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# A Global Symbol for the Small b-Calculus on Manifolds with Boundary

John Christian Hower
*University of Colorado Boulder*, john.hower@colorado.edu

# A Global Symbol for the Small $b$-Calculus on Manifolds With Boundary

by

**John Hower**

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Markus Pflaum

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Florian Sobieczky

Date ____________________

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Hower, John (Ph.D., Mathematics)

A Global Symbol for the Small $b$-Calculus on Manifolds With Boundary

Thesis directed by Prof. Markus Pflaum

Starting with the normal symbold of M. Pflaum ([Pfl98]), we generalize the machinery of global symbol calculi from the familiar setting of a compact Riemannian manifold to the $b$-Calculus on a compact manifold-with-boundary, endowed with an exact $b$-metric. We define a notion of $b$-linearization, crucial to the symbol calculus. and show that an exact $b$-metric can be used to construct a $b$-linearization. Using the $b$-linearization, we define the global symbol of a $b$-pseudodifferential operator as a fiberwise Fourier transform over the $b$-tangent bundle. We prove that the global symbol is truly a symbol, of the same order of its operator, and define a quantization map with which one can recover the operator (modulo $b$-smoothing operators.) We compute the global symbol for a $b$-Laplacian, and give a formula for the $b$-trace in terms of the global symbol.

## Dedication

For my parents, Tom and Fil Hower.

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# Chapter 1

# Introduction

The small $b$-calculus, or calculus of *totally characteristic operators*, is a class of operators for compact manifolds-with-boundary which appears naturally in many problems of global singular analysis. However, at the current time it lacks a full global symbol calculus, a feature which the author believes every pseudodifferential calculus ought to posess. Our purpose is to develop such a symbol, and apply it to the analysis of such operators.

Why a global symbol? Well, this question can perhaps be answered best by seeing the applications that global symbols have found on boundaryless manifolds. Perhaps the most notable application of a global symbolic calcului can be found in Widom's paper, [Wid80]. Here, a functional calculus for (families of) elliptic pseudodifferential operators is developed by drawing upon ideas of [Str72], which allows one to compute the symbol for $f(A)$. This computation, when applied to the exponential of the Laplacian, gives an asymptotic expansion for the heat kernel. This approach was used later by Getzler in [Get86] to give a proof of the local Atiyah-Singer index theorem.

Another important field in which symbol calculi play an important role is deformation quantization. It is well known that one of the primary differences between classical and quantum physics is that quantum replaces the commutative algebra of classical observables, usually functions on a symplectic manfold known as phase space, with a noncommutative algebra of quantum observables, usually operators on some Hilbert space. The passage from the classical to the quantum is known as *quantization*, or in more specific contexts, *deformation quantization*. Roughly speaking, a deformation quantization is a linear isomorphism $Op$ between the classical and quantum algebras of observables which, to first order in the Planck length, converts the Poisson bracket $\{a, b\}$ to the commutator $[Op(a), Op(b)]$. This is precisely what global symbol calculi do. We believe that our symbol for the $b$-calculus enjoys an analagous property, and thus provides a concrete example of what Nest and Tsygan call a *formal deformation of the $b$-cotangent bundle.* For details on deformations of symplectic manifolds-with-boundary, see [NT96]

Index theory is another place where a global symbol calculus can find applications. It has been established in [NT95] that formal deformations will give rise to algebraic index theorems. As our global symbol for the $b$-calculus is an example of a formal deformation, it is our hope that an associated algebraic index theorem can be proved.

On the analytic side of index theory, the $b$-calculus carries its own version of the classical Atiyah-Singer theorem called the Atiyah-Patodi-Singer index theorem, which computes the index of a $b$-Dirac operator in terms of topological invariants of the underlying manifold, as well as a spectral invariant called the $\eta$-invariant. The proof of this theorem relies heavily on properties of a generalized trace, the so-called *$b$-trace*, (which is not really a trace as it does not vanish on commutators.) Our global symbol calculus will give us a new formula for the $b$-trace.

Aside from their wealth of applications, global symbols allow for simplifications and beautification of the general theory. Without a global symbol, the proofs of certain basic analytic facts of pseudodifferential operator theory are rather cumbersome and rely on coordinates. Contrast this to the local case, where an appeal to the operator's symbol allows one to reduce operator-theoretic questions to simple questions about bounds on the symbol. Global symbols allow direct, coordinate-free generalizations these proofs to the manifold case. For example, in [Pfl98], Pflaum uses his normal symbol calculus to construct a parametrix for an elliptic pseudodifferential operator, using a Neumann series argument that appeared, in the local case, as early as [Hör85]. As a bonus, this proof allowed for a new, more general notion of ellipticity than was previously considered.

This brings us to the main point of this paper. We believe that the *point* of a pseudodifferential calculus is to reduce operator-theoretic questions to questions about symbols. Without a global correspondence between operators and symbols, the $b$-calculus lacks this ability (on a global scale). Granted, this may be more of an aesthetic justification than a practical one, but it is the one that motivated the work in the first place and kept us going.

We will begin by reviewing the necessary local material. This will include the analysis of pseudodifferential operators on $\mathbb{R}^n$, including definitions, asymptotic expansions of symbols, and the definition of the symbol in this setting. Then, we will give an account of the normal symbol on a Riemannian manifold, due to M. Pflaum. We will present it in a different style, with definitions and proofs tailored so that they generalize immediately to the $b$-calclulus. Once the normal symbol is established, we will define the $b$-calculus, and its global symbol, which will be a direct generalization of the normal symbol. We will derive the basic properties of this new symbol, and show how the $b$-trace of a $b$-pseudodifferential operator can be found from it's symbol. Finally, we will finish by suggesting further applications and generalizations.

# Chapter 2

# Local Results and Definitions

The purpose of this chapter is to describe the symbol calculus in the local case; that is, on an open subset $U \subseteq \mathbb{R}^n$. We'll define pseudodifferential operators and symbols in this context, and describe the process by which one associates to every operator, $A$, its full symbol, $\sigma_A$. In addition, we will also develop a theory of *Fourier integral distributions* and a method for estimating them. These distributions will appear several times in subsequent chapters, and the estimation method (essentially stationary phase) will be used to prove our fundamental results. Our exposition follows that of [GS94], though it is necessarily lighter on details, and is tailored for our specific needs.

## 2.1 Symbols

Throughout this paper we will use multi-index notation. A *multi-index* $\alpha$ is a tuple of natural numbers which we use to shorten formulas in which a large or arbitrary number of partial differentiations take place. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, and $u(x) = u(x_1, \ldots, x_n) \in C^\infty(\mathbb{R}^n)$ then we write

$$\partial_x^\alpha u = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

**Definition 1.** *Let $U$ be an open subset of $\mathbb{R}^n$. A* ***symbol*** *of order $m$ on $U$ is a smooth function $a(x, \xi) \in C^\infty(U \times \mathbb{R}^N)$ with the property that, for any compact subset $K \subseteq U$, and any multi-indices $\alpha, \beta$, there exists a constant $C = C_{K,\alpha,\beta}(a)$ such that for all $(x, \xi) \in K \times \mathbb{R}^N$,*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C(1+|\xi|)^{m-|\beta|} \tag{2.1}$$

When speaking of symbols, it is important to draw a distinction between the $x$ and the $\xi$ variables. Accordingly, we will refer to the $x$'s as *base variables* and the $\xi$'s as *fiber variables.* Note that the number of base variables and the number of fiber variables do not have to agree, though often they will.

At this time, it will be useful to introduce a notational convention, to clear up the following potential source of ambiguity. In many cases, either the open subset $U$ or the fiber space $\mathbb{R}^N$ (or both) will arise as a product of copies of $\mathbb{R}^n$. For example, we may have a symbol, $a(x,y,\sigma)$, on $U = \mathbb{R}^n_x$ with fiber space equal to $\mathbb{R}^n_y \times \mathbb{R}^n_\sigma$. When describing the symbol space which $a$ belongs to, the notation would dictate that we write

$$a \in S^m(\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^n_\sigma)$$

which is ambiguous. It does not specify whether or not $y$ is a fiber variable. To clear this up, we will group the fiber variables together with parentheses as follows:

$$a \in S^m(\mathbb{R}^n_x \times (\mathbb{R}^n_y \times \mathbb{R}^n_\sigma))$$

Similarly, if the base space arises as a product, we will group the base variables together with parentheses.

Sometimes, we will be looking at a function $a$ which satisfies a bound of the form

$$|a(x,\xi)| \leq C(1+|\xi|)^m$$

for some particular value of $m$, but we won't necessarily know that all of its derivatives satisfy the corresponding estimates. In this case, it would be incorrect to call $a$ a symbol, so we will just say that *$a$ has order $m$* in this situation. This teminology is nonstandard, but it will simplify the wording of a couple of proofs.

Now that we know what a symbol is, we will describe the structure of the symbol spaces $S^m$. The first observation we make is that, for any value of $m$, the space $S^m$ is closed under addition and scalar multiplication. In fact, $S^m$ is a topological vector space, for its elements are defined by the symbol bounds:

$$\frac{|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|}{(1+|\xi|)^{m-|\beta|}} \leq C$$

The smallest constant $C$ which satisfies this inequality is a seminorm, $\rho_{\alpha,\beta,K}(a)$. Among the totality of all such seminorms, we can choose a countable subset in the standard way, by taking an increasing sequence of compact sets $K_n$ which exhausts $U$, and choosing the set of all $\rho_{\alpha,\beta,K_n}$. This countable family of seminorms gives $S^m$ the topology of a Fréchet space, independent of the choice of $K_n$. Under this topology, the inclusion and multiplication maps

$$S^m \hookrightarrow S^{m+\epsilon}$$

$$S^m \times S^{m'} \rightarrow S^{m+m'}$$

are continuous. In particular, we can form the limit

$$S^{-\infty} = \bigcap S^m$$

The symbols belonging to $S^{-\infty}$ will be called *symbols of order* $-\infty$, or *smoothing symbols.* The space $S^{-\infty}$ inherits seminorms from all of the finite order symbol spaces, and is itself a Fréchet space. (In fact, it is a topological algebra, though *not* a Fréchet algebra. The seminorms are not submultaplicative.) In practical terms, $S^{-\infty}$ can be defined as the space of smooth functions $a(x,\xi)$ for which each of the quantities

$$\frac{|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|}{(1+|\xi|)^{M}}$$

is *uniformly bounded on compact $x$-sets.* (By "uniform on compact $x$-sets" we mean that, given a compact set $K \subseteq U$, the same bound will work for all $x \in K$. Different bounds may be needed for different multi-indices $\alpha, \beta$ and different values of $M \in \mathbb{R}$. We will be using this terminology frequently.) The smallest such bound is a seminorm, and this family of seminorms defines the Fréchet space topology on $S^{-\infty}$ (again, after choosing a sequence of $K_n$ which cover $U$, and also restricting attention to integer values of $M$.)

One point of view which we will consistently take in this paper is that the space of smoothing symbols can be thought of as a space of "smoothly varying Schwartz test functions". More precisely, a symbol $a$ associates to every point $x$ in the base space $U$ a rapidly decreasing function $a_x \in \mathscr{S}(\mathbb{R}^N_\xi)$ defined in the obvious way:

$$a_x(\xi) = a(x, \xi)$$

Thus a symbol can be thought of as a smooth map $a : U \to \mathscr{S}(\mathbb{R}^N)$, so in a sense, $a$ is a section of a trivial vector bundle over $U$ whose fibers are isomorphic to the Fréchet space $\mathscr{S}(\mathbb{R}^N)$. We will make this notion more precise later.

The analogy between $S^{-\infty}$ and a space of smoothly varying Schwartz test functions can be pushed quite far. For example, there is a notion of Fourier transform on $S^{-\infty}$:

**Definition 2.** *The* ***fiberwise Fourier transform*** $\mathcal{F}_{z\to\xi}$ *is the integral transform which acts on symbols $a(x, z)$ as follows:*

$$[\mathcal{F}_{z\to\xi} a](x, \xi) = (2\pi)^{-N/2} \int e^{-iz\cdot\xi} a(x, z)\, dz$$

*We will denote the fiberwise Fourier transform of $a(x, z)$ by $\hat{a}(x, \xi)$ whenever there is no confusion as to which variable is being transformed.*

It should come as no surprise that the fiberwise Fourier transform is a (continuous) Fréchet space isomorphism on $S^{-\infty}$. It's inverse, $\mathcal{F}^{-1}_{\xi\to z}$, is identical except that the phase $-iz \cdot \xi$ is replaced by $iz \cdot \xi$. In addition, the Fiberwise transform satisfies all of the properties that one would hope for, including Parseval's formula and the all-important bijection between differential operators and multiplication by polynomial symbols. Proofs of these facts are identical to the usual ones, except that one must always check that eny estimate made along the way holds uniformly on compact $x$-sets. We'll be using these theorems below, so we'll state them formally.

**Theorem 3.** *The fiberwise Fourier transform is an isomorphism of Fréchet spaces*

$$\mathcal{F}_{z\to\xi} : S^{-\infty}(U \times \mathbb{R}^n_z) \to S^{-\infty}(U \times \mathbb{R}^n_\xi)$$

**Theorem 4.** *If $a \in S^{-\infty}(U \times \mathbb{R}^n_z)$ and $b \in S^{-\infty}(U \times \mathbb{R}^n_\xi)$,*

$$\int b(x,\xi)\,[\mathcal{F}_{z\to\xi}a]\,(x,\xi)\,d\xi = \int a(x,z)\,[\mathcal{F}_{\xi\to z}b]\,(x,z)\,dz$$

**Theorem 5.** *Let $p(x,\partial_z) = \sum_{|\alpha|\le k} a_a(x)\partial_z^\alpha$ be a differential operator of order $k$ on $U$. Then for any $a \in S^{-\infty}(U \times \mathbb{R}^n_z)$,*

$$p(x,\partial_z)a = \mathcal{F}^{-1}_{\xi\to z} p(x,i\xi)\mathcal{F}_{z\to\xi}a$$

We should note at this point that the fiberwise Fourier transform can act on symbols of finite order, as well. This is because every $a \in S^m(U \times \mathbb{R}^N)$ can be thought of as as smoothly varying *tempered function*. By a "tempered function" (of order $m$) we mean a smooth function $u \in C^\infty(\mathbb{R}^N)$ whose every derivative $\partial^\beta u(\xi)$ is $O(|\xi|^{m-|\beta|})$ as $\xi \to \infty$. This is equivalent to the existence of a constant, $C_\beta$, which satisfies

$$|\partial^\beta u(\xi)| \le C_\beta(1+|\xi|)^{m-|\beta|}$$

The smallest such constant is a seminorm, and so $\mathscr{S}^m(\mathbb{R}^N)$ is another Fréchet space. Thus it makes sense to talk about differentiable mappings into $\mathscr{S}^m(\mathbb{R}^N)$, and every symbol $a \in S^m(U \times \mathbb{R}^N)$ can be thought of as a smooth map

$$a : U \to \mathscr{S}^m(\mathbb{R}^N)$$

On the other hand, every tempered function can be identified with a tempered distribution in the usual way. Thus it is possible to take the Fourier transform of tempered functions, hence the fiberwise Fourier transform of a symbol $a \in S^m$ is possible. One must be careful, though, for the Fourier transform of a tempered function is *not* a tempered function. In general, it will be singular at the origin, and so it must be thought of as a distribution. Therefore, the fiberwise transform of a symbol is not a symbol, or even smooth. Despite this, theorem (3) still hold for symbols of finite order, provided it is interpreted correctly in a distributional context. Here is the precise statement:

**Theorem 6.** *The fiberwise Fourier transform is an isomorphism of Fréchet spaces*

$$\mathcal{F}_{z\to\xi} : S'(U \times \mathbb{R}^n_z) \to S'(U \times \mathbb{R}^n_\xi)$$

*where $S'$ is the space of smooth maps $a : U \to \mathscr{S}'(\mathbb{R}^N)$.*

Formally, we think of $S'$ as the space of continuous linear maps $S^{-\infty}(U \times \mathbb{R}^n) \to C^\infty(U)$.

The next theorem, the distributional analogue of Parseval's formula, is really the definition of the Fourier transform on tempered distributions. We will be using it below, so we will state it as a theorem.

**Theorem 7.** *If $a \in S^m(U \times \mathbb{R}^n_z)$ and $b \in S^{-\infty}(U \times \mathbb{R}^n_\xi)$, then*

$$\int b(x,\xi) \left[\mathcal{F}_{z\to\xi} a\right](x,\xi)\, d\xi = \int a(x,z) \left[\mathcal{F}_{\xi\to z} b\right](x,z)\, dz$$

When studying symbols, we will be most interested in their asymptotic behavior as the fiber variables go to infinity. It turns out that this behavior can always be captured quite precisely by an asymptotic summation of symbols of lower order. The following theorem makes this notion precise.

**Definition 8.** *Let $\{a_k\}_{k=0}^\infty$ be a sequence of symbols whose orders, $m_k$ decrease to $-\infty$. Then there exists a symbol $a$, unique modulo $S^{-\infty}$, such that for each $M$, the difference*

$$s_M = a - \sum_{k=0}^{M-1} a_k$$

*is a symbol of order $m_M$. In this situation, we call $a$ the* ***asymptotic sum*** *of the $a_k$, and write*

$$a \sim \sum_{k=0}^{\infty} a_k \tag{2.2}$$

Practically any smooth function possessing an asymptotic representation of the form (2.2) is a symbol. So if we are looking at a function and want to prove that it is a symbol, we will seek to find such a summation. More precisely, we will have to check that the hypotheses of the following theorem are satisfied. The question of how to produce such asymptotic summations will be answered in section 2.2.

**Theorem 9.** *Let $a \in C^\infty(U \times \mathbb{R}^N)$. Suppose that*

1. *For any multi-indices $\alpha, \beta$, there exists $M_{\alpha,\beta} \in \mathbb{R}$ such that $\partial_x^\alpha \partial_\xi^\beta a$ has order $M_{\alpha,\beta}$.*
2. *There exists a sequence of symbols $a_k$, whose orders $m_k$ decrease to $-\infty$, and a sequence of real numbers $m'_k \to -\infty$ such that the difference*

$$s_M = a - \sum_{k=0}^{M-1} a_k$$

*has order $m'_M$.*

*Then $a$ is the asymptotic summation of the $a_k$.*

## 2.2 Estimation of Fourier Integral Distributions

Symbols will appear frequently in integrals of the following form:

$$I(a, \varphi)(x) = \int e^{i\varphi(x,\theta)} a(x, \theta)\, d\theta \tag{2.3}$$

Such integrals are called *Fourier integral distributions.* In this context, the function $a$ is called the *amplitude*, and the function $\varphi$ appearing in the exponent is called a *phase function.* A phase function is required to satisfy the following three properties:

(1) The imaginary part of $\varphi$ is nonnegative.

