_T; and a dielectric coating around the sheath of index n_a and radius a as shown in Fig. 2. Provided the amount of leakage is small, the surface impedance can be found by calculating the quantity

$$Z(\alpha) = \left.\frac{E_z(\rho,\alpha)}{2\pi\rho H_\phi(\rho,\alpha)}\right|_{\rho=a} \tag{7}$$

when the cable is driven by an axially symmetric source, where E_z is the z-directed electric field and H_ϕ is the ϕ-directed magnetic field. According to Wait [5], Casey [6] and Wait ahd Hill [7], this transfer impedance is given by

$$Z(\alpha) = Z_a(\alpha) + Z_T(\alpha)[Z_b(\alpha) + Z_i(\alpha)]/[Z_T(\alpha) + Z_b(\alpha) + Z_i(\alpha)] \tag{8}$$

where

$$Z_a(\alpha) = \frac{(n_a^2 - \alpha^2)k_o^2}{2\pi i\varepsilon_o\omega n_a^2} \ln(\frac{a}{b}) \tag{9a}$$

$$Z_b(\alpha) = \frac{(n_b^2 - \alpha^2)k_o^2}{2\pi i\varepsilon_o\omega n_b^2} \ln(\frac{b}{c}) \tag{9b}$$

$$Z_i(\alpha) = \frac{i\zeta_w k_o}{2\pi c\varepsilon_o\omega n_w^2} \frac{J_o(\zeta_w k_o c)}{J_1(\zeta_w k_o c)} \tag{10}$$

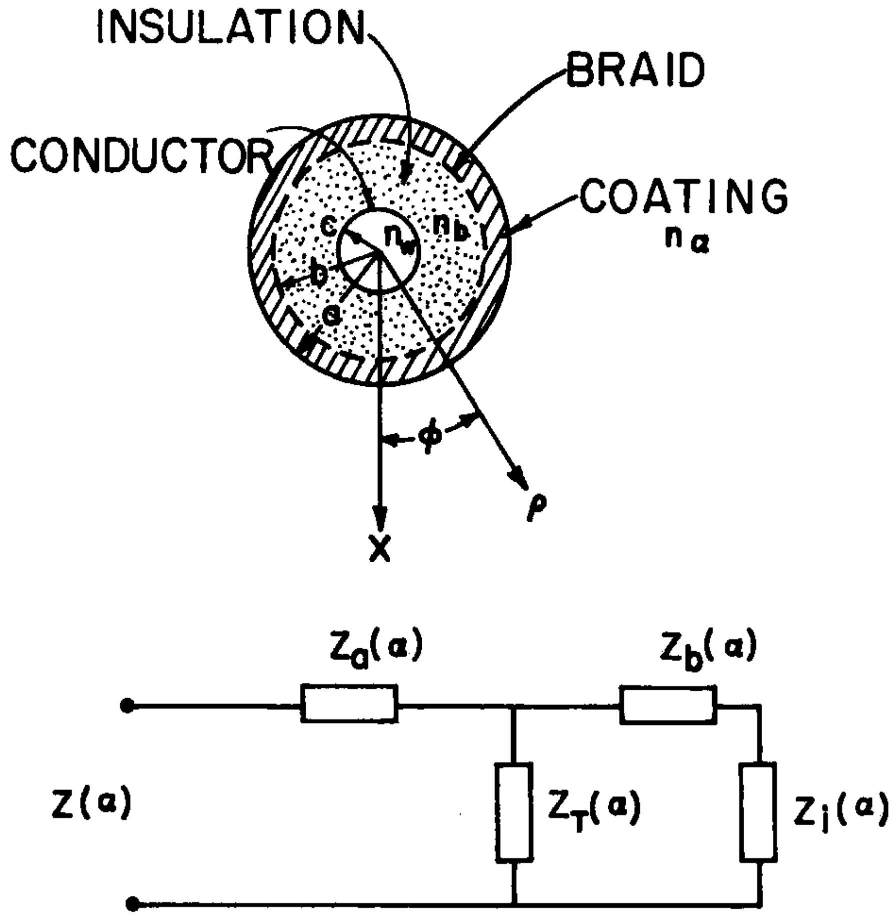

Figure 2. Cross section of a leaky cable and its equivalent circuit representation

$$\zeta_w = (n_w^2 - \alpha^2)^{\frac{1}{2}}$$

The equivalent circuit representation of (8) is also shown in Fig. 2. We note that $Z_{a,b}$ in the case represents respectively the series impedance due to the dielectric layer between a and b, and between b and c; Z_i is the internal impedance of the inner conductor, and Z_T is the shunt transfer impedance of the braided sheath. For a highly conducting inner conductor $Z_i(\alpha)$ reduces to

$$Z_i = \frac{ik_o}{2\pi c\varepsilon_o \omega n_w} \quad \frac{J_o(n_w k_o c)}{J_1(n_w k_o c)} \tag{11}$$

which is then independent of α.

The transfer impedance of the braided sheath is given in [5] to be

$$Z_T(\alpha) = \frac{-ik_o^2 L_T}{\varepsilon_o \omega \mu_o} \left(1 - \frac{\alpha^2}{n_a^2 + n_b^2}\right) \tag{12}$$

where L_T is the transfer inductance. It is particularly noteworthy that the total surface impedance as defined in (8), is a function of the propagation constant α. Furthermore, it is easy to show that denominator of (8) may be rewritten as

$$(Z_T(\alpha) + Z_b(\alpha) + Z_i) = Z'(\alpha^2 - \alpha_Z^2) \tag{13}$$

where

$$Z' = \frac{ik_o^2}{2\pi\varepsilon_o\omega}\left[\frac{1}{n_b^2}\ln\left(\frac{b}{c}\right) + \frac{L_T}{\mu_o(n_a^2 + n_b^2)}\right] \tag{14}$$

$$\alpha_Z^2 = \frac{n_b^2(n_a^2 + n_b^2)[\mu_o \ln(b/c) + L_T + i2\pi Z_i/\omega]}{[(n_a^2 + n_b^2)\mu_o \ln(b/c) + n_b^2 L_T]} \tag{15}$$

This shows that the $Z(\alpha)$ has a pair of poles at $\alpha = \pm\alpha_Z$. As will be shown in a later section of this report, these poles have a strong influence on the location of one of the roots of the modal equation.

3. Approximate Expression for the Integrals $P(\alpha;H)$ and $Q(\alpha;H)$

The integrals $P(\alpha;H)$ and $Q(\alpha;H)$ as given in (3) and (4) may be computed numerically in order to find the roots of $M(\alpha)$. This can be a time consuming process considering that $M(\alpha)$ must be evaluated several times for each root found. It is desirable to find approximations to $P(\alpha;H)$ and $Q(\alpha;H)$ that are valid in the regions of interest and are more efficient to compute than direct numerical integration. These approximations may also be used to find limiting forms of $M(\alpha)$ in special cases.

