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A Nonexpert Introduction to Rational Homotopy Theory

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What is Rational Homotopy Theory, and Why Do We Care?

It is well known that homotopy theory, while well developed theoretically, is challenging to make practical use of in many situations. In particular, it is notoriously difficult to compute homotopy groups (and, more generally, homotopy classes of maps) for many, even most topological spaces. Indeed, we have not computed all homotopy groups for all spheres, among the simplest of spaces. This is the fundamental value of rational homotopy theory — it is remarkably computable. In fact, I will finish this introduction by computing the rational homotopy groups of the spheres as a demonstration.

But what are the rational homotopy groups? Phrased most algebraically, they are the regular homotopy groups ”mod torsion”, or, equivalently, times $\mathbb{Q}$. What effect does this really have? It is twofold; first, we learn nothing about any homotopy group elements that have finite order. These elements are elements of the torsion subgroup — which disappears when we multiply by $\mathbb{Q}$. More geometrically, this means that any loop that can be made homotopic to the constant loop by going around it several times is not represented in the rational homotopy groups. The second difference is that we gain a notion of division; it is clear what it means to multiply a loop by an integer (go around it that many times), but not all loops can be divided by an integer (have a loop which, when gone around n times is equivalent to the numerator).

In general, we will call a space rational if its rational homotopy groups are the same as its homotopy groups — if you loose no information when disregarding loops with finite order. As is shown later in this document, every space has a rationalization, a continuous map from another space which induces an isomorphism of rational homotopy groups. Moreover, we have a construction of such a rationalization for any given space, which can be helpful when doing computations.

Far more important computationally, however, is the main tool of rational homotopy theory, a functor which takes topological spaces into commutative differential graded algebras. CDGAs are quite well behaved objects, and for any given CDGA we can often find a simpler one that has the same cohomology, which means it also preserves the rational homotopy. These so called minimal models of a CDGA, and we once again prove that any CDGA has such a minimal model, and give its description. Once we have such a model, we are often immediately able to compute the cohomology and the rational homotopy quite quickly.

Still, not all spaces have easily computable rational homotopy groups, even if they are more easily computed than the full homotopy groups. Thankfully, there are many important examples that are. Lie groups have immediate minimal models, and there is a broader class, called formal spaces, which have models that are isomorphic to their homologies, generally facilitating easier computation. This class includes the spheres and compact Kähler manifolds, as well as many compact manifolds with positive sectional curvature.[2] Homotopy theory can tell us many interesting and important things about a space, like the number of periodic geodesics (of considerable importance in GR) or whether two objects are homeo (thus diffeo) morphic (which is interesting in and of itself, but also in topics like topological field theories, when topological irregularities can beget special pseudo-particles.) [4] As such, having a more computationally practical route to such information can be important.

This piece is guided by Felix, Halperin, and Thomas's book, reference [1]. It is my hope that I can clarify some of the arguments presented in their work for those without intimate knowledge of algebraic topology and category theory. As such, the statements, and often, though not always, the main idea of the proof of theorems, will be borrowed from their work. I have also borrowed some of the organizational principles, especially in the early chapters, from Katheryn Hess' article [3]. Although less thorough by far than [1], this is a useful article for understanding the broader context of RHT.

I will now close off this introduction by using some of the tools introduced in the body of the thesis to compute the rational homotopy groups of the spheres. This begins with noting that the cohomology of the n-sphere is $H^k(S^n, \mathbb{Q}) = \mathbb{Q}$ when k=0 or n, and is trivial otherwise. Then we proceed with the construction given in Theorem 3.2. As we will see is part of the definition of a minimal (Sullivan) model, we will construct V, the graded vector space we want inductively. In the general case, this is fairly involved. Thanks to the simplicity of the cohomology of the spheres, however, it is simpler in this case. We begin with $V(n)$ (with all elements of degree n), which we take to be equal to $H^n(0) = \mathbb{Q}$ and define the differential on $V(n)$ to be zero. From here, our construction varies depending on the parity of n. First, take n to be odd. Then $\Lambda V(n)$ has no elements of higher degree, since ΛV^{odd} is the exterior algebra, and there is a single basis element (so $e \wedge e = 0$). Thus, in the odd case, we are done! When n is even, however, this generates terms of higher order; now ΛV is the polynomial or symmetric algebra, so it has elements of all degrees that are multiples of n. We have to cancel the cohomology contributions these would otherwise make. Since there is still only a single generator of $V(n)$, we have the $(\Lambda V(n))_{kn} = \mathbb{Q}$. But theorem 3.2 once again guides our construction; Since we have a non-trivial cohomology at degree $2n$, we take $W(2n-1)$ to be $\mathbb{Q}e'$ where e' is now an element of degree $2n-1$, and take $V(2n-1) = V(n) \otimes W(2n-1)$. Then we extend the differential to $V(2n-1)$ by declaring that $de' = e^2$. Now e' is of odd degree, so it will not create any higher order terms by itself. However, we will now have that $(\Lambda V(2n-1))_{(k+1)n-1} = \mathbb{Q}e'e^{kn}$. However, the differential has trivial kernel from degree $(k+1)n-1$ to degree $(k+1)n$, so there are no new contributions to the cohomology, and d is a surjection from $(\Lambda V(2n-1))_{(k+1)n-1}$ to $(\Lambda V(2n-1))_{(k+1)n}$, so the errant cohomology groups are killed. We then take $V = V(2n-1)$.

Now that we have the minimal models for the spheres, we can use theorem 4.4 to find that the map there called ζ_n is an isomorphism from the rational homotopy groups of S^n and the $\mathbb{Q}$ linear maps from $V(2n-1)$ to $\mathbb{Q}$. More precisely, it says that $\pi_k'(S^n) = Hom_{\mathbb{Q}}(V^k, \mathbb{Q})$. For even n, V^k is only nonzero when $k = n$, and we have that $\pi_n'(S^n) = \mathbb{Q}$. For odd n we have the same result, and the additional result that $pi_{2n-1}'(S^n) = \mathbb{Q}$. Thus all higher homotopy groups of spheres have only elements of finite order, with the exception of the $2n-1$th homotopy group when n is odd, which has one generating element with infinite order.

Chapter 1

Rational Spaces and Rationalizations of Topological Spaces

In order to present our definitions of a rational space and the rationalization of a space, we must first understand what it means for a group to be a vector space over $\mathbb{Q}$. To begin with, let's consider the definition of a Module. An R-module is a set V equipped with a binary operation $+ : V \times V \to V$ and the action $\cdot : R \times V \to V$ of a unital ring R on V, which satisfy:

1. $+$ is associative, i.e. $(a + b) + c = a + (b + c)$ for all $a,\ b,\ c \in V$.
2. $+$ has a two sided identity, 0. That is $a + 0 = 0 + a = a$.
3. Every element a of V has an inverse with respect to $+$, which we will denote $-a$. That is $v + (-v) = 0$.
4. $+$ is commutative, i.e. $a + b = b + a$ for all $a,\ b \in V$
5. $1 \cdot a = a$ for all $a \in V$.
6. $(\lambda + \mu) \cdot a = \lambda \cdot a + \mu \cdot a$ for all $\lambda,\ \mu \in R$ and $a \in V$.
7. $\lambda \cdot (a + b) = \lambda \cdot a + \lambda \cdot b$ for all $\lambda \in R$ and $a,\ b \in V$.
8. $(\lambda\mu) \cdot a = \lambda \cdot (\mu \cdot a)$ for all $\lambda,\ \mu \in R$ and $a \in V$.

Then a vector space is a module where R is a field. Since the first four of these axioms are the axioms of an abelian group, it is natural to consider modules as abelian groups equipped with a compatible action of a unital ring. Therefore, when we want to consider a group as a potential vector space over $\mathbb{Q}$ we need only consider what action of $\mathbb{Q}$ we can define on it. Thankfully, half the work is already done for us, since we can define the action of $\mathbb{Z}$ on any abelian group as follows: $0g = e$, $(n+1)g = ng + g$ and $(-n)g = -(ng)$. To consider the group as a vector space over $\mathbb{Q}$ we need only figure out how to divide group elements by integers.

It is not hard to find conditions on our group that will allow us to do so. First, by axioms 8 and 5 we require $n(\frac{1}{n}g) = 1g = g$.That is, we can only hope to define $\frac{g}{n}$ when there is an element h of the group that satisfies $nh = g$. Thus, for division to always be well defined, we require that the group be divisible. Moreover, for division to be defined we must have that the h is always unique, which ultimately requires that the group be torsion free (it is this requirement that informs the common remark "rational homotopy theory is homotopy theory modulo torsion"). Moreover, the converse is true as well: any torsion free divisible group is a vector space over $\mathbb{Q}$. This gives us our first baby theorem, which may be useful to some readers when considering the forthcoming definition.

Theorem 1.1 *An abelian group G is a vector space over $\mathbb{Q}$ if and only if it is torsion free and divisible.*

Now, with that out of the way, we can present the definition of a rational topological space.

Definition 1.1 *A simply connected topological space X is rational if any of the following equivalent conditions hold:*

1. *$\pi_\bullet X$ is a $\mathbb{Q}$ vector space.*
2. *$\widetilde{H}_\bullet(X;\mathbb{Z})$ is a $\mathbb{Q}$ vector space.*
3. *$\widetilde{H}_\bullet(\Omega X,\mathbb{Z})$ is a $\mathbb{Q}$ vector space.*

where $\widetilde{H}_\bullet$ is the reduced homology. I should now define the Moore loop space ΩX.

We begin with the unrestricted Moore path space $MX \subset X^{\mathbb{R}_{\geq 0}}$ the subset of continuous functions from $\mathbb{R}_{\geq 0}$ that are constant after a certain time. That is, a Moore path is a pair (γ, l) with $\gamma : \mathbb{R}_{\geq 0} \to X$ and $l \in \mathbb{R}$ such that $\gamma(t) = \gamma(l)$ whenever $t \geq l$. We then call this a path of length l. We can then define the Moore path space PX for a topological space with base point x_0 as $\{(\gamma, l) \in MX : \gamma(l) = x_0\}$. Finally, we define the Moore loop space $\Omega X \subset PX$ as the subset of paths which also begin at x_0.

Our next goal is to define the rationalization of a space, which is the rational topological space which in some sense best represents it. However, before doing so, we will need to briefly digress into some group theory once again. Readers familiar with the tensor product of abelian groups and its properties can skip this discussion.

Tensor products are defined, essentially, as the most general structure which respects multilinear maps. For this to make sense, we must have a notion of linearity, meaning we must have a module structure. Thankfully, as noted earlier, all abelian groups can be equipped with a module structure over $\mathbb{Z}$ in a canonical way. With this additional, but free, structure, we define the tensor product of abelian groups to be their tensor product when viewed as $\mathbb{Z}$-modules. To be explicit, take G and H to be groups. We then construct $G \otimes H$ as the free module over $G \times H$ "mod linearity". That is, we take $(g+g') \otimes h \sim g \otimes h + g' \otimes h$, $g \otimes (h' + h) \sim g \otimes h' + g \otimes h$, and $(ng) \otimes h \sim g \otimes (nh) \sim n(g \otimes h)$.

In particular, the tensor product of a module of dimension n and a module of dimension m is a new module of dimension nm (when all dimensions are defined). With this construction, we need to understand $G \otimes \mathbb{Q}$ and its properties. In particular, we will show that this is a vector space over $\mathbb{Q}$.

Every element of $G \otimes \mathbb{Q}$ is of the form $\sum n_i g_i \otimes r_i$, a finite linear combination of simple tensors. We need to define an action of $\mathbb{Q}$ on this group, but as above, this reduces to understanding how to divide. Thankfully, by distributivity it is enough to define division on the simple tensors, which we can do in the natural way: $\frac{1}{k}(g \otimes r) = g \otimes (r/k)$. It is easy to verify this yields a vector space over $\mathbb{Q}$. Moreover, we will now show that if G is already a vector space over $\mathbb{Q}$ then $G \otimes \mathbb{Q} \simeq G$. The natural isomorphism to consider is $g \otimes r \mapsto r \cdot g$. That this is an isomorphism follows from

$$b(\frac{a}{b} g \otimes 1) \sim b(\frac{g}{b} \otimes a) \sim (b\frac{g}{b}) \otimes a = g \otimes a = g \otimes (\frac{a}{b} b) \sim b(g \otimes \frac{a}{b})$$

Then, since $G \otimes \mathbb{Q}$ is a vector space over $\mathbb{Q}$, we can divide the first and last terms by b to find $(\frac{a}{b} g) \otimes 1 \sim g \otimes \frac{a}{b}$, which shows the natural map is an isomorphism.

Finally, we can use these comments to define a rationalization of a topological space:

Definition 1.2 *Let X be a simply connected space and Y a rational simply connected space. Then a map $r : X \to Y$ is a rationalization of X if either of the (equivalent) conditions hold:*

1. $\pi_\bullet r \otimes \mathbb{Q} : \pi_\bullet X \otimes \mathbb{Q} \xrightarrow{\simeq} \pi_\bullet Y \otimes \mathbb{Q} \simeq \pi_\bullet Y$
2. $H_\bullet(r, \mathbb{Q}) : H_\bullet(X, \mathbb{Q}) \xrightarrow{\simeq} H_\bullet(Y, \mathbb{Q})$

Our next goal is to show that a space always has a rationalization, and that this rationalization is in some sense unique. The good news is that we will actually be able to construct this rationalization; the bad news is that the construction is, at least at first glance, somewhat complicated and unintuitive. However, if the reader is already familiar with the proof from regular homotopy theory that all spaces have a CW-complex which represents their homotopy groups, then the proof becomes much more transparent. Indeed, the method we will use in the rational case has the following steps:

1. Find a rational analog of S^n and D^m in the construction of CW-complexes.
2. Find the rational analog of CW-complexes, $CW_{\mathbb{Q}}$-complexes.

3. Using a very similar constructing to the case for CW-complexes, glue together these rational cells so the conditions in definition 2.2 can be satisfied.