(2) $\varphi(x, \lambda\theta) = \lambda\varphi(x, \theta)$ for $\lambda > 0$.

(3) There is no point $(x, \theta)$ where the partial derivatives of $\varphi$ all vanish, except possibly when $\theta = 0$.

During our development of the global symbol calculus, we will encounter many Fourier integral distributions. In general, such integrals are not smooth, or even continuous. In fact, there is no reason to believe that the integral (2.3) converges at all, and so such an integral is to be regarded as a distribution, until proven otherwise. As it turns out, the smoothness of a Fourier integral is determined by the *critical set* of $\varphi$, $C_\varphi$:

$$C_\varphi = \left\{(x, \theta) \in U \times \mathbb{R}^N : \theta \neq 0, \frac{\partial\varphi}{\partial\theta_1} = \cdots = \frac{\partial\varphi}{\partial\theta_N} = 0\right\} \tag{2.4}$$

The following lemma makes this relationship more precise, and provides a sufficient condition for smoothness which is easy to check.

**Theorem 10.** *Let $a \in S^m(U \times \mathbb{R}^n)$ and let $\varphi$ be a smooth phase function on $U \times \mathbb{R}^N$. If $a(x, \theta)$ vanishes in a conical neighborhood of $C_\varphi$, then $I(a, \varphi)$ is smooth.*

(The term *conical neighborhood* refers to an open set $V$ containing $C_\varphi$ to which $(x, t\theta)$ belongs whenever $(x, \theta) \in V$ and $t > 0$.)

Not all of the Fourier integral distributions that we encounter will be smooth. In fact, the Schwartz kernels of pseudodifferential operators are themselves examples of singular Fourier integral distributions! When this is the case, we will be interested in locating the singularities of the given distribution. We will do this with the help of the following lemma.

**Lemma 11.** *Let $I(a,\varphi)$ be a Fourier integral distribution with phase function $\varphi$. $I(a,\varphi)$ is a distribution on $U$, whose singular support can be located as follows:*

$$\operatorname{sing\,supp} I(a,\varphi) \subseteq \left\{x \in U : \exists \theta \in \mathbb{R}^N (x,\theta) \in C_\varphi\right\}$$

Many of the Fourier integral distributions that we will encounter will have the following form:

$$b(x,\lambda) = \int e^{i\lambda\langle z,Qz\rangle/2} a(x,z)\,dz \tag{2.5}$$

here, the integral is understood to run over all of $\mathbb{R}^N_z$, $Q$ is a nonsingular symmetric $N \times N$ matrix, $a(x,z)$ is a symbol, and $\lambda$ is a real parameter. Note that, for integrals of this form, the critical set of the phase function is empty. Therefore, $b(x,\lambda)$ is smooth. We will be interested in obtaining asymptotic expansions of such integrals, as $\lambda \to \infty$. The standard approach is to use the method of stationary phase.

**Theorem 12.** *Let $a(x,z) \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^N)$. Fix a nonsingular symmetric $N \times N$ matrix $Q$ and define an oscillatory integral $b(x,\lambda)$ by (2.5).*

*Then as $\lambda \to \infty$, the following asymptotic expansion for $b(x,\lambda)$ holds uniformly on compact $x$-sets:*

$$b(x,\lambda) \sim C_Q \sum_{k=0}^{\infty} \frac{\lambda^{-N/2-k}}{2^k k!} \langle \partial_z, Q^{-1}\partial_z\rangle^k a(x,z)\Big|_{z=0}$$

*Proof.* We are are going to use Parseval's formula (theorem (7)) to write $b$ out in terms of Fourier transforms:

$$b(x,\lambda) = \int_{\mathbb{R}^N} \mathcal{F}_{z\to\xi}\left(e^{i\lambda\langle z,Qz\rangle/2}\right)\hat{a}(x,\xi)\,d\xi$$

It can be shown that the oscillating term transforms as follows:

$$\mathcal{F}_{z\to\xi}\left(e^{i\lambda\langle z,Qz\rangle/2}\right) = C_Q \lambda^{-N/2} e^{-i\langle \xi,Q^{-1}\xi\rangle/2\lambda}$$

where $C_Q$ is a constant depending only on $Q$:

$$C_Q = (2\pi)^{n/2} e^{i\frac{\pi}{4}\operatorname{sgn}(Q)} |\det Q|^{-1/2} \tag{2.6}$$

(Here, sgn($Q$), the signature of $Q$, is the difference between the number of positive and negative eigenvalues of Q.) Thus

$$b(x,\lambda) = C_Q \lambda^{-N/2} \int e^{-i\langle \xi, Q^{-1}\xi\rangle/(2\lambda)} \hat{a}(x,\xi)\, d\xi$$

The Taylor expansion of the oscillating term is

$$e^{-i\langle \xi, Q^{-1}\xi\rangle/(2\lambda)} = \sum_{k=0}^{m-1} \frac{1}{k!} \left( \frac{1}{2\lambda i} \langle \xi, Q^{-1}\xi\rangle \right)^k + R_m(\xi,\lambda)$$

Where the remainder $R_m$ satisfies the Lagrange error bound:

$$R_m(\xi,\lambda) \le \frac{|\langle \xi, Q^{-1}\xi\rangle|^m}{m!(2\lambda)^m}$$

Substituting the Taylor expansion into $b(x,\lambda)$ and using the fact that polynomials correspond to differential operators under the Fourier transform, we arrive at the following approximation for $b$:

$$b(x,\lambda) = C_Q \lambda^{-N/2} \left[ \sum_{k=0}^{m-1} \frac{1}{k!} \left( \frac{1}{2\lambda i} \langle D_z, Q^{-1} D_z \rangle \right)^k a(x,z) \bigg|_{z=0} + S_m(x,\lambda) \right]$$

where

$$S_m(x,\lambda) = \int R_m(\xi,\lambda) \hat{a}(x,\xi)\, d\xi$$

We want to show that this approximation is asymptotic in $\lambda$ as $\lambda \to \infty$, and that the estimates inherent in this expansion are uniform on compact $x$-sets. Explicitly, this means that for any compact $K \subseteq \mathbb{R}^n_x$, and any $m \ge 0$, there exists a constant $C = C_{K,m}$ such that for all $x \in K$ and all large enough $\lambda$,

$$|S_m(x,\lambda)| \le C\lambda^{-m}$$

This is easy to do once you recognize that $\hat{a}(x,z)$ is a symbol of order $-\infty$. Using the Lagrange error bound, we write

$$|S_m(x,\lambda)| \leq \frac{1}{m!(2\lambda)^m} \int |\langle \xi, Q^{-1}\xi\rangle|^m |\hat{a}(x,\xi)|\, d\xi$$

Choose a constant $C$ so that

$$|\hat{a}(x,\xi)| \leq C(1+|\xi|)^{-(N+1+2m)}$$

holds for all $\xi$ and all $x \in K$. Substituting this estimate into the integral shows that $|\langle \xi, Q^{-1}\xi\rangle|^m |\hat{a}(x,\xi)|$ grows like $|\xi|^{-(N+1)}$ and hence is integrable. After performing this integration, and absorbing all of the constants together, we arrive at the desired result.

□

In practice, we will need to estimate integrals of the form (2.5) when the amplitude $a(x,z)$ does not decrease rapidly. The following generalization allows us to do this.

**Theorem 13.** *Let $a(x,z) \in S^m(\mathbb{R}^n \times \mathbb{R}^N)$. Fix a nonsingular symmetric $N \times N$ matrix $Q$ and define an oscillatory integral $b(x,\lambda)$ by*

$$b(x,\lambda) = \int_{\mathbb{R}^N} e^{i\lambda\langle z, Qz\rangle/2} a(x,z)\, dz$$

*Then as $\lambda \to \infty$, the following asymptotic expansion for $b(x,\lambda)$ holds uniformly on compact $x$-sets:*

$$b(x,\lambda) \sim C_Q \sum_{k=0}^{\infty} \frac{\lambda^{-N/2-k}}{2^k k!} \langle \partial_z, Q^{-1}\partial_z\rangle^k a(x,z)\Big|_{z=0}$$

*Proof.* We're going to reduce this to the previous theorem by introducing a smooth cut-off function $\chi : [0,\infty) \to [0,1]$, supported in $[0,1]$, with the property that $\chi(x) = 1$ for $x \leq 1/2$. We use $\chi$ to break $b$ into two parts, $b = b_1 + b_2$ where

$$b_1(x,\lambda) = \int e^{i\lambda\langle z,Qz\rangle/2}\chi|z|a(x,z)\,dz$$
$$b_2(x,\lambda) = \int e^{i\lambda\langle z,Qz\rangle/2}\,(1-\chi|z|)\,a(x,z)\,dz$$

We'll show that $b_2$ decreases rapidly as $\lambda \to \infty$ (uniformly on compact $x$-sets) and hence has no contribution to the asymptotic expansion. The expansion can then be derived from $b_1$ via stationary phase.

To show that $b_2$ decreases rapidly, we use a trick called "formal integration by parts." This involves finding a differential operator, $L$, which fixes the oscillating term:

$$L\left(e^{i\lambda\langle z,Qz\rangle/2}\right) = e^{i\lambda\langle z,Qz\rangle/2}$$

This allows us to introduce an arbitrarily high power of $L$ into the integrand as follows:

$$b_2(x,\lambda) = \int L^k(e^{i\lambda\langle z,Qz\rangle/2})\,(1-\chi|z|)\,a(x,z)\,dz$$

We'll trade this for the following integral involving $L^t$, the transpose of $L$:

$$b_2(x,\lambda) = \int e^{i\lambda\langle z,Qz\rangle/2}(L^t)^k\,[(1-\chi|z|)\,a(x,z)]\;dz$$

and then show that each application of $L^t$ lowers the order of $a$, in both $z$ and $\lambda$. This is easy to do once we have written down an explicit formula for $L$:

$$L = \frac{-i}{\lambda|Qz|^2}\sum_{j,l} Q_{jl}z_l\partial_{z_j}$$

Checking that $L$ fixes the oscillating term is elementary. The transpose of $L$ is easier to compute if we change variables to $w_i = Q_{ij}z_j$. In terms of $w$, $L$ takes the form

$$\begin{aligned} L &= \frac{-i}{\lambda|w|^2}\sum_{j,l} Q_{jl}w_j\partial_{w_l} \\ &= \sum_{j,l}\left(\frac{-i}{\lambda}\right)Q_{jl}A_{jl} \end{aligned}$$

where$A_{jl} = |w|^{-2} w_j \partial_{w_l}$. We can find $A^t_{jl}$, the transpose of $A_{jl}$, as follows:

$$\begin{aligned}
A^t_{jl} &= -\partial_{w_l} \circ (|w|^{-2} w_j) \\
&= -\left(\partial_{w_l}\left(|w|^{-2} w_j\right) + |w|^{-2} w_j \partial_{w_l}\right) \\
&= -\left(-(|w|^{-2})^{-2} 2 w_l w_j + |w|^{-2}\delta_{jl} + |w|^{-2} w_j \partial_{w_l}\right) \\
&= \frac{2 w_l w_j}{|w|^4} - \frac{\delta_{jl}}{|w|^2} - A_{jl}
\end{aligned}$$

Multiplying by $\left(\frac{-i}{\lambda}\right) Q_{jl}$ and summing over $j, l$, we see that $L^t$ is given by

$$\begin{aligned}
L^t &= \sum_{j,l} \left(\frac{-i}{\lambda}\right) Q_{jl} A^t_{jl} \\
&= \left(\frac{-i}{\lambda}\right)\left(\frac{2}{|w|^4}\sum_{j,l} Q_{jl} w_l w_j - \frac{1}{|w|^2}\sum_{j,l} Q_{jl}\delta_{jl}\right) - \left(\frac{-i}{\lambda}\right)\sum_{j,l} Q_{jl} A_{jl} \\
&= \left(\frac{-i}{\lambda}\right)\left(\frac{2\langle Qw, w\rangle}{|w|^4} - \frac{\operatorname{Tr} Q}{|w|^2}\right) - L
\end{aligned}$$

It is not difficult to see that

$$L^t((1-\chi|z|)\, a(x,z)) = \lambda^{-1} a_1(x,z)$$

where $a_1$ is a symbol of order $m-2$ which vanishes in a neighborhood of $z = 0$. This can be iterated to conclude that

$$(L^t)^k((1-\chi|z|)\, a(x,z)) = \lambda^{-k} a_k(x,z)$$

where $a_k$ is a symbol of order $m-2k$ which vanishes in a neighborhood of $z = 0$. (This vanishing condition is important because without it $L^t a_k$ would be undefined at $z = 0$.) We now have the following bound on $b_2(x,\lambda)$:

$$|b_2(x,\lambda)| \le \lambda^{-k} \int |a_k(x,z)|\, dz$$

We're home free now. Since $a_k(x,\cdot)$ does not blow up at the origin, we can use the symbol bound:

$$|b_2(x,\lambda)| \le C\lambda^{-k} \int (1+|z|)^{m-k}\, dz$$

As long as k is large enough, this integral converges, so it can be absorbed into the constant. Since $C$ was obtained by a symbol bound, it is uniform on compact $x$-sets, so we have proven that $b_2$ is rapidly decreasing in $\lambda$, uniformly on compact $x$-sets.

Now the amplitude for $b_1$ is a symbol of order $-\infty$, so the first stationary phase lemma can be applied to it. The asymptotic expansion that we obtain is

$$b_1(x,\lambda) \sim C_Q \sum \frac{\lambda^{-N/2-k}}{2^k k!} \langle \partial_z, Q^{-1}\partial_z\rangle^k \left[\chi|z|a(x,z)\right]\Big|_{z=0}$$

However, $\chi(z) = 1$ for $z$ sufficiently close to 0, so this is actually equivalent to the original expansion:

$$b_1(x,\lambda) \sim C_Q \sum \frac{\lambda^{-N/2-k}}{2^k k!} \langle \partial_z, Q^{-1}\partial_z\rangle^k a(x,z)\Big|_{z=0}$$

Since $b$ is equivalent to $b_1$ modulo a rapidly decreasing term, this expansion holds for $b$ as well.

$\square$

The previous theorem will be referred to informally as the "estimation theorem." A particularly important special case of theorem (13) is when the symbol $a$ has the form

$$a(x,y,\sigma) \in S^m\left(\mathbb{R}^n_x \times \left(\mathbb{R}^n_y \times \mathbb{R}^n_\sigma\right)\right)$$

and the matrix $Q$ is given by

$$Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where $I$ is the $n \times n$ identity matrix. The oscillatory integral $b$ will also have a factor of $\lambda^n$ in front of it:

$$b(x,\lambda) = \lambda^n \iint e^{i\lambda y\cdot\sigma} a(x,y,\sigma)\, dyd\sigma \tag{2.7}$$

so that the asymptotic expansion which we derive for $b$ will be

$$b(x,\lambda) \sim (2\pi)^n \sum \frac{\lambda^{-k}}{2^k k!} \left(\partial_y \cdot \partial_\sigma\right)^k a(x,y,\sigma)\Big|_{y=\sigma=0} \tag{2.8}$$

## 2.3 Pseudodifferential Operators and the Local Symbol

Throughout this paper, we will make frequent use of the *Schwartz kernel theorem.* Recall that this theorem tells us that we can identify continuous linear maps $A : C^\infty(U) \to \mathcal{D}'(U)$ with distributions $K_A \in \mathcal{D}'(U \times U)$ via the pairing:

$$\iint K_A(x,y)u(x)v(y)\, dx\, dy = \int [Av](x)u(x)\, dx$$

The distribution $K_A$ is known as the *Schwartz kernel* of the operator $A$. Pseudodifferential operators, then, can be defined by specifying their Schwartz kernels, and we will do this below.

**Definition 14.** *Let $U$ be an open subset of $\mathbb{R}^n$. A **pseudodifferential operator** of order $m$ on $U$ is a continuous linear operator*

$$A : C_0^\infty(U) \to \mathcal{D}'(U)$$

*whose Schwartz kernel $K_A$ is a Fourier integral distribution given by*

$$K_A(x,y) = \int e^{-i(y-x)\cdot\theta} a(x,y,\theta)\, d\theta$$

*for some $a \in S^m\left((U \times U) \times \mathbb{R}^n\right)$. The space of all pseudodifferential operators of order $m$ on $U$ will be denoted by $\Psi^m(U)$.*

Note that, by lemma (11), the singularities of $K_A$ all lie on the diagonal, $x = y$.

As there are symbols of order $-\infty$, so are there operators of this order $-\infty$. Such operators are called smoothing operators, because they smooth out everything that they act on. That is, for all $A \in \Psi^{-\infty}$, and all distributions $u$, $Au \in C^\infty(U)$. The reason this happens is that the Schwartz kernel of $A$, being the fiberwise Fourier transform of a rapidly decreasing symbol, is a smooth function. In fact, smoothing pseudodifferential operators are the only pseudodifferential operators with smooth Schwartz kernels.

In the previous paragraph, we hinted that pseudodifferential operators can be extended to act on distributions. This is done in the usual way, as the transpose of a pseudodifferential operator is pseudodifferential (it's Schwartz kernel can be obtained by interchanging $x$ and $y$.)

An important class of pseudodifferential operators are the *properly supported* operators. Here, the word "support" refers to the support of the Schwartz kernel $K_A \subseteq U \times U$, and it means that the inverse image of every compact set $K \subseteq U$, under the projections $\pi_1, \pi_2 : \operatorname{supp} K_A \to U$, is compact.

One reason that properly supported operators are important is that the composition of two pseudodifferential operators is, in fact, not guaranteed to be pseudodifferential. However, if at least one of them is properly supported, then the composition will be pseudodifferential. This is not a big restriction, for every pseudodifferential operator is properly supported modulo $\Psi^{-\infty}$.

For us, the main reason that we care about proper supports is that one can define the global symbol for a properly supported operator.

**Definition 15.** *Let $A \in \Psi^m(U)$ be properly supported. The* ***full symbol*** *of $A$, $\sigma_A$, is the Fourier integral distribution defined by*

$$\begin{aligned} \sigma_A(x,\xi) &= \int e^{iv\cdot\xi} K_A(x, x+v)\, dv \\ &= \iint e^{iv\cdot(\xi-\theta)} a(x, x+v, \theta)\, dv d\theta \end{aligned} \tag{2.9}$$

Since $K_A$ is properly supported, the integral appearing in (2.9) can be interpreted as running over all of $\mathbb{R}^n_v$. Thus the full symbol $\sigma_A$ is the (inverse) fiberwise Fourier transform of the tempered distribution $a(x, v) = K_A(x, x + v)$.