In [8], approximations to $P(\alpha;H)$ and a slightly different form of $Q(\alpha;H)$ are found that are valid in the region $|\zeta_1| \ll |\zeta_2|$. This corresponds to α in the neighborhood of n_1. Any root in this region would be highly attenuated, so even though roots may exist in this region, they are not extremely important. Instead, we attempt, in Appendix A, to find approximations that are valid in the region $|\zeta_1| \gg |\zeta_2|$, because any mode found in this region has a relatively low attenuation. The approximations need only be valid for $|n_1|H < 1$ because the cable usually is placed within a skin depth of the surface in order to insure sufficient penetration of the wave into the air region. In such a situation we have shown in Appendix A that $P(\alpha;H)$ can be approximated by

$$P(\alpha;H) = \frac{1}{i\pi(\zeta_1 H)^2} [4(1 - i\zeta_1 H) + (\zeta_2\zeta_1 H/2N)^2 \ln(\zeta_2/\zeta_1)] \exp(i\zeta_1 H)$$
$$+ 2H_o^{(1)}(\zeta_1 H) - 2H_1^{(1)}(\zeta_1 H)/(\zeta_1 H), \qquad (16)$$

$$Q_3'(\alpha;H) = 2\exp(-u_{1p}H)\{\ell n\, 2 - \ell n(-i\zeta_2) - (u_{2p}/\lambda_p)[\ell n(-iu_{2p}-i\lambda_p) - \ell n(-\zeta_2)]\} \quad (22)$$

where $u_{1p} = (\lambda_p^2 - \zeta_1^2)^{\frac{1}{2}}$ and $u_{2p} = (\lambda_p^2 - \zeta_2^2)^{\frac{1}{2}}$; $H_x^{(1)}$ is the Schwarz or Lipschitz-Hankel integral defined in [12] as

$$H_x^{(1)}(a,z) = \int_0^z \exp(iat)H_o^{(1)}(t)dt$$

and can be computed from a series expansion given in (A.23). Ein is the modified exponential integral given in [10] as

$$\mathrm{Ein}(z) = -\sum_{k=1}^{\infty} \frac{(-z)^k}{k(k!)}$$

Finally, the logarithmic function $\ell n(z)$ is specified by its principle value. It should be noted that the behavior of $Q'(\alpha;H)$ given in (19) is not singular at $\alpha = \alpha_B$, and its value at α_B may be found using (A.27b), (A.36b) and (A.40b). However, the first term in (18) does contain explicitly the term $(\alpha^2 - \alpha_B^2)^{-\frac{1}{2}}$ so that the expression of $Q(\alpha;H)$ will blow up when $\alpha = \alpha_B$.

4. Classification of Modes

Because the leakage of a braided cable is usually very small (i.e. $L_T << \mu_o/k_o$) it is logical to view the propagating modes of a buried leaky cable as resulting from coupling of modes between the coaxial cable and the external waveguiding system of a buried and insulated wire located near the earth surface. Thus, by inserting (8) and (13) into (1), the modal equation can be rearranged to reflect such a viewpoint:

$$(\alpha^2 - \alpha_Z^2)[M_o(\alpha) + (4\omega\varepsilon_o/k_o^2)Z_a(\alpha)] + \Delta(\alpha) = 0; \quad (23)$$

where

$$\Delta(\alpha) = \left(\frac{4\omega\varepsilon_o}{k_o^2}\right)[Z_b(\alpha) + Z_i(\alpha)]Z_T(\alpha)/Z' \tag{24}$$

and $|\Delta(\alpha)| << 1$ because Z_T is assumed to be small. For simplicity, we let $Z_i(\alpha) = 0$ so that the center conductor of the leaky coaxial is perfectly conducting. As is evident in (23) and the definition of Z_T in (12), if we now let the transfer inductance of the braid L_T approach zero then we have $\alpha = \alpha_Z = n_b$, which obviously corresponds to the TEM-mode of a coaxial line with inner and outer radii c and b. Provided the leakage remains small, we can find the solution of the modal equation by perturbation; the zeroth-order solution is $\alpha = \alpha_Z$; the first-order solution is found by inserting α_Z for α in $M_o(\alpha)$, $Z_a(\alpha)$ and $\Delta(\alpha)$; the second-order solution is found by inserting the first-order solution in $M_o(\alpha)$, $Z_a(\alpha)$ and $\Delta(\alpha)$; etc. Similar to the TEM-mode in a coaxial line, we expect the current in this case to be almost equal and opposite on the inner conductor c and the inner surface of conductor b. Such a mode is designated as a bifilar mode. On the other hand, other acceptable solutions of (23) when $\Delta(\alpha) \to 0$ may be found from

$$M_o(\alpha) + (4\omega\varepsilon_o/k_o^2)Z_a(\alpha) = 0 \tag{25}$$

which obviously represents the modes supported by an insulated conducting wire, or a Goubau-line buried near the earth surface, and with a surface impedance of Z_a. As shown in [2], such a line supports at least one mode with a known propagation constant in the low frequency limit. The current is now highly concentrated on the outer surface of conductor b, and consequently, the mode is designated as a monofilar mode. Obviously, the value of α may be found again from a perturbative scheme based upon a zero-order solution obtained from (25).

In addition to the monofilar mode, it is known that a second mode near $\alpha = \alpha_B$ also exists in the case of an elevated Goubau-line in air and located near the earth surface [9], [12]. Mathematically, the occurrence of this mode is heavily influenced by the inverse square-root singularity of the Q-integral in the modal equation; such a singularity is displayed explicitly in the approximate form of Q given in (18). The dominance of the Q-integral also means physically the field distribution of such a mode will be much more spread out along the air-earth interface than that of the monofilar mode, and consequently, is designated as a surface-attached mode. In Appendix B, it is shown that this mode indeed can exist for a sufficiently small buried depth H, but may cross the branch cut $(\mathrm{Re}(\alpha^2 - \alpha_B^2)^{\frac{1}{2}} = 0)$ and become an improper mode for a larger H.

Before we present the numerical evaluation of modes for a general case, another special case of interest is when the wire is located at the air-earth interface. Consider the limit as H approaches zero, while keeping the ratio A:H sufficiently small so that the thin wire approximation continues to hold. If we keep the relative cable geometry the same then the transfer impedances $Z_a(\alpha)$, $Z_b(\alpha)$, $Z_T(\alpha)$, and quantity Z' will remain constant. We will again assume that $Z_i(\alpha)$ is zero for simplicity. The two Hankel functions in the expression for $M_o(\alpha)$ in (2) diverge individually; however, their difference approaches a constant. By taking the limit as H approaches zero in (16) and (18) we find that both $P(\alpha;H)$ and $Q(\alpha;H)$ diverge logarithmically. Hence in this limit $M_o(\alpha)$ becomes

$$M_o(\alpha) \sim \frac{4\,i\,\ln(H)}{\pi(n_1^2 + n_2^2)}\left[\frac{n_1^2 + n_2^2}{2} - \alpha^2\right] \tag{26}$$

By inserting (26) into (23) we find that we can neglect $Z_a(\alpha)$. Hence we obtain

$$(\alpha^2 - \alpha_z^2)\left(\frac{n_1^2 + n_2^2}{2} - \alpha^2\right) = \left[\frac{i\pi(n_1^2 + n_2^2)}{4\ \ln H}\right]\Delta(\alpha) \tag{27}$$

As H approaches zero the term on the right side of (27) vanishes so in this limit, the value of α for the monofilar mode reduces to

$$\alpha = [(n_1^2 + n_2^2)/2]^{\frac{1}{2}}$$

which agrees with the well-known result obtained by Coleman [11] for a thin-wire located in the air-earth interface.

5. Numerical Results

We have developed a computer program to compute the roots of the modal equation (1). This program computes $P(\alpha;H)$ and $Q(\alpha;H)$ either by direct numerical integration of (3) and (4) or by the approximations given in section 3. Unless specified otherwise, the refractive index of the earth is taken to be $n_1 = 5.3 + i0.95$, and for the air $n_2 = 1.0$. The frequency is $f = 100$ MHz, and the dimensions of the cable in reference to figure 2 are: $a = 1.15$ cm; $b = 1.0$ cm; $c = 0.4$ cm. The braid inductance is $L_T = 40$ nH, and the inner conductor is assumed to be perfectly conducting so that $Z_i = 0$. The refractive index of the coating was $n_a = 1.449$ which corresponds to Teflon. The insulator refractive index was varied from $n_b = 1.0$ to 1.449, and the depth at which the cable was buried was varied from $h = 0.1$ m to 1.0 m. The skin depth of the earth is about 0.5 m so that the root locations should be relatively independent of h when $h > 0.5$ m. In all cases, we found two and sometimes three distinct roots for any given set of parameters. The first mode is a bifilar mode in which the currents on the inner conductor and the braid are approximately equal but have opposite signs. This mode has the least attenuation of any of the modes since the fields are concentrated on the inside of the cable. The location of this root tended to follow α_Z when the value of n_b was changed. A plot of the location of this mode as a function of h is given in figure 3.