Let's proceed. For the first point, let's consider what is actually important about S^n and D^m in the theory of CW-complexes, namely that $\pi_n(S^n) = \mathbb{Z}$ and $\pi_j(S^n) = 0$ when $j < n$, which allows us to construct groups containing any other group (in particular, containing the nth homotopy group of a space we're interested in) by gluing together n-spheres, and then that $\pi_j(D^n) = 0$ for all j, which, together with the fairly natural method of gluing disks onto their boundaries, allows us to kill off unnecessary elements of the nth homotopy group of our CW complex. In the rational case, given the equivalent conditions in definition 2.1, we will consider the homology groups instead of the homotopy groups. Thankfully, the rational n-sphere and n-disk end up simply being the rationalizations of S^n and D^n, henceforth referred to as $S^n_{\mathbb{Q}}$ and $D^n_{\mathbb{Q}}$. I will now give a construction of $S^n_{\mathbb{Q}}$ and $D^n_{\mathbb{Q}}$ and compute their reduced homology.

$$S^n_{\mathbb{Q}} = \left(\bigvee_{i=0}^{\infty} S^n_i\right) \bigcup_h \left(\bigsqcup_{j=1}^{\infty} D^{n+1}_j\right)$$

where the attaching maps h_j are given by a map $S^n \to S^n_{j-1} \vee S^n_j$ representing $[1_{j-1}] - j[1_j]$ where $[1_i] \in \pi_n(S^n_i)$ is the generator of the nth homotopy group of S^n. For example, in the case of the circle these maps would go once around S^1_{i-1} and then go backwards around S^1_i i times. What is the intuition behind this approach? We need a way to divide by integers, and since the disk is contractible, we would have $[1_{j-1}] - j[1_j] = 0$, or $\frac{1}{j}[1_{j-1}] = [1_j]$. If we do this enough times, we can get a denominator with any factors we want. (it may bother the astute reader that the circle is not simply connected, and they would be right; we do not construct $S^1_{\mathbb{Q}}$ for this reason, but the maps h_j are easiest to picture in this case). We can then define the the rational $n+1$-disk as

$$D^{n+1}_{\mathbb{Q}} = S^n_{\mathbb{Q}} \times I / S^n_{\mathbb{Q}} \times \{0\}$$

I.e. as the cone of $S^n_{\mathbb{Q}}$. This definition should seem reasonable, since D^n is homeomorphic to the cone of S^{n-1}. Now we must show that this space is rational. To do so, consider

$$X(r) = \left(\bigvee_{i=0}^{r} S^n_i\right) \bigcup_h \left(\bigsqcup_{j=1}^{r} D^{n+1}_j\right)$$

Let's take a moment to digest this with a picture of $X(1)$ when we are dealing with circles. We begin with a figure 8, and connect the left lobe to the right by gluing a disk's edge first around the left side then twice around the right. We can then see that the rightmost circle, S^1_1 is a strong deformation retract of $X(1)$ by noticing that the square is homepmorphic to the disk, and taking the attaching map to be constant on the top and bottom edges, the identity map (from S^1 to

S^1) on the left edge and twice the identity on the right edge. Since the edge of the square is a strong deformation retract of the square, we have the previous claim. Indeed, this is true for every $S_j^n \vee S_{j+1}^n \cup_h D_{j+1}^{n+1}$, which allows us to inductively show that S_r^n is a strong deformation retract of $X(r)$. Since this is true for all r, cohomology respects homotopy equivalence, and the cohomology of the spheres is trivial in all degrees except their dimension, we have that $H^k(S_{\mathbb{Q}}^n) = 0$ for all k with the possible exception of $k = n$. Then to compute the nth cohomology group we observe that the inclusion $X(r) \subset X(r+1)$ induces a morphism of the homology groups that amounts to multiplication by $r+1$. This implies that

$$\widetilde{H}_i(X(r); \mathbb{Q}) = \begin{cases} \mathbb{K} & i = n \\ 0 & else \end{cases}$$

where $\mathbb{K}$ is the subset of the rationals which have denominators that divide $r!$. Thus, once we have glued all the disks in (aka, taking the direct limit of the $X(r)$), we have

$$\widetilde{H}_i(S_{\mathbb{Q}}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = n \\ 0 & else \end{cases}$$

while the reduced homology groups of $D_{\mathbb{Q}}^n$ are all trivial. Thus both $S_{\mathbb{Q}}^n$ and $D_{\mathbb{Q}}^n$ are rational topological spaces. The homomorphism induced in the cohomology groups by the inclusion of S^n and D^n within them (within the first copy, e.g. S_1^n) are isomorphisms, as any cohomology representative in them is equal to an integer times a representative in a later sphere. Thus, for this inclusion we can pick both an integer and a rational number, but $\mathbb{Z} \otimes \mathbb{Q} = \mathbb{Q}$ so we have that it is an isomorphism.

We can now proceed to the second point above and define a relative $CW_{\mathbb{Q}}$-complex as a topological pair (X, A) such that A is simply connected, $X = \bigcup_{-1}^{\infty} X_{(i)}$ where $X_{(i)} \subset X_{(i+1)}$, $X_{(1)} = X_{(0)} = X_{(-1)} = A$, and

$$X_{(n+1)} = X_{(n)} \cup_{f_n} \left(\bigsqcup_{\alpha} D_{\mathbb{Q},\alpha}^{n+1} \right)$$

where, as in the case of CW-complexes, f_n is a function from $\sqcup D_{\mathbb{Q},\alpha}^{n+1} \supset \sqcup S_{\mathbb{Q},\alpha}^n \to X_{(n)}$. Then the images of the $D_{\mathbb{Q},\alpha}^{n+1}$ are called the rational $(n+1)$ cells of the relative $CW_{\mathbb{Q}}$-complex (the parenthesis here are important; we will use them to distinguish the rational cells from the traditional cells— we will see in a moment that a relative $CW_{\mathbb{Q}}$-complex is a relative CW-complex). Finally, we call a relative $CW_{\mathbb{Q}}$-complex a $CW_{\mathbb{Q}}$-complex if it is of the form (X, pt).

Since $S_{\mathbb{Q}}^n$ and $D_{\mathbb{Q}}^n$ are CW-complexes and the f_n are cellular maps in the usual sense, we find that $CW_{\mathbb{Q}}$ complexes are indeed CW-complexes. Moreover, since $S_{\mathbb{Q}}^n$ has only $n+1$ cells, and $D_{\mathbb{Q}}^{n+1}$ has only $n+1$ and $n+2$ cells, we find that $X_n \subset X_{(n)} \subset X_{n+1}$, where those terms without parentheses in the subscript are the skeleta of the CW-complex structure. This containment tells us that $X_1 = X_0 = pt$ so the $CW_{\mathbb{Q}}$ complexes are simply connected. Moreover, by the previous calculation, we have that the homology is a $\mathbb{Q}$ vector space.

This finally leads us to the big theorem of this section!

Theorem 1.2 *1. For every simply connected space X there is a relative CW-complex $(X_{\mathbb{Q}}, X)$ with no zero-cells or one-cells such that the inclusion is a rationalization.*

2. If $(X_{\mathbb{Q}}, X)$ is as in (1.) then any continuous map f from X to a rational space Y extends to a map $g : X_{\mathbb{Q}} \to Y$. Moreover, given a homotopy between $f, f' : X \to Y$, this extends to a homotopy of their extensions.

An immediate corollary of this is that the rationalization of a space is unique up to homotopy equivalence.

Rather than simply present a proof of this theorem, I will try to motivate the proof along the way. To begin with, we already know that there is a CW-complex, Z, which shares the same homotopy groups as X, as well as a weak homotopy equivalence $\psi : Z \to X$. Given this, it seems only natural that if we can rationalize Z, then we will have essentially rationalized X. While this is largely true, there is a slight complication resulting from the fact that ψ goes from Z to X, while a rationalization goes from X to $X_{\mathbb{Q}}$. The fix to this issue is to consider X and $Z_{\mathbb{Q}}$ as disjoint topological spaces with $Z \times I$ connecting them. In particular, once we have an inclusion $i : Z \to Z_{\mathbb{Q}}$ we will take $X_{\mathbb{Q}} = X \cup_{\psi} Z \times I \cup_i Z_{\mathbb{Q}}$, where the first identification is made between $(z, 0)$ and $\psi(z)$ and the second is made between $(z, 1)$ and $i(z)$.

This leaves us to find this inclusion. Since Z is a CW-complex, there is really only one natural notion for what its rationalization can be: replacing each n-cell of Z with a rational n-cell. The only particular challenge here is determining how to extend each gluing map $f_\alpha : D^n_\alpha \to Z_{n-1}$ to a rational gluing map $F_\alpha : D^n_{\mathbb{Q},\alpha} \to Z_{(n-1)}$. We formalize this as follows.

Suppose, by way of induction, that we have already defined $Z_{(n-1)}$ and i on Z_{n-1}. Further, take $Z_n = Z_{n-1} \cup_{f_\alpha} (\sqcup D^n_\alpha)$, we will define $Z_{(n)} = Z_{(n-1)} \cup_{F_\alpha} \left(\sqcup D^n_{\mathbb{Q},\alpha}\right)$. To define the F_α we begin by recognizing that $if_\alpha : S^{n-1} \to Z_{(n-1)}$ is a representative of an element of $\pi_{n-1}(Z_{(n-1)})$, which is a $\mathbb{Q}$-vector space. We can offer a preliminary extension of f_α to $G_\alpha : S^{n-1}_{\mathbb{Q}} \to Z_{(n-1)}$ by first defining $G_\alpha|_{S^n_0}$ as simply if. Then we define $G_\alpha|_{S^{n-1}_r}$ to be a representative of $[if]/r!$ in $\pi_{n-1}(Z_{(n-1)})$. To define $G_\alpha|_{D^n_r}$ we first note that D^n_r is attached to $S^{n-1}_{r-1} \vee S^{n-1}r$ by a representative h of $[id_{S^{n-1}_{r-1}}] - r[id_{S^{n-1}_r}]$. Therefore, $[G_\alpha h] = [if]/(r-1)! - r[if]/r! = 0$. Hence, $G_\alpha h$ is homotopic to the constant map, which can be extended to the entire disk and then homotoped back (this is because of the whitehead lifting lemma). This defines G_α on all of $S^{n-1}_{\mathbb{Q}}$. This is then a map between CW-complexes $S^{n-1}_{\mathbb{Q}}$ and $Z_{(n-1)}$ and so is homotopic to a cellular map. It is this cellular map that we define as F_α. We then define i in the obvious way, as the inclusion $D^n \to D^n_{\mathbb{Q}}$ when restricted to D^n_α.

Since we now have a rationalization $i : Z \to Z_{\mathbb{Q}}$ of the cellular model of X, we have a candidate for the rationalization of X as the relative CW-complex $(X_{\mathbb{Q}}, X)$ where we remind ourselves that

$$X_{\mathbb{Q}} = X \cup_{\psi} Z \times I \cup_i Z_{\mathbb{Q}}$$

What remains is to show that the inclusion of X into $X_{\mathbb{Q}}$ is a rationalization. That is, that it induces an isomorphism of the homologies. While we need to show that $H_\bullet(X, \mathbb{Q}) \simeq H_\bullet(X_{\mathbb{Q}}, \mathbb{Q})$ we will for now content ourselves to show that $H_\bullet(X, \mathbb{Z}) \simeq H_\bullet(X_{\mathbb{Q}}, \mathbb{Z})$, which will ultimately be sufficient. First, we can use excision to find

$$H_\bullet(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; \mathbb{Z}) \simeq H_\bullet(X \cup_{\psi} Z \times I, Z \times \{1\}; \mathbb{Z})$$

Then since ψ (being a weak homotopy equivalence) induces an isomorphism of homologies we find

$$H_\bullet(X \cup_{\psi} Z \times I, Z \times \{1\}; \mathbb{Z}) = 0$$

In turn, this implies that $H_\bullet(X_{\mathbb{Q}}, pt; \mathbb{Z}) \simeq H_\bullet(Z_{\mathbb{Q}}, pt; \mathbb{Z})$. Then because the latter is a $\mathbb{Q}$ vector space, the former is as well, and their homologies with rational coefficients are the same. Thus, the inclusion of X in $X_{\mathbb{Q}}$ is a rationalization.

In proving (2) our objective will be to use the Whitehead lifting lemma to define g. In particular, we wish to use the white head lifting lemma on the diagram

$$\begin{array}{ccc} Y & \xrightarrow{id} & Y \\ \downarrow & & \downarrow j \\ Y \cup_f X_{\mathbb{Q}} & \xrightarrow{\sigma} & (Y \cup_f X_{\mathbb{Q}})_{\mathbb{Q}} \end{array}$$

where σ is essentially the inclusion. To do so, however, we need to show that j is a weak homotopy equivalence. In doing so, we will invoke the Whitehead-Serre theorem, which we will not prove. In essence, it shows a continuous map between simply connected spaces induces an equivalence of the homotopy groups (tensored with $\mathbb{Q}$) iff it induces an equivalence of the Homology groups with rational coefficients. For this reason, we will need to show that the morphism of homology groups induced by j is an equivalence. To whit:

Define $(Y \cup_f X_{\mathbb{Q}}, Y)$ to be the CW-complex obtained by identifying X with its image in Y. The construction from part 1 gives a rationalization $Y \cup_f X_{\mathbb{Q}} \to (Y \cup_f X_{\mathbb{Q}})_{\mathbb{Q}}$. However, since Y is rational, we can consider this as a rationalization $\sigma : (Y \cup_f X_{\mathbb{Q}}, Y) \to ((Y \cup_f X_{\mathbb{Q}})_{\mathbb{Q}}, Y)$. This gives us that $H_\bullet(\sigma, \mathbb{Q})$, the induced map on the rational homologies, is an isomorphism. However, $H_\bullet(i, \mathbb{Q}) : H_\bullet(X; \mathbb{Q}) \to H_\bullet(X_{\mathbb{Q}}; \mathbb{Q})$ is also an isomorphism, so we have that $H_\bullet(Y \cup_f X_{\mathbb{Q}}, Y; \mathbb{Q}) \simeq H_\bullet(X_{\mathbb{Q}}, X; \mathbb{Q}) = 0$. Therefore $H_\bullet(Y; \mathbb{Q}) \simeq H_\bullet(Y \cup_f X_{\mathbb{Q}}; \mathbb{Q})$. Indeed, if we take j to be the inclusion $Y \to Y \cup_f X_{\mathbb{Q}} \to (Y \cup_f X_{\mathbb{Q}})_{\mathbb{Q}}$ the previous result, combined with the definition of the rationalization tells us that j induces an isomorphism of the Homologies. Therefore, by the Whitehead-Serre Theorem, it is a weak homotopy equivalence (since both the domain and target are rational, their homotopy groups do not change when tensored with the rationals.)