Of course, the full symbol of a pseudodifferential operator really is a symbol, of the same order as $A$; $\sigma_A \in S^m(U \times \mathbb{R}^n)$. Roughly, this is proven by applying the following coordinate change to (2.9):

$$\lambda = |\xi|$$

$$\sigma = \frac{\xi - \theta}{\lambda}$$

This transforms (2.9) to the following integral:

$$(-\lambda)^n \iint e^{i\lambda v \cdot \sigma} a(x, x + v, \xi - \lambda\sigma)\, dv d\sigma$$

This oscillatory integral has the form of (2.7). (In particular, it is smooth.) Thus, we may apply stationary phase to obtain an asymptotic expansion for $\sigma_A$ in decreasing powers of $\lambda = |\xi|$, similar to (2.8). Such an expansion guarantees that $\sigma_A$ is a symbol, by theorem (9). Details of this proof can be found in [GS94]. This technique will be used for both the normal symbol and $b$-normal symbol.

# Chapter 3

## The Normal Symbol on Riemannian Manifolds Without Boundary

It's time to put all of the local machinery to work, and define a global symbol calculus for a compact Riemannian manifold $M$ (without boundary.) This has been done several times before, in varying degrees of generality, by [Sha05], [Pfl98], [Wid80], and [BH69]. Our approach does not differ in substance from [Pfl98]. We will restrict our attention to a Riemannian manifold, and use the geometry therein to define our symbol, which will be equivalent to the normal symbol of M. Pflaum.

Though the mathematical content below is equivalent to [Pfl98], our viewpoint is slightly different. We will define the normal symbol directly from the operator's Schwartz kernel. This is done with an outlook to the $b$-calclulus, where everything is done in terms of Schwartz kernels. The proofs we present here will generalize immediately to the $b$-setting, where a boundary is present. In fact, they will even be valid over the interior, unaltered, so that when we define the $b$-normal symbol, we will only need to give proofs at the boundary. These will be slightly more complicated, but they follow the same general plan of attack as the ones appearing in this section.

### 3.1 Basic Definitions and Preliminaries

Before proceeding with a symbol calculus, we are going to review the basic elements of a theory of pseudodifferential operators on a compact manifold. We will start by defining the notion of a symbol on a manifold.

Now when we abstract the definition of the local symbols $a \in S^m(U \times \mathbb{R}^n)$, we are going to let the open set $U$ play the part of a manifold $M$. Therefore, the proper generalization of symbol is not a function on $M$, but rather a smooth function on a *vector bundle* on $M$, but which one? Recall that one obtains the symbol of a differential operator by replacing each partial derivative $\partial/\partial x_j$ with the variable $i\xi_j$. Therefore, if we want any hope of a symbol transforming properly under coordinate changes, $\xi_j$ must transform in the same way that the partial derivative does. Of course, we know of a system of variables that has this transformation property: the coefficients of cotangent vectors. Thus a symbol ought to be a special type of smooth function on the cotangent bundle, which looks like an ordinary symbol in each trivialization.

**Definition 16.** *Let $M$ be a compact closed manifold. A* ***symbol*** *of order $m$ on $T^*M$ is a smooth function $a \in C^\infty(T^*M)$ with the property that, given any local coordinate chart $U$ and trivialization of $T^*M|U$, the induced function $a(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ obtained by composing with the trivialization is a symbol of order $m$ in the local sense. Symbols on $TM$ are defined similarly.*

The space of all symbols of order $m$ on $T^*M$ is denoted $S^m(T^*M)$, and the same remarks that followed the local definition still apply. $S^m(T^*M)$ is a Fréchet space under the same seminorms, differentiation and multiplication of symbols is continuous with respect to these seminorms, there is a corresponding notion of $S^{-\infty}(T^*M)$ and of asymptotic sum, etc.

When we check that a given function is a symbol, we will not need to consider every possible local trivialization of $T^*M$. Rather, we only need to check an open covering of $M$ by trivializations, because transition functions do not affect the validity of symbol estimates. This is a local statement, and it amounts to the following claim.

**Claim 17.** *Let $\phi : \tilde{U}_x \to U_y$ be a diffeomorphism between open subsets of $\mathbb{R}^n$. Let $\psi : \tilde{U} \to GL(n,\mathbb{R})$ be smooth. Given $a(\cdot,\cdot) \in S^m(U \times \mathbb{R}^n)$, define $\tilde{a} \in C^\infty(\tilde{U} \times \mathbb{R}^n)$ by $\tilde{a}(x,\xi) = a(\phi(x), \psi(x)\xi)$. Then $\tilde{a} \in S^m(\tilde{U} \times \mathbb{R}^n)$.*

*Proof.* Let's begin by showing that $\tilde{a}$ has order $m$. This will be done by directly verifying the symbol bound. To this end, choose a compact set $K \subseteq \tilde{U}$. We need a bound on $|\tilde{a}(x,\xi)|$ which holds for all $x \in K$. Upon writing

$$|\tilde{a}(x,\xi)| = |a(\phi(x),\psi(x)\xi)|$$

it becomes clear that we have the following bound for $\tilde{a}$:

$$|\tilde{a}(x,\xi)| \leq \rho(a)\left(1+|\psi(x)\xi|\right)^m$$

(Here $\rho$ is the seminorm on the symbol space $S^m(U\times\mathbb{R}^n)$ which is associated to the compact set $\phi(K)$ and zero differentiations.) Since $\psi$ is smooth, we can bound $(1+|\psi(x)\xi|)^m$ by a constant multiple of $(1+|\xi|)^m$ uniformly on $K$, so $\tilde{a}$ has order $m$, indeed.

Next we'll show that $\partial_{x_i}\tilde{a}$ has order $m$. This amounts to writing

$$\partial_{x_i}[\tilde{a}(x,\xi)] = \partial_{x_i}\left[a(\phi(x),\psi(x)\xi)\right]$$

and thinking about the chain rule. The chain rule produces two types of terms. The first type is a product of a $y$-derivative of $a$, (order $m$) with an $x$-derivative of $\phi$ (order 0). The second type of term is a product of an $\xi_j$-derivative of $a$, (order $m-1$) with an $x$-derivative of $\psi$, (order 0) and $\xi_j$ itself (order 1). Hence both types of term have order $m$, so $\partial_{x_i}\tilde{a}$, being a sum of such terms, has order $m$. A similar analysis shows that $\partial_{\xi_i}[\tilde{a}(x,\xi)]$ has order $m-1$. Iterating these arguments, we see that every possible partial derivative $\partial_x^\alpha\partial_\xi^\beta\tilde{a}$ has order $m-|\beta|$, and so $\tilde{a}$ is a symbol of order $m$. □

**Definition 18.** *Let $M$ be a Riemannian manifold. A **pseudodifferential operator** of order $m$ is a continuous linear map $A: C_0^\infty(M)\to C^\infty(M)$ which satisfies the following two properties:*

*(1) For any coordinate chart $(\mathbf{y},U)$, the Schwartz kernel of $A$, when viewed as a distribution on $\mathbf{y}(U)\subseteq\mathbb{R}^n$, is an oscillatory integral of the form*

$$K_A\left(\mathbf{y}^{-1}(x),\mathbf{y}^{-1}(y)\right) = \int_{\mathbb{R}^n} e^{-i(y-x)\cdot\theta}a(x,y,\theta)\,d\theta$$

*for some $a(x,y,\theta)\in S^m\left((\mathbf{y}(U)\times\mathbf{y}(U))\times\mathbb{R}^n\right)$.*

*(2) $K_A$ is smooth away from the diagonal.*

Condition (2) above is equivalent to a property posessed by the operator $A$ called *pseudolocality*. Roughly speaking, a pseudolocal operator is one which doesn't create any new singularities. More precisely, if a distribution $u$ is smooth at a point $p$, then $Au$ will be smooth at $p$ as well. Or in symbols, $\operatorname{sing\,supp} Au \subseteq \operatorname{sing\,supp} u$.

Later, we will need to check that a given operator is pseudodifferential. The task would be impossible to carry out if we just had the above definition to work with, for the Schwartz kernel may look quite complicated in a general coordinate chart. Instead, we will choose a covering by *normal* coordinate charts and check that (1) holds there. This will be sufficient, because the following theorem guarantees that transition maps preserve the form of the Schwartz kernel:

**Theorem: (The Kuranishi Trick) 19.** *Let $K(x, y)$ be an oscillatory integral distribution of the form*

$$K(x, y) = \int e^{-i(\kappa(y)-\kappa(x))\cdot\theta} a(x, y, \theta)\, d\theta$$

*where $a \in S^m(U \times U \times \mathbb{R}^n)$ and $\kappa$ is a diffeomorphism onto its image. Then $K(x, y)$ is the Schwartz kernel of a pseudodifferential operator of order $m$ on $U$. That is, there exist coordinates $\tilde{\theta}$ on $\mathbb{R}^n$ (depending on $x$ and $y$) such that, after changing variables in the integral, $K(x, y)$ has the form*

$$K(x, y) = \int e^{-i(y-x)\cdot\tilde{\theta}} \tilde{a}(x, y, \tilde{\theta})\, d\tilde{\theta}$$

*for some $\tilde{a} \in S^m(U \times U \times \mathbb{R}^n)$.*

*Proof.* This is really just the fundamental theorem of calculus in disguise. We begin by considering a line segment $\gamma_{x,y}$ joining $x$ to $y$:

$$\gamma_{x,y}(t) = tx + (1-t)y$$

Inserting this path into $\kappa$ and using the fundamental theorem of calculus, we see that

$$\kappa(x) - \kappa(y) = \int_0^1 \frac{d(\kappa \circ \gamma_{x,y})}{dt} dt$$

Now we use the chain rule, and the fact that the time derivative of the path $\gamma$ is given by $\dot{\gamma}_{x,y} = x - y$:

$$\kappa(x) - \kappa(y) = \int_0^1 D\kappa|_{\gamma_{x,y}(t)}\, dt \cdot (x - y)$$

Here, $D\kappa$ is the Jacobian matrix of $\kappa$, and the vertical bar signifies that we are evaluating at the points along the path $\gamma_{x,y}$. Thus

$$\kappa(x) - \kappa(y) = F(x, y) \cdot (x - y)$$

where $F(x, y)$ is the smooth matrix-valued function given by integrating $D\kappa$ along the path $\gamma_{x,y}$. Using an elementary property of the matrix transpose, we define

$$\tilde{\theta} = {}^t F(x, y)\theta$$

.

Finally, we define

$$\tilde{a}(x, y, \tilde{\theta}) = a(x, y, {}^t F(x, y)^{-1}\tilde{\theta})|\det {}^t F(x, y)^{-1}| \tag{3.1}$$

and change variables to arrive at our desired result. Notice that $\tilde{a}$ is still a symbol, by claim (17). $\square$

The Kuranishi trick will be used again later, in the following modified form. The proof is identical, except that one replaces the path $\gamma_{x,y}$ with the path $\gamma_y(t) = (1-t)y$.

**Theorem: (The Kuranishi Trick) 20.** *Let $K(x,y)$ be an oscillatory integral distribution of the form*

$$K(x,y) = \int e^{-i(\kappa_x(y)-x)\cdot\theta} a(x,y,\theta)\, d\theta$$

*where $a \in S^m\left((U \times U) \times \mathbb{R}^n\right)$ and $\kappa_x$ is a smoothly varying family of diffeomorphisms such that $\kappa_x(x) = 0$ for all $x$. Then $K(x,y)$ is the Schwartz kernel of a pseudodifferential operator of order $m$ on $U$. That is, there exist coordinates $\tilde{\theta}$ on $\mathbb{R}^n$ (depending on $x$ and $y$) such that, after changing variables in the integral, $K(x,y)$ has the form*

$$K(x,y) = \int e^{-iy\cdot\tilde{\theta}} \tilde{a}(x,y,\tilde{\theta})\, d\theta$$

*for some $\tilde{a} \in S^m((U \times U) \times \mathbb{R}^n)$*

This trick will appear in other modified forms as well, but as long as one chooses the correct path $\gamma$, the proof always carries through. Whenever we use a different version of the Kuranishi trick, we will always indicate which path we choose.

Our task is to associate to every pseudodifferential operator $A \in \Psi^m(M)$ a global symbol $\sigma_A \in S^m(T^*M)$ with which we can recover $A$ modulo $\Psi^{-\infty}(M)$. The first hope is that the local symbols $a(x,\xi)$ in the above definition glue together in an appropriate manner to a global object, but this does not work. To see why not, consider the following simple example: the symbol of the Laplacian on $\mathbb{R}^2$. In Cartesian and polar coordinate systems, the Laplacian takes the following two forms:

| Cartesian | Polar |
| --- | --- |
| $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ | $\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}$ |

and so the symbol has the form

| Cartesian | Polar |
| --- | --- |
| $-(\xi_x^2 + \xi_y^2)$ | $-\left(\xi_r^2 + \frac{1}{r^2}\xi_\theta^2\right) + \frac{1}{r} i\xi_r$ |

However, if we start with the cartesian symbol, and use the transformation laws

$$\xi_x = \xi_r \cos\theta - \xi_\theta \frac{\sin\theta}{r}$$
$$\xi_y = \xi_r \sin\theta + \xi_\theta \frac{\cos\theta}{r}$$

then the symbol transforms to

$$-\left(\xi_r^2 + \frac{1}{r^2}\xi_\theta^2\right)$$

which is not the polar symbol! The reason for this discrepancy is obvious: when we transformed the operator to its polar form, we used the transformation laws as well as the product rule for derivatives, but when we transformed the symbol, we did not use the product rule. Of course this happens in general, so defining a global symbol by gluing together the symbols from each coordinate chart does not work; the symbols simply do not transform well.

The observant reader will have noticed that the symbol transforms well if one ignores the terms of order 1 or less. This phenomenon is not special to the Laplacian. The local symbols $a(x, \xi)$ of any pseudodifferential operator of order $m$ will always transform into one another *modulo symbols of order* $(m-1)$. Therefore, a notion of global *principal symbol* can be had easily, by modding out $S^{m-1}(T^*M)$. The principal symbol is an object of importance in it's own right, for the index of an elliptic operator only depends on the homotopy type of it's principal symbol. However, we are interested in going another direction, and will have little use for principal symbols below.

The approach we take to define a global symbol will be a direct generalization of the method we mentioned in the local section: we take a Fourier transform of the Schwartz kernel. Before we can state this precisely, we have to re-consider the full symbol, as defined in the local case:

$$\sigma_A(x,\xi) = \int_U e^{i(x'-x)\cdot\xi} K_A(x,x')\,dx'$$
$$\sigma_A(x,\xi) = \int_U e^{iv\cdot\xi} K_A(x,v+x)\,dv \qquad\qquad v = x'-x$$

Obviously, we cannot carry this definition verbatim to the manifold case; the difference $x'-x$ doesn't make sense. This issue arises from the following fact: on $\mathbb{R}^n$, one has a natural way to identify pairs of points $(x,x')$ with tangent vectors $v$, $(v = x'-x)$ and so it makes sense to pair $x'-x$ with the cotangent vector $\xi$. Such an identification, called a *linearization*, can be imposed on a general smooth manifold, but it is non-canonical. The need for a linearization was first recognized by Juliane Bokobza-Haggiag, in [BH69], where she gave the following definition.

**Definition 21.** *Let $M$ be a smooth manifold. A smooth map $\nu : M \times M \to TM$ is called a* ***linearization*** *of $M$ if the following conditions are satisfied:*

*(1) For all $p,q \in M$, $\nu(p,q) \in T_pM$. We'll write $\nu_p : M \to T_pM$ when we fix the first coordinate at $p$.*

*(2) For all $p \in M$, $\nu(p,p) = 0$.*

*(3) The differential of $\nu_p : M \to T_pM$ at a point $q$ is a linear map $[D\nu_p]_q : T_qM \to T_pM$. We require $[D\nu_p]_p$ to be the identity map on $T_pM$.*

Condition (3) guarantees that every linearization is a diffeomorphism in some neighborhood $W \subseteq M \times M$ of the diagonal. Therefore, it has an "inverse" $\iota : \nu(W) \to W$. For us, the map $\iota$ will play an equally prominent role as the true linearization $\nu$, and for this reason, we will also call $\iota$ a linearization. When we restrict $\iota$ to a particular tangent space $T_pM$, we will use the notation $\iota_p$.

Outside of the neighborhood $W$, the linearization need not be a diffeomorphism, and so its action there does not preserve structure or contain any relevant information. To eliminate the irrelevant parts, we introduce the notion of a *cut-off function*, $\psi$. This is a smooth function on

$M \times M$ which is supported in $W$ and is identically equal to 1 in a smaller neighborhood which still contains the zero section. Sometimes, it will be convenient to abuse notation and consider $\psi$ to be a smooth function on $TM$, via the linearization; we will do this frequently below.

Linearizations can be found on every smooth manifold, because every connection induces a linearization via its exponential map:

**Definition 22.** *Let $M$ be a smooth manifold with connection $\nabla$ on $TM$. The **linearization** on $M$ induced by $\nabla$ is the smooth map $\iota : TM \to M \times M$ defined as follows:*

$$\iota(v) = (\rho(v), \exp(v))$$

*Where $\rho : TM \to M$ is the basepoint map and* $\exp : TM \to M$ *is the exponential map induced by $\nabla$.*

Below, we will be concerned only with Riemannian manifolds, and we will only ever deal with the linearization induced by the Levi-Civita connection. The symbol calculus which results from this choice is called the *normal symbol*, and is due to M. Pflaum. All of the results which follow are his, though we will present them in a slightly different fashion, in order that the proofs generalize immediately to the $b$-setting.

## 3.2 Global Symbols

We will now devote some time to the definition of a global symbol on compact Riemannian manifolds with boundary. Essentially, this will be done by using the linearization $\iota$ associated to the Levi-Civita connection to lift Schwartz kernels to the tangent bundle, and then performing a fiberwise Fourier transform. We will begin by setting up the rigorous foundations for this idea. Once this is done, we will define the normal symbol of a pseudodifferential operator, and prove that the operator can be recovered from its normal symbol modulo smoothing operators. As a first step, we must introduce the setting where fiberwise Fourier transforms can take place: the Fréchet bundles.