The second mode is the monofilar mode in which almost all the current is on the braid, so this mode is similar to the case of a buried coated wire. A plot of the location of this mode is given in figure 4. Note that this mode has an attenuation of about 12 to 15 dB/m. Although it is heavily attenuated along the line, such a mode has the major part of field distribution located outside of the cable and, hence, is capable of interacting with surrounding objects in earth.

The third mode is the surface attached mode which has its fields concentrated near the interface between the air and earth. As shown in figure 5 this mode only exists for certain values of h. For $h > 0.34$ or $h < 0.19$ m, the root crosses the branch cut onto the improper Riemann surface in the complex α plane. The mode becomes improper in these cases and is absorbed in the surface wave radiation. Even when this mode does exist it is located very close to the branch point α_B given by (17) . For

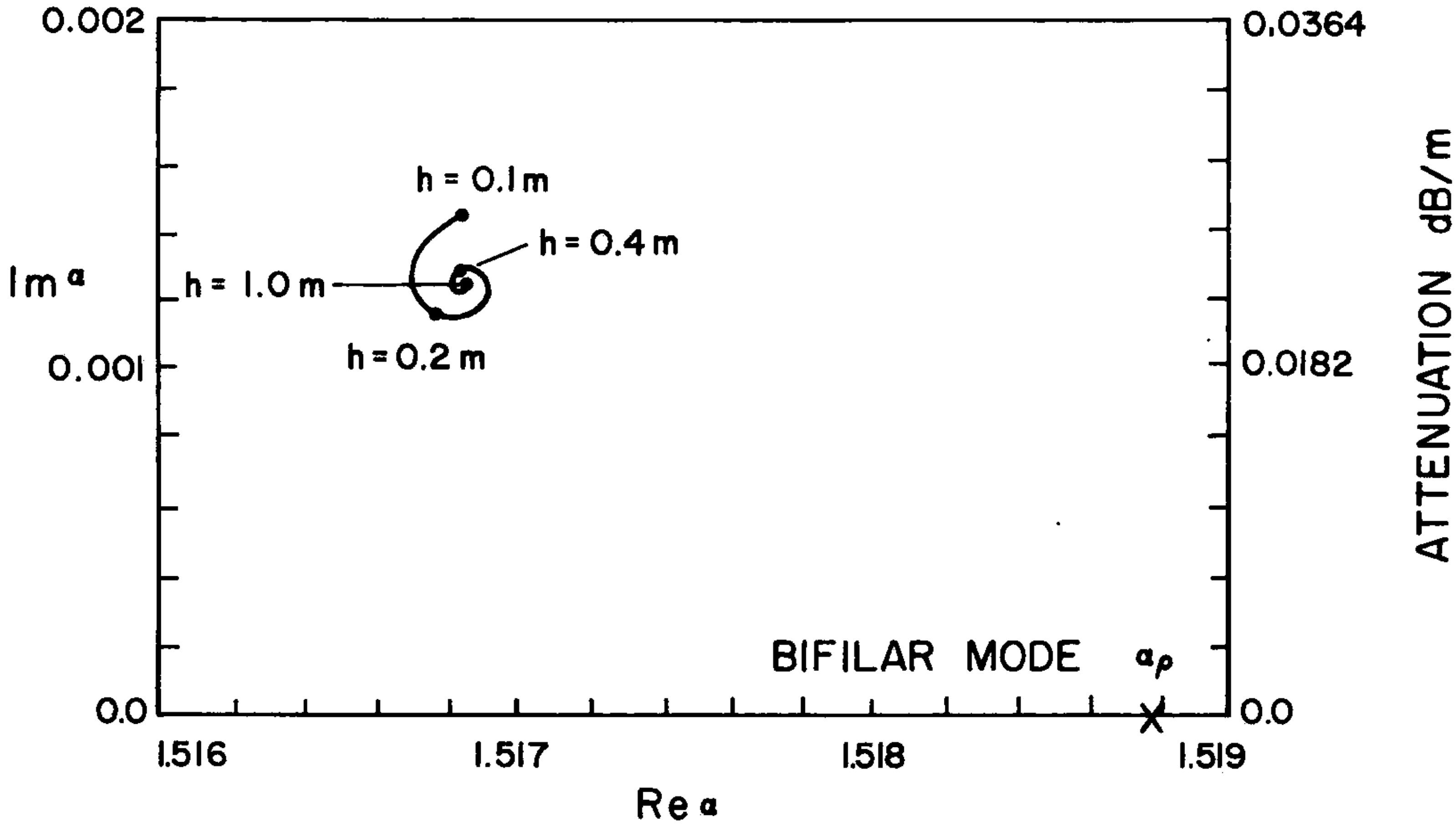

Figure 3. Complex propagation constant for the bifilar mode; $a = 0.0115$ m, $b = 0.01$ m, $c = 0.004$ m, $n_1 = 5.3 + i0.95$, $n_a = 1.449$, $n_b = 1.449$, $L_T = 40$ nH, $f = 100$ MHz.