We can now apply the Whitehead lifting lemma to the aforementioned diagram, providing us with a map $g : Y \cup_f X_{\mathbb{Q}} \to Y$, which, when restricted to $X_{\mathbb{Q}}$ gives us the desired map.

To see then that homotopies lift, we must consider the space $K = X_{\mathbb{Q}} \times \{0\} \cup X \times I \cup X_{\mathbb{Q}} \times \{1\}$. Given $g, g' : X_{\mathbb{Q}} \to Y$ that restrict to homotopic maps f and f', we can easily define a $G : K \to Y$. Then, the calculation below shows that the inclusion $i : K \to X_{\mathbb{Q}} \times I$ is a rationalization, so G extends to a homotopy in $X_{\mathbb{Q}}$.

Chapter 2

Differential Graded Algebras and Sullivan Algebras

Although the reader is probably familiar with them from their study of homology, I think it is useful to include a brief discussion of graded complexes, and chain and cochain complexes. A graded complex $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is a set of abelian groups (or more generally, a coproduct of objects from any abelian category) which are indexed by the integers equipped with a map $\partial : A \to A$ that satisfies $\partial(A_i) \subset A_{i-1}$ and $\partial \circ \partial = 0$. These are the central object of study in homology theory, and we require both that the category be abelian and the second condition on ∂ so we can take the quotient object $\ker(\partial_{i-1})/\mathrm{Im}(\partial_i)$. We generally consider chain complexes to have only non-negatively indexed entries (i.e. $A_i = 0$ whenever i is negative). We then have two equivalent notions of a cochain complex. The first is the natural definition given by the categorical dual: a set $A = \bigoplus A^i$ which have a map $d : A \to A$ such that $d(A^i) \subset A^{i+1}$ and $d \circ d = 0$. We can also consider a cochain complex as a graded complex which has nontrivial entries only in the negative values. We generally use A_i when we are referring to a graded complex (i.e. one where the boundary map reduces degree) and A^i when we are referring to a cochain complex, and when we switch between these two views we use the convention $A^i = A_{-i}$.

A morphism of (co)chain complexes is a set of maps $f_i : A_i \to B_{i+j}$ such that $\partial f_i = f_{i+1}\partial$ (so that there is a well defined induced map of the homology groups). The j is constant and is called the degree of the map f. In particular, we can thus define $\partial : A \to A$ as a chain map of degree -1, and $d : A \to A$ as a cochain map of degree 1. If $a \in A_i$ we define the degree of a to be i.

We can now define a commutative differential graded algebra. As a first, most abstract definition, we define it to be a commutative monoid in the category of cochain complexes over $\mathbb{Q}$ (that is, the objects A_i are now taken to be vector spaces over $\mathbb{Q}$). Let's untangle this definition a bit. To begin with, a monoid needs a monoidal category—a category C equipped with a functor $F : C \times C \to C$. Furthermore, this functor must be associative (up to natural isomorphism, for those who care) and have an identity object. Put symbolically, that means $F(A, F(B, C)) = F(F(A, B), C)$ and that there is an object I of C such that $F(I, A) = A = F(A, I)$.

Does the category of cochain complexes over $\mathbb{Q}$ satisfy these conditions? It does, as we can see by taking $F = \otimes_{\mathbb{Q}}$. That is we take $F(A, B)$ to be $A \otimes_{\mathbb{Q}} B$, which is the cochain complex which satisfies $(A \otimes_{\mathbb{Q}} B)^i = \bigoplus_{j+k=i} A^j \otimes_{\mathbb{Q}} B^k$ and $d(a \otimes b) = da \otimes b + (-1)^{deg\ a} a \otimes db$. However, since A^j and B^k are taken to be vector spaces over $\mathbb{Q}$ this amounts essentially to the usual tensor product. We also know that $\mathbb{Q} \otimes V = V$ when V is a vector space over $\mathbb{Q}$, so we can take $\mathbb{Q}$ to be the cochain complex whose only nontrivial entry is $\mathbb{Q}$ in degree 0 and find that $(A \otimes \mathbb{Q})^i = (\mathbb{Q} \otimes A)^i = \bigoplus \mathbb{Q}^j \otimes A^k = \mathbb{Q}^0 \otimes A^i = A^i$ and $d(r \otimes a) = dr \otimes a + (-1)^{deg\ r} r \otimes da = 0 + r \otimes da = 1 \otimes d(ra)$. Thus, the tensor product exhibits the category of cochain complexes as a monoidal category.

Equipped with the previous discussion, we can quickly define a monoid in the category of cochain complexes. The general definition of a monoid, M, is an object equipped with a map from the identity object and a map from $M \otimes M \to M$ that is associative and respects the identity. In particular, that means a differential graded algebra is a cochain complex with a map $\eta : \mathbb{Q} \to M$ and $\mu : M \otimes M \to M$ such that μ is associative and $\mu(\eta \otimes Id_A) = Id_A = \mu(Id_A \otimes \eta)$. We call such a differential graded algebra commutative (a CDGA for short) if it also satisfies $\mu(a \otimes b) = (-1)^{deg\ a \cdot deg\ b} \mu(b \otimes a)$.

Now, that was a lot of decoding category theory, so let's take a moment to comment on what this really means. In particular, we all know what an algebra is, so how is the above actually an algebra? Well, since we are interested in vector spaces over $\mathbb{Q}$ and cochain maps are linear, all η is really doing is selecting a particular element $\eta(1) \in A^0$ which we take to be the identity of the multiplication μ, which is of degree 0. Indeed, when we do this, we can see that it makes A as a whole into an algebra in the usual sense. In fact, we would equally well define a CDGA as graded vector space with a unital product structure and a differential which satisfies the (graded) Leibniz rule (or, equivalently, a derivation which is nilpotent). In saying this, it is worth noting that our definition does satisfy the Leibniz rule because of the definition of the differential in the tensor product.

We can now define a category of CDGAs by defining morphisms between them to be cochain morphisms $f : (A, \eta, \mu) \to (B, \eta', \mu')$ that satisfy $f\mu = \mu'(f \otimes f)$ (i.e. $f \circ \mu(a, b) = \mu'(f(a), f(b))$) and $f\eta = \eta'$ (or f(1)=1').

The next object of interest is the notion of a relative Sullivan algebra, which we define as a step on the way to defining a Sullivan model. However, to do even this we will need to understand the notion of a free CDGA. The prototypical model for a free CDGA is any CDGA whose underlying algebra is isomorphic to ΛV where V is a graded vector space, and ΛV is defined to include graded commutativity; in particular, we have that $\Lambda V = S(V^{even}) \otimes E(V^{odd})$ where $S(-)$ represents the symmetric algebra, and $E(-)$ represents the exterior algebra. Readers comfortable with the category theoretic notion of a free object should skip the next two paragraphs, in which I will show that this is a free object in CDGA (over the category of graded vector spaces) and thus that any morphism from ΛV is determined by its value on V and, conversely, that any morphism from V extends to a unique morphism from λV.

The usual definition of a free object over a set A in a concrete category is defined to be one that satisfies the following universal property: Given a function of sets $f : A \to B$ where B is a CDGA, there is a unique morphism of CDGAs $u : \Lambda A \to B$ such that f is the restriction of u to A. Phrased diagrammatically, this means:

$$\begin{array}{ccc} A & \xhookrightarrow{i} & \Lambda A \\ \downarrow f & \swarrow \exists!\ u & \\ B & & \end{array}$$

where the diagram commutes, f and i are maps of sets and u is a morphism of CDGAs. However, in this case we want a free object generated by a set that already has some structure (the graded vector space structure) so we must make the slight modification of restricting f and i to be linear functions of a fixed degree. Once we can prove that this new universal property holds, the two other results are immediate; the first follows from the uniqueness in the universal property and the second follows from existence.

To show this, we need to begin by understanding the structure of ΛV. The symmetric algebra is linear combinations of elements of the form $\prod v_i^{n_i}$, that is, the polynomial algebra with a basis of V as indeterminate. The degree of ab where a and b are such monomials is deg(a)+deg(b). Much of the same is true for the exterior algebra. The degree of a product is still the sum of the degrees, the only real difference is that instead of polynomials, we have the wedge product. Take v and w in V, then $v \wedge w = (-1)^{deg\ w \cdot deg\ v} w \wedge v$. Therefore, an arbitrary element of ΛV is a linear combination of elements of the form $v_1...v_n \otimes w_1 \wedge ... \wedge w_k$, and the degree of this object is $\sum \text{deg}(v_i) + \sum \text{deg}(w_j)$. Since we are usually only interested in the degree modulo 2, we can see that this is congruent to $\sum \text{deg}(w_j) \equiv k \mod 2$ since all the v_i have even degree and all the w_j have odd degree. Now we can verify that this is a commutative DGA: take $x_1 = v_1...v_{n_1} \otimes w_1 \wedge ... \wedge w_{k_1}$ and $x_2 = y_1...y_{n_2} \otimes z_1 \wedge ... \wedge z_{k_2}$. Then

$$x_1 \cdot x_2 = v_1...v_{n_1} y_1...y_{n_2} \otimes w_1 \wedge ... \wedge w_{k_1} \wedge z_1 \wedge ... \wedge z_{k_2}$$
$$(-1)^{k_1 k_2} y_1...y_{n_2} v_1...v_{n_1} \otimes z_1 \wedge ... \wedge z_{k_2} \wedge w_1 \wedge ... \wedge w_{k_1} = (-1)^{deg\ x_1 \cdot deg\ x_2} x_2 x_1$$

We can then define the u in our diagram in the obvious way: $u(1) = 1_B$, $v \in V \Rightarrow u(v) = f(v)$, and $u(\alpha \cdot \beta) = u(\alpha) \cdot u(\beta)$. We then extend this map via linearity. This map is well defined — $u(\alpha \cdot \beta) = u(\alpha) \cdot u(\beta) = (-1)^{\text{deg} u(\alpha) + \text{deg} u(\beta)} u(\beta) \cdot u(\alpha)$ $= (-1)^{\text{deg}\alpha + \text{deg}\beta + 2j} u(\beta) \cdot u(\alpha) = (-1)^{deg\ \alpha \cdot deg\ \beta} u(\beta \cdot \alpha)$ where j is the degree of the map.

With this, we are equipped to define a relative Sullivan algebra.

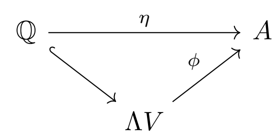
Definition 2.1 *A relative Sullivan Algebra is an inclusion of CDGAs* $A \to A \otimes \Lambda V$ *satisfying*

1. *There exists an increasing (ordered by containment) sequence of graded subspaces* $V(k) \subset V(k+1)$ *such that* $V = \cup_0^\infty V(k)$.
2. $dV(0) = \{0\}$ *and* $dV(k) \subset \Lambda V(k-1)$.
3. $V^0 = 0$ *(this is the degree zero factor of V, not the space V(0)).*

We further define a Sullivan Algebra to be a relative Sullivan algebra with $A = \mathbb{Q}$ and a minimal (relative) Sullivan algebra as a Sullivan algebra satisfying $\text{im}(d) \subset \Lambda^{\geq 2} V$.

As the last thing to be introduced in this chapter, we can define a (relative) Sullivan minimal model of a CDGA.

Definition 2.2 *[3] A Sullivan minimal model of a CDGA, A, which satisfies* $H^0(A) = \mathbb{Q}$ *is a minimal Sullivan algebra paired with a quasi-isomorphism* ϕ *for which the following diagram commutes.*



This is a marvelously succinct definition, and it reflects the technical definition of a Sullivan algebra, as both an object and a map, but it may obfuscate an otherwise simple meaning. A Sullivan minimal model of A is simply a minimal Sullivan algebra which has the same cohomology as A.

There is are two important theorem about these objects, namely that they always exist and any two are isomorphic. Astute readers will notice that this means that Sullivan models bear a striking resemblance to rationalizations of a topological space; a model of the algebraic structure that is unique up to some kind of homotopy equivalence. There is good reason; these will be the fundamental algebraic objects we use to discuss rational homotopy theory and, in particular, the homotopy and homology of rational spaces. Thus, the remainder of the chapter will be devoted to their proofs.

Theorem 2.1 *For any CDGA A satisfying* $H^0(A) = \mathbb{Q}$ *there is a map* $\phi : \Lambda V \xrightarrow{\simeq} A$ *which is a Sullivan model for A.*

Proof: We will construct this model inductively. We begin with $V(0) = H^+(A;\mathbb{Q})$, and $d|_{\Lambda V(0)} = 0$. Since $d = 0$, we have that $H(\Lambda V(0)) = \Lambda V(0)$, so we define $H(\phi)_0' : V(0) \to H^+(A)$ to be the identity. Then we define $\phi_0' : V(0) \to A$ such that $\phi'(v)$ is a representative of $H(\phi')(v)$. It is worth noting that this requires the axiom of choice; if we have a basis v_α for $V(0)$ then for each α we must pick an element of the equivalence class of $H(\phi')(v_\alpha)$. Since we have not restricted the dimension of any degree in V, we cannot do this without choice in general. In any event, once we have defined $\phi'_0 : V(0) \to A$ we use the universal property for $\Lambda V(0)$ to extend it to $\phi_0 : \Lambda V(0) \to A$.