A *Fréchet bundle* is a fiber bundle whose fibers and transition maps belong to the category of Fréchet spaces. Thus if $F \to E \to M$ is a Fréchet bundle, then the total space $E$ is a manifold modeled on the Fréchet space $\mathbb{R}^n \times F$. This allows us to speak of smooth sections $C^\infty(E)$, and their partial derivatives. Using this structure, we can define a countable family of seminorms on $C^\infty(E)$ as follows. We first fix a coordinate chart $U$ on which $E$ is trivial. We then consider a section $s : U \to E$ to be a smooth map $U \to F$ via this trivialization. Then for each seminorm, $p$, on $F$, each multi-index $\alpha$, and each compact $K \subseteq U$, we define a seminorm $p_{\alpha,K}$ by

$$p_{\alpha,K}(s) = \sup_{x \in K} p\left(\partial^\alpha s(x)\right)$$

If $s$ is a smooth section of $E$, then the above seminorms are automatically finite. Therefore, the smooth sections of $E$ form a Fréchet space under the seminorms $p_{\alpha,K}$.

As with finite dimensional vector bundles, Fréchet bundles can be constructed as associated bundles of principal bundles. This construction leads us to our first important example of a Fréchet bundle: the rapidly decreasing functions on $T^*M$, $\mathscr{S}(T^*M)$. The fiber of this bundle is the space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions on $\mathbb{R}^n$. The structure group $GL(n, \mathbb{R})$ acts through precomposition:

$$u \cdot A = u \circ A \qquad\qquad u \in \mathscr{S}(\mathbb{R}^n), A \in GL(n, \mathbb{R})$$

Note that this action is on the right. This is the opposite of the usual convention, where the structure group acts on $F$ on the left. Therefore, to properly glue together an associated bundle, we have to reverse our convention for the $GL(n, \mathbb{R})$-action on the frame bundle and consider $GL(n, \mathbb{R})$ to act on frames on the left. Under this convention, we define

$$\mathscr{S}(T^*M) = \mathscr{S}(\mathbb{R}^n) \times_{GL(n,\mathbb{R})} F(T^*M)$$

$$\mathscr{S}(TM) = \mathscr{S}(\mathbb{R}^n) \times_{GL(n,\mathbb{R})} F(TM)$$

where $F(TM)$ is the frame bundle of $TM$. (We have reversed the standard notational convention and written $F \times_G P$ to accommodate the fact that the structure group $G$ acts on the principal bundle $P$ on the left and on the fiber $F$ on the right.)

The fiber of $\mathscr{S}(TM)$ over $p \in M$, which we will denote $\mathscr{S}_p(TM)$, should be thought of as the space of rapidly decreasing functions on $T_pM$. Indeed, elements of $\mathscr{S}_p(TM)$ are equivalence classes of pairs $(\phi, e_i)$, where $\phi$ is a rapidly decreasing function on $\mathbb{R}^n$, and $e_i$ is a basis of $T_pM$. Letting $\alpha : T_pM \to \mathbb{R}^n$, be the isomorphism induced by choosing the basis $e_i$, we see that $\phi \circ \alpha^{-1}$ is a rapidly decreasing function on $T_pM$. Moreover, the equivalence relation imposed on $\mathscr{S}(\mathbb{R}^n) \times F(TM)$ which produces $\mathscr{S}(TM)$ guarantees that the function $\phi \circ \alpha^{-1}$ does not depend on the representative chosen for $u$.

The reason that we have introduced these two bundles is that their spaces of smooth sections coincide with the smoothing symbols $S^{-\infty}(T^*M)$ and $S^{-\infty}(TM)$. Thus smoothing symbols are nothing more than smoothly varying test functions, indexed by points of $M$. This point of view will allow us to use elementary facts about rapidly decreasing functions to prove some of our foundational results. Particularly, it allows us to speak of fiberwise Fourier transforms in a global sense.

**Definition 23.** *Let $M$ be a compact Riemannian manifold. The* ***fiberwise Fourier transform*** *is a continuous linear isomorphism*

$$\mathcal{F} : S^{-\infty}(TM) \to S^{-\infty}(T^*M)$$

$$\mathcal{F}^{-1} : S^{-\infty}(T^*M) \to S^{-\infty}(TM)$$

*Defined by*

$$\mathcal{F}u(\xi) = (2\pi)^{-n/2} \int_{T_{\pi(\xi)}M} e^{-i\langle v,\xi\rangle} u(v)\, dv$$
$$\mathcal{F}^{-1}a(v) = (2\pi)^{-n/2} \int_{T^*_{\rho(v)}M} e^{i\langle v,\xi\rangle} a(\xi)\, d\xi$$

*where $d\xi$ is the canonical density on $T^*M$ restricted to the fiber, and $dv$ is the density on $T_{\rho(v)}M$ corresponding to $d\xi$ under the Riemannian metric.*

The Fourier inversion theorem implies that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are inverse bundle isomorphisms, and in fact, when acting on sections, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous with respect to the Fréchet space topologies on $S^{-\infty}(TM)$ and $S^{-\infty}(T^*M)$. This is a local statement, its proof is identical to the standard one; one just needs to make sure that all estimates hold uniformly on compact $x$-sets.

When computing the Fourier transform, we will identify the tangent space $T_pM$ with $\mathbb{R}^n$ via the choice of a basis, $e_i$. If $\alpha^i$ are the components of $v \in T_pM$, and $g$ is the metric tensor $g_{ij} = \langle e_i, e_j\rangle$, then the volume form $dv$ will transform to $(\det g)^{1/2} d\alpha^1 \wedge \ldots \wedge d\alpha^n$. If we choose the dual basis $e^i$ for the cotangent space, and let $\beta_i$ be the components of $\xi \in T^*_pM$, then the volume form $d\xi$ will transform to $(\det g)^{-1/2} d\beta_1 \wedge \ldots d\beta_n$. Thus one must insert a correction factor of $(\det g)^{\pm 1/2}$ when calculating fiberwise Fourier transforms.

Our goal now is to define the symbol of a pseudodifferential operator $A$ on $M$. First, we need to extend $\mathcal{F}$ to the bundle $\mathscr{S}'(TM)$ of tempered distributions on $TM$. $\mathscr{S}'(TM)$ is just the (continuous) dual bundle to $\mathscr{S}(TM)$, and its sections can be thought of as smoothly varying tempered distributions, or as continuous linear maps $S^{-\infty}(TM) \to C^\infty(M)$. We extend $\mathcal{F}$ to $\mathscr{S}'(TM)$ in the standard way to obtain an isomorphism $\mathcal{F} : \mathscr{S}'(TM) \to \mathscr{S}'(T^*M)$.

We are now ready to define the global symbol of $A \in \Psi^m(M)$. First, we multiply $K_A$ by the cut-off function $\psi(x,y)$ to obtain a distribution $\psi K \in \mathcal{D}'(M^2)$. The support of $\psi K$ is contained in $W$, so we can compose with the linearization $\iota$ induced by the Levi-Civita connection to obtain a section $\iota^*(\psi K) \in \Gamma(\mathscr{S}'(TM))$. We then take the Fourier transform to obtain a section $\mathcal{F}(\iota^*(\psi K)) \in \Gamma(\mathscr{S}'(T^*M))$. This section is called the *$\psi$-cut normal symbol of $A$*, and is denoted $\sigma_{A,\psi}$.

**Definition 24.** *Let $A \in \Psi^m(M)$, and let $\psi$ be a cut-off function on $M \times M$. The **$\psi$-cut normal symbol of** $A$, $\sigma_{A,\psi}$, is the section of the tempered distribution bundle of $TM$ defined as follows:*

$$\sigma_{A,\psi} = \mathcal{F}(\iota^*(\psi K_A))$$

As the name suggests, $\sigma_{A,\psi}$ is more than a smoothly varying tempered distribution. It is a symbol of order $m$.

**Theorem 25.** *Let $A \in \Psi^m(M)$ be a scalar pseudodifferential operator on $M$, and let $\sigma_{A,\psi}$ be the $\psi$-cut normal symbol of $A$. Then $\sigma_{A,\psi} \in S^m(T^*M)$*

*Proof.* The heart of this argument is an application of theorem (13) but in order to use this tool, we have to reduce to the local setting. That is, given $r \in M$, we must exhibit an open neigborhood, $U$, of $r$ and a trivialization of $T^*M$ over $U$. Once we compose $\sigma_{A,\psi}$ with this trivialization, we will be able to express it as an oscillatory integral which will work with our local machinery. The neighborhood we choose will be a (uniformly) normal neighborhood, and the trivialization will be induced by a normal coordinate chart, based at $r$. This proof requires a lot of notation, so for the reader's convenience we have laid it all out in the following dictionary:

$p, q, r$ **-** Points of $U$ ($r$ is the fixed point, $p$ and $q$ are arbitrary.)

$x, y, z$ **-** Points of $\mathbb{R}^n$

$e_i$ **-** Arbitrarily chosen orthonormal basis for $T_rM$

$e^i$ **-** Dual coframe to $e_i$

$\alpha_r : T_rM \to \mathbb{R}^n$ **-** isomorphism induced by choosing the basis $e_i$

$\beta_r : T_r^*M \to \mathbb{R}^n$ **-** isomorphism induced by choosing the basis $e^i$

$\mathbf{y} = \alpha_r \circ \nu_r : U \to \mathbb{R}^n$ **-** Normal coordinates based at $r$, with respect to the orthonormal basis $e_i$

$\alpha_p : T_pM \to \mathbb{R}^n$ **-** isomorphism induced by choosing the basis $\partial_{y^i}\big|_p$ associated to the coordinate chart $\mathbf{y}$.

$\beta_p : T_p^*M \to \mathbb{R}^n$ **-** isomorphism induced by choosing the basis $dy^i_p$ associated to the coordinate chart $\mathbf{y}$.

$\mathbf{y}_p = \alpha_p \circ \nu_p : U \to \mathbb{R}^n$ **-** Coordinates based at $p$. (Not normal.)

It will also help to keep the following diagrams in mind, in order to visualize how all of these maps relate to each other:

$$\begin{tikzcd}
 & U \arrow[dl, "\nu_p"'] \arrow[dr, "\nu_r"] & \\
T_pM \arrow[d, "\alpha_p"'] & & T_rM \arrow[ll, "\nu_p\nu_r^{-1}"'] \arrow[d, "\alpha_p"] \\
\mathbb{R}^n & & \mathbb{R}^n \arrow[ll, "\mathbf{y}_p\mathbf{y}_r^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & U \arrow[dl, "\nu_p"'] \arrow[dr, "\hat{\nu}_p"] & \\
T_pM \arrow[rr, shift left, "\flat"] \arrow[dr, "\alpha_p"'] & & T_p^*M \arrow[ll, shift left, "\sharp"] \arrow[dl, "\beta_p"] \\
 & \mathbb{R}^n &
\end{tikzcd}$$

Now we can write down the trivialization explicitly:

$$\begin{array}{rcl} T^*M|_U & \to & \mathbf{y}_r(U) \times \mathbb{R}^n \\ \xi & \mapsto & \left(\mathbf{y}_r(\pi(\xi)), \beta_{\pi(\xi)}(\xi)\right) \end{array}$$

We define $\sigma(x,\theta) \in C^\infty\left(\mathbf{y}_r(U) \times \mathbb{R}^n\right)$ to be the composition of $\sigma_{A,\psi}$ with the trivialization. Our goal is to show that $\sigma(x,\theta)$ is a symbol of order $m$.

To this end, write out the definition of $\sigma_{A,\psi}$:

$$\sigma_{A,\psi}(\xi) = \int_{T_pM} e^{i\langle v,\xi\rangle} \left[\psi K_A\right](\iota(v))\ dv \qquad\qquad p = \pi(\xi)$$

Recall that this integral is defined by using a basis to identify $T_pM$ with $\mathbb{R}^n$, and then integrating as usual, including the correction factor $(\det g)^{1/2}$ (which is determined by the basis.) Of course, we will use the basis $\left.\partial_{y^i}\right|_p$:

$$\sigma_{A,\psi}(\xi) = (\det g)^{1/2} \int_{\mathbb{R}^n} e^{i\langle \alpha_p^{-1} y,\xi\rangle} \left[\psi K_A\right]\left(\iota\left(\alpha_p^{-1}y\right)\right)\ dy \qquad\qquad y = \alpha_p(v)$$

(Rather than writing out $(\det g)^{1/2}$ repeatedly, we're going to absorb it into the $dy$ term and write $d\bar{y}$ from now on.)

By definition, $\iota\left(\alpha_p^{-1}y\right) = (p, \mathbf{y}_p^{-1}y)$. Thus

$$\sigma_{A,\psi}(\xi) = \int_{\mathbb{R}^n} e^{i\langle \alpha_p^{-1} y,\xi\rangle} \left[\psi K_A\right]\left(p, \mathbf{y}_p^{-1}y\right)\ d\bar{y}$$

Letting $x = \mathbf{y}_r(p)$ and $\theta = \beta_p(\xi)$, we find the following expression for $\sigma(x,\theta)$:

$$\sigma(x,\theta) = \int_{\mathbb{R}^n} e^{i\langle \alpha_p^{-1} y, \beta_p^{-1}\theta\rangle} \left[\psi K_A\right] \left(\mathbf{y}_r^{-1}x, \mathbf{y}_p^{-1}y\right) \, d\bar{y}$$
$$= \int_{\mathbb{R}^n} e^{iy\cdot\theta} \left[\psi K_A\right] \left(\mathbf{y}_r^{-1}x, \mathbf{y}_p^{-1}y\right) \, d\bar{y}$$

(Here, $y \cdot \theta$ denotes the standard dot product on $\mathbb{R}^n$.) Now since $A$ is a pseudodifferential operator, $\psi K_A$ can be written as an oscillatory integral:

$$\left[\psi K_A\right] \left(\mathbf{y}_r^{-1}x, \mathbf{y}_r^{-1}z\right) = \int e^{-i(z-x)\cdot\eta} a(x,z,\eta) \, d\eta$$

where $a \in S^m \left(\mathbf{y}_r(U) \times \mathbf{y}_r(U) \times \mathbb{R}^n\right)$. Letting $z = \mathbf{y}_r \mathbf{y}_p^{-1} y$, we see that

$$\left[\psi K_A\right] \left(\mathbf{y}_r^{-1}x, \mathbf{y}_p^{-1}y\right) = \int e^{-i(\mathbf{y}_r\mathbf{y}_p^{-1}y - x)\cdot\eta} a(x, \mathbf{y}_r\mathbf{y}_p^{-1}y, \eta) \, d\eta$$

which can be rewritten as

$$\left[\psi K_A\right] \left(\mathbf{y}_r^{-1}x, \mathbf{y}_p^{-1}y\right) = \int e^{-iy\cdot\eta} \hat{a}(x,y,\eta) \, d\eta$$

by the modified Kuranishi trick, with the path $\gamma(t) = (1-t)y$. Thus $\sigma(x,\theta)$ can be written as an oscillatory integral

$$\sigma(x,\theta) = \iint e^{iy\cdot(\theta-\eta)} \hat{a}(x,y,\eta) \, d\bar{y} d\eta$$

where $a$ is a symbol of order $m$, compactly supported in $y$ for each fixed $x$. Bringing $(\det g)^{1/2}$ out of $\bar{d}y$, we see that

$$\sigma(x,\theta) = \iint e^{iy\cdot(\theta-\eta)} \hat{a}(x,y,\eta) (\det g)^{1/2} \, dy d\eta$$

Now $(\det g)^{1/2}$ only depends on $x$, so it is a symbol of order 0. Thus we can absorb it into the symbol $\hat{a}$ without affecting its order. We finally arrive at

$$\sigma(x,\theta) = \iint e^{iy\cdot(\theta-\eta)}\hat{a}(x,y,\eta)\,dyd\eta \tag{3.2}$$

We want to apply our local machinery to estimate integral (3.2). To do this, we first make the following change of variables:

$$\lambda = |\theta|,\ \mu = (\theta-\eta)/\lambda$$

This turns (3.2) into the following integral:

$$\sigma(x,\theta) = (-\lambda)^n \iint e^{i\lambda y\cdot\mu}\hat{a}(x,y,\theta-\lambda\mu)\,dyd\mu$$

Note that $\hat{a}$ is still a symbol when we consider $y$ to be a fiber variable, because it is compactly supported in $y$ for each fixed $x$. Therefore, this Fourier integral has the same form as (2.7). In particular, it is smooth, and the estimation theorem (more precisely, its corollary, equation (2.8)) can be applied to obtain the following asymptotic expansion:

$$\sigma(x,\theta) \sim (-2\pi)^n \sum_k \frac{\lambda^{-k}}{2^k k!}\left(\partial_y\cdot\partial_\mu\right)^k \left[\hat{a}(x,y,\theta-\lambda\mu)\right]\big|_{y=\mu=0}$$

After using the chain rule to clean this up, we arrive at

$$\sigma(x,\theta) \sim (-2\pi)^n \sum_k \frac{(-1)^k}{2^k k!}\left(\partial_y\cdot\partial_\theta\right)^k \left[\hat{a}(x,y,\theta)\right]\big|_{y=0} \tag{3.3}$$

The meaning of this asymptotic expansion is that $\sigma_M$, the difference between $\sigma$ and the sum of the first $M$ terms on the right hand side, is uniformly bounded on compact $x$-sets by a constant multiple of $\lambda^{-M} = |\theta|^{-M}$ as $\theta \to \infty$. By theorem 9, we see that $\sigma(x,\theta)$ is indeed a symbol of order $m$.

□

In the above proof, if we keep track of how the amplitude transforms at each step, we can express the final symbol $\hat{a}$ appearing in equation (3.3) in terms of the original symbol $a$ as follows:

$$\hat{a}(x, y, \theta) = \frac{a\left(x, \mathbf{y}_r \mathbf{y}_p^{-1}(y), {}^t G(x, y)^{-1}\theta\right)}{|\det G(x, y)|} (\det g)^{1/2} \tag{3.4}$$

where $G(x, y)$ is the transformation from the Kuranishi trick given by

$$G(x, y) = \int_0^1 \left[D(\mathbf{y}_r \mathbf{y}_p^{-1})\right]((1-t)y)\, dt \tag{3.5}$$

(Note that, even though the cut-off function $\psi$ does not explicitly appear in formula (3.4), it is present in the symbol $a$, which was chosen as a local symbol for the operator $\psi A$.)