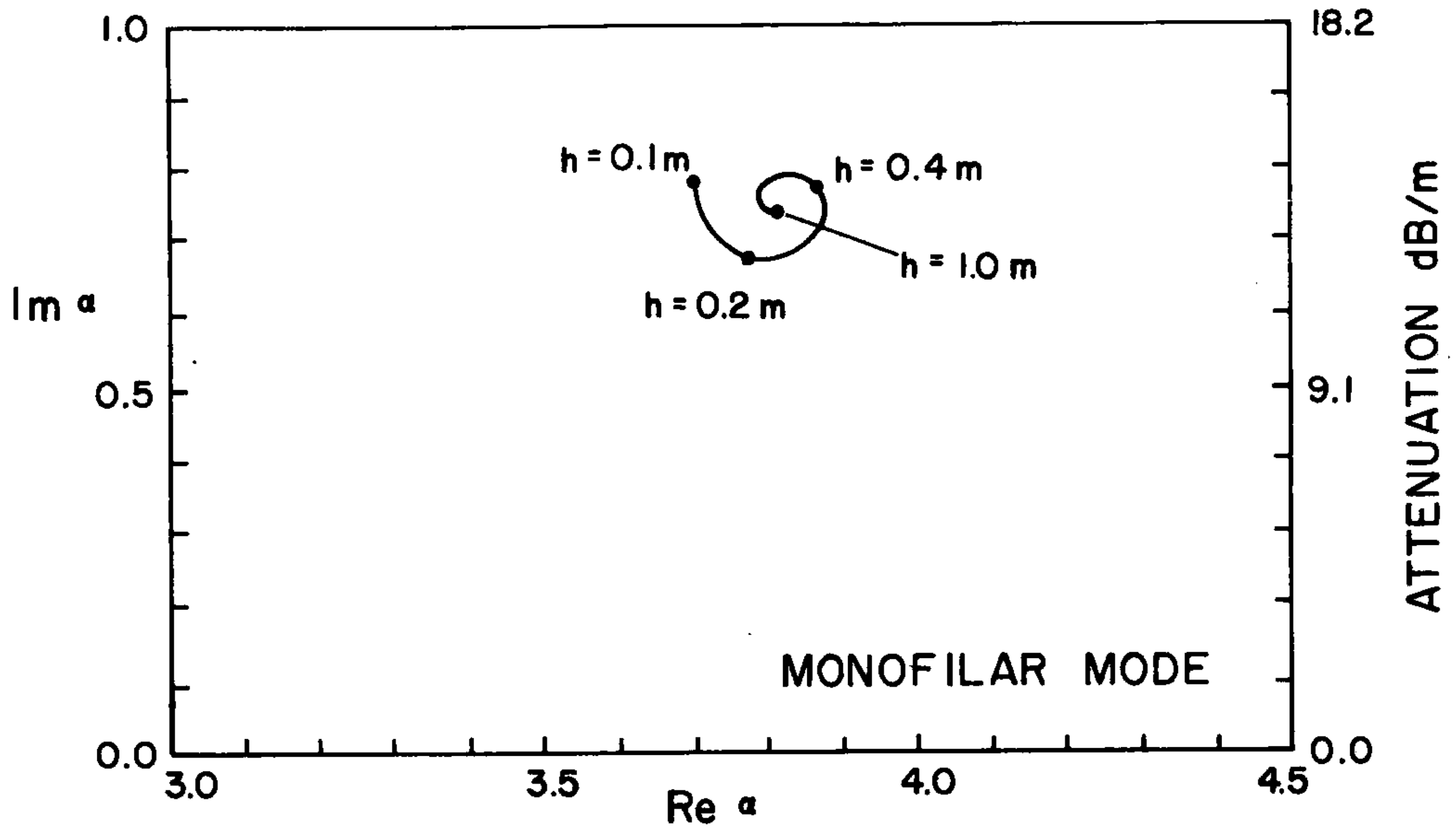

Figure 4. Complex propagation constant for the monofilar mode; $a = 0.0155$ m, $b = 0.01$ m, $c = 0.004$ m, $n_1 = 5.3 + i0.95$, $n_a = 1.449$, $n_b = 1.449$, $L_T = 40$ nH, $f = 100$ MHz.

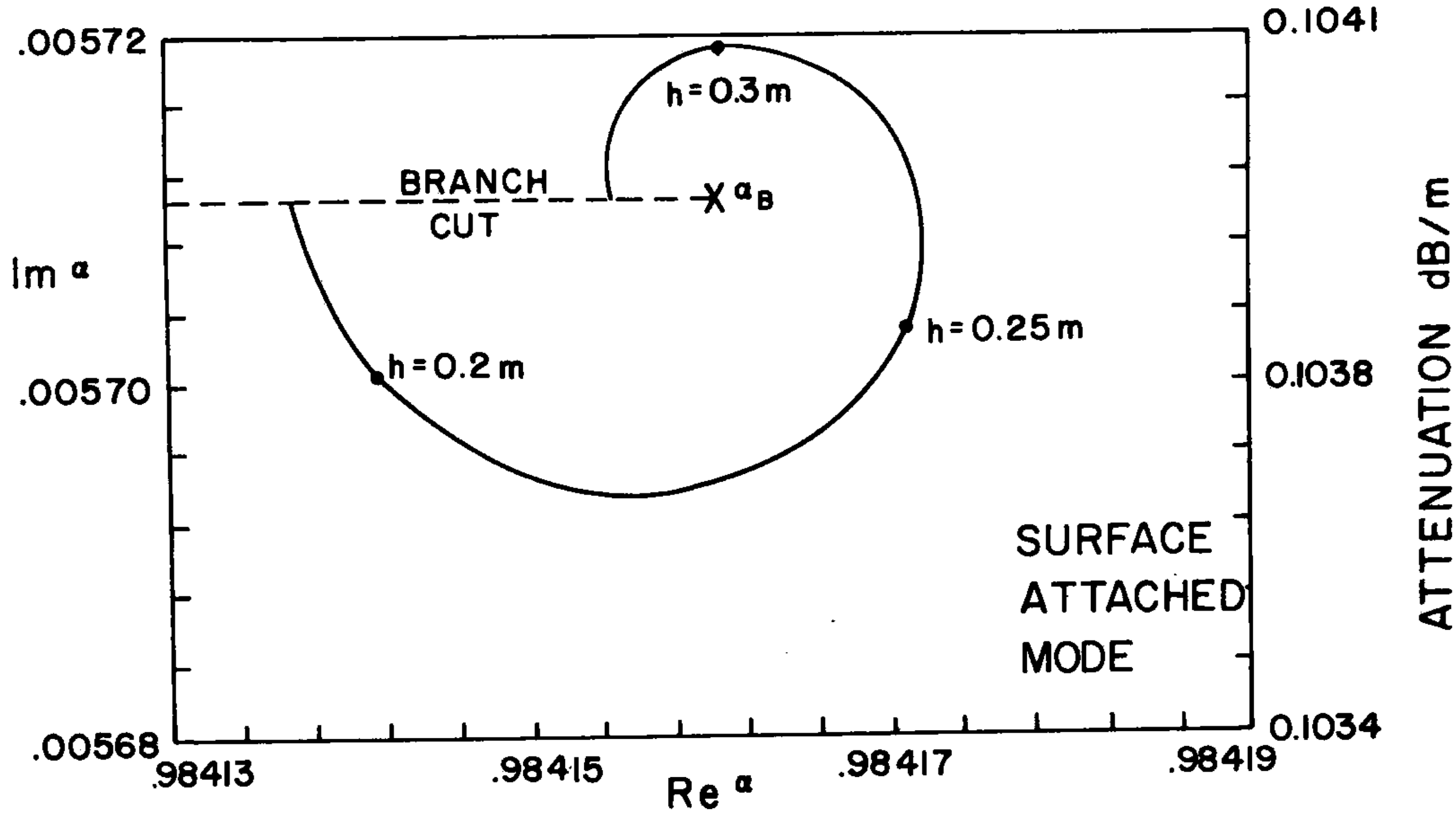

Figure 5. Complex propagation constant for the surface-attached mode. $a = 0.0115$ m, $b = 0.01$ m, $c = 0.004$ m, $n_1 = 5.3 + i0.95$, $n_a = 1.449$, $n_b = 1.449$, $L_T = 40$ nH, $f = 100$ MHz.

this reason the fields of this mode will be very similar to the ground wave fields which are spread out along the entire surface.

The accuracy of the approximations to $P(\alpha;H)$ and $Q(\alpha;H)$ was determined by comparing the values of the roots found using the approximations to the values obtained using the numerical integration. For the bifilar mode at $h = 0.1$ m the values of the roots agrees to within 10^{-5} and at $h = 0.1$ m, they agreed to within 8×10^{-5}. For the surface attached mode the roots agreed within 10^{-9} for all values of h. For the monofilar mode the approximations gave completely inaccurate results except for very small h say $h < .01$ m, because the assumption that $|\zeta_1| >> |\zeta_2|$ is no longer valid in the region where α of this mode is located. Thus, only direct numerical integration is used in this case.

6. Concluding Remarks

In this report we have investigated the roots of the modal equation for a leaky coaxial cable buried in a lossy earth. We have found that three distinct modes do exist for most cable depths of practical interest. We have given perturbation formulas to locate the roots of two of these modes.

Of the three modes the bifilar mode is probably the easiest mode to excite since the fields are similar to a TEM mode in a lossless coaxial cable. This mode has the least attenuation but also has only a small amount of field penetration into the air region because the total current on the cable is nearly zero. The surface-attached mode has more field strength in the air for a given amount of current on the inner conductor than the bifilar mode. For this reason, such a mode is ideally suited for the design of a groundwave detection system. But since the location of this mode is close to the branch point at $\alpha = \alpha_B$, one may not be able to excite such a mode independently without substantial surface radiation. The monofilar mode has fields concentrated in the earth region so that this mode is highly attenuated.