We now presume that we have defined ϕ_k for some k, so that we can extend it to ϕ_{k+1}. Take z_α to be cocycles in $\Lambda V(k)$ (that is, $dz_\alpha = 0$) such that $[z_\alpha]$ is a basis for $\ker H(\phi_k)$. Then take $W(k+1)$ to be the free vector space on $\{z_\alpha\}$ (for clarity we will henceforth call them z_α when we are referring to the elements of $V(k)$ and y_α when we are referring to the generators of $W(k+1)$). We then assign a grading on $W(k+1)$ by declaring that $deg\ y_\alpha = deg\ z_\alpha - 1$. Finally, we take $V(k+1) = V(k) \bigoplus W(k+1)$. We then extend the differential to $\Lambda V(k+1)$ by declaring $dy_\alpha = z_\alpha$. Then, since $d^2 y_\alpha = dz_\alpha = 0$ this extension remains a differential. We can then use commutativity of ϕ and d to extend ϕ_k to $\phi_{k+1} : \Lambda V(k+1) \to A$ by $\phi(y_\alpha) = a_\alpha$ where $da_\alpha = \phi_k(z_\alpha)$ (we know such an a_α exists by the definition of z_α as satisfying $H(\phi_k)([z_\alpha]) = 0$).

This completes the construction of ϕ, but we must still prove that it is a quasi-isomorphism. It is clear that $H(\phi)$ is surjective in the positive degrees of $H^\bullet(A)$, since $H(\phi)|_{V(0)}$ is already surjective. Moreover, since $V(0)^0 = 0$, we have that the degree zero component of $\Lambda V(0) = \mathbb{Q} = H^0(A)$ we have that it is actually surjective over all of $H^\bullet(A)$. Then to see that it is an injection we note $H(\phi)[x] = H(\phi)[y] \Rightarrow H(\phi)([x-y]) = 0$. Then $V = \cup_0^\infty V(k) \Rightarrow x - y \in \Lambda V(k)$ for some k. However, by construction this implies there is some $y_\alpha \in V(k+1)$ such that $dy_\alpha = x - y$ so $x - y$ is in the image of d, so $[x - y] = 0$. Thus $H(\phi)$ is an isomorphism.

Now we have constructed a V and ϕ almost as in the definition of a Sullivan model. All we must do now is check that $V^0 = 0$. Clearly $V^0(0) = 0$, so if $V^0 \neq 0$ there must be a smallest $V(k)$ such that $V^0(k) \neq 0$. In turn, this requires that $W^0(k) \neq 0$ which require that there is at least one $0 \neq z \in V^1(k-1)$ such that $H(\phi)_{k-1}[z] = 0$. However, this requires that $z = dy$ for some $y \in V^0(k-1)$ which contradicts minimality of k. This proves that any space satisfying $H^0(A) = \mathbb{Q}$ has a Sullivan model. □

We can do better than this, but doing so requires more machinery than I think is purely appropriate for this text; in particular, we would need a lengthy discussion into the properties of general relative Sullivan models and would allow us to add the word minimal to the above theorem. Instead, I will present the simply connected version of the proof and direct the reader to Theorem 14.12 of [1]. The simply connected case will present a different construction of the model, but I have decided to keep both constructions in since it is the above construction that is used in the more general case.

Theorem 2.2 *For any CDGA A satisfying $H^0(A) = \mathbb{Q}$ and $H^1(A) = 0$ there is a map $\phi : \Lambda V \xrightarrow{\simeq} A$ which is a minimal Sullivan model for A.*

Proof: We proceed by a slightly different method in this case. We begin with $V(2)$ which is concentrated in degree 2, and an isomorphism $H(\phi_2) : V(2) \to H^2(A)$; defining the differential to be zero on $V(2)$ then identifies $\Lambda V(2)$ with $H(\Lambda V(2))$ as before, so we can use the same procedure to define

$\phi_2 : \Lambda V \to A$. Then suppose we have constructed ϕ_k through some k to be extended to a $\phi_{k+1} : \Lambda V(k+1) \to A$. Begin by selecting $a_\alpha \in A$ and $z_\beta \in (\Lambda V(k))^{k+2}$ satisfying $da_\alpha = dz_\beta = 0$ and so that the $[a_\alpha]$ form a basis of $H^{k+1}(A)/Im\ H(\phi_k)$ and the $[z_\beta]$ form a basis of ker $H^{k+2}(\phi_k)$. As before, we will take $W(k+1)$ to be the free vector space generated by the z_β and a_α this time concentrated in degree $k+1$ and define $V(k+1) = V(k) \bigoplus W(k+1)$ (I will henceforth use primes to denote the copies of z_β and a_α in W(k+1)). Next we extend the differential according to $da'_\alpha = 0$ and $dz'_\beta = z_\beta$ and extend ϕ_k to ϕ_{k+1} by defining $\phi_{k+1}(a'_\alpha) = a_\alpha$ and $\phi_{k+1}(z'_\beta) = b_\beta$ where $b_\beta \in A$ satisfies $db_\beta = \phi_k z_\beta$. Then $d^2 z'_\beta = dz_\beta = 0$ so the extension of d is still a differential. Likewise, $\phi_{k+1} da'_\alpha = \phi_{k+1} 0 = 0 = da_\alpha = d\phi_k a'_\alpha$ and $\phi_{k+1} dz'_\beta = \phi_{k+1} z_\beta = 0 = db_\beta = d\phi_{k+1} z_\beta$. So we have indeed constructed a CDGA and a morphism of the same.

Before proving that it is a minimal Sullivan algebra, I will discuss why it should be, that is, why we would expect our construction to yield a minimal Sullivan algebra. First, we are concerned with the bijectivity of $H(\phi)$. The selection of the a_α, as a spanning set of that part of $H^{k+1}(A)$ which is not yet in the image of $H(\phi)$ is intended to fix surjectivity, since now all the a_α are in the image of ϕ, and the selection of z_β is intended to kill the kernel of $H(\phi)$; in particular, the z_β form a basis for the kernel of $H^{k+2}(\phi)$, but we define the extension of d so that $z_\beta \in \text{Im } d$, i.e. $H^{k+1}(\Lambda V) \ni [z_\beta] = 0$ for all beta. Second, we are concerned with why this should be minimal, which is a property of the differential, namely that Im $d \subset \Lambda^{\geq 2} V$. This follows quickly from the definition of z_β; V(k) is concentrated in degrees $\leq k$, so there are no elements of degree $k+2$ of word length 1 or 0, hence the $z_\beta \in \Lambda^{\geq 2} V(k)$.

Indeed, this second comment is actually enough to prove minimality, and since the differential takes $V(k+1)$ into $\Lambda V(k)$ we have that ΛV is indeed a Sullivan algebra. Once we formalize the first comment a little bit we will have that it is also a minimal Sullivan model. Begin by noting that $H^i(\phi) = H^i(\phi_{i+1})$ since there are no new elements of degree less than or equal to $i+1$ added by any subsequent steps in the construction. Therefore it is sufficient to show that $H^i(\phi_{i+1})$ is an isomorphism. To this end it will be useful to note that $H^i(\phi_{i-1})$ is injective for all i. As a base case, observe that $H^1(\Lambda V) = 0$ since ΛV is nontrivial only in even degrees, so $H^1(\phi_2)$ is an isomorphism. Furthermore, we have that $H^3(\phi_2)$ is injective since $(\Lambda V)^3 = 0$. Next we consider the general case: take $H^{i+1}(\phi_i)$ to be injective and $H^i(\phi_i)$ is bijective, and consider first $H^{i+1}(\phi_{i+1})$, which is surjective by the definition of a_α and $\phi(a'_\alpha)$. To see that it is injective we need to know first that $H(\phi_{i+1})|_{W(i+1)}$ is injective and then that $H^{i+1}(\phi_{i+1})(W(i+1)) \cap \text{Im } H^{i+1}(\phi_i) = \emptyset$. The first point is clear since $[a_\alpha]$ is a basis of a subspace of $H^{i+1}(A)$ and since $\phi_{i+1}(z'_\beta) \in \text{Im } d$ so $H^{i+1}(\phi_{i+1})([z_\beta]) = 0$. The second point follows from the fact that $[a_\alpha]$ is a basis of $H^{i+1}(A)/\text{Im } H^{i+1}(\phi_i)$, i.e. $H^{i+1}(A) \equiv \text{Im } H^{i+1}(\phi_i) \oplus \bigoplus \mathbb{Q} a_\alpha$, so the image of $[a'_\alpha]$ was not already hit by $H^{i+1}(\phi_i)$. That $H^{i+2}(\phi_{i+1})$ is injective follows exactly from the comment in the previous paragraph. $\square$

Proving the uniqueness result still requires some machinery, but it is interesting machinery in and of itself. In particular, we will need a description of homotopy in the CDGA category. Of course, this is not a topological category (by which I mean a category with a faithful functor to **Top**) so the usual definition of homotopy does not apply. Instead we need a more algebraic definition, and I will borrow motivation from [1]. In the next chapter I will introduce a contravariant functor A_{pl} from **Top** to $\mathbf{CDGA}_{\mathbb{Q}}$ which will be our principle connection between the topology and the algebra. The image of I, the unit interval/standard 1 simplex, under this functor will be the $CDGA$ $\Lambda(t, dt)$ which is defined by $\Lambda(t, dt)^0 = \mathbb{Q}t$ and $\Lambda(t, dt)^1 = \mathbb{Q}dt$. Now, we would want an algebraic notion of homotopy to respect the topological notion. To clarify what I mean, consider a diagram (in which both the lower and upper triangles separately commute) representing a homotopy H from $f : X \to Y$ to $g : X \to Y$:

$$\begin{array}{ccc} X \times I & \xrightarrow{r_0} & X \\ \big\downarrow{\scriptstyle r_1} & \searrow^{H} & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

where r_i is the restriction of H to $t = i$. How would A_{pl} act on this diagram? It would reverse all the arrows, and replace all the spaces by suitable CDGAs. Most importantly, it will end up that $A_{pl}(X \times I) = A_{pl}(X) \otimes \Lambda(t, dt)$. This gives rise to our approximate definition of a homotopy of function of CDGAs; we say the functions f and g are homotopic if there is a diagram with the top and bottom triangles commuting

$$\begin{array}{ccc} A \otimes \Lambda(t, dt) & \xrightarrow{id \cdot \epsilon_0} & A \\ \big\downarrow{\scriptstyle id \cdot \epsilon_1} & \nwarrow^{H} & \big\uparrow{\scriptstyle f} \\ A & \xleftarrow[g]{} & B \end{array}$$

To make this precise we only need to figure out what ϵ_0 and ϵ_1 are. They turn out to be the augmentations (maps from the algebra to the base field) $\epsilon_i(t) = i$.

An important result, which we will use in the remaining two lemmas before this chapters final theorem is the lifting lemma for Sullivan algebras.

Lemma 2.1 *Given a diagram*

$$\begin{array}{ccc} & & A \\ & & \big\downarrow{\scriptstyle \eta} \\ \Lambda V & \xrightarrow{\psi} & B \end{array}$$

where ΛV is a Sullivan algebra and η is a surjective quasi-isomorphism, there is a map $\phi : \Lambda V \to A$ such that $\phi\eta = \psi$.

Proof: We will construct ϕ inductively over the $V(k)$. We suppose we have constructed ϕ in V(k). Then, because $d(V(k+1)) \subset \Lambda V(k)$, we know that $phi(dv)$ is defined for $v \in V(k)$. Moreover, $\eta\phi(dv) = \psi(dv) = d\psi(v)$ so $\phi(dv)$ is a coboundary (since eta is a quasi-isomorphism). Moreover, since η is surjective there is an a in the preimage of $\psi(v)$ and $da = \phi dv$. Performing this construction for a basis and then extending via linearity will give the final construction. This leaves only the construction of a base case. This is actually very simple, however, since the above construction goes through if we consider $V(-1) = 0 \subset V$ with the trivial linear map. Then since $d|_{V(0)} = 0$ we still have that $d(V(0)) \subset \Lambda V(-1)$. $\square$

The reader should be able to convince themselves that this forms an equivalence relation.

The next result is an extension of the lifting lemma to a more general case, in which η is not necessarily surjective. We will use it to find an isomorphism between Sullivan models.

Lemma 2.2 *Given a diagram*

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle \eta} \\ \Lambda V & \xrightarrow{\psi} & B \end{array}$$

where ΛV is a Sullivan algebra and η is a quasi-isomorphism, there is a map $\phi : \Lambda V \to A$ such that $\phi\eta \sim \psi$. Moreover, if ϕ_1 and ϕ_2 are two maps satisfying this constraint, then they are homotopic.

Proof: The guiding technique will be to find a surjective map that we can factor appropriately. Thus we will begin by showing this in the case that η is surjective. Existence is an immediate result of the previous lemma, so we are only interested in showing that any two solutions are homotopic. So suppose we have ϕ_0 and ϕ_1 such that $\eta\phi_0 \sim \eta\phi_1$. We will attempt to construct a diagram that allows us to find a homotopy $H : \Lambda V \to A \otimes \Lambda(t, dt)$ from ϕ_0 to ϕ_1. We will start with the pullback diagram

$$\begin{array}{ccc} (B \otimes \Lambda(t, dt)) \times_{B \times B} (A \times A) & \longrightarrow & A \times A \\ \downarrow & & \downarrow{\scriptstyle \eta \times \eta} \\ B \otimes \Lambda(t, dt) & \xrightarrow{id \cdot \epsilon_1 \times id \cdot \epsilon_0} & B \times B \end{array}$$

onto which we will introduce the map $(\eta\otimes id, id_A\cdot\epsilon_1, id_A\cdot\epsilon_0) : A\otimes\Lambda(t,dt) \to (B\otimes\Lambda(t,dt))\times_{B\times B}(A\times A)$. This map is now a surjective quasi-isomorphism. That means we can take a homotopy $K : \Lambda V \to B\otimes\Lambda(t,dt)$ from $\eta\phi_0$ to $\eta\phi_1$, combine it with ϕ_0 and ϕ_1 to form a map $(K,\phi_1,\phi_0) : \Lambda V \to (B\otimes\Lambda(t,dt))\times_{B\times B}(A\times A)$ and lift it to a morphism $H : \Lambda V \to A\otimes\Lambda(t,dt)$, which is the desired homotopy from ϕ_0 to ϕ_1.