It will simplify the notation if we let $H(x, y) = {}^t G(x, y)^{-1}$. This reduces (3.4) to the following form:

$$\hat{a}(x, y, \theta) = a\left(x, \mathbf{y}_r \mathbf{y}_p^{-1}(y), H(x, y)\theta\right) |\det H(x, y)| (\det g)^{1/2} \tag{3.6}$$

Formulas (3.3) and (3.6) allow us to give explicit expressions for normal symbols, in normal coordinate charts. Let's compute an example:

**Example 1.** The normal symbol of a second order differential operator $A$.

Let $a(x, \xi) = a^{ij}(x)\xi_i \xi_j + a^k(x)\xi_k + a^0(x)$ be the local symbol of $A$ in a normal coordinate chart. Then the symbol $\hat{a}$ which appears in (3.6) is given by

$$\hat{a}(x, y, \theta) = a(x, H(x, y)\theta) J(x, y)$$

where $J(x, y) = |\det H(x, y)| (\det g(x))^{1/2}$.

Since $a$ is a second-degree polynomial, the asymptotic expansion for the normal symbol of $A$, in normal coordinates, contains only three terms, which are constant multiples of the following:

$$\hat{a}(x,0,\theta) = \left[a^{ij}(x)H^l_j H^k_i \theta_l\theta_k + a^i H^k_i \theta_k + a^0\right] J$$
$$\frac{\partial^2 \hat{a}}{\partial y^\mu \partial \theta_\mu}(x,0,\theta) = \left[2a^{ij}\left(\frac{\partial H^\mu_i}{\partial y^\mu} H^l_j + \frac{\partial H^l_i}{\partial y^\mu} H^\mu_j\right)\theta_l + a^k \frac{\partial H^\mu_k}{\partial y^\mu}\right] J$$
$$+ \left[a^{ik} H^\mu_k \theta_i + a^k H^\mu_k\right] \frac{\partial J}{\partial y^\mu}$$
$$\frac{\partial^4 \hat{a}}{\partial y^\mu \partial \theta_\mu \partial y^\nu \partial \theta_\nu}(x,0,\theta) = 2a^{jk} \frac{\partial^2}{\partial y^\mu \partial y^\nu}\left(H^\mu_j H^\nu_k J\right)$$

To save space, we've suppressed all of the arguments on the right hand side. All functions are understood to be evaluated at $(x, y) = (x, 0)$.

By inspecting $G$, we see that $H^i_j(x,0) = \left.\frac{\partial y^i_p}{\partial y^j_r}\right|_p$.

Further derivatives of $H$ can be computed as follows. Using the product rule and the fact that $HG = I$, we see that

$$\frac{\partial H}{\partial y^\mu} G = -H \frac{\partial G}{\partial y^\mu}$$
$$\frac{\partial H}{\partial y^\mu} = -H \frac{\partial G}{\partial y^\mu} H \tag{3.7}$$

The derivative of $G$ can be evaluated by differentiating under the integral sign in (3.5). Higher order derivatives of $H$ can be evaluated by differentiating (3.7).

The choice of cut-off function $\psi$ in the definition of the normal symbol is rather arbitrary and non-canonical, so one would hope that $\sigma_{A,\psi}$ does not depend on this choice. In fact it does, but only up to smoothing symbols.

**Theorem 26.** *Let $A \in \Psi^m(M)$, and let $\psi$ and $\psi'$ be cut-off functions on $M \times M$. Then $\sigma_{A,\psi} - \sigma_{A,\psi'} \in S^{-\infty}(T^*M)$.*

*Proof.* Observe that $\sigma_{A,\psi} - \sigma_{A,\psi'}$ is the Fourier transform of $\iota^*((\psi - \psi')K_A)$. But $\iota^*((\psi - \psi')K_A)$ is smooth because the singularities of $K_A$ can only lie on the diagonal, where $(\psi - \psi')$ vanishes to infinite order. Furthermore, $\iota^*((\psi - \psi')K_A)$ is compactly supported in each fiber, and hence is a smoothing symbol. Therefore, it's Fourier transform $\sigma_{A,\psi} - \sigma_{A,\psi'}$ is also a smoothing symbol.

□

Since the symbol mapping only depends on the cut-off function modulo $S^{-\infty}$, we can obtain a symbol map which is independent of $\psi$ by modding out $S^{-\infty}$. This mapping will be denoted by $\sigma_A$, and will simply be called the *normal symbol* of $A$:

$$\sigma_A : S^m(T^*M)/S^{-\infty} \to \Psi^m(M)/\Psi^{-\infty}$$

Before defining the quantization map, we should address an issue that we glossed over in the construction of the $\psi$-cut normal symbol. This occurs at the point where we use $\iota$ to pull back the distribution $\psi K \in \mathcal{D}'(M^2)$ to a section of the tempered distribution bundle $\iota^*(\psi K) \in \Gamma(\mathscr{S}'(TM))$. By pointing this out, we hope that the issue is now obvious: strictly speaking, $\iota^*(\psi K)$ is a distribution on the total space $TM$, not a smoothly varying family of tempered distributions. We need to make sure that we can identify these two types of objects.

So, we must consider a distribution $K \in \mathcal{D}'(M^2)$ whose support is contained in $W$. We wish to identify $K$ with a smoothly varying family of tempered distributions $K_x \in \mathscr{S}'(T_x^*M)$. To do this, we will first associate to each $x \in M$ a distribution $K_x \in \mathcal{D}'(M)$. The support of $K_x$ will be contained in $W_x$, and so we will be able to pull $K_x$ back to a distribution $\iota_x^* K_x$ on $T_xM$ with compact support. Since every compactly supported distribution is tempered, the association $x \mapsto \iota_x^* K_x$ will thus be a section of $\mathscr{S}'(TM)$ (modulo questions of continuity and smoothness, of course.) To remain consistent, we will abuse notation and call this section $\iota^*(K)$.

Heuristically, we think of $K_x$ as the "function" obtained by freezing the first coordinate at $x$:

$$K_x(y) = K(x, y)$$

Of course, this doesn't make sense if $K$ is singular. It is akin to restricting $K$ to a non-open subset of $M^2$, a procedure which is, in general, not well-defined for distributions. To make sense of $K_x$, let $u, v \in C^\infty(M)$ and observe the following formal manipulations:

$$\begin{aligned}
\int u(x)\langle K_x, v\rangle_M \, dx &= \int u(x) \int K(x,y)v(y)\,dy\,dx \\
&= \iint K(x,y)u(x)v(y)\,dy\,dx \\
&= \langle K, u\otimes v\rangle_{M^2} \\
&= \langle Au, v\rangle_M \\
&= \langle u, A^* v\rangle_M \\
&= \int u(x)[A^*v](x)\,dx
\end{aligned}$$

Here, $A$ is the operator corresponding to $K$ under the Schwartz kernel theorem, and $A^*$ is its formal adjoint with respect to the volume form induced by the Riemannian metric. It is now clear that we ought to define $K_x$ to be the distribution on $M$ which associates to each $v \in C^\infty(M)$ the number $[A^*v](x)$.

Of course, this discussion was only necessary to provide a rigorous interpretation for the pullback $\iota^*[\psi K_A]$. It doesn't change the way that the normal symbol is actually computed. It's still the same oscillatory integral.

Our next goal is to show that one can recover the operator $A$ from its normal symbol modulo smoothing operators. This will be done by defining a map $Op : \Psi^m \to S^m$, going in the opposite direction, which will provide an inverse for the symbol mapping. To do this, we are going to have to take the Fourier transform of symbols on $T^*M$, and so we will begin our discussion by making sense of this notion. The key is to realize $S^m$ as the space of sections of a Fréchet bundle $\mathscr{S}^m(T^*M)$ which is naturally contained in the tempered distribution bundle $\mathscr{S}'(T^*M)$. The fibers of $\mathscr{S}^m(T^*M)$ are isomorphic to the Fréchet space $\mathscr{S}^m(\mathbb{R}^n)$ of temperate growth functions of order $m$. Recall that these are the smooth functions $a(\xi)$ satisfying

$$|\partial_\xi^\beta a(\xi)| \leq C(1+|\xi|)^{m-|\beta|}$$

for all multi-indices $\beta$. (Here $C$ is a constant depending only on $\beta$). In words, $a(\xi)$ grows more slowly than $|\xi|^m$ as $|\xi| \to \infty$. Allowing $GL(n,\mathbb{R})$ to act on $\mathscr{S}^m(\mathbb{R}^n)$ by precomposition, and using the same associated bundle construction as before, we produce the bundles $\mathscr{S}^m(T^*M)$ and $\mathscr{S}^m(TM)$:

$$\mathscr{S}^m(T^*M) = \mathscr{S}^m(\mathbb{R}^n) \times_{GL(n,\mathbb{R})} F(T^*M)$$

$$\mathscr{S}^m(TM) = \mathscr{S}^m(\mathbb{R}^n) \times_{GL(n,\mathbb{R})} F(TM)$$

It is immediate from this definition that the space of sections $\Gamma(\mathscr{S}^m(T^*M))$ coincides with the standard symbol space $S^m(T^*M)$. The advantage to this viewpoint is that it is clear that $\mathscr{S}^m(T^*M)$ is contained in $\mathscr{S}'(T^*M)$ Indeed, the product of a temperate growth function with a rapidly decreasing function must rapidly decrease, and in particular must be integrable. Therefore, if $a \in \mathscr{S}^m(\mathbb{R}^n)$, then we can identify $a$ with a tempered distribution via

$$\langle a, b \rangle = \int \bar{a} b$$

This gives us a natural inclusion $\mathscr{S}^m(T^*M) \subseteq \mathscr{S}'(T^*M)$ and so the Fourier transform of a symbol $a \in S^m(T^*M) = \Gamma(\mathscr{S}^m(T^*M))$ is well-defined.

Equipped with the Fourier transform of symbols, we can define the quantization map by (almost) reversing the steps that we used to define the normal symbol. First, we take the inverse Fourier transform of $a \in S^m(T^*M)$ to obtain a tempered distribution, $\mathcal{F}^{-1}(a) \in \Gamma(\mathscr{S}'(TM))$. Then, we multiply by $\psi$ to get a distribution $\psi\mathcal{F}^{-1}(a)$ supported in $W$. Since the support is contained in $W$, we can use the linearization $\nu$ to identify $\psi\mathcal{F}^{-1}(a)$ with a distribution $\nu^*(\psi\mathcal{F}^{-1}(a)) \in \mathcal{D}'(M^2)$. This distribution will be shown to be the Schwartz kernel of a pseudodifferential operator $Op_\psi(a) \in \Psi^m(M)$.

**Definition 27.** *Let $M$ be a compact Riemannian manifold, and let $\iota$ be the linearization associated to the Levi-Civita connection. If $a \in S^m(T^*M)$, we define the $\psi$-**cut quantization** of $a$, $Op_\psi(a)$, by specifying its Schwartz kernel as follows:*

$$K_{Op_\psi(a)} = \nu^*(\psi \mathcal{F}^{-1}(a))$$

**Theorem 28.** *Let $a \in S^m(T^*M)$ be a symbol of order $m$ on $M$. Then $Op_\psi(a) \in \Psi^m(M)$.*

*Proof.* We'll begin by showing that $Op_\psi(a)$ has the form of a pseudodifferential operator, locally. For this, we will use the same notational set-up that we used in Theorem (25). Thus, we are given a point $r \in M$, and we choose a uniformly normal neighborhood $U$ of $r$. We must show that $K_{Op_\psi(a)}$, when considered as a distribution on $\mathbf{y}_r(U) \times \mathbf{y}_r(U)$, has the correct form. To save space below, we will write $K$ instead of $K_{Op_\psi(a)}$.

Consider the Schwartz kernel:

$$K(p, q) = \psi\,(p, q) \int_{T_p^* M} e^{i\langle \nu(p,q), \xi \rangle} a(\xi)\, d\xi$$

By identifying $U$ with an open subset of $\mathbb{R}^n$ via $\mathbf{y}_r$, and identifying $T_p^*M$ with $\mathbb{R}^n$ via $\beta_p$, we transform the above integral to

$$\int_{\mathbb{R}^n} e^{i\langle \nu\left(\mathbf{y}_r^{-1}(x), \mathbf{y}_r^{-1}(y)\right), \beta_p^{-1}(\theta)\rangle} a\left(\beta_p^{-1}(\theta)\right) (\det g)^{-1/2}\, d\theta$$

where $x = \mathbf{y}_r(p)$ and $y = \mathbf{y}_r(q)$. By considering diagram (1), one sees that the phase function simplifies as follows:

$$\left\langle \nu\left(\mathbf{y}_r^{-1}(x), \mathbf{y}_r^{-1}(y)\right), \beta_p^{-1}(\theta) \right\rangle = \left[\mathbf{y}_p \circ \mathbf{y}_r^{-1}\right](y) \cdot \theta$$

(The dot on the right hand side denotes the usual dot product in $\mathbb{R}^n$.) Letting $\kappa_x = \mathbf{y}_p \circ \mathbf{y}_r^{-1}$, and noting that $\kappa_x(x) = 0$, we see that

$$K(\mathbf{y}_r^{-1}x, \mathbf{y}_r^{-1}y) = \int e^{i(\kappa_x(y)-\kappa_x(x))\cdot\theta}\bar{a}(x,y,\theta)\,d\theta$$

(We have pushed all of the nastiness into the single symbol $\bar{a}$:

$$\bar{a}(x,y,\theta) = \psi(\mathbf{y}_r^{-1}(x), \mathbf{y}_r^{-1}(y))\,a\left(\beta^{-1}_{\mathbf{y}_r^{-1}(x)}(\theta)\right)(\det g)^{-1/2}$$

Since $\psi$ and $(\det g)^{-1/2}$ do not depend on $\theta$, $\bar{a}$ is a symbol of order $m$, which is all that matters right now.)

Finally, we appeal to the Kuranishi trick (with $\gamma_{x,y} = ty + (1-t)x$) to conclude that

$$K_{Op_\psi(a)}(\mathbf{y}_r^{-1}x, \mathbf{y}_r^{-1}y) = \int e^{iy\cdot\tilde{\theta}}\tilde{a}(x,y,\tilde{\theta})\,d\tilde{\theta} \tag{3.8}$$

where $\tilde{a} \in S^m((V\times V)\times\mathbb{R}^n)$. Thus $Op_\psi(a)$ has the form of a pseudodifferential operator, locally.

To see that $Op_\psi(a)$ is pseudolocal, we begin by noting that the only singularities of $\mathcal{F}^{-1}(a)$ occur at the zero section of $TM$. This can be seen by writing out $\mathcal{F}^{-1}(a)$ as an oscillatory integral, and then appealing to lemma (11):

$$\mathcal{F}^{-1}(a)(x,v) = (2\pi)^{-n/2}\int_{T_x^*M} e^{i\langle v,\xi\rangle}a(\xi)\,d\xi$$

In this case, the critical set of the phase function is given by $v = 0$. Thus the singular support of $\mathcal{F}^{-1}(a)$ is contained in the zero section of $TM$, hence the singular support of $\nu^*(\psi\mathcal{F}^{-1}(a))$ is contained in the diagonal.

$\square$

In the above proof, if we keep track of how the amplitude transforms at each step, we can express the final symbol $\tilde{a}$ appearing in equation (3.8) in terms of the original symbol $a$ as follows:

$$\tilde{a}(x,y,\theta) = \frac{\psi(\mathbf{y}_r^{-1}(x), \mathbf{y}_p^{-1}(y))\, a\left(\beta^{-1}_{\mathbf{y}_r^{-1}(x)}({}^tF(x,y)^{-1}\theta)\right)}{|\det F(x,y)|}(\det g)^{-1/2}$$

Here, $F(x,y)$ is the transformation from the Kuranishi trick given by

$$F(x,y) = \int_0^1 \left[D(\mathbf{y}_p\mathbf{y}_r^{-1})\right](ty + (1-t)x)\, dt$$

As in the case of the symbol map, the operator $Op_\psi(a)$ only depends on the cutoff function modulo order $-\infty$.

**Theorem 29.** *Let $a \in S^m(T^*M)$. If $\psi, \psi'$ are cut-off functions associated to the same linearization $\iota$, then $\nu^*(\psi\mathcal{F}^{-1}(a)) - \nu^*(\psi'\mathcal{F}^{-1}(a))$ is smooth, and hence $Op_\psi(a) - Op_{\psi'}(a) \in \Psi^{-\infty}(M)$.*

*Proof.* Writing $\nu^*(\psi\mathcal{F}^{-1}(a)) - \nu^*(\psi'\mathcal{F}^{-1}(a))$ as $\nu^*\left((\psi - \psi')\mathcal{F}^{-1}(a)\right)$, and noting that $\mathcal{F}^{-1}(a)$ is a Fourier integral operator whose singular support is contained in the zero section, we see that the Schwartz kernel of $Op_\psi(a) - Op_{\psi'}(a)$ is smooth, and hence $Op_\psi(a) - Op_{\psi'}(a) \in \Psi^{-\infty}$, as required. $\square$

As with the normal symbol, we can mod out by smoothing operators to obtain a map $Op : \Psi^m/\Psi^{-\infty} \to S^m/S^{-\infty}$ which is independent of the chosen cut-off function. This map will be referred to simply as the *quantization* map. As we have said before, we aim to prove that it is an inverse for the normal symbol mapping. This is done in the following theorem.

**Theorem 30.** *Let*

$$Op_\psi : S^m(T^*M) \to \Psi^m(M)$$

$$\sigma_\psi : \Psi^m(M) \to S^m(T^*M)$$

*be the $\psi$-cut normal symbol and quantization maps, as defined above. Then for any $a \in S^m(T^*M)$, and for any $A \in \Psi^m(M)$,*

*(1)* $a - \sigma_{Op_\psi(a),\psi} \in S^{-\infty}(T^*M)$

*(2)* $A - Op_\psi(\sigma_{A,\psi}) \in \Psi^{-\infty}(M)$

*Proof.* Let $\tilde{a} = a - \sigma_{Op_\psi(a),\psi}$. Pick $r \in M$, and take the usual trivialization of $T^*M$ near $r$. Let $b(x,u) = \tilde{a}\left(\beta^{-1}_{\mathbf{y}_r^{-1}x}u\right) \in S^m(\mathbf{y}_r(U) \times \mathbb{R}^n)$ be the composition of $\tilde{a}$ with this trivialization. As an oscillatory integral, $b$ is given by

$$b(x,u) = \int_{T_pM}\int_{T_p^*M} e^{i\langle v,\theta-\beta_p^{-1}u\rangle}\left(1-\psi(v)^2\right)a(\theta)\,d\theta dv$$

(Here $p = \mathbf{y}_r^{-1}x$). Writing $\theta = \beta_p^{-1}w$, $v = \alpha_p^{-1}t$, we convert this to the following integral over $\mathbb{R}^n$:

$$\begin{aligned} b(x,u) &= \iint e^{it\cdot(w-u)}\left(1-\psi\left(\alpha_p^{-1}t\right)^2\right)a\left(\beta_p^{-1}w\right)\,dwdt \\ &= \iint e^{it\cdot(w-u)}\left(1-\bar{\psi}\left(x,t\right)^2\right)\bar{a}\left(x,w\right)\,dwdt \end{aligned}$$

We make the change of variables $\lambda = |u|$, $\sigma = (w-u)/\lambda$ to transform this integral into

$$\lambda^n \iint e^{i\lambda t\cdot\sigma}\left(1-\bar{\psi}\left(x,t\right)^2\right)\bar{a}\left(x,u+\lambda\sigma\right)\,d\sigma dt$$

Note that the amplitude does not depend on $t$ when $t$ is large. Therefore, it is a symbol even when $t$ is considered as a fiber variable. Using this fact, we can apply the estimation theorem to find the following asymptotic expansion:

$$b(x,u) \sim C_Q \sum \frac{\lambda^{-k}}{2^k k!}\left(\partial_t\cdot\partial_\sigma\right)^k\left[\left(1-\bar{\psi}\left(x,t\right)^2\right)\bar{a}\left(x,u+\lambda\sigma\right)\right]\Big|_{t=\sigma=0}$$

Since $1-\bar{\psi}(x,t)^2$ vanishes identically for $t$ sufficiently small, all of the terms in this expansion are zero, and hence $b \in S^{-\infty}$, as needed.