From the above discussion it is quite obvious that there are several conflicting considerations in the design of a wave guiding system for detecting anomalies (or intruders). On the one hand, the bifilar mode is easy to excite and once excited, it is capable of propagating along the line with least amount of attenuation. However, its field distribution is highly concentrated to the interior coaxial region so that it is not particularly sensitive to anomalies located above the earth surface. On the other hand, the monofilar mode suffers higher attenuation and the surface-attached mode can not be excited easily. The question then is how to bring about an optimized system which provides the best compromise to these conflicting considerations. To this end, it may be useful to first define a performance index of a specific mode as

$$\text{P.I.} = \frac{\mathcal{E}}{P_T \mathrm{Im}(\alpha)} \tag{28}$$

where

$\mathcal{E} = \int_{\infty}^{\infty} |E_n|^2 dy$ is the integration of the magnitude square of the vertical electric field component along the earth surface at a given cross-section;

P_T is the total amount of time-average power flow across the same cross-section;

$\mathrm{Im}(\alpha)$ is the imaginary part of α or the attenuation constant. Although such a definition is not at all unique, it is reasonable to assume that the set of parameters that yields the highest performance index according to (28) provides the "optimum" design for a detection system.

In Appendix C expression for the field components for each discrete mode are given. However, the actual computation and intercomparison of the performance indices of various modes will be presented separately in a supplemental report at a later date.

Appendix A

In this Appendix, approximate expressions are derived for the integrals $P(\alpha;H)$ and $Q(\alpha;H)$ under the assumption that $|\zeta_1| >> |\zeta_2|$ and $|n_2|H < |n_1|H < 1$. It is easy to see from (3) that the main contribution to the P-integral comes from the range of λ where u_2 can be approximated simply by λ. We therefore divide the integral in (3) into two parts,

$$P(\alpha;H) = P_1(\alpha;H) + P_2(\alpha;H) \tag{A.1}$$

where $P_1(\alpha:H)$ represents the dominant contribution, i,e,,

$$P_1(\alpha;H) = \frac{4}{i\pi}\int_0^\infty \frac{\exp(-u_1H)}{u_1 + \lambda} \quad d\lambda$$

$$= \frac{4}{i\pi\zeta_1^2}\left\{\int_0^\infty \lambda\exp(-u_1H)d\lambda - \int_0^\infty u_1e^{-u_1H}d\lambda\right\}$$

which is known analytically to be

$$P_1(\alpha;H) = \frac{4}{i\pi\zeta_1^2}\left[\frac{-i\zeta_1}{H} + \frac{1}{H^2}\right]\exp(i\zeta_1H)$$

$$+ 2H_0^{(1)}(\zeta_1H) - \frac{2}{\zeta_1H} \; H_1^{(1)}(\zeta_1H) \tag{A.2}$$

The remainder term $P_2(\alpha;H)$ is now given by

$$P_2(\alpha;H) = \frac{4}{i\pi}\int_0^\infty \left[\frac{1}{u_1+u_2} - \frac{1}{u_1+\lambda}\right] \exp(-u_1H)d\lambda \tag{A.3}$$

Note that the term in the brackets decays as λ^{-3} for large λ so the important part of the integral where λ is small. For H small enough the exponential term is constant for λ less than some λ_o so the exponential term may be replaced by its value at $\lambda = 0$ for the integration of λ between 0 and λ_o. Since the integrand decays as λ^{-3} for large λ, the integration from λ_o to ∞ is insignificant even without the exponential decay. This allows us to let λ_o approach infinity without affecting the results, The exponential term is replaced by its value at $\lambda = 0$ instead of unity so that $P_2(\alpha;H)$ remains small when compared to $P_1(\alpha;H)$ as n_1 approaches infinity. Hence $P_2(\alpha;H)$ is approximately

$$P_2(\alpha;H) \approx \frac{4}{i\pi} \exp(i\zeta_1 H) \lim_{\lambda_o \to \infty} \int_o^{\lambda_o} \left[\frac{1}{u_1 + u_2} - \frac{1}{u_1 + \lambda}\right] d\lambda$$

$$= \frac{2\zeta_2^2}{i\pi N^2} \exp(i\zeta_1 H) \ln(\zeta_2/\zeta_1) \tag{A.4}$$

where $N^2 = n_1^2 - n_2^2$, and the principal branch of the logarithm is chosen.

Substituting (A.2) and (A.4) into (A.1) yields

$$P(\alpha;H) \approx \frac{4}{i\pi\zeta_1^2 H^2} \left[1 - i\zeta_1 H + \frac{\zeta_2^2 \alpha_1^2 H^2}{2N^2} \ln\left(\frac{\zeta_2}{\zeta_1}\right)\right] \exp(i\zeta_1 H)$$

$$+ 2H_o^{(1)}(\zeta_1 H) - \frac{2}{\zeta_1 H} H_1^{(1)}(\zeta_1 H) \tag{A.5}$$

This approximation for $P(\alpha;H)$ is valid for $|\zeta_2| << |\zeta_1|$ or for $H|\zeta_1| << 1$. In the latter case (A.5) may be further approximated by using the small argument expansion of the Hankel functions to obtain

$$P(\alpha;H) \approx \frac{2i}{\pi} [\ln(\zeta_1 H) + \gamma - \ln 2] + 1 - \frac{i}{\pi} - \frac{2\zeta_2^2 i}{\pi N^2} \ln\left(\frac{\zeta_2}{\zeta_1}\right) \tag{A.6}$$

where $\gamma = 0.577216$ is Euler's constant. The result given in (A.6) agrees with the approximate formula derived earlier by Chang and Wait in [2].

On the other hand, by multiplying and dividing the integrand by the factor $n_1^2 u_2 - n_2^2 u_1$, the integral $Q(\alpha;H)$ as given in (4) may be rewritten as

$$Q(\alpha;H) = \frac{2}{i\pi(n_1^4 - n_2^4)} \int_{\infty}^{\infty} \frac{n_1^2 u_2 - n_2^2 u_1}{\lambda^2 - \lambda_p^2} \exp(-u_1 H) d\lambda \tag{A.7}$$

where

$$\lambda_p = (\alpha_B^2 - \alpha^2)^{\frac{1}{2}}$$

$$= (\zeta_2^2 - n_2^4/\hat{n}^2)^{\frac{1}{2}} \qquad 0 \leq \arg \lambda_p < \pi \tag{A.8}$$

$$\hat{n} = (n_1^2 + n_2^2)^{\frac{1}{2}} \qquad 0 \leq \arg \hat{n} < \pi \tag{A.9}$$

$$\alpha_B = n_1 n_2/\hat{n} \tag{A.10}$$

The integrand in (A.7) appears to have a pair of poles at $\lambda = \pm\lambda_p$, the location of which are a function of α. According to [8] the discontinuity of the residue calculation at $\lambda = \lambda_p$ when λ_p crosses the real axis causes a branch cut in the α-plane of $Q(\alpha;H)$. In addition, $Q(\alpha;H)$ becomes unbounded as α approaches the branch point α_B because λ_p tends to zero and the integral in (22) does not converge when λ_p equals zero.