Now, to treat the case when η is not surjective we require a small trick. We introduce the CDGA $E(B) = \Lambda(B\oplus\delta B)$ where $\delta : B \to \delta B$ is an isomorphism of vector spaces (to a copy of B), and we take δ to be the differential (nilpotency of the differential then tells us that the differential is uniformly 0 on δB. We can therefore define a surjective map, ρ of CDGAs from $E(B)$ to B by $b \mapsto b$ and $\delta b \mapsto db$. We can also define an augmentation on $E(B)$ simply as $\epsilon(B) = 0$. Moreover, it is easy to see that $H(E(B)) = \mathbb{Q}$, so $H(A\otimes E(B)) = H(A)$. Thus we can say that $\eta\cdot\rho : A\otimes E(B) \to B$ is a quasi-isomorphism, and since ρ is surjective, it is as well. Moreover, $id\cdot\epsilon : A\otimes E(B) \to A$ is also a surjective quasi-isomorphism. By part one this tells us that $\eta\cdot\rho\psi_1 \sim \eta\cdot\rho\psi_0 \Rightarrow \psi_0 \sim \psi_1$ where $\psi_i : \Lambda V \to A\otimes E(B)$. It also tells us that $id\cdot\epsilon\psi_1 \sim id\cdot\epsilon\psi_0 \Rightarrow \psi_1 \sim \psi_0$. In fact, since there is an inclusion $i : A \to A\otimes E(B)$ given by $a \mapsto a\otimes 1$, we find that i is a section of $id\cdot\epsilon$, i.e. $id\cdot\epsilon i = id_A$. Therefore, $\phi_1 \sim \phi_0 \iff i\phi_1 \sim i\phi_0 \iff \eta i\phi_0 \sim \eta i\phi_1$. □

The final lemma we need to prove that all minimal Sullivan models of a CDGA are isomorphic regards the linear part of a CDGA morphism $\phi : \Lambda V \to \Lambda W$. We define the linear part of ϕ, $L(\phi) : V \to W$ as the word length one component of ϕ. That is, we require that $\phi(v) - L(\phi)(v) \in \Lambda^{\geq 2}w$.

Lemma 2.3 *1. If $\phi_0 \sim \phi_1 : \Lambda V \to A$ then $H(\phi_0) = H(\phi_1)$.*

2. If $\phi_0 \sim \phi_1 : \Lambda V \to \Lambda W$ are morphisms between minimal Sullivan algebras, and if $H^1(\Lambda V) = 0$ then $L(\phi_0) = L(\phi_1)$.

Proof of 1: We begin with the observation that any element of $\Lambda(t,dt)$ has a unique representation of the form $a + bt + cdt + x + dy$ where x and y are polynomials in t which are divisible by $t(1-t)$ (which we choose so they evaluate to 0 under both ϵ_0 and ϵ_1. Unless you've taken the time to explore $\Lambda(t,dt)$ this may seem slightly mystifying. The anti-commuting nature of the exterior algebra is why there is only ever one power of dt. The more interesting question is why any element of the form $t^n\otimes dt$, of higher order terms times dt is always representable as $d(y)$ where y is purely some polynomial in t. Examining the differential gives us our answer quite quickly, however; the definition of the symmetric algebra as a quotient of the tensor algebra tells us that the differential must obey the Leibniz rule, so $d(t^n) = nt^{n-1}dt$. Since we are working over a field, we can "integrate" this to yield $t^n dt = d(t^{n+1}/n+1)$. As such, given a homotopy $H : \Lambda V \to A\otimes\Lambda(t,dt)$ we can define a linear map $h : \Lambda V \to A$ essentially as the c in the above expression. More specifically, we define it using

$$H(z) = \phi_0(z) + (\phi_1(z) - \phi_0(z))t - (-1)^{deg\ z}h(z)dt + higher\ powers\ of\ t$$

We know that H can be expressed in this way since $id \cdot \epsilon_0$, which essentially replaces t with 0, must yield ϕ_0 and $id \cdot \epsilon_1$), which essentially replaces t with 1, must yield ϕ_1 (remember that $\epsilon_i(dt) = 0$). Now let's examine $dH - Hd$. On the one hand, $dH = Hd$, since they are morphisms of CDGAs, but we can also expand the definition on the right hand side to see if this might tell us anything interesting about h. First,

$$dH(z) = d\phi_0 z + (d\phi_1 z - d\phi_0 z)t + (-1)^{deg\ z}(\phi_1 z - \phi_0 z)dt - (-1)^{deg\ z} dhz\ dt \\ + d(higher\ order)$$

and

$$Hd(z) = \phi_0 dz + (\phi_1 dz - \phi_0 dz)t - (-1)^{deg\ z} hdzdt + higher\ order$$

Subtracting the two equations yields

$$0 = (-1)^{deg\ z}(\phi_1 z - \phi_0 z)dt - (-1)^{deg\ z}(dh + hd)z\ dt$$

Thus, $\phi_1 - \phi_0 = dh + hd$. Thus $H(\phi_1) - H(\phi_0) = 0$, since dh is clearly in the image of d, and $hd[z] = h[0] = [0]$.

Proof of 2: Our goal is to exhibit the linear parts as quotients of CDGA maps. In particular, if H is a homotopy between ϕ_0 and ϕ_1 we will consider the induced map

$$H' : \Lambda^+ V/\Lambda^{\geq 2} V \to (\Lambda^+ W/\Lambda^{\geq 2} W) \otimes \Lambda(t, dt)$$

since $\Lambda^+ V/\Lambda^{\geq 2} V = V$. To begin with, however, we need to understand why this map should be well defined, starting with why we are interested only in $\Lambda^+ V$ and not all of ΛV. A homotopy is always of degree 0, and we know that $V^0 = 0$, so the image of H will not include any elements of the form $r \otimes t$, the degree zero part of $\Lambda W \otimes \Lambda(t, dt)$. However, it may include terms of the form $r \otimes dt$, an element of $\Lambda^0 W \otimes \Lambda(t, dt)$, unless we can show that V^1 is also 0. We can do this inductively. For a base case we can use the trick above and introduce $V(-1) = 0$. The general case proceeds as follows; suppose $V^1(k) = 0$. Then, because ΛV is minimal, we have that $d(V(k+1) \subset \Lambda^{\geq 2} V(k)$, which contains elements only of degree 4 or more (or zero). Thus there are no degree 3 elements, and d must be zero. However, again since Im $d \subset \Lambda^{\geq 2} V$ no element of $V^1(k+1)$ can be a coboundary, but since $H^1(\Lambda V) = 0$, $0 = kerd = V^1(k+1)$.

Now that we know that $V = V^{\geq 2}$, we know that $H : V \to \Lambda^+ W \otimes \Lambda(t, dt)$. When we look at H', we find that Im $d \subset \Lambda^{\geq 2}$ implies the differential is trivial in $\Lambda^+ V/\Lambda^{\geq 2} V$ (and likewise for W). Thus H' can be considered as a map

$$H' : V \to W \otimes \Lambda(t, dt)$$

and we can recover $L(\phi_0)$ and $L(\phi_1)$ as $id \cdot \epsilon_0 H'$ and $id \cdot \epsilon_1 H'$ respectively. Then since the differential in V is zero we have that all elements in the image of H' must be cocycles, but since the differential in W is zero, we have that all cocycles are elements of w times a cocycle in $\Lambda(t, dt)$. That is to say, no element in the image of H' can have any component which is not either a pure value of W or multiplied by dt. Thus, all values where ϵ_0 and ϵ_1 disagree are excluded from the image of H', and we have that $L(\phi_0) = L(\phi_1)$. □

With these lemmas we can prove the final result of this chapter.

Theorem 2.3 *1. A quasi-isomorphism $\psi : \Lambda V \to \Lambda W$ between minimal Sullivan algebras with vanishing first cohomologies is an isomorphism.*

2. Every minimal Sullivan model of a CGDA A satisfying $H^0(A) = \mathbb{Q}$ and $H^1(A) = 0$ is isomorphic.

Proof of 1: Lemma 3.2 gives us a $\phi : \Lambda W \to \Lambda V$ such that $\phi\psi \sim id$. We can therefore conclude that $\psi\phi\psi \sim \psi$, which in turn gives us $\phi\psi \sim id$ by the second part of lemma 3.2. Now lemma 3.3 part 1 tells us that $L(\phi\psi) = L(id) = id$. Since $L(fg) = L(f)L(g)$ these statements give us that $L(\phi)$ and $L(\psi)$ are inverse linear maps, and thus inverse isomorphisms of V and W. Thus the linear part of ψ is surjective, so W^0 is contained in the image of ψ and $W^k \subset \mathrm{Im}\psi + \Lambda W^{\leq k-1}$. Thus induction gives that ψ is surjective. Since ψ is now a surjective quasi-isomorphism, we can use the lifting lemma to choose ϕ' such that $\psi\phi' = id$. This ϕ' is a monomorphism, and thus an injection. However, as before $\psi\phi'\psi = \psi$ so $\phi'\psi \sim id$, and the previous argument gives that ϕ is surjective. Therefore ψ is the inverse of an isomorphism, and so is an isomorphism.

Proof of 2: Take $\phi : \Lambda V \to A$ and $\phi' : \Lambda W \to A$ to be two minimal Sullivan models of A. Then we can use lemma 3.2 to produce a $\psi : \Lambda V \to \Lambda W$ such that $\phi'\psi \sim \phi$. Now part 1 of lemma 3.3 gives us that $H(\phi'\psi) = H(\phi')H(\psi) = H(\phi)$ so $H(\psi)$ is an isomorphism. Then part one implies ψ is an isomorphism and we are done. □.

Chapter 3

The Functors

3.1 From Top to CGDA: $A_{pl}(-)$

The main theorem, which will be stated more carefully later, is that there is a bijection between rational homotopy types and equivalence classes of minimal Sullivan algebras when spaces are simply connected and have homologies of finite type. To show this, we want a functor from topological spaces to CDGAs, and another in the reverse direction.

The first, which we will call $A_{pl}(-)$, the polynomial differential forms, will be the composition of 2 functors. The first should be familiar — $S_* : Top \to sSet$ (simplicial sets), which assigns to a topological space X $S_*(X)$, the set of singular simplices $\sigma : \Delta^n \to X$ along with face and degeneracy maps, and which maps a function $f : X \to Y$ to $S_*(f) : S_*(X) \to S_*(Y)$ given by $S_*(f)(\sigma) = f\sigma$. The second is a functor $sSet^{op} \times sCDGA \to CDGA$, which we will notate as $(K, A) \mapsto A(K)$. We will also need a particular $sCDGA$ (simplicial CDGA), called A_{pl}.

To properly define A_{pl} and $A(K)$, and to remind the reader of the definition of $sSet$ and $sCDGA$, it is useful to define a simplicial object with values in a category C. A simplicial object (or an object in the category sC) is a sequence of objects S_n in C along with morphisms $\partial_i : S_{n+1} \to S_n$ $i = 0, 1, ..., n+1$, called face maps, and $s_i : S_n \to S_{n+1}$ $i = 0, ..., n$, called degeneracy maps. We model these objects on the set $S_*(X)$, the set of simplices $\sigma : \Delta^n \to X$, which comes equipped with face maps ∂_i defined by composition with the inclusion $f_i : \Delta^n \to \Delta^n$ which maps $\langle e_0, ..., e_{n-1}\rangle$ to $\langle e_0, ..., \hat{e}_i, ..., e_n\rangle$ (the hat indicates exclusion, as usual). It also comes equipped with degeneracy maps s_i which are again defined by composition with the function $D_i : \Delta^n \to \Delta^{n+1}$ given by $\langle e_0, ..., e_n\rangle \mapsto \langle e_0, ..., e_i, e_i, ..., e_n\rangle$. We can use this example to motivate the commutation relations that constitute the final condition for a simplicial object. First, let's consider the composition of two face maps $\partial_i \partial_j$ which first removes the jth vertex and then removes the ith vertex. However, removing one vertex reduces the index of all subsequent vertices by one, so we find that $\partial_i \partial_j = \partial_{j-1} \partial_i$

whenever $i \leq j$. Similarly, since s_i adds a vertex, and thus increases the index of all vertices after the ith, we find $s_i s_j = s_{j+1} s_i$ whenever $i \leq j$. Similar reasoning then gives that $\partial_i s_j$ is the identity when $i = j, j+1$ (either duplicate of e_j), is $s_{j-1}\partial_i$ when i is less than j, and is $s_j \partial_{i-1}$ when i is greater than $j+1$.

We can then finish construct the category sC by defining morphisms between simplicial objects, which are simply a sequence of morphisms $f_n : S_n \to T_n$ which commute with the face and degeneracy maps.

We can now define the second functor. Given a simplicial set, K, and a simplicial CDGA, A, we define the (non-simplicial) CDGA $A(K)$ as follows. The sets $A(K)^m = Hom_{sSet}(K, A^m)$, that is $A(K)^m$ is the set of simplicial set functions from K_n to the simplicial set A^m_n. Then we can define addition, scalar multiplaition, algebra multiplication and the differential in the obvious ways: $(f+g)(k) = f(k) + g(k)$, $(\lambda f)(k) = \lambda f(k)$, $(fg)(k) = f(k)g(k)$ and $(df)(k) = d(f(k))$. We then define the action of the functor component wise; take $\phi : A \to B$ a CDGA map and $\psi : K \to L$ an sSet map, then define $A(\psi) : A(L) \to A(K)$ by $f \mapsto f\psi$ and $\phi(K) : A(K) \to B(K)$ by $f \mapsto \phi f$.