The other composition is even easier:

$$K_{A-Op_\psi(\sigma_{A,\psi})} = (\psi^2-1)K_A$$

As in previous proofs, we see that $(\psi^2 - 1)K_A$ is smooth, because $\psi^2 - 1$ vanishes in a neighborhood of the diagonal. $\square$

This concludes our presentation of the normal symbol calculus. We have presented the bare minimum of the subject; global symbol calcului have many applications to the theory of pseudodifferential operators which we do not have the time to discuss here. We have only included those elements of the theory which will be needed in subsequent chapters, either to provide proofs which hold in the interior, or to hint at how generalizations should proceed.

# Chapter 4

## The Small $b$-Calculus on Manifolds With Boundary

We will now give a brief introduction to the small $b$-calculus for a compact manfold-with-boundary, $M$. We will begin by defining two objects, the $b$-tangent bundle ${}^bTM$ and the stretched product $M_b^2$ which are at the heart of this calculus. As one might expect, these $b$-objects play the same role as the ordinary tangent bundle $TM$ and product $M^2$ play in the standard calculus of pseudodifferential operators. Once they are defined, we will be able to define the notions of $b$-symbol and $b$-pseudodifferential operator, and will then turn our attention to the problem of defining a symbol calculus.

Detailed expositions of the $b$-calculus can be found in [Mel93], [Loy05], [Hör85], and [Gri01]. We should note that there is an equivalent, and equally important viewpoint on the $b$-calculus, which differs from the one we take here. Instead of working on a compact manifold-with-boundary, one can push the boundary away to infinity using a logarithmic scale. All of the important definitions can be given and all of the major theorems can be proved in this language. [Loy05] contains a very nice development of the theory from this viewpoint.

### 4.1 The $b$-Tangent Bundle

Before getting into the world of $b$-geometry, it will be necessary to establish a few notational conventions. Throughout this section $M$ denotes an $(n+1)$-dimensional compact manifold-with-boundary $\partial M$, and $p$ will denote a point on $M$. When $p$ is on the boundary, we will often work in *adapted coordinates at* $p$: $x, y^1, \ldots, y^n$. These are coordinate systems for which $p$ corresponds to the origin, and with the property that $x \geq 0$, and $x^{-1}(0) = \partial M$. In this situation, $x$ is often called a *boundary defining function*.

It can be shown that the boundary defining function $\hat{x}$ from a different set of adapted coordinates $\hat{x}, \hat{y}^1, \ldots, \hat{y}^n$, must vanish at $x = 0$ to no higher than first order:

$$\hat{x} = a(x, y)x,\ a(0, 0) > 0$$

This can easily be seen by writing out the Jacobian:

$$\begin{pmatrix} \frac{\partial \hat{x}}{\partial x} & \frac{\partial \hat{x}}{\partial y^1} & \cdots & \frac{\partial \hat{x}}{\partial y^n} \\ \frac{\partial \hat{y}^1}{\partial x} & \frac{\partial \hat{y}^1}{\partial y^1} & \cdots & \frac{\partial \hat{y}^1}{\partial y^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \hat{y}^n}{\partial x} & \frac{\partial \hat{y}^n}{\partial y^1} & \cdots & \frac{\partial \hat{y}^n}{\partial y^n} \end{pmatrix}$$

Since $\hat{x}$ vanishes on the boundary, all of the derivatives $\frac{\partial \hat{x}}{\partial y^j}$ vanish when evaluated at $p$. If $\hat{x}$ vanished at $x = 0$ to second or higher order, then $\frac{\partial \hat{x}}{\partial x}$ would vanish too, and the Jacobian matrix would be singular at $p$. (For the same reason, we can conclude that the minor $\det(\frac{\partial \hat{y}^i}{\partial y^j})_{i,j=1}^n$ must be nonzero at $p$, although the fact that $y^i$ and $\hat{y}^i$ are both coordinate systems on $Y$ is enough to reach the same conclusion.) These facts will be used later, to prove that the $b$-tangent bundle is a smooth vector bundle.

Before defining the $b$-tangent bundle, we must introduce the concept of a $b$-vector field. Geometrically, a $b$-vector field is a smooth vector field on $M$ whose values at the boundary are tangent to $\partial M$. The precise definition follows:

**Definition 31.** *A* ***b-vector field*** *on $M$ is a smooth vector field, $V$, on $M$ which satisfies $V(f) = 0$ for any $f \in C^\infty(M)$ which vanishes identically on $\partial M$. The set of all $b$-vector fields is denoted $\mathcal{V}_b(M)$.*

The $b$-tangent bundle can now be constructed by localizing the spaces of $b$-vector fields. We begin by defining its fibers:

**Definition 32.** *Let $\mathcal{I}_p(M)$ denote the set of smooth functions on $M$ that vanish at $p$. The **b-tangent space** at $p$, ${}^bT_pM$, is the vector space quotient*

$$^bT_pM = \frac{\mathcal{V}_b(M)}{\mathcal{I}_p(M)\mathcal{V}_b(M)}$$

Elements of ${}^bT_pM$ will be denoted by brackets; so if $V$ is a $b$-vector field, then the element of the $b$-tangent space that it represents will be denoted $[V]$ or $[V]_p$ when we want to emphasize the basepoint.

Of course, the $b$-tangent bundle, ${}^bTM$, is the bundle whose total space is given by the disjoint union of all of the $b$-tangent spaces. To prove that this is a smooth vector bundle,we must identify it's local frames, and their transition matrices. On the interior, this is easy: the evaluation map $\mathcal{V}_b(M) \to T_pM$ is surjective, and it's kernel is $\mathcal{I}_p(M)\mathcal{V}_b(M)$. Using this identification, we see that the equivalence classes $[\frac{\partial}{\partial y^i}]_p$ form a basis for ${}^bT_pM$, and so the transition matrices for ${}^bTM$ on the interior will be identical to those of the regular tangent bundle. In particular, ${}^bTM$ and $TM$ are naturally isomorphic over the interior of $M$.

Now to the boundary. If $p \in \partial M$, choose adapted coordinates $\bar{z} = (x, y^1, \ldots, y^n)$ in a neighborhood of $p$. It is easy to see that the vector fields $x\frac{\partial}{\partial x}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}$ are $b$-vector fields. We aim to show that their equivalence classes form a basis for ${}^bT_pM$. First, a lemma.

**Lemma 33.** *Let $p \in \partial M$, and let $(x, y^1, \ldots, y^n)$ be adapted coordinates in a neighborhood of $p$. Then $[x\frac{\partial}{\partial x}]_p$ is not the zero element of ${}^bT_pM$.*

*Proof.* Suppose to the contrary. Then by definition, there exist $f_1, \ldots, f_m \in \mathcal{I}_p(M)$ and $V_1, \ldots, V_m \in \mathcal{V}_b(M)$ such that

$$x\frac{\partial}{\partial x} = \sum_{j=1}^{m} f_j V_j$$

Let the coordinates of $p$ be named $\bar{z}(p) = (0, \bar{y}(p))$. Now for small $h > 0$, define points $q_h \in M$ by $\bar{z}(q_h) = (h, \bar{y}(p))$. Evaluating both sides of $(*)$ at $q_h$ gives

$$h\frac{\partial}{\partial x}\bigg|_{q_h} = \sum_{j=1}^{m} f_j(q_h)V_j|_{q_h}$$

Dividing both sides of this equation by $h$ and taking the limit as $h \to 0$, we see that

$$\frac{\partial}{\partial x}\bigg|_p = \sum_{j=1}^{m} \frac{\partial f_j}{\partial x}\bigg|_p V_j|_p$$

The right hand side is tangent to the boundary, whereas the left hand side clearly is not. This is a contradiction, and so the lemma is proven. $\square$

We now prove the required result.

**Theorem 34.** *Let $p \in \partial M$, and let $(x, y^1, \ldots, y^n)$ be adapted coordinates in a neigborhood of $p$. Then the equivalence classes $[x\frac{\partial}{\partial x}], [\frac{\partial}{\partial y^1}], \ldots, [\frac{\partial}{\partial y^n}]$ form a basis for ${}^bT_pM$.*

*Proof.* (1) Linear Independence:

Let $a^x, a^1, \ldots, a^n$ be scalars such that

$$a^x\left[x\frac{\partial}{\partial x}\right] + a^1\left[\frac{\partial}{\partial y^1}\right] + \cdots + a^n\left[\frac{\partial}{\partial y^n}\right] = 0$$

in ${}^bT_pM$. By definition, there exist $f_1, \ldots, f_m \in \mathcal{I}_p(M)$ and $V_1, \ldots, V_m \in \mathcal{V}_b(M)$ such that

$$a^x x\frac{\partial}{\partial x} + a^1\frac{\partial}{\partial y^1} + \cdots + a^n\frac{\partial}{\partial y^n} = f_1V_1 + \cdots + f_mV_m$$

Plugging in $p$, we see that

$$a^1\left.\frac{\partial}{\partial y^1}\right|_p + \cdots + a^n\left.\frac{\partial}{\partial y^n}\right|_p = 0$$

and so all the $a^i$'s are zero. We arrive at $a^x[x\frac{\partial}{\partial x}] = 0$, and so $a^x = 0$ by the lemma.

(2) Spanning:

Let $V = V^x\frac{\partial}{\partial x} + V^1\frac{\partial}{\partial y^1} + \cdots + V^n\frac{\partial}{\partial y^n} \in \mathcal{V}_b(M)$. Set $a^x = \frac{\partial V^x}{\partial x}\big|_p$, and $a^i = V^i(p)$ for $1 \leq i \leq n$. Then

$$V - \left(a^x x\frac{\partial}{\partial x} + a^1\frac{\partial}{\partial y^1} + \cdots + a^n\frac{\partial}{\partial y^n}\right) = \left(\frac{V^x}{x} - a^x\right)x\frac{\partial}{\partial x} + \sum_{j=1}^{n}(V^j - a^j)\frac{\partial}{\partial y^j}$$

Since $V^x(p) = 0$, $\frac{V^x}{x} \to a^x$ as $x \to 0$, so the right hand side belongs to $\mathcal{I}_p(M)\mathcal{V}_b(M)$. Thus

$$[V]_p = \frac{\partial V^x}{\partial x}\bigg|_p \left[x\frac{\partial}{\partial x}\right]_p + a^1\left[\frac{\partial}{\partial y^1}\right]_p + \cdots + a^n\left[\frac{\partial}{\partial y^n}\right]_p$$

which proves the theorem.

$\square$

An important feature of the $b$-tangent bundle is that its smooth sections are in a natural one-to-one correspondence with the space of $b$-vector fields. We are now in a position to prove this. Given a $b$-vector field $V$, the corresponding section $[V] \in C^\infty({}^bTM)$ is had by quotienting out: the value of $[V]$ at $p \in M$ is the equivalence class $[V]_p$ of $V$ modulo $\mathcal{I}_p\mathcal{V}_b$. It is easy to see that $V \mapsto [V]$ is globally injective. Simply note that every member of $\mathcal{I}_p(M)\mathcal{V}_b(M)$ must necessarily vanish at $p$. So, if $[V]_p = 0$ for all $p \in M$, then $V(p) = 0$ everywhere, as well.

In order to prove surjectivity, we appeal to the proof of Theorem (34) to get a coordinate expression for the quotient mapping. If the $b$-vector field $V$ can be written $V = V^x\frac{\partial}{\partial x} + V^1\frac{\partial}{\partial y^1} + \cdots + V^n\frac{\partial}{\partial y^n}$ in adapted coordinates, the coefficients of $[V]$ are exactly what one would expect naively:

$$[V] = \frac{V^x}{x}\left[x\frac{\partial}{\partial x}\right] + V^1\left[\frac{\partial}{\partial y^1}\right] + \cdots + V^n\left[\frac{\partial}{\partial y^n}\right] \tag{4.1}$$

(This equation even holds at the boundary, as long as we agree to extend $V^x/x$ by continuity.)

Equation (4.1) implies that the pointwise quotient map $V \mapsto [V]$ is locally bijective. Using the fact that $\mathcal{V}_b$ is a sheaf, we can now quickly prove that $V \mapsto [V]$ is globally surjective. To be more explicit about this point, let $X$ be a smooth section of ${}^bTM$. Choose a covering $U_i$ of $M$ by coordinate charts, using adapted coordinates whenever at the boundary. In each chart, choose the unique $b$-vector field $V_i$ such that $[V_i] = X$ on $U_i$. Since such a $V_i$ is unique, we see that, whenever $U_i$ and $U_j$ intersect, $V_i$ must equal $V_j$ on the overlap. By the sheaf property, we can glue the $V_i$ together to form a unique $V \in \mathcal{V}_b$ such that $V = V_i$ on $U_i$. Clearly, $[V] = X$ globally.

We now complete the proof that ${}^bTM$ is a smooth vector bundle. We have written down the local trivializations already, so it remains to show that their transition matrices are smooth on overlapping neighborhoods. There are three possibilities that must be checked:

(1) Both neighborhoods are in the interior

(2) One neighborhood is in the interior, one on the boundary

(3) Both neighborhoods are on the boundary.

Since the $b$-tangent bundle is identical to the tangent bundle over the interior, the first case is well-known to be smooth. The second case will also be smooth, because the intersection of an interior neighborhood with a boundary neighborhood is still contained in the interior. Thus we only need to check the third case.

To this end, let $(x, y^1, \ldots, y^n)$ and $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^n)$ be adapted coordinates in a neighborhood of $p \in \partial M$. Using equation (4.1), we compute:

$$\left[\frac{\partial}{\partial \hat{y}^i}\right] = \left(\frac{1}{x}\frac{\partial x}{\partial \hat{y}^i}\right)\left[x\frac{\partial}{\partial x}\right] + \frac{\partial y^j}{\partial \hat{y}^i}\left[\frac{\partial}{\partial y^j}\right]$$
$$\left[\hat{x}\frac{\partial}{\partial \hat{x}}\right] = \left(\frac{\hat{x}}{x}\frac{\partial x}{\partial \hat{x}}\right)\left[x\frac{\partial}{\partial x}\right] + \hat{x}\frac{\partial y^j}{\partial \hat{x}}\left[\frac{\partial}{\partial y^j}\right]$$

Thus in order to prove that ${}^bTM$ is a smooth vector bundle, we need only show that the functions $\left(\frac{\hat{x}}{x}\frac{\partial x}{\partial \hat{x}}\right)$ and $\left(\frac{1}{x}\frac{\partial x}{\partial \hat{y}^i}\right)$ are smooth. This is straightforward, once we write

$$\hat{x} = xa(x, y)$$

$$x = \hat{x}\hat{a}(\hat{x}, \hat{y})$$

where $a, \hat{a}$ are smooth and $\hat{a}(0,0), a(0,0) > 0$. This immediately shows that $\hat{x}/x$, and hence $\left(\frac{1}{x}\frac{\partial x}{\partial \hat{y}^i}\right)$, is smooth. Furthermore, since

$$\frac{\partial \hat{x}}{\partial x} = a(x, y) + x\frac{\partial a}{\partial x}(x, y)$$

we see that $\frac{\partial x}{\partial \hat{x}}$ is smooth, since $a(0,0)$ is nonzero:

$$\frac{\partial x}{\partial \hat{x}}(0,0) = \frac{1}{a(0,0)}$$

So, the $b$-tangent bundle really is a smooth vector bundle.

It is worth noting that the structure of the $b$-tangent bundle near the boundary is actually quite simple. This is due to the fact that every compact manifold-with-boundary has a collar neighborhood, where it is diffeomorphic to a product $[0,1) \times Y$, where $Y = \partial M$ is a compact closed manifold. Letting $x \in C^\infty([0,1) \times Y)$ be the boundary defining function which projects onto the leftmost coordinate, we can decompose every tangent space $T_{(x,y)}M$ as $T_x[0,1) \oplus T_yY$. Thus a vector field on $M$ assigns to each $(x, y)$ a pair $(U_{(x,y)}, V_{(x,y)}) \in T_x[0,1) \oplus T_yY$. Viewed this way, a $b$-vector field is simply a vector field with the property that $U_{(0,y)} = 0$ for all $y$. Now ${}^bT_{(x,y)}M$ is equal to the set of $b$-vector fields modulo a linear equivalence relation. Since linear equivalence relations distribute over direct sums, we have that

$${}^bT_{(x,y)}M = {}^bT_x[0,1) \oplus {}^bT_yY = {}^bT_x[0,1) \oplus T_yY$$

So, to understand the $b$-tangent bundle of a general manifold-with-boundary, it suffices to understand the $b$-tangent bundle of the interval $[0,1)$. Of course, $[0,1)$ is a contractible topological space, so it's $b$-tangent bundle is a trivial, real, rank one vector bundle. The $b$-vector field $[x\partial_x]$ provides our favorite example of a global nonvanishing section.

A bundle which is just as important as the $b$-tangent bundle is its dual, the $b$-cotangent bundle, denoted ${}^bT^*M$. In any system of adapted coordinates, ${}^bT^*M$ has a frame $\frac{dx}{x}, dy^1, \ldots, dy^n$ which is just the dual coframe to $x\frac{\partial}{\partial x}, dy^1, \ldots, dy^n$. The $b$-cotangent bundle splits in the same way that the $b$-tangent bundle does in a collar neighborhood, and the $b$-one-form $\frac{dx}{x}$ is a global nonvanishing section of ${}^bT^*[0,1)$.