In order to insure that the poles do exist, the numerator in (A.7) needs to be evaluated at $\lambda = \lambda_p$, and verified that it is non-zero there. In the physical case that region 1 is the earth and region 2 is air, the following relationships hold:

$$\arg n_1 > \arg \hat{n} > \arg n_2 \geq 0 \tag{A.11}$$

Under these constraints it can be shown that

$$u_1\big|_{\lambda=\lambda_p} = u_{1p} = -in_1^2/\hat{n} \tag{A.12}$$

If in addition we can assume that

$$\arg \hat{n} > 2 \arg n_2 \tag{A.13}$$

then it can also be shown that

$$u_2\big|_{\lambda=\lambda_p} = u_{2p} = in_2^2/\hat{n} \tag{A.14}$$

Using (A.12) and (A.14) the numerator of the integrand of (22) is evaluated at $\lambda = \lambda_p$ to be

$$n_1^2u_2 - n_2^2u_1\big|_{\lambda=\lambda_p} = 2in_1^2n_2^2/\hat{n} \tag{A.15}$$

which is non-zero, hence the integrand does have poles at $\lambda = \pm\lambda_p$. Note however that the relationship (A.13) may not always hold if both regions 1 and 2 are lossy, in which case the sign in (A.14) should be reversed and the poles would disappear.

An approximation for $Q(\alpha;H)$ will now be found in which the singularity at $\alpha = \alpha_B$ is accounted for exactly. In addition the approximations will have the proper limit as ζ_2 approaches zero, or as H approaches zero. Rewrite (A.7) as

$$Q(\alpha;H) = \frac{2}{i\pi(n_1^4 - n_2^4)} \{- n_2^2Q(\alpha;H) + n_1^2Q_2(\alpha;H) + n_1^2Q_3(\alpha;H)\} \tag{A.16}$$

where

$$Q_1(\alpha;H) = \int_{-\infty}^{\infty} \frac{u_1}{\lambda^2-\lambda_p^2} \exp(-u_1H)d\lambda \tag{A.17}$$

$$Q_2(\alpha;H) = 2\int_{0}^{\infty} \frac{\lambda}{\lambda^2-\lambda_p^2} \exp(-u_1H)d\lambda \tag{A.18}$$

$$Q_3(\alpha;H) = 2\int_{\infty}^{\infty} \frac{u_2 - \lambda}{\lambda^2 - \lambda_p^2} \exp(-u_1 H)\,d\lambda \tag{A.19}$$

The integral $Q_1(\alpha;H)$ is evaluated as follows.

$$Q_1(\alpha;H) = \frac{\partial^2}{\partial H^2}\left\{\int_{-\infty}^{\infty} \frac{\exp(-u_1 H)}{(u_1^2 - u_{1p}^2)u_1}\, d\lambda\right\}$$

$$= \frac{1}{2u_{1p}} \frac{\partial^2}{\partial H^2} \{Q_{11}(\alpha;H,u_{1p}) - Q_{11}(\alpha;h,-u_{1p})\} \tag{A.20}$$

where

$$Q_{11}(\alpha;H,u_{1p}) = \int_{-\infty}^{\infty} \frac{\exp(-u_1 H)}{(u_1 - u_{1p})u_1}\, d\lambda$$

$$= \exp(-u_{1p}H)\left\{\int_{-\infty}^{\infty} \frac{[\exp(-(u_1 - u_{1p})H) - 1]}{(u_1 - u_{1p})u_1}\, d\lambda + \int_{-\infty}^{\infty} \frac{d\lambda}{(u_1 - u_{1p})u_1}\right\}$$

$$= \exp(-u_{1p}H)\left\{-\int_0^H \exp(u_{1p}t)\int_{-\infty}^{\infty} \frac{\exp(-u_1 H)}{u_1}\, d\lambda\, dt + \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2 - \lambda_p^2} + u_{1p}\int_{-\infty}^{\infty} \frac{d\lambda}{(\lambda^2 - \lambda_p^2)u_1}\right\}$$

$$= \exp(-u_{1p}H)\left\{-i\pi\int_0^H \exp(u_{1p}t) H_o^{(1)}(\zeta_1 t)\,dt + \frac{i\pi}{\lambda_p} + u_{1p}\int_{-\infty}^{\infty} \frac{d\lambda}{(\lambda^2 - \lambda_p^2)u_1}\right\} \tag{A.21}$$

In order to evaluate the finite integral in (A.21) we can use the incomplete Lipschitz-Hankel integral or Schwarz function which is defined as

$$He^{(1)}(a,Z) = \int_0^Z \exp(iat)H_0^{(1)}(t)dt \tag{A.22}$$

This function may be evaluated using the following expansion [6]

$$He^{(1)}(a,Z) = Z\left\{H_0^{(1)}(Z)\sum_{m=0}^{\infty}\frac{(iZ)^m}{m!}C_m(\alpha)\right.$$

$$\left.+ ZH_1^{(1)}(Z)\sum_{m=0}^{\infty}\frac{(iZ)^m}{(m+1)!}C_m(a)\right\} \tag{A.23}$$

where $C_m(a)$ is given by the recursion formula

$$C_o(a) = 1; \quad C_1(a) = a/2; \quad C_m(z) = \frac{1}{m+1}[mC_{m-2}(a) + a^m], \quad m \geq 2$$

The infinite integral in (A21)has been evaluated in [9] to yield

$$\int_{-\infty}^{\infty}\frac{d\lambda}{(\lambda^2-\lambda_p^2)u_1} = \frac{2i}{\lambda_p(\zeta_1^2-\lambda_p^2)^{\frac{1}{2}}}\left\{\frac{i\pi}{2} + \ell n[(\zeta_1^2 - \lambda_p^2)^{\frac{1}{2}} - i\lambda_p] - \ell n\ \zeta_1\right\} \tag{A.24}$$

where the principal branches of the ℓn terms are chosen, and the square roots have positive real parts.

Using (A.22) and (A.24) we can write $Q_{11}(\alpha;H,u_{1p})$ as

$$Q_{11}(\alpha;H,u_{1p}) = \exp(-u_{1p}H)\left\{\frac{-i\pi}{\zeta_1}He^{(1)}\left(\frac{-iu_{1p}}{\zeta_1},\zeta_1 H\right) + \frac{i\pi}{\lambda_p}\right.$$

$$\left.+\frac{2iu_{1p}}{\lambda_p(\zeta_1^2-\lambda_p^2)^{\frac{1}{2}}}\left\{\frac{i\pi}{2} + \ell n[(\zeta_1^2-\lambda_p^2)^{\frac{1}{2}}-i\lambda_p] - \ell n\ \zeta_1\right\}\right\} \tag{A.25}$$

We obtain $Q_{11}(\alpha;H,-u_{1p})$ by replacing u_{1p} with $-u_{1p}$ in (A.25). We may then substitute iu_{1p} for $(\zeta_1^2 - \lambda_p^2)^{\frac{1}{2}}$. By inserting these expressions into (A.20) and differentiating, we obtain the following expression for $Q_1(\alpha;H)$

$$Q_1(\alpha;H) = Q_1'(\alpha;H) + \frac{i\pi u_{1p}}{\lambda_p} \exp(-u_{1p}H) \tag{A.26}$$

where

$$Q_1'(\alpha;H) = -\frac{i\pi u_{1p}}{2\zeta_1} \exp(-u_{1p}H)He^{(1)}(-\frac{iu_{1p}}{\zeta_1}, \zeta_1 H)$$

$$+ \frac{i\pi u_{1p}}{2\zeta_1} \exp(u_{1p}H)He^{(1)}(\frac{iu_{1p}}{\zeta_1}, \zeta_1 H) + i\pi H_o^{(1)}(\zeta_1 H)$$

$$+ \frac{u_{1p}}{\lambda_p} [\exp(-u_{1p}H) + \exp(u_{1p}H)][\ell n(iu_{1p} - i\lambda_p) - \ell n\, \zeta_1] \tag{A.26a}$$