The last thing we need is the particular simplicial CDGA A_{pl}, called the polynomial differential forms on Δ^n. In particular, to construct $A_{pl,n}$ we begin with the CDGA $\Lambda(t_0, ..., t_n, y_0, ..., y_n)$, where all the t_i are taken to be of degree zero, and $dt_i = y_i$. We then divide by the differential ideal generated by $\sum t_i - 1$, or the regular ideal generated by both $\sum t_i - 1$ and $d(\sum t_i - 1) = \sum y_i$. We then assign face maps $\partial_i : A_{pl,n+1} \to A_{pl,n}$ which take t_i to zero (and all t_j, $j > i$ to t_{j-1}) and $s_i : A_{pl,n} \to A_{pl,n+1}$ which takes t_i to $t_i + t_{i+1}$ and t_j to t_{j+1} when $j > i$. Now, a term in this algebra is represented by a sum of terms $t_0^{a_0} ... t_n^{a_n} dt_{i_1} \wedge dt_{i_k}$. This gives rise to the name polynomial differential forms; if we take the t_i to be the coordinate functions on the n-simplex then $y_i = dt_i$ is the standard frame of the cotangent bundle, and instead of general $C^\infty(\Delta^n)$ coefficients, we allow only coefficients which are polynomials in the coordinate functions. Then ideals we have modded out reflect the condition for a point in $\mathbb{R}^{n+1}$ to be in Δ^n and for a covector in $\mathbb{R}^{n+1}$ to be tangent to Δ^n. It is worth noting that, since Δ^n is contained in a hyperplane in $\mathbb{R}^{n+1}$, we can recoordinatize it to find that $A_{pl,n} \cong \Lambda(t_1, ..., t_n, y_1, ..., y_n)$.

Finally, the functor we want, $A_{pl}(-) : Top \to CDGA$ is given by $A_{pl}(S_*(X))$, that is, the second functor evaluated at $(S_*(X), A_{pl})$. There are two important facts about $A_{pl}(-)$: it preserves the cohomology, and it maps rational homotopy equivalences to quasi-isomorphisms of CDGAs. Proving this requires a fair amount of work and the following two technical notions.

First, a simplicial object is extendable if for all $n \geq 1$ and any subset, I, of $\{0, ..., n\}$ we have that $a_i \in A_{n-1}$, $i \in I$ and $\partial_i a_j = \partial_{j-1} a_i$ whenever $i < j$ implies the existence of $\alpha \in A_n$ such that each a_i is the image of α under the ith face map. Going back to the prototypical example of singular simplices, this essentially says that if you have a collection of n-simplices that could feasibly be the faces (or a some of the faces) of an $n+1$ simplex, then that $n+1$ simplex is actually included in the set.

Second, we need the simplicial sets $\Delta(n) \subset S_*(\Delta^n)$ which are essentially the faces of Δ^n. More precisely, we allow only those singular simplices that are linear images of Δ^k which map vertices to vertices and maintain orientation in the sense that $\langle e_0, ..., e_k \rangle$ can only be mapped to $\langle e_{i_0}, ..., e_{i_k} \rangle$ where $i_0 \leq i_1 \leq ... \leq i_k$. Whenever there is an equality among the i_j the simplex is degenerate (the image of a degeneracy map). Additionally, there is a single non-degenerate n simplex, the identity on Δ^n (henceforth i_n), and any non-degenerate simplex can be obtained by repeated application of the face maps to i_n. Because of this, we can specify any simplicial set map from $\Delta(n)$ by its value on i_n.

We will now need several lemmas on our way to our first theorem of the section. We begin with

Lemma 3.1 *Suppose A is a simplicial CDGA, then*

1. *For all n, there is an ismorphism of CDGAs from $A(\Delta(n))$ to A_n given by $x \mapsto x(i_n)$.*
2. *If A is extendable and $i : L \to K$ is an inclusion of simplicial sets, then $A(i)$ is a surjection.*

Proof of 1: We have defined the additive, multiplicative and differential structure of $A(K)$ so that this function is a morphism of CDGAs. Moreover, since each function from $\Delta(n)$ is specified by its value on i_n, this is a bijection.

Proof of 2: Given an $f \in A(L)$ we need to find a $g \in A(K)$ that extends f. We will proceed by induction. Beginning with K_0, we notice that there are no face maps that we need to consider, so we look instead at the degeneracy maps. For this to be a simplicial map, $gs = sg$. Since there is no s into K_0, we can take $g|_{K-L}$ to be any element we want without contradicting this yet. In the general case, we first consider degenerate elements of K_n; that is elements of the form $x = s(t)$ for some t in K_{n-1}. In this case we use $gs = sg$ to define $g(x)$. The commutation relations satisfied by the face and degeneracy maps tell us that sg is independent of our choice of s or t. Then, for x a nondegenerate element of K_n we will use extendability of A and $\partial g = g\partial$. In particular, we consider the $n-1$ simplices $g(\partial_j x)$. When $i < j$, we have that $\partial_i g(\partial_j x) = g\partial_i\partial_j x = g\partial_{j-1}\partial_i s = \partial_{j-1} g(\partial_i x)$. Thus the $g(\partial_j x)$ are a set that satisfies the assumptions of extendability, and there exists a y such that $g(\partial_j x) = \partial_j y$, and we take $g(x) = y$. □

An important step in the theorem will invoke that a quasi-isomorphism of simplicial CDGAs E and D will map to an isomorphism of the homology groups $H(E(K))$ and $H(D(K))$. Proving this requires lemma 3.1 and the coming lemma 3.2. To properly state lemma 3.2, we need to understand the notion of an m-skeleton of a simplicial set. Take K to be a simplicial set, then we can most concisely define another simplical set, the m-skeleton of K (denoted $K(m)$), as the simplicial set generated by the nondegenerate simplices in K_1 through K_m. Put more explicitly,

$$K(m)_n = \begin{cases} K_n & n \leq m \\ \bigcup_i s_i(K(m)_{n-1}) & n > m \end{cases}$$

We will also define NK, where K is still a simplicial set, to be the non-degenerate simplices of K. For the next lemma, let $i : K(n-1) \to K(n)$ and $j : \Delta(n-1) \to \Delta(n)$ be inclusions. Then

Lemma 3.2 *Define a function* F :*ker* $A(i) \to \prod_{\sigma \in NK_n} kerA(j) = kerA(j)^{NK_n}$ *by* $F(f) = (A(\alpha_\sigma)(f), A(\alpha_{\sigma'})(f), ...)$ *where* α_σ *is the unique map* $\alpha_\sigma : \Delta(n) \to K$ *that takes* $\alpha_\sigma(i_n) = \sigma$. *Then* F *is an isomorphism.*

Proof: To see injectivity, consider the kernel of F; $F(f) = 0$ implies $A(\alpha_\sigma)(f) = 0$ for all non- degenerate σ, i.e. $f(\alpha_\sigma(x)) = 0$ for all x in $\Delta(n)$ and non-degenerate σ. Thus, $f(\alpha_\sigma(i_n)) = f(\sigma) = 0$ for all non-degenerate σ. This carries over to all sigma since the commutation relations of ∂ and s imply that ∂ is surjective. For surjectivity, consider a set $f_\sigma \in kerA(j) \subset A(\Delta(n)) \cong A_n$. When we consider the f_σ in this way, the condition that they be in the kernel of ∂_j for some j (which reflects how we have chosen to embed $\Delta(n-1)$ into $\Delta(n)$). We can now define f by $f(\sigma) = f_\sigma(i_n)$ for the non-degenerate n-simplices, and $f(\sigma) = 0$ for degenerate n-simplices or any m-simplices (given m¡n). Then $F(f) = (f_\sigma)$. $\square$

Now lemmas 3.1 and 3.2 lead us to

Lemma 3.3 *Take* $\theta : A \to B$ *to be a morphism of extendable simplicial CDGAs satisfying* $H(\theta_n) = H(A_n) \to H(B_n)$ *is an isomorphism for all* n. *Then for all simplicial sets* K, $H(\theta(K))$ *is also an isomorphism.*

Proof: To begin with we recall the theorem from homological algebra which states: given a row exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow & F' & \longrightarrow & 0 \end{array}$$

whenever two of f, g, h are quasi-isomorphisms, so is the third (I will refer to this as the RED theorem). Taking i to be an inclusion of L into M, we can take $A' = kerA(i)$, $B' = A(M)$, $C' = A(L)$, $D' = kerB(i)$, $E' = B(M)$, and $F' = B(L)$. Then lemma 4.1 part 2 is all that is required for this diagram to be row exact after these substitutions. Further, lemma 4.1 part 1 identifies A_n with $A(\Delta(n))$ and B_n with $B(\Delta(n))$. Thus $\theta(\Delta(n)) = \theta_n$ is a quasi-isomorphism by hypothesis. Now we will use induction to show that $\theta(K(n))$ is an isomorphism for all n. To begin with, define $j : K(n-1) \to K(n)$ as an inclusion. Next, we can see that $\theta(\partial\Delta(n)) = \partial\theta(\Delta(n))$ so $\theta(\Delta(n))$ is a quasi-isomorphism. Then, taking $M = \Delta(n)$ and $L = \partial\Delta(n)$, the RED theorem gives us that $\theta(i)$ is a quasi-isomorphism as well. Then identifying $\partial\Delta(n)$ with $\Delta(n-1)$ and applying theorem 4.2 gives that $\theta(j)$ is a quasi-isomorphism. Then we can apply the RED theorem to arrive at the conclusion that if $\theta(K(n-1))$ is a quasi-isomorphism, then so is $\theta(K(n))$. Then, taking $K(-1)$ to be 0, then $\theta(K(-1))$ is clearly a quasi-isomorphism, so $\theta(K(n))$ is a quasi-isomorphsim for all n, as is $\theta(j)$.

Our final goal is to show that $\theta(K)$ is a quasi-isomorphism given that $\theta(K(n))$ is for all n. To do so, we will show that $f \in A(K)$ and $g(B(K))$ such that $df = 0$ and $\theta(K)(f) = dg$ allows us to find $x \in A(K)$ and $y \in B(K)$ such that $f = dx$ and $g = \theta(K)x + dy$. We begin with $x_{-1} = y_{-1} = 0$. Then, given x_i and y_i, we will set $f' = f - \sum_{i<n} dx_i$ and $g' = g - \sum_{i<n}(\theta(K)(x_i) + dy_i)$. Then we define f'' to be the restriction of f' to $K(n)$ and likewise g'' to be the restriction of g' to $K(n)$. Suppose we have constructed the x_i and y_i so that f' and g' vanish on $K(n-1)$, then the results of the previous paragraph (that $\theta(j)$ is a quasi-isomorphism) allow us to find $x' \in kerD(j)$ and $y' \in kerB(j)$, such that $f'' = dx'$ and $g'' = \theta(K)(x') + dy'$. Then $f' - dx'$ vanishes on the domain of f'', K(n), and $g' - \theta(K)(x') - dy_i$ does as well. Finally, since A and B are extendable, we can find x_n and y_n that restrict to x' and y'. Then we define $x = \sum_n x_n$ and $y = \sum y_n$. Both sums are well defined since $K = \cup_n K(n)$ and only finitely many terms contribute in any $K(n)$. □

Next, we turn the attention of our lemmas to A_{pl} in particular.

Lemma 3.4 *1.* $A_{pl,0} = \mathbb{Q}$

2. $H(A_{pl,n}) = \mathbb{Q}$ *for all* n.

3. Each A^p_{PL} *is extendable.*

Proof of 1: The earlier comment finds that $A_{pl,0}$ is isomorphic to $\Lambda(\emptyset)$, which is simply $\mathbb{Q}$ (since there are no elements to make polynomials or differential forms out of).

Proof of 2: The same comment gives that $A_{pl,n} \cong \bigotimes_1^n \Lambda(t_i, y_i)$. Then since all elements of $\Lambda(t_i, y_i)$ have degree 0 or 1, we have that all homology groups are 0 with the possible exceptions of the first and zeroth. In the first case, $d(t^k y) = kt^{k-1}y \wedge y = 0$ so the kernel of d_1 is everything, while $d(t^k) = kt^{k-1}y$, so the image is also everything, giving $H^1(A_{pl,n}) = 0$. In the second case, we have the $d(t^k) = kt^{k-1}y$, which is zero only when $k = 0$. Thus the kernel of d_0 is the "constant functions", which are naturally isomorphic to $\mathbb{Q}$. Since the cohomology of a tensor product is equal to the tensor product of the cohomologies, we are done.

Proof of 3: As per the definition of extendability, we begin with $I \subset \{0, ..., n\}$ and $\omega_i \in A_{pl,n-1}$ satisfying $\partial_i \omega_j = \partial_{j-1}\omega_i$ whenever $i < j$. We need $\Omega \in A_{pl,n}$ such that $\omega_i = \partial_i \Omega$. As we so often do, we will proceed inductively beginning with the trivial case ($\Omega_{-1} = 0$) to construct a sequence of Ω_i such that $\partial_i \Omega_j = \omega_i$ whenever $i < j$ and $i \in I$. Thus Ω_n will be the desired Ω.

To whit, suppose we have Ω_i as above and that $i + 1 \notin I$, then we define $\Omega_{i+1} = \Omega_i$. Otherwise, take F to be the field of fractions of $A^0_{pl,n}$ (i.e. F is the rational functions) and let B be the subalgebra generated by $\frac{1}{1-t_j}$ (i.e. those rational functions with a power of $(1-t_j)$ in the denominator). We can then turn B into a CDGA by defining (the natural definition) $d(1/1-t_j) = dt_j/(1-t_j)^2$.

This CDGA contains $A_{pl,n}$ once we identify dt_i with y_i. Moreover, we can define a morphism of CDGAs $f : A_{pl,n-1} \to B$ using commutativity with the differential and

$$f(t_i) = \begin{cases} \frac{t_i}{1-t_j} & i < j \\ \frac{t_{i+1}}{1-t_j} & i \geq j \end{cases}$$

However, since ∂_i essentially maps t_i to zero, we can also naturally extend ∂_j to a map $\partial_r B \to A_{pl,n-1}$ by defining $\partial_j(1/1-t_j) = 1$. Then $\partial_r f = id_{A_{pl,n}}$ (remember that ∂_j also decreases the index of t_i when i is greater than j).