With the $b$-cotangent bundle defined, we can speak of $b$-metrics. These are exactly what they sound like: positive-definite sections of the second symmetric power of the $b$-cotangent bundle. Among the $b$-metrics are the class of *exact* $b$-metrics, which take the form

$$g = \left(\frac{dx}{x}\right)^2 + h$$

in a specific collar neighborhood with boundary defining function $x$. (Here, $h$ is a Riemannian metric on the boundary.) The exact $b$-metrics will play the same role in our global symbol calculus which Riemannian metrics play in M. Pflaum's normal symbol calculus.

The $b$-tangent and $b$-cotangent bundles also have Fréchet bundles of rapidly decreasing functions, and tempered distributions. Exactly as in the boundaryless case, the Fréchet bundles $\mathscr{S}({}^bT^*M)$ and $\mathscr{S}({}^bTM)$ are $\mathscr{S}(\mathbb{R}^n)$-bundles associated to the frame bundles of ${}^bT^*M$ and ${}^bTM$. The $b$-cotangent bundle also possesses a canonical $b$-volume form. In adapted coordinates it is given by

$$dV = \frac{dx}{x} dy^1 \cdots dy^n d\alpha_x d\alpha_1 \cdots d\alpha_n$$

(Here, $\alpha_x$ is the $\frac{dx}{x}$ coefficent, and $\alpha_j$ is the $dy^j$ coefficient.) When an exact $b$-metric is given, this form can be transferred to the $b$-tangent bundle, exactly as in the boundaryless case. This allows us to define the fiberwise Fourier transform $\mathcal{F} : \Gamma\left(\mathscr{S}({}^bT^*M)\right) \to \Gamma\left(\mathscr{S}({}^bTM)\right)$. It extends to the tempered distribution bundle $\mathscr{S}'({}^bT^*M)$ in the usual way. When actually computing an integral over a fiber of ${}^bTM$, one inserts a correction factor of $(\det h)^{1/2}$. On the cotangent bundle, one inserts $(\det h)^{-1/2}$.

## 4.2 The $b$-stretched product, $M_b^2$

We will begin by describing *blow-ups*, a general concept which includes the stretched product as a special case. The situation is as follows. $X$ is a smooth manifold, possibly with boundary, and $Y$ is a smooth closed submanifold, possibly contained in $\partial X$. We want to form a new smooth manifold, the *blow up* of $Y$ in $X$, which we will denote $[X, Y]$. This is done by replacing each point $p$ of $Y$ with the set of all possible directions that one can approach $p$ orthogonally from within $X$.

For example, when we blow up the origin in $\mathbb{R}^2$, we replace the origin with a copy of $\mathbb{R}P^2$, the set of all possible lines through the origin. When we blow up the $x$-axis in $\mathbb{R}^3$, we replace each point $p$ of the $x$-axis with a copy of $\mathbb{R}P^2$, which we think of as the set of all possible lines through $p$ which are orthogonal to the $x$-axis. In general, when we blow up a submanifold, we replace each point by a copy of a projective space, of dimension equal to the codimension of the submanifold (minus 1).

This notion is made precise by the *projective normal bundle* $\mathbb{P}N(Y)$ of $Y$ in $X$, which is obtained by modding out the normal bundle $NY = (TX|_Y)/TY$ by the action of nonzero scalars. Thus we consider two tangent vectors $u, v \in TX|_Y$ to be equivalent if there exists $\alpha \neq 0$ such that $u - \alpha v$ is tangent to $Y$. As they are equivalence classes of tangent vectors, points on the projective normal bundle will often be thought written as $[v]$, or $[\dot{\gamma}(0)]$ if we have a specific curve in mind to represent $v$. In a situation where coordinates $x^1, \ldots, x^n$ are given, we may also write $[\dot{x}^1(0), \ldots, \dot{x}^n(0)]$.

As a set, the blow-up $[X, Y]$ is obtained by replacing $Y$ with its projective normal bundle. To topologize $[X, Y]$, we take as a subbasis all sets of the form

$$V_g = \{x \in X \setminus Y : g(x) \neq 0\} \cup \{[v] \in \mathbb{P}N(Y); \langle dg, v\rangle = 0\}$$

where $g \in C^\infty(X)$ and $g = 0$ on $Y$, along with all open sets of $X$ which are disjoint from $Y$. That these sets cover $[X, Y]$ can be seen by choosing coordinates $(x^1, \dots x^n)$ on $X$ such that $Y = \{x^1 = \cdots = x^k = 0\}$, and considering the sets $V_{x^1}, \dots, V_{x^k}$. In fact, these sets can be taken as coordinate charts for a smooth structure. The appropriate coordinates to use are projective coordinates:

$$\begin{array}{rcl} V_{x^j} & \to & \mathbb{R}^n \\ (x^1, \dots, x^n) & \mapsto & (x^1/x^j, \dots, x^n/x^j) \\ \left[a_{k+1}\frac{\partial}{\partial x^{k+1}} + \cdots + a_n \frac{\partial}{\partial x^n}\right] & \mapsto & (0, \dots, 0, a_{k+1}/a_j, \dots, a_n/a_j) \end{array}$$

For future reference, there is a natural *blow-down* map $\beta : [X, Y] \to X$ which projects the normal bundle onto it's base, and identifies points of $X \setminus Y$ with themselves. It is smooth with respect to the aforementioned smooth structure on $[X, Y]$.

When working with blow-ups, it will be important to lift curves, functions and other objects from $X$ to $[X, Y]$. Of course, now that we know what the blow-up *is*, it should be obvious how a curve is lifted. If $\gamma$ passes through a point $p \in Y$ at time 0, for example, then the lifted curve $\tilde{\gamma}$ will pass through the point $[\dot{\gamma}(0)]$ on the projective normal bundle at time 0:

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & \gamma(t) \notin Y \\ [\dot{\gamma}(t)] & \gamma(t) \in Y \end{cases}$$

In other words, the value of the lifted curve at the blow-up is the *direction* in which the original curve hit the submanifold.

One nice feature of blow-ups is that some functions which are smooth on $X \setminus Y$, which can't be smoothly extended to all of $X$, may be lifted to a smooth function on $[X, Y]$. For example, the function $f(x, y) = (2xy)/(x^2 + y^2)$ is not smooth at the origin, yet it has a unique smooth extension $\bar{f}$ to the blown-up space $[\mathbb{R}^2, 0]$. This can easily be checked by expressing $f$ in terms of the coordinate systems on $V_x$ and $V_y$. If one expresses $f$ in terms of polar coordinates, then it is clear what is happening geometrically: $f(r, \theta)$ is nothing more than $\sin(2\theta)$. Therefore, $f$ posesses directional limits at the origin in all possible directions. These limits are the values that $\bar{f}$ takes on the projective normal bundle.

The procedure of lifting *distributions* to the blow-up is more delicate. Lifting a distribution $K \in \mathcal{D}'(X)$ to $\bar{K} \in \mathcal{D}'[X,Y]$ necessarily involves a pushforward of functions:

$$\beta_* : C_0^\infty[X,Y] \to C_0^\infty(X)$$

But clearly, $\beta_* f$ can't be defined for a general $f \in C_0^\infty[X,Y]$, for $f$ may not be constant on $SN^+(Y)$. However, $\beta_*$ is well-defined on $\mathcal{J}_0^\infty([X,Y], SN^+(Y))$, the space of compactly supported smooth functions on $[X,Y]$ which vanish to infinite order at $SN^+(Y)$:

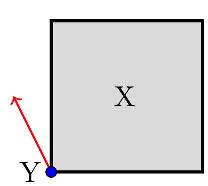
$$\beta_* : \mathcal{J}_0^\infty([X,Y], SN^+(Y)) \to \mathcal{J}_0^\infty(X,Y)$$

Therefore, there is a well-defined lift, going the opposite way, between the continuous duals:

$$\beta^* : \mathcal{J}_0^{-\infty}(X,Y) \to \mathcal{J}_0^{-\infty}([X,Y], SN^+(Y))$$

which is actually an isomorphism. (See [Mel93])

The only blow-ups we will need will belong to the case where $X$ is a manifold-with-corner and $Y$ is a submanifold of $\partial X$. In this case, the projective normal bundle no longer captures the notion of orthogonal approach from within $X$, because there may be tangent vectors which point out of $X$:



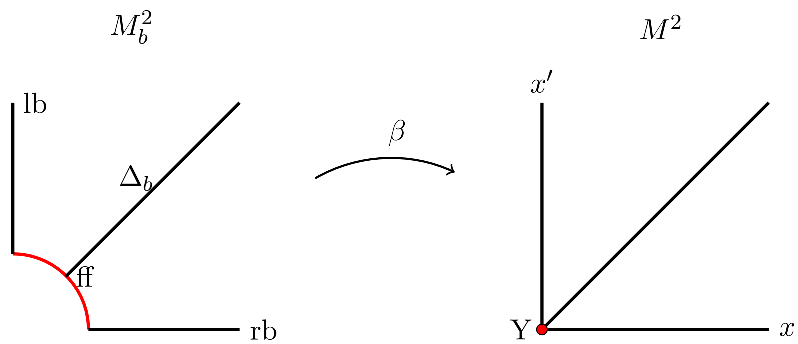

To remedy this, we replace the projective normal bundle with the *inward spherical normal bundle* $S_+N(Y)$, which is obtained by modding out the *inward* normal bundle by the action of $\mathbb{R}_{>0}$. The topology and smooth structure can be defined in a completely analagous manner, and so blow-ups in this situation are not more complicated.

We are now able to define the stretched product:

**Definition 35.** *Let $M$ be a manifold-with-boundary, and let $B \subseteq M \times M$ be the corner, $B = \partial M \times \partial M$. The blow-up $[M \times M, B]$ is called the* ***stretched product*** *of $M$ and is denoted $M_b^2$.*

The stretched product plays a very prominent role in our understanding of the $b$-calculus, so we need to introduce notational conventions for working with it. First, we examine the most basic example of a stretched product:

**Example 2.** Let $M = [0, 1)$, and $Y = \{(0, 0)\}$ be the corner of $M \times M$. The stretched product $[0, 1)_b^2$ is often pictured as follows:




The labeled submanifolds in this diagram are named the *left boundary*, *right boundary front face*, and *b-diagonal*, respectively. The left and right boundaries are the lifts of the axes $\{0\} \times [0, 1)$ and $[0, 1) \times \{0\}$. The $b$-diagonal is the lift of the ordinary diagonal, and the front face is another name for the inward spherical normal bundle of $Y$.

Just as with the ordinary product, there are natural projection maps $\pi_{1b} : M_b^2 \to M$, $\pi_{2b}$ : $M_b^2 \to M$ and a diffeomorphism $\Delta_b \simeq M$ whose definitions are obvious.

Points on $[0, 1)^2$ will be denoted as pairs, $(x, x')$. Thus points on the spherical normal bundle can be thought of as equivalence classes of curves $\gamma(t) = (x(t), x'(t))$. We will denote such an equivalence class as $[x(t), x'(t)]$. Note that the $x'$ denotes the second coordinate, and $x'(t)$ means the second coordinate of the parameterization, not a derivative. When we want to take a time derivative,we will use the dot notation.

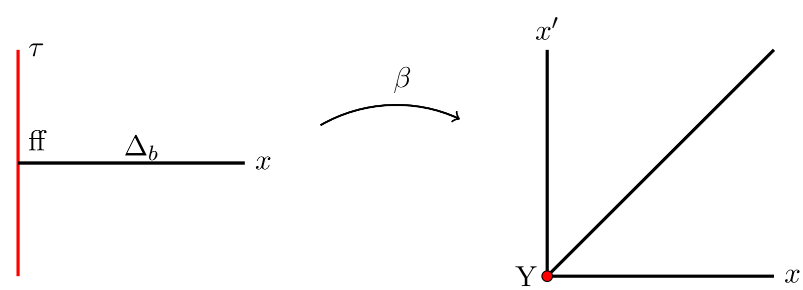
Speaking of time derivatives, we can characterize the labeled submanifolds of $[0,1)^2_b$ in terms of the derivatives of $x(t)$ and $x'(t)$. For example, the left boundary consists of all points of $[0,1)^2_b$ which are either $(a)$ away from the corner, and have $x = 0$, or $(b)$ on the spherical normal bundle and represented by a curve $[x(t), x'(t)]$ with $\dot{x}(t) = 0$ (we will sometimes say that such a point *corresponds to a slope of* $\infty$). Similarly, points on the right boundary correspond to zero slope, and points on the $b$-diagonal correspond to a slope of 1.

$[0,1)^2_b$ posesses a particular coordinate system which will be very convenient for our purposes. We call it the *logarithmic projective chart* and denote it by $\Phi$. It is a coordinate system defined on the open subset $[0,1)^2_b \setminus (lb \cup rb)$, and it identifies $[0,1)^2_b \setminus (lb \cup rb)$ with $[0,1)_x \times \mathbb{R}_\tau$ as follows:

$$\Phi(x, x') = \left( x, \log\left(\frac{x'}{x}\right)\right) \qquad \text{(interior)}$$

$$\Phi[x(t), x'(t)] = \left(0, \log\left(\frac{\dot{x}'(0)}{\dot{x}(0)}\right)\right) \qquad \text{(front face)}$$

We will use the variables $x$ and $\tau$ for this coordinate system. With an abuse of notation, we will often write $\tau = \log(x'/x)$, understanding this to mean that $\tau$ is the unique smooth lift of $\log(x'/x)$ to the blown-up space. Under logarithmic projective coordinates, the stretched product looks as follows. Notice that the left and right boundaries are pushed away to $\infty$ and $-\infty$.




When $M$ is a general manifold-with-boundary, the structure of its stretched product $M^2_b$ is no more complicated than the structure of $[0,1)^2_b$. This is due to the existence of collar neighborhoods.

Near the corner, $M_b^2$ is diffeomorphic to the stretched product $([0,1)\times Y)_b^2$, which can easily be identified with $[0,1)_b^2\times Y^2$ by the following map:

$$(x,y,x',y')\mapsto(x,x',y,y') \qquad \text{(Interior, left boundary, right boundary)}$$

$$[x(t),y(t),x'(t),y'(t)]\mapsto\big([x(t),x'(t)],y(0),y'(0)\big) \qquad \text{(front face)}$$

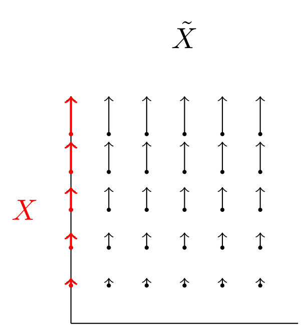
Given this decomposition, we can easily define the analagous labeled submanifolds of $M_b^2$. The left boundary is the product of the left boundary of $[0,1)_b^2$ with $Y^2$, the front face and right boundaries are defined the same way, and the $b$-diagonal is the product of the $b$-diagonal of $[0,1)_b^2$ with the diagonal of $Y\times Y$.

For the purposes of defining linearizations, it is important to note that the tangent bundle of $M_b^2$ contains a subbundle which is naturally isomorphic to the $b$-tangent bundle. This is the *normal bundle to the $b$-diagonal*, $N\Delta_b$, defined by

$$N\Delta_b=\frac{TM_b^2\big|_{\Delta_b}}{T\Delta_b}$$

The isomorphism ${}^bTM\to N\Delta_b$ is given as follows. Start with a $b$-tangent vector $v\in{}^bT_pM$. Choose a $b$-vector field $X$ to represent $v$. Then lift $X$ to a vector field $\tilde{X}$ on $M_b^2$. We will see exactly how this lift is accomplished below, and in the process we will see that $\tilde{X}$ is tangent to the $b$-diagonal if and only if $v=0$. Thus we can restrict $\tilde{X}$ to $\Delta_b$ to obtain a vector in $N\Delta_b$.

Lifting $X$ to $M_b^2$ is a two-step process. First, we define $\tilde{X}$ on $M^2\setminus Y$ by having it act on the second coordinate:



To be more precise, we can define $\tilde{X}$ on smooth functions by

$$\tilde{X}_{p,q} f = X_q f_p$$

where $(p, q) \in M^2$, and $f_p \in C^\infty(M^\circ)$ is given by fixing the first coordinate at $p$:

$$f_p(q) = f(p, q)$$

Then we extend $\tilde{X}$ to the front face by continuity, using the topology of the blow-up. That is, if $\alpha$ belongs to the front face of $M_b^2$, then we define

$$\tilde{X}_\alpha f = \lim_{(p,q)\to\alpha} \tilde{X}_{p,q} f$$

Here, the limit is taken over all points $(p, q) \in M_b^2 \setminus \text{ff}$.