The second term on the right side of (A.26) contains all of the singular parts of $Q_1(\alpha,H)$ near the branch point α_B. At $\alpha = \alpha_B$, $Q_1(\alpha,H)$ is finite and reduces to

$$Q_1'(\alpha_B;H) = i\pi H_o^{(1)}(iu_{1p}H) + u_{1p}H\pi H_1^{(1)}(iu_{1p}H) \tag{A.27}$$

The integral $Q_2(\alpha;H)$ may be evaluated as follows:

$$Q_2(\alpha;H) = 2\int_o^\infty \frac{\lambda}{u_1^2 - u_{1p}^2} \exp(-u_1H)d\lambda$$

$$= \frac{1}{u_{1p}} \{Q_{22}(\alpha;H,u_{1p}) - Q_{22}(\alpha;H,-u_{1p})\} \tag{A.28}$$

where

$$Q_{22}(\alpha;H,u_{1p}) = \int_o^\infty \frac{\lambda}{u_1 - u_{1p}} \exp(-u_1H)d\lambda$$

$$= \exp(-u_{1p}H)\int_H^\infty \exp(u_{1p}t) \int_o^\infty \lambda \exp(-u_1t)d\lambda\, dt \tag{A.29}$$

This last step is valid only if $\mathrm{Real}(u_1-u_{1p}) > 0$ for all λ. This is true if $\mathrm{Real}(-i\zeta_1 - u_{1p}) > 0$. We will derive an expression for $Q_{22}(\alpha;H,u_{1p})$

assuming that α is in the region in which the above inequalities hold. We will then analytically continue the resulting expression for all values of α. Note that this problem does not exist in the derivation of $Q_{22}(\alpha;H,-u_{1p})$ because $\mathrm{Real}(u_1+u_{1p}) > 0$ for all λ and all α.

Doing the λ integration in (A.29) yields

$$Q_{22}(\alpha;H,u_{1p}) = \exp(-u_{1p}H)\int_H^\infty \left[\frac{-i\zeta_1}{t} + \frac{1}{t^2}\right]\exp[(i\zeta_1 + u_{1p})t]dt$$

$$= \frac{1}{H}\exp(i\zeta_1 H) + u_{1p}\exp(-u_{1p}H)E_1(H[-i\zeta_1 - u_{1p}]) \tag{A.30}$$

$E_1(Z)$ and $E_2(Z)$ are the exponential integrals of order 1 and 2 respectively [9]. Alternately $E_1(Z)$ may be expressed as

$$E_1(Z) = \mathrm{Ein}(Z) - \ell n(Z) - \gamma \tag{A.31}$$

where γ is Euler's constant (=0.577216), and $E_{in}(Z)$ is an entire function which has the expansion [9]

$$\mathrm{Ein}(Z) = -\sum_{k=1}^{\infty}\frac{(-Z)^k}{k\,k!} \tag{A.32}$$

Inserting (A.31) into (A.30) we obtain an expression for $Q_{22}(\alpha;H,u_{1p})$

$$Q_{22}(\alpha;H,u_{1p}) = \frac{1}{H}\exp(i\zeta_1 H) + u_{1p}\exp(-u_{1p}H)\mathrm{Ein}(H[-i\zeta_1 - u_{1p}])$$
$$- u_{1p}\exp(-u_{1p}H)[\ell n\, H + \ell n(-i\zeta_1 - u_{1p}) + \gamma] \tag{A.33}$$

$$= \frac{1}{H}\exp(i\zeta_1 H) + u_{1p}\exp(-u_{1p}H)\mathrm{Ein}(H[-i\zeta_1 - u_{1p}])$$
$$- u_{1p}\exp(-u_{1p}H)[\ell n\, H - \ell n(-i\zeta_1 + u_{1p}) + \gamma]$$
$$- 2u_{1p}\exp(-u_{1p}H)\ell n(-i\lambda_p) \tag{A.34}$$

The only singularities in (A.34) are the branch cuts due to the square root in ζ_1 and the logarithm of λ_p, hence this form is the proper analytic continuation for all α.

Inserting (A.34) and (A.33) with $-u_{1p}$ substituted for u_{1p} into (43) we obtain the following expression for $Q_2(\alpha;H)$

$$Q_2(\alpha;H) = Q_2'(\alpha;H) - 2\exp(-u_{1p}H)\ln(-i\lambda_p) \tag{A.35}$$

where

$$\begin{aligned} Q_2'(\alpha;H) = {} & \exp(-u_{1p}H)\mathrm{Ein}(H[-i\zeta_1 - u_{1p}]) + \exp(u_{1p}H)\mathrm{Ein}(H[-i\zeta_1 + u_{1p}]) \\ & - [\exp(-u_{1p}H) + \exp(u_{1p}H)][\ln(H) + \gamma] \\ & + [\exp(-u_{1p}H) - \exp(u_{1p}H)]\ln(-i\zeta_1 + u_{1p}) \end{aligned} \tag{A.36a}$$

$Q_2'(\alpha;H)$ has a logarithmic singularity at $\alpha = \alpha_B$ due to the last term in (A.35). $Q_2'(\alpha;H)$ has no singularity at $\alpha = \alpha_B$ and may be evaluated at this point to obtain

$$\begin{aligned} Q_2'(\alpha_B;H) = {} & \exp(u_{1p}H)\mathrm{Ein}(2u_{1p}H) - [\exp(-u_{1p}H) + \exp(u_{1p}H)][\ln(H) + \gamma] \\ & + [\exp(-u_{1p}H) - \exp(u_{1p}H)]\ln(2u_{1p}) \end{aligned} \tag{A.36b}$$

The integral $Q_3(\alpha;H)$ may be rewritten here as

$$Q_3(\alpha;H) = -2\zeta_2^2 \int_0^\infty \frac{\exp(-u_1H)}{(u_2+\lambda)(\lambda^2-\lambda_p^2)}\, d\lambda \tag{A.37}$$

From this expression it is apparent that $Q_3(\alpha;H)$ vanishes as ζ_2 approaches zero. The integral also converges even when H is set to zero whereas both $Q_1(\alpha;H)$ and $Q_2(\alpha;H)$ diverge logarithmically as H approaches zero.

We therefore expect the major part of the integration to come from the region near the poles $\pm\lambda_p$ in the complex λ-plane. Since the integrand decays as λ^{-3} for large λ, we will approximate $Q_3(\alpha;H)$ by replacing the exponential term by its value at $\lambda = \lambda_p$. Hence $Q_3(\alpha;H)$ becomes

$$\begin{aligned} Q_3(\alpha;H) &\approx 2\exp(-u_{1p}H)\int_0^\infty \frac{u_2-\lambda}{\lambda^2-\lambda_p^2}\,d\lambda \\ &= 2\exp(-u_{1p}H)\lim_{\lambda_o\to\infty}\left\{\int_0^{\lambda_o}\frac{d\lambda}{u_2}-\int_0^{\lambda_o}\frac{\lambda\,d\lambda}{\lambda^2-\lambda_p^2}+u_{2p}^2\int_0^\infty\frac{d\lambda}{u_2(\lambda^2-\lambda_p^2)}\right\} \\ &= 2\exp(-u_{1p}H)\left\{\ln(2)-\ln(-i\zeta_2)+\ln(-i\lambda_p)\right. \\ &\qquad \left.+\frac{u_{2p}^2}{2}\int_{-\infty}^\infty\frac{d\lambda}{u_2(\lambda^2-\lambda_p^2)}\right\} \end{aligned} \tag{A.38}$$