Now take $f(\omega_j - \partial_j \Omega_{j-1}) = x/(1-t_j)^N$ with $x \in A_{pl,n}$. Then by the commutation relations obeyed by the ∂_k and the assumptions placed on ω_i and Ω_i we have $\partial_i(\omega_j - \partial_j \Omega_{j-1}) = \partial_{j-1}(\omega_i - \partial_i \Omega_{j-1}) = \partial_{j-1}(\omega_i - \omega_i) = 0$ for all $i \in I$ less than j. Thus $\omega_i - \partial_j \Omega_{j-1}$ has a factor of either t_i of dt_i so, by the definition of f and $i < j$ x must as well, meaning $\partial_i x = 0$. By our earlier comments, however, we also have $\partial_j x = \omega_j - \partial_j \Omega_{j-1}$. This gives that $\partial_j(x + \Omega_{j-1}) = \omega_j$ so we take Ω_j to be $\Omega_{j-1} + x$. $\square$

This brings us very close to being done. Alas, we must introduce an additional piece of machinery to facilitate the proof of the first theorem. Thankfully, it is reasonably simple. In particular, we seek a simpler functor which yields the same CDGA as the singular cohomology of a space.

To begin with, this discussion relies on the notion that the singular CDGA for a space is actually given by a functor from sSet to sCDGA applied to $S_*(X)$, the set of singular simplices of that space. We can therefore try to replace this functor by a simpler one. For the sake of completeness, I will begin by describing the functor $C^*(K)$, the singular cochain functor.

As a set, it is defined by $C^p(K) \subset Hom_{sSet}(K_p, \mathbb{Q})$ given by those maps which vanish on degenerate simplices (this differs from the perhaps more obvious definition which does not require that the functions vanish on the degenerate simplices, but this version is used because it gives the same cohomology and is computationally simpler).

We further define a product (usually called the cup product) as follows. For $f \in C^l(K)$ and $g \in C^m(K)$, $f \cup g(\sigma) = (-1)^{lm} f(\partial_{l+1} ... \partial_{l+m}(\sigma)) \cdot g(\partial_0 ... \partial_0(\sigma))$. From this definition, we can see that $f \cup g \in C^{l+m}(K)$ so $\sigma \in K_{l+m}$, which in turn tells us that there must be l copies of ∂_0 in the argument of g. Finally, the differential is given by $(df)(\sigma) = \sum_{i=0}^{l+1} (-1)^{l+i+1} f(\partial_i \sigma)$.

Our second functor will be constructed using the second functor from above evaluated at a new simplicial CDGA, C_{pl}. C_{pl} is simply the CDGA $C^*(\Delta(n))$. Since the $\Delta(n)$ have as their face and degeneracy maps linear maps, $\Delta(n)$ is the *piecewise linear* subset of $S_*(\Delta^n)$, and C_{pl} is called the piecewise linear CDGA. The next lemma further expounds on their relationship.

Lemma 3.5 *There is a natural isomorphism $C_{pl}(K) \to C^*(K)$ for all simplicial sets K.*

Proof: Take $\alpha \in C^n_{pl}(K)$. Define this isomorphism by $\alpha \mapsto f \in C^p(K)$ where f is given by the only natural option: $f(\sigma) = \alpha(\sigma)(i_n)$ (that we need $\alpha(\sigma)(\textit{something})$ is made clear simply analyzing the domains and codomains of both functors, and that the something should be i_n is clear from the earlier discussions of $\Delta(n)$. As a first check, we need to know that this is a CDGA morphism. Graded linearity is clear. Suppose $\alpha \mapsto f$ and $\beta \mapsto g$, then $\alpha\beta \mapsto h$ where $h(\sigma) = (\alpha\beta)(\sigma)(i_n) = (\alpha(\sigma) \cup \beta(\sigma)(i_n)$. That it commutes with the differential is similarly clear from the definitions (since C_{pl} is defined using C^* we do not even need to reference the more detailed definition of the differential). Now we need to show that this is a bijection. For injectivity, we once again examine the kernel. That is, suppose $\alpha \mapsto 0$. Then $\alpha(\sigma)(i_n) = 0$ for all σ, but since every simplicial map from $\Delta(n)$ is fixed by its value at i_n we have that $\alpha(\sigma) = 0$ for all σ. That is, we have $\alpha = 0$. We can then see surjectivity by recalling that for any element $\sigma \in K$ there is a unique map $\Sigma : \Delta(n) \to K$ such $\Sigma(i_n) = \sigma$. Thus, given $f \in C^n(K)$ we can define $\alpha(\sigma)$ to be this unique map for $f(\sigma)$. □

This leaves us with only a single lemma before the first theorem, namely. It is a technical lemma whose sole use is to let us apply the previous theorems to the coming construction which we will use in the first theorem.

Lemma 3.6 *1.* $H(C_{pl,n}) = \mathbb{Q} = H(C_{pl,n} \otimes A_{pl,n})$.

2. C_{pl} *is extendable.*

3. $C_{pl} \otimes A_{pl}$ *is extendable.*

Proof of 1: We already know that cohomology respects tensor products and that $H(A_{pl,n}) = \mathbb{Q}$, so this reduces to showing that $H(C_{pl,n}) = H^*(\Delta(n)) = \mathbb{Q}$. Since this is a fairly simple exercise in cohomology, it is left to the reader.

Proof of 2: The inclusion of Δ^{n-1} into Δ^n as the ith face induces an inclusion of simplicial sets $\Delta(n-1)$ into $\Delta(n)$. This inclusion also identifies $f \in C^n(\Delta(n-1))$ with a map f' from the ith face to $\mathbb{Q}$. Given a sequence of $f_i \in C^n(\Delta(n-1))$ the condition $\partial_i f_j = \partial_{j-1} f_i$ is equivalent to saying that f'_i and f'_j restrict to the same map on their overlap. We can therefore define a function $f : \Delta(n)\mathbb{Q}$ face-wise by simply taking it to be equal to any f_i defined on a given face.

Proof of 3: As per the definition take $x_i \in C^k(\Delta(n-1)) \otimes A^l_{pl,n-1}$ satisfying the appropriate conditions on ∂x. Express x_i as $\sum_\alpha c_{i\alpha} \otimes a_{i\alpha}$ and define x'_i from the ith inclusion of $\Delta(n-1)$ into $\Delta(n)$ to $A^l_{pl,n-1}$ by $x'_i(\gamma|_{\Delta(n-1)}) = (-1)^{kl} \sum_\alpha c_{i\alpha}(\gamma) a_{i\alpha}$. Then whenever these maps are well defined (i.e. whenever σ is in the ith and jth inclusions) we have $\partial_i(x'_j(\sigma)) = \partial_{j-1}(\sigma)$. Since we know A_{pl} is extendable, we can find a $w \in A^l_{pl,n}$ such that $\partial_i(w) = x'_i$. After we identify $C^k(\Delta(n)) \otimes A^l_{pl,n}$ with the set of functions $\Delta(n)_k \to A^l_{pl,n}$ we can define $x \in C^k(\Delta(n)) \otimes A^l_{pl,n}$ by $x(\sigma) = w(\sigma)$ so that x is our desired simplex. □

Now, at long last, we are able to say prove

Theorem 3.1 *1. $\theta : C_{pl,n} \to C_{pl,n} \otimes A_{pl,n}$ given by $c \mapsto c \otimes 1$ and ϕ : $A_{pl,n} \otimes C_{pl,n} \otimes A_{pl,n}$ given by $a \mapsto 1 \otimes a$ are quasi-isomorphisms.*

2. A_{pl} preserves cohomology! That is $H(A_{pl}(X)) = H^*(X)$.

Proof of 1: This is close to trivial since lemmas 4.4 and 4.6 give that $H(C_{pl,n}) = H(A_{pl,n}) = H(C_{pl} \otimes A_{pl}) = \mathbb{Q}$. Thus $H(\theta_n)$ and $H(\phi_n)$ are isomorphisms.

Proof of 2: This follows from the slightly more general result that $H(A_{pl}(K)) = H(K)$ for any simplicial set K. We arrive at this result first by recognizing that lemmas 4.4 and 4.6 give us that A_{pl}, C_{pl} and $C_{pl} \otimes A_{pl}$ are all extendable and part 1 shows they satisfy the other hypotheses of lemma 4.3. $\square$.

Earlier I claimed that $A_{pl}(-)$ maps rational homotopy equivalences to quasi-isomporphisms. This is a corollary to theorem 4.1, since a rational homotopy equivalence is simply a map which preserves (rational) cohomology, and $A_{pl}(-)$ also preserves rational cohomology.

3.2 The Adjoint Functor: $|\langle - \rangle|$

Once again the functor in the reverse direction is the composition of two other functors, one due to Sullivan and one to Milnor. The first takes CDGAs to simplicial sets and the second take simplicial sets to topological spaces.

Let's begin with Sullivan's realization functor, which maps A to $\langle A \rangle$. As a set we have $\langle A \rangle_n = Hom_{dga}(A, A_{pl,n})$. We then define the face and degeneracy maps quite simply via composition with the face and degeneracy maps on A_{pl}. That is, if $\sigma : A \to A_{pl}$, then $\partial_i \sigma = \partial_i \circ \sigma$ and similarly for the degeneracy maps. At this point there is only one fact to show about $\langle - \rangle$ — that it is adjoint to $A_{pl}(-)$. Given the nature of the definitions, this should come as no surprise. One way of phrasing the meaning of this is to say that there is a bijection between $Hom_{DGA}(A, A_{pl}(K))$ and $Hom_{sSet}(K, \langle A \rangle)$. In this case this bijection is easy to see; take $f \in Hom_{DGA}(A, A_{pl}(K))$ then $f \mapsto g$ where $f(a)(k) = g(k)(a)$, which is obviously a bijection.

The second functor, Milnor's spacial realization functor, is slightly more complicated. First, give each K_n the discrete topology. Then Milnor's functor is given by $K \mapsto |K| = \left(\bigsqcup_n K_n \times \Delta^n \right) / \sim$, where $\sim$ is an equivalence relation which says that taking faces or degeneracies in either component is equivalent. I.e. $\partial_i \sigma \times x \sim \sigma \times \lambda_i x$ and $s_i \sigma \times x \sim \sigma \times \rho_i x$. Then, given $f : K \to L$ we take $|f| : |K| \to |L|$ to be the map given by $|f|([\sigma \times x]) = [f(\sigma) \times x]$. Well definedness of this map is ensured by the fact that simplicial maps must commute with the face and degeneracy maps.

Take $\partial \Delta^n$ to be the boundary of Δ^n in $\mathbb{R}^{n+1}$ (i.e. the union of the images of the face maps). Then define the interior $\mathring{\Delta}^n$ to be $\Delta^n - \partial \Delta^n$. Then

Lemma 3.7 *The quotient map $q : \sqcup K_n \times \Delta^n \to |K|$ restricts to a bijection $q : \sqcup N K_n \times \mathring{\Delta}^n \to |K|$.*

Proof: Take $\sigma \times x \in K_n \times \Delta^n$, and find the smallest k such that there is a $\tau \times y \in K_k \times \Delta^k$ satisfying $q(\sigma \times x) = q(\tau \times y)$. Then neither τ is non-degenerate and y is in the interior of Δ^n, since otherwise there would be a smaller k satisfying the same condition. The commutation relations of the face and degeneracy maps then give injectivity. □

It is also useful to know

Lemma 3.8 *Take K to be a simplicial set. Then $|K|$ is a CW complex with n-skeleton $|K(n)|$, whose n-cells are the non-degenerate n-simplices. The attaching map for σ is the restriction of the quotient to $\sigma \times \partial\Delta^n$.*

Proof: Take $\sigma \in K_n$ to be nondegenerate. Then, since the quotient map satisfies $q(\partial_i\sigma \times x) = q(\sigma \times \lambda_i x)$ we have that $q(\sigma \times \partial\Delta^n) = \cup_i q(\partial_i \times \Delta^{n-1}) \subset |K(n-1)|$. Thus, the restrictions of the quotient maps are valid attaching maps. Then, since Δ^n is homeomorphic to D^n and $\partial\Delta^n$ is homeomorphic to S^{n-1} we have that $NK_n \times \Delta^n = K(n)_n \times \Delta^n = \sqcup_{\sigma \in NK_n}\{\sigma\} \times \Delta^n \cong \sqcup_{\sigma \in NK_n} D^n_\sigma$ is clearly a family of n-cells. □

Another useful feature of the Milnor functor is that it respects both products and fiber products. Since the construction of a fibre product in sSet mirrors exactly the construction of a fibre product in set, the second assertion follows from the first.

Theorem 3.2 *Take K and L to be simplicial sets. Then define $K \times L$ to be the simplicial set whose underlying set is $K \times L$ and whose face and degeneracy maps act component-wise. Then take $p_K : K \times L \to K$ to be the projection onto K and likewise p_L to be the projection onto L. Then $f : |K \times L| \to |K| \times |L|$ given by $k \times l \times x \mapsto (k \times x, l \times x)$ is a homeomorphism.*

Proof: Since f restricted to a particular $k \times l$ is proper, f is a continuous proper map between compact Hausdorff spaces, so if it is a bijection (it is clearly a surjection) then is it a homeomorphism. To show this, we begin by noting that and simplex $\sigma \in (K \times L)_n$ can be written as $(s_\alpha k, s_\beta l)$ where α and β are increasing multi-indices (that is, $\alpha = \alpha_0 \leq \alpha_1 ... \alpha_m$ and likewise for β) and k is a non-degenerate p simplex and l is a non-degenerate q simplex. These multi-indices act from right to left, i.e. $s_{\alpha_0,...,\alpha_m} = s_{\alpha_0} \circ ... \circ s_{\alpha_m}$. Since the degeneracy maps in $K \times L$ act component wise, we have that $(s_\alpha k, s_\beta l)$ is degenerate exactly when α and β have some element in common.

Now f restricts to $f : (s_\alpha k, s_\beta l) \times \mathring{\Delta}^n \to (k \times \mathring{\Delta}^p) \times (l \times \mathring{\Delta}^q) = (k \times l) \times (\mathring{\Delta}^p \times \mathring{\Delta}^q)$. So we can use this to define an $f_{\alpha,\beta} : \mathring{\Delta}^n \to \mathring{\Delta}^p \times \mathring{\Delta}^q$ where $n = p + |\alpha| = q + |\beta|$. We need to know that each $f_{\alpha,\beta}$ is injective.

To see this, then, we consider the simplices as subsets of $\mathbb{R}^{p+q+2} = \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$. Then $f_{\alpha,\beta}$ is the linear map which first generates two sequences of length n. The first, S, is given by taking $e_0, ..., e_p$ and duplicating all the indices in α and the second, T, is given by doing the same to $e_0, ..., e_q$ and β. Then f is the linear map which takes e_i to (S_i, T_i), but since α and β share no elements, we have that these are all linearly independent. □

There are four more foundational results that I will discuss in this chapter, all of which are focused on the relationship of Sullivan algebras to homotopy. I will begin with the simplest to prove and perhaps most expected. Henceforth, I will abreviate $|\langle - \rangle|$ simply by $| - |$.

Theorem 3.3 *Given two maps $\phi_0, \phi_1 : \Lambda V \to \Lambda W$ which are homotopic, then we have $|\phi_1| \sim |\phi_0|$.*

Proof: Recall that a homotopy from ϕ to ψ is a map $\Phi : \Lambda V \to \Lambda W \otimes \Lambda(t, dt)$ such that $id \times \epsilon_i = \phi_i$ where ϵ_i maps t to i, where i is 0 or 1. Since $|K \otimes L| = |K \times L| = |K| \times |L|$ we find the $|\Phi| : |\Lambda W| \times |\Lambda(t, dt)| \to |\Lambda V|$. Let's first examine $|\Lambda(t, dt)| = |\langle \Lambda(t, dt) \rangle|$; I claim it is equal to I, the unit interval. For this to be true two conditions must be satisfied. First, there can be no simplices of dimension greater than 1 in $\langle \Lambda(t, dt) \rangle$ and second, there should be only two 0-simplices and one 1-simplex. The zero simplices are DGA morphisms of $\Lambda(t, dt) \to A_{pl,0} = \mathbb{Q}$. That is to say, they are augmentations of $\Lambda(t, dt)$. All such morphisms are specified uniquely by their value on t. Only two such maps are actually independent — ϵ_i. Then the one simplices are DGA morphisms from $\Lambda(t, dt)$ to $\Lambda(t, dt)$, of which there is only one, the identity. Thus $\Lambda(t, dt) = I$. Now, this identifies $|\Phi| : |\Lambda W| \times I \to |\Lambda V|$ which satisfies $|\Phi|(x, \epsilon_i) = |\phi_i|(x)$, exhibiting it as a homotopy. $\square$

Next we consider a Sullivan algebra ΛV. Since $\langle - \rangle$ and $A_{pl}(-)$ are adjoint, we know that there is a function $\eta : \Lambda V \to A_{pl}(\langle \Lambda V \rangle)$. Moreover, we can construct a surjective quasi-isomorphism $A_{pl}(\xi) : A_{pl}(|\Lambda V|) \to A_{pl}(\langle \Lambda V \rangle)$. The necessary ξ is the inclusion of simplicial sets $\langle \Lambda V \rangle \to S_*(|\Lambda V|)$. Remember that $S_*(X)$ is the set of singular simplices into X, and $|\Lambda V| = \sqcup N(\langle \Lambda V \rangle)_n \times \mathring{\Delta}^n$ so we can take $f \in (\langle \Lambda V \rangle)_n$ to be $q|_{f \times \Delta^n}$. Thus we have a diagram

$$\begin{array}{ccc} & & A_{pl}(|\Lambda V|) \\ & & \downarrow{\scriptstyle A_{pl}(\xi)} \\ \Lambda V & \xrightarrow{\eta} & A_{pl}(\langle \Lambda V \rangle) \end{array}$$

which, by the lifting lemma in the previous chapter, implies there is a map $m : \Lambda V \to A_{pl}(|\Lambda V|)$ such that $A_{pl}(\xi)m = \eta$ and any other map satisfying this constraint is homotopic to m.

Moreover, we would like to compare the homotopy groups of $|\Lambda V|$ to $Hom_{\mathbb{Q}}(V, \mathbb{Q})$. To do so, it is first useful to note that there is a natural base point of $|\Lambda V|$ since V is concentrated in degrees greater than or equal to 2, it has a unique augmentation ϵ, whose image is the base point of $|\Lambda V|$. We will define a function $\zeta_n : \pi_n(|\Lambda V|) \to Hom_{\mathbb{Q}}(V, \mathbb{Q})$ by noting $\alpha : S^n \to |\Lambda V|$ is mapped to a function $A_{pl}(\alpha) : A_{pl}(|\Lambda V|) \to A_{pl}(S^n)$. Composing this with the m from before yields a map $\phi_\alpha : \Lambda V \to A_{pl}(S^n)$, which we can lift to the minimal model of $A_{pl}(S^n)$. This model is always of the form $\Lambda(e)$ or $\Lambda(e, e'), de' = e^2$ so we can restrict ϕ_α

to a linear map $V^n \to \mathbb{Q}e$ which is dependent exclusively on α. Thus we can determine a map $\langle -; - \rangle : V^n \times \pi_n(|\Lambda V|) \to \mathbb{Q}$ by $\langle v; \alpha \rangle e = \phi_\alpha(v)$. Then $\langle -; - \rangle$ is linear in V^n and, for $n \geq 2$ is a group homomorphism in $\pi_n(|\Lambda V|)$. Finally we can define the promised $\zeta_n : \pi_n(|\Lambda V|) \to Hom_{\mathbb{Q}}(V^n, \mathbb{Q})$ by $\zeta_n(\alpha) = (-1)^n \langle v; \alpha \rangle$.

This leads to an important theorem. Unfortunately, the proof of the theorem is too involved to fit here and maintain readability. Instead I will state the theorem and off a description of the proof. A full treatment can be found in [1] as Theorem 17.10.

Theorem 3.4 *If ΛV is a simply connected Sullivan algebra of finite type (all degrees are finite dimensional). Then*

1. *$|\Lambda V|$ is simply connected and ζ_n is an isomorphism.*
2. *The afforementioned $m : \Lambda V \to A_{pl}(|\Lambda V|)$ is also an isomorphism.*

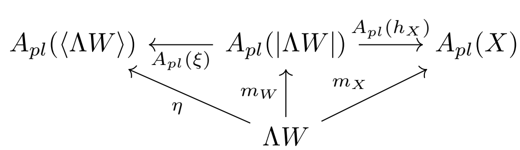
Sketch of Proof: If ΛV is not minimal we will find that $|\Lambda V|$ is the product of $|\Lambda V'|$ and a contractible CW-Complex, where $\Lambda V'$ is minimal, so we only really need to consider the minimal case. To prove part one, we begin by proving it for Eilenberg-Maclane spaces. This we accomplish using the lifting lemma and starting with Eilenberg-MacLane spaces concentrated in the first homotopy group, and extend the result to those concentrated in higher homotopy groups. We can then use induction to show that it is true for finite dimensional V, and then for the general case (by constructing a clever fibration). We prove 2 we look at rational spaces, and begin by looking at V^n where n is order of the homotopy group we are considering. We then extend this using induction to the case when V is concentrated in degrees less than n and then finally to the general case. □

This leads us to the next foundational theorem of this chapter. Let $m_X : \Lambda W \to A_{pl}(X)$ be a minimal Sullivan model. Since we have defined $A_{pl}(X) = A_{pl}(S_*(X))$ the adjointness of $A_{pl}(-)$ and $\langle - \rangle$ produces a simplicial map $\gamma_X : S_*(X) \to \langle \Lambda W \rangle$ which is adjoint to m_X. However, we can also show that the map $s_X : \cup S_n(X) \times \Delta^n \to X$ defined by $s_X(\sigma \times y) = \sigma(y)$ is a homotopy equivalence. Take t_X to be its inverse, and

$$h_X = |\gamma_X| \circ t_X : X \to |\Lambda W|$$

then

Theorem 3.5 1. *The diagram*



is homotopy commutative.

2. *All the maps in the diagram are quasi-isomorphisms.*

An immediate corollary of the second point is that h_X is a rationalization of X.

Proof of 1: The left triangle genuinely commutes by the definition of m_W. We find that $id_{|S_*(X)|} = |S_*(s_X)| \circ |\xi_{S_*}(X)| \sim t_X s_X$. which implies that for any $\phi : \Lambda W \to A_{pl}(|S_*(X)|)$ $A_{pl}(\xi_{S_*(X)}) \circ \phi \sim A_{pl}(t_X) \circ \phi$. In particular, we have that $A_{pl}(h_X) \circ m_W \sim A_{pl}(\xi_{S_*(X)}) \circ A_{pl}(|\gamma_X|) \circ m_W$ but this equals $A_{pl}(\gamma_X) \circ A_{pl}(\xi_{\langle \Lambda W \rangle}) \circ m_W = A_{pl}(\gamma_X) \circ \eta = m_X$.

Proof of 2: m_X is defined to be a quasi-isomorphism, and the previous theorem gives m_W is as well, so $A_{pl}(h_X)$ must be too. We also know that $C_*(\xi)$ is a quasi-isomorphism so $A_{pl}(\xi)$ is as well. □

Finally, let's recall lemma 2.2. It states that whenever η is a quasi-isomorphism and ΛV a Sullivan algebra, we can add a diagonal to the diagram:

$$\begin{array}{ccc} & \overset{\exists \phi}{\nearrow} & A \\ & & \downarrow \eta \\ \Lambda V & \xrightarrow{\psi} & B \end{array}$$

such that $\phi\eta \sim \psi$ (and that this is unique up to homotopy). In turn, this gives us that we can complete this diagram

$$\begin{array}{ccc} \Lambda W & \xrightarrow{m_W} & A \\ \downarrow \exists \phi & & \downarrow \eta \\ \Lambda V & \xrightarrow{m_V} & B \end{array}$$

such that $\phi m_V \sim m_W \eta$ whenever η is a quasi-isomorphism, and m_V, m_W are Sullivan models (apply lemma 2.2 to $\eta \circ m_W$). Such a ϕ is called a Sullivan representative of η. Our final theorem of this chapter can thus be stated as

Theorem 3.6 *Take f to be a continuous map $f : X \to Y$, and take ϕ to be a Sullivan representative of $A_{pl}(f)$. Then*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h_X & & \downarrow h_Y \\ |\Lambda W| & \xrightarrow{|\phi|} & |\Lambda V| \end{array}$$

is homotopy commutative, i.e. $h_Y f \sim |\phi| h_X$.

Proof: Recall the definition of h_X (and h_Y); $h_X = |\gamma_X| \circ t_X$ where γ_X is a map $S_*(X) \to \langle \Lambda W \rangle$ which we know exists by adjointness of $A_{pl}(-)$ and $\langle - \rangle$ and t_X is the homotopy inverse to $s_x : |S_*(X)| \to X$. Next consider the definition of s_X as $\sigma \times \delta \mapsto \sigma(\delta)$ and the action of the functor $|S_*(-)|$ on f, namely, $|S_*(f)|(\sigma) = f \circ \sigma$. Thus, $fs_X(\sigma \times \delta) = f(\sigma(\delta)) = |S_*(f)|(\sigma)(\delta)$ or equivalently, $fs_X = s_Y|S_*(f)|$. Composing with the respective homotopy inverses on the appropriate sides then gives $t_Y f \sim t_Y f s_X t_X = t_Y s_Y |S_*(f)| t_X \sim |S_*(f)| t_X$. However, if we have $|\gamma_Y| \circ |S_*(f)| \sim |\phi| \circ |\gamma_X|$, then we have $h_Y f = |\gamma_Y| \circ t_Y f \simeq |\gamma_Y| \circ |S_*(f)| t_X \simeq |\phi| \circ |\gamma_X| t_X = |\phi| h_X$, so the problem is reduced to showing that $|\gamma_Y| \circ |S_*(f)| \sim |\phi| \circ |\gamma_X|$.

This we do by noting that since ϕ is a Sullivan representative of $A_{pl}(f)$, there is a CDGA map $\Phi : \Lambda V \to A_{pl}(X) \otimes \Lambda(t, dt)$ such that $(id \otimes \epsilon_0)\Phi = A_{pl}(f) m_Y$ and $(id \otimes \epsilon_1)\Phi = m_X A_{pl}(f)$. Then compose this with the natural map $o : A_{pl}(X) \otimes \Lambda(t, dt) \to A_{pl}(S_*(X) \times \Delta(1))$. Then use adjointness once again to find $\omega : S_*(X) \times \Delta(1) \to \langle \Lambda V \rangle$. Then we have that $|w|$ is a homotopy $|\gamma_Y S_*(f)| \sim |\phi \gamma_X|$. $\square$

Chapter 4

The Biggest Theorem

Admittedly, the biggest theorem is a quick when equipped with the results of the previous chapter. Still, I think it is important enough to warrant it's own place in the table of contents. It states

Theorem 4.1 *If two spaces have the same rational homotopy types then their minimal Sullivan models will be isomorphic, and vice versa. Additionally, every minimal Sullivan algebra is the minimal model of some space.*

Proof: The first conclusion follows from three facts. First, $A_{pl}(-)$ preserves cohomology, second, our construction of the minimal Sullivan models depends only on the cohomology, and third, any two minimal Sullivan models of the same space are isomorphic. The second conclusion follows from theorems 3.6 and 3.3. The last follows from Theorem 3.4 part 2.

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