The infinite integral in (A.38) is given by (A.24) with ζ_2 substituted for ζ_1. We may also replace $(\zeta_2^2-\lambda_p^2)^{\frac{1}{2}}$ with $-iu_{2p}$ so that $Q_3(\alpha;H)$ becomes

$$Q_3(\alpha;H) = Q_3'(\alpha;H) + \left\{\frac{i\pi u_{2p}}{\lambda_p} - 2\ln(-i\lambda_p)\right\}\exp(-u_{1p}H) \tag{A.39}$$

where

$$Q_3'(\alpha;H) = 2\exp(-u_{1p}H)\left\{\ln(2)-\ln(-i\zeta_2)-\frac{u_{2p}}{\lambda_p}[\ln(-iu_{2p}-i\lambda_p)-\ln(-\zeta_2)]\right\} \tag{A.40a}$$

$Q_3'(\alpha;H)$ does not have a singularity at $\alpha=\alpha_B$ and may be evaluated at this point to obtain

$$Q_3'(\alpha_\beta;H) = 2\exp(-u_{1p}H)[\ln(2)-\ln(u_{2p})-1] \tag{A.40b}$$

The logarithmic singularity at $\alpha = \alpha_B$ in the last term of (A.39) will exactly cancel the singularity of $Q_2(\alpha;H)$ in (A.35). By inserting (A.26), (A.35) and (A.39) into (A.16) we may write $Q(\alpha;H)$ as

$$Q(\alpha;H) = \frac{4i\, n_2^2 n_1^2}{\lambda_p \hat{n}(n_1^4 - n_2^4)} \exp\left(\frac{i n_1^2 H}{\hat{n}}\right) + Q'(\alpha;H) \tag{A.41}$$

where

$$Q'(\alpha;H) = \frac{2}{i\pi(n_1^4 - n_2^4)} \{-n_2^2 Q_1'(\alpha;H) + n_1^2 Q_2'(\alpha;H) + n_1^2 Q_3'(\alpha;H)\} \tag{A.42}$$

Q_1', Q_2', and Q_3' are defined in (A.27), (A.36) and (A.39) respectively. It should be noted that $Q'(\alpha;H)$ is not singular at $\alpha = \alpha_B$ and its value at this point may be found using (A.27), (A.36b), and (A.40b).

Appendix B

In order to demonstrate that a surface-attached mode may exist in the buried cable problem, we first insert (18) into (1) and (2) to yield an alternative form of the modal equation as

$$\lambda_p = \frac{4i\alpha^2 n_1^2 n_2^2 \exp(in_1^2 H/\hat{n})}{\hat{n}(n_1^4 - n_2^4)[M_o'(\alpha) + 4\omega\varepsilon_o Z(\alpha)/k_o^2]} \tag{B.1}$$

where

$$M_o'(\alpha) = (\zeta_1^2/n_1^2)[H_o^{(1)}(\zeta_1 A) - H_o^{(1)}(\zeta_1 H)] + P(\alpha;H) - \zeta^2 Q'(\zeta;H) \tag{B.2}$$

and $Q'(\alpha;H)$ is defined in (19). The right side of (B.1) is a smooth function of α near α_B, so we may insert α_B for α and calculate λ_p. We may then calculate α using the expression

$$\alpha^2 = \alpha_B^2 - \lambda_p^2 = n_1^2 n_2^2/(n_1^2 + n_2^2) - \lambda_p^2 \tag{B.3}$$

This value of α may be inserted back into (B.1) to recalculate a more accurate value for α .

The imaginary part of λ_p must be positive, so that if the right side of (28) has a negative imaginary part then the mode does not exist. We have found that this depends on the value of H. In the limit as H approaches zero, (B.1) at $\alpha = \alpha_B$ reduces to

$$\lambda_p = \frac{2\pi n_1^4 n_2^4}{\hat{n}(n_1^2 - n_2^2)(n_1^4 + n_2^4)\ln(H)} \quad ; \quad \hat{n} = (n_1^2 + n_2^2)^{\frac{1}{2}} \quad . \tag{B.4}$$

If $|n_1|$ is greater than $|n_2|$ and they satisfy the condition that $\arg \hat{n} > 2 \arg n_1$, then the right side of (B.4) has a positive imaginary part. So for very small H a root of the modal equation exists that is close to $\alpha = \alpha_B$. As H is increased, this root may or may not exist depending on the imaginary part of the right side of (B.1). In all cases, as will be shown in a later section, when the root does exist it is very close to $\alpha = \alpha_B$.

Appendix C

In this Appendix, expressions for the fields of the discrete modes are given. To be consistent with the thin wire approximation, all higher order terms of $k_o a$ are dropped. Generalizing the results of [3] we can express the fields in terms of the z-components of the electric and magnetic Hertz potentials to obtain

$$E_x(x,y,\alpha) = ik_o[\alpha\partial U_{1,2}/\partial x + \eta_o \partial V_{1,2}/\partial y]$$

$$E_y(x,y,\alpha) = ik_o[\alpha\partial U_{1,2}/\partial y - \eta_o \partial V_{1,2}/\partial x]$$

$$E_z(x,y,\alpha) = \zeta_{1,2}^2 k_o^2 U_{1,2}$$

$$H_x(x,y,\alpha) = ik_o[\alpha\partial V_{1,2}/\partial x - (n_{1,2}^2/\eta_o)\partial U_{1,2}/\partial y]$$

$$H_y(x,y,\alpha) = ik_o[\alpha\partial V_{1,2}/\partial y + (n_{1,2}^2/\eta_o)\partial U_{1,2}/\partial x]$$

$$H_z(x,y,\alpha) = \zeta_{1,2}^2 k_o^2 V_{1,2}$$

The subscript 1 is used for the earth region $(x > 0)$ and 2 is for the air region $(x < 0)$, see Figure 1. The potentials are found in [1] or [2] to be

$$U_1(x,y,\alpha) = \frac{-\eta_o I}{4\zeta_1^2 k_o}\left\{\frac{\zeta_1^2}{n_1^2} H_o^{(1)}\left[\zeta_1 k_o((x-h)^2 + y^2)^{\frac{1}{2}}\right] - \frac{\zeta_1^2}{n_1^2} H_o^{(1)}\left[\zeta_1 k_o((x+h)^2+y^2)^{\frac{1}{2}}\right]\right.$$

$$\left. + \frac{2}{i\pi}\int_{\infty}^{\infty}\left[\frac{1}{u_1+u_2} - \frac{\alpha^2}{n_1^2 u_2 + n_2^2 u_1}\right]\exp(-u_1 k_o(x+h) - i\lambda k_o y)\,d\lambda\right\}$$

$$U_2(x,y,\alpha) = \frac{i\eta_o I}{2\pi k_o \zeta_2^2} \int_{\infty}^{\infty} \left[\frac{1}{u_1+u_2} - \frac{2}{n_1^2 u_2 + n_2^2 u_1}\right] \exp(-u_1 k_o h + u_2 k_o x - i\lambda k_o y)\, d\lambda$$

$$V_1(x,y,\alpha) = \frac{\alpha I}{2\pi k_o \zeta_1^2} \int_{\infty}^{\infty} \left[\frac{1}{u_1+u_2} - \frac{n_1^2}{n_1^2 u_2 + n_2^2 u_1}\right] \frac{\lambda}{u_1} \exp(-u_1 k_o (x+h) - i\lambda k_o y)\, d\lambda$$

$$V_2(x,y,\alpha) = \frac{\alpha I}{2\pi k_o \zeta_2^2} \int_{\infty}^{\infty} \left[\frac{1}{u_1+u_2} - \frac{n_1^2}{n_1^2 u_2 + n_2^2 u_1}\right] \frac{\lambda}{u_1} \exp(-u_1 k_o h + u_2 k_o x - i\lambda k_o y)\, d\lambda$$

In these expressions I is the total current on the cable. $U_{1,2}$ and $\zeta_{1,2}$ are defined in (5) and (6).

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