**Example 3.** Let's work out the lift in a simple case. We'll take $M = [0, 1)$, as usual, and we'll lift the $b$-vector field $X = g(x)x\partial_x$, working in logarithmic projective coordinates. In other words, our goal is to compute $\tilde{X}_{\Phi^{-1}(x_0,\tau_0)}$ for all pairs $(x_0, \tau_0) \in [0, 1) \times \mathbb{R}^n$. (We'll start by assuming that $x_0 \neq 0$, and then take a limit to find out what happens when $x_0 = 0$.) Begin by noting that, if $\phi_h(x)$ denotes the flow of $X$, then $\tilde{X}_{\Phi^{-1}(x_0,\tau_0)} f$ can be computed as the following derivative:

$$\tilde{X}_{\Phi^{-1}(x_0,\tau_0)} f = \left.\frac{\partial}{\partial h}\right|_{h=0} f(x, \phi_h(x_0 e^{\tau_0}))$$

By inserting the logarithmic projective coordinate system as follows,

$$f(x_0, \phi_h(x_0 e^{\tau_0})) = \left[f \circ \Phi^{-1}\right] \left( x_0, \log\left(\frac{\phi_h(x_0 e^{\tau_0})}{x_0}\right)\right)$$

we can use the chain rule to compute that

$$\tilde{X}_{\Phi^{-1}(x_0,\tau_0)} f = \frac{\partial[f \circ \Phi^{-1}]}{\partial\tau}(x_0, \tau_0) \frac{\frac{\partial}{\partial h}\big|_{h=0} \phi_h(x_0 e^{\tau_0})}{\phi_0(x_0 e^{\tau_0})}$$

Now by definition of flow, we have

$$\phi_0(x_0 e^{\tau_0}) = x_0 e^{\tau_0}$$
$$\left.\frac{\partial}{\partial h}\right|_{h=0} \phi_h(x_0 e^{\tau_0}) = x_0 e^{\tau_0} g(x_0 e^{\tau_0})$$

so we arrive at the following simple formula for $\tilde{X}$:

$$\tilde{X}_{\Phi^{-1}(x_0,\tau_0)} f = \frac{\partial [f \circ \Phi^{-1}]}{\partial \tau}(x_0, \tau_0) g(x_0 e^{\tau_0})$$

Or, if we drop the argument $f$,

$$\tilde{X}_{\Phi^{-1}(x,\tau)} = g(xe^{\tau}) \left.\frac{\partial}{\partial \tau}\right|_{(x,\tau)}$$

Since $g(xe^{\tau})$ is smooth in $x$ and $\tau$, $\tilde{X}$ extends naturally to the front face as:

$$\tilde{X}_{\Phi^{-1}(0,\tau)} = g(0) \left.\frac{\partial}{\partial \tau}\right|_{(0,\tau)}$$

Notice that $\partial_\tau$ is not tangent to $\Delta_b$ anywhere. Therefore, the only way that $\tilde{X}_{\Phi^{-1}(x,\tau)}$ can lie in $T\Delta_b$ is if $g(xe^{\tau}) = 0$, which means precisely that $v = 0$ in ${}^bT_x[0,1)$. This shows that the map ${}^bTM \to N\Delta_b$ is well-defined and injective. Since ${}^bTM$ and $\Delta_b$ have the same fiber dimension, we see that this map is indeed an isomorphism. We will denote it by $\phi$:

$$\begin{aligned} \phi : N\Delta_b &\to^b TM \\ \left[\frac{\partial}{\partial \tau}\right] &\to \left[x\frac{\partial}{\partial x}\right] \end{aligned} \tag{4.2}$$

If we move to the general case when $M = [0,1) \times Y$, then it is easy to compute the lifts of the coordinate basis $Y_i = \partial_{y^i}$:

$$\tilde{Y}_i = \frac{\partial}{\partial y^{i\prime}}$$

Again, we see that the lifts are smooth on $M_b^2$, and never tangent to the $b$-diagonal, so the isomorphism between $N\Delta_b$ and ${}^bTM$ holds in the general case.

## 4.3 The Schwartz Kernel Theorem

The point of introducing the stretched product is that the Schwartz kernels of $b$-pseudodifferential operators naturally live on $M_b^2$. In order for this analogy to be meaningful, we ought to have a Schwartz kernel theorem which is adapted to this situation. We do have such a theorem, and its statement is as follows:

**Theorem 36.** *Let $X$ and $Y$ be compact manifolds with boundary, with $b$-metrics $g_X$ and $g_Y$. Then there is a one-to-one correspondence between continuous linear maps*

$$A : \mathcal{J}_0^{\infty}(X) \to \mathcal{J}_0^{-\infty}(Y)$$

*and the space of distributions $K \in \mathcal{J}_0^{-\infty}(X \times Y)$, the continuous dual of $\mathcal{J}_0^{\infty}(X \times Y)$. This correspondence is given, as usual, by integration against the $b$-metric.*

(Here, $\mathcal{J}^{\infty}(X)$ denotes the space of compactly supported smooth functions on $X$ that vanish to infinite order at the boundary.)

This result is most useful to us when $X = Y = M$. In this case, it states that linear operators on $\mathcal{J}_0^{\infty}(M)$ can be identified with a certain class of distributions on $M \times M$; those which are dual to the smooth functions which vanish to infinite order at the boundary. As discussed above, this class of distributions can be identified with a class of distributions on the stretched product. Thus, there is a natural one-to-one correspondence between continuous linear maps

$$\mathcal{J}_0^{\infty}(M) \to \mathcal{J}_0^{-\infty}(M)$$

and the space of distributions $\mathcal{J}_0^{-\infty}(M_b^2)$. By singling out a special class of these distributions, we will define the $b$-pseudodifferential operators.

## 4.4 The small $b$-calculus

We are now ready to define the small $b$-calculus. We'll begin with symbols:

**Definition 37.** *A* ***symbol*** *of order $m$ on ${}^bT^*M$ is a smooth function $a \in C^\infty(T^*M)$ such that in any coordinate neighborhood $U \subseteq M$ and with respect to any trivialization of ${}^bT^*M$ over $U$, the induced function $a(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ is a symbol of order $m$ in the local sense.*

Of course, symbols on ${}^bTM$ are defined similarly, and there is the usual notion of a symbol of order $-\infty$. Symbols of order $-\infty$ coincide with sections of the Fréchet bundle $\mathscr{S}({}^bT^*M)$, and symbols of order $m$ coincide with sections of the Fréchet bundle $\mathscr{S}^m({}^bT^*M)$.

The operators in the small $b$-calculus are defined in a similar fashion to the operators in the boundaryless calculus. However, there is an additional condition: the Schwartz kernel must vanish to infinite order at the left and right boundaries.

**Definition 38.** *Let $M$ be a compact $n$-dimensional manifold-with-boundary, equipped with a $b$-metric. A $b$-pseudodifferential operator on $M$ of order $m$ is a continuous linear operator $A : \mathcal{J}^\infty(M;\partial M) \to \mathcal{J}^{-\infty}(M;\partial M)$ whose Schwartz kernel, $K_A \in \mathcal{J}^{-\infty}(M_b^2, SN^+(B))$, satisfies*

*(1) $K_A$ is smooth away from the $b$-diagonal*

*(2) $K_A$ vanishes to infinite order at the left and right boundaries.*

*(3) For any coordinate neighborhood $(U, \mathbf{x})$ on $M_b^2$ which intersects the $b$-diagonal, such that $\Delta_b \cap U = \{x^{n+1} = \cdots = x^{2n} = 0\}$, $K_A$ must have the form of an oscillatory integral*

$$K_A\left(\mathbf{x}^{-1}(x', x'')\right) = \int e^{ix''\cdot\theta} a(x', x'', \theta)\, d\theta$$

*where $x' = (x^1, \ldots, x^n)$, $x'' = (x^{n+1}, \ldots, x^{2n})$, and $a(x', x'', \xi)$ is a symbol of order $m$ on $U \times \mathbb{R}^n$.*

We'll sometimes call operators of order $-\infty$ *$b$-smoothing operators*. The terminology is appropriate; such operators have smooth Schwartz kernels which vanish to infinite order at the left and right boundaries, and hence they smooth out any distribution that they act on. Despite this, the nature of $b$-smoothing operators is quite different than their boundaryless counterparts. In the boundaryless case, smoothing pseudodifferential operators are also compact, and even trace class. This is not the case in the $b$-setting. Nonetheless, a sort of "regularized trace" for $b$-smoothing operators has been defined, and plays a crucial role in the index theory of manifolds with boundary. This "trace" is known as the $b$-trace, and we will describe it now.

If $A$ is a $b$-smoothing operator, one might attempt to define the trace of $A$ by integration: restrict the Schwartz kernel of $A$ to the $b$-diagonal to obtain a smooth function on $M$, and integrate this against the $b$-volume form:

$$\mathrm{Tr}(A) = \int_M K_A|_{\Delta_b}\, dV$$

The problem with doing this, is that such an integral might not converge. This is because of the $dx/x$ term which appears in the $b$-volume form. To be explicit, let's suppose that $K_A|_{\Delta_b}$ is supported in the collar neighborhood, and identify it with a smooth function $K(x, y)$. The integral above becomes

$$\int_M K_A|_{\Delta_b}\, dV = \int_0^1 \left( \int_Y K(x, y)\, dV_y \right) \frac{dx}{x}$$

The inner integral gives a smooth function of $x$, which might not vanish at $x = 0$, and hence the outer integral blows up.

The procedure we use to fix this singularity is called *regularizing* the integral, and it is very straightforward. We just modify $K(x, y)$ so that it vanishes at zero, and then integrate that. This regularized integral is denoted by appending a "$b$" onto the standard integral notation:

$${}^b\!\!\int_M K\, dV = \int_0^1 \left( \int_Y \left(K(x, y) - K(0, y)\right)\, dV_y \right) \frac{dx}{x}$$

This works because $K(x, y) - K(0, y)$, being smooth, is $O(x)$ as $x \to 0$.

To define a regularized integral in the case when $K$ is not supported inside the collar neighborhood, merely introduce a smooth function $\phi \in C^\infty(M)$ which is identically 1 near the boundary and is supported in the collar neighborhood. Use that to split $K$ into $K\phi + K(1-\phi)$, and proceed in the obvious fasion.

**Definition 39.** *Let $K \in C^\infty(M)$. The **regularized integral** of $K$ over $M$, denoted ${}^b\int_M K\,dV$, is defined as follows:*

$$ {}^b\int_M K\,dV = \int_0^1 \left( \int_Y (\phi(x,y)K(x,y) - K(0,y))\,dV_y \right) \frac{dx}{x} + \int_M (1-\phi)K\,dV $$

The $b$-trace of $A$ is the regularized integral of $K_A|_{\Delta_b}$:

**Definition 40.** *Let $A \in \Psi_b^{-\infty}(M)$. The $b$-**trace** of $A$ is given by*

$$ {}^b\operatorname{Tr}(A) = {}^b\int_M K_A|_{\Delta_b}\,dV $$

After defining the global symbol calculus, we will give a new formula for the $b$-trace.

# Chapter 5

## A Global Symbol for the Small $b$-Calculus

Our next goal is to generalize the normal symbol to the $b$-setting. To do this, we will simply replace all of the boundaryless geometric entities with their $b$-counterparts. The $b$-tangent bundle replaces the tangent bundle, the stretched product replaces the ordinary product, and an exact $b$-metric takes the place of the Riemannian metric. Only the most crucial part of the machinery still remains to be generalized: the linearization $\iota$. It is to this task which we now turn our attention.

### 5.1 Linearization

The method for definining $\iota : {}^bTM \to M_b^2$ is straightforward. Since the $b$-tangent bundle and the stretched product reduce to the tangent bundle and the ordinary product over the interior, we simply define $\iota(v) = (\rho(v), \exp(v))$ whenever the basepoint of $v$ belongs to the interior of $M$. When $v$ is based at the boundary, we define $\iota(v)$ by a limiting process. To be more explicit, we choose a $b$-vector field $X$ which represents $v$, and then define $\iota(v)$ by

$$\iota(v) = \lim_{q \to p} (q, \exp(X_q))$$

where $p$ is the basepoint of $v$, and $q$ runs over all interior points of $M$.

Of course, we ought to check that $\iota(v)$ doesn't depend on the $b$-vector field $X$ that we chose to represent $v$. More importantly, we must define what the word "linearization" means in the $b$-context, and show that $\iota$ satisfies this definition. This will be done in due time, but first let's investigate how $\iota$ works in a simple case so that we can get a feel for it.

**Example 4.**

$$M = [0,1)_x$$
$$g = \left(\frac{dx}{x}\right)^2$$

In order to compute $\iota$ we must first determine what the geodesics are. This can be accomplished by computing the Christoffel symbol $\Gamma$:

$$\Gamma(x) = -\frac{1}{x}$$

and then solving the geodesic equation

$$\ddot{x} - \frac{\dot{x}^2}{x} = 0$$

to obtain the path

$$x(t) = c_1 e^{c_2 t}$$

Alternatively, we can change coordinates to $s = \ln(x)$ and note that in the new coordinate system, the $b$-metric takes the form $g = ds^2$. Therefore, the geodesics are just linear functions

$$s(t) = c_1 + c_2 t$$

Converting back to the $x$-system, we obtain the same general solution.

Now let $x_0$ be an interior point of $M$, $\tau$ a real number, and consider the tangent vector $\mathbf{u} = \tau x_0 \frac{\partial}{\partial x}\big|_{x_0}$. In order to compute $\exp(\mathbf{u})$, we consider the unique geodesic $x(t)$ which passes through $x_0$ at time $t = 0$ and whose velocity vector at that time is $\mathbf{u}$. One easily computes

$$x(t) = x_0 e^{\tau t}$$

and therefore $\exp(\mathbf{u}) = x(1) = x_0 e^{\tau}$. Writing this in terms of the linearization $\iota$, we see that

$$\iota\left(\tau x\frac{\partial}{\partial x}\right) = (x, xe^{\tau})$$

for $x$ in the interior. Let $\mathbf{v} = \left[\tau x\frac{\partial}{\partial x}\right]_0$ be an arbitrary $b$-tangent vector based at 0. We can now compute $\iota(\mathbf{v})$:

$$\iota(\mathbf{v}) = \lim_{x\to 0}(x, xe^{\tau})$$

The limit here is taken in the stretched product. Since

$$\left.\frac{d}{dx}\right|_{x=0} xe^{\tau} = e^{\tau}$$
$$\left.\frac{d}{dx}\right|_{x=0} x = 1$$

the limiting value is the point on the front face corresponding to a slope of $e^{\tau}$. In particular, $\iota$ takes a very simple form when expressed in terms of the logarithmic projective coordinate system on $M_b^2$:

$$\iota\left(\left[\tau x\frac{\partial}{\partial x}\right]\right) = \Phi^{-1}(x, \tau)$$

Or, if we identify ${}^bTM$ with $[0,\infty)\times\mathbb{R}$ via the global section $\left[x\frac{\partial}{\partial x}\right]$, and the stretched product with $[0,\infty)\times\mathbb{R}$ via logarithmic projective coordinates, then we obtain

$$\iota(x, \tau) = (x, \tau)$$

It is now fairly obvious that $\iota$ is a diffeomorphism in a neighborhood of the zero-section, and that it identifies the zero-section with the $b$-diagonal. (In fact, $\iota$ is a diffeomorphism everywhere except for the left and right boundaries.)

Now it's time to check that this computation did not depend on the representative that we chose for $\mathbf{v}$. So, we need to add an arbitrary element of $\mathcal{I}_0\mathcal{V}_b$ to $\tau x\frac{\partial}{\partial x}$, exponentiate the result, and then take the limit as $x \to 0$, hopefully arriving again at $\Phi^{-1}(x, \tau)$. The fact that $M$ is one-dimensional makes this quite simple, for every $b$-vector field is of the form $g(x)x\frac{\partial}{\partial x}$. Thus every element of $\mathcal{I}_0\mathcal{V}_b$ also has this form, except that $g(x)$ necessarily vanishes at $x = 0$. Let $X = \tau x\frac{\partial}{\partial x} + g(x)x\frac{\partial}{\partial x}$. The exponential of $X$, over the interior, is given by

$$\begin{aligned}\exp(X) &= \exp\left((\tau + g(x))\, x\frac{\partial}{\partial x}\right)\\ &= xe^{\tau+g(x)}\end{aligned}$$

In particular,

$$\left.\frac{d}{dx}\right|_{x=0} \exp(X) = e^{\tau}$$

By the same reasoning as before, we see that the limit $\lim_{x\to 0}\iota(X_x)$ is the point on the front face corresponding to a slope of $e^{\tau}$. Our goal is accomplished. We have unambiguously defined $\iota(\mathbf{v})$.

These results generalize immediately to a case of greater interest: the exact $b$-metric on a compact manifold-with-boundary. For in this case, $M$ has the form of a product $[0,1)\times Y$ near the boundary, and the exact $b$-metric $g$ is a product metric: $g = \left(\frac{dx}{x}\right)^2 + h$. This implies that geodesics on $M$ are of the form $(x(t), y(t))$, where $x(t)$ is a $\left(\frac{dx}{x}\right)^2$-geodesic and $y(t)$ is an $h$-geodesic. Since the $b$-tangent bundle of $M$ naturally splits as ${}^bT[0,1)\oplus TY$ on the collar, the exponential map will be a product of the two respective exponential maps. Computations can thus be carried out in each component exactly as above, and the same results go through.

Now that we have a legitimate candidate for a "linearization", it is time for a precise definition of the word. Recall that $\pi_{1b} : M_b^2 \to M$ is the $b$-projection onto the first factor.

**Definition 41.** *A smooth map $\iota : N\Delta_b \to M_b^2$ is called a* ***linearization*** *if it satisfies*

*(1) For all $v \in N_p\Delta_b$, $\iota(v) \in \pi_{1b}^{-1}(p)$. We will write $\iota_p$ for the restriction of $\iota$ to the fiber $N_p\Delta_b$.*

*(2) $\iota$ maps the zero-section of $N\Delta_b$ to the b-diagonal.*

*(3) By the previous two properties, the differential of $\iota_p$, at the zero vector, is a linear map*

$$[D\iota_p]_0 : N_p\Delta_b \to T_{\iota_p(0)}M_b^2 \subseteq TM_b^2\big|_{\Delta_b}$$

*After composing this with the projection $T_{\iota_p(0)}M_b^2\big|_{\Delta_b} \to N_p\Delta_b$, we obtain a linear map $N_p\Delta_b \to N_p\Delta_b$. We require this to be the identity map.*

We do not require $\iota$ to be a global diffeomorphism, or even injective. However, property (3) above guarantees that $\iota$ is a diffeomorphism in a neighborhood, $W$, of the zero section. As before, a map $\nu : \iota(W) \to N_p\Delta_b$ which is an inverse for $\iota$ on $W$ will also be called a linearization.

Next, we'll prove that the $\iota$ which is induced by an exact $b$-metric is a linearization. As it is defined, $\iota$ maps ${}^bTM$ to $M_b^2$, so we have to compose it with the isomorphism $\phi : N\Delta_b \simeq {}^bTM$ to make sense of this. To see that $\iota \circ \phi$ is a linearization, just note that, by equation (4.2),

$$\iota \circ \phi \left( c[\partial_\tau|_{\Phi^{-1}(x,0)}] \right) = \Phi^{-1}(x, c)$$

Thus, if we identify $N\Delta_b$ with $[0,1) \times \mathbb{R}$ via the global section $[\partial_\tau]$, and we identify $M_b^2$ with $[0,1) \times \mathbb{R}$ via logarithmic projective coordinates, then $\iota \circ \phi$ is just the identity map. Certainly the differential of the identity map is the identity map, so $\iota \circ \phi$ is indeed a linearization. For obvious reasons, when we speak of this map below, we will abuse notation and call it $\iota$.

## 5.2 The Symbol and Quantization

With all of the pieces in place, the definition of the global symbol of a $b$-pseudodifferential operator now follows verbatim from the definition for the normal symbol. The quantization map is defined analagously, and as you can guess, the symbol and quantization maps are inverses of one another, modulo elements of order $-\infty$. The proofs are identical to those given in the boundaryless case, except that we will use the logarithmic projective coordinates to carry out all of our computations. The precise statements follow.

**Definition 42.** *Let $M$ be a compact manifold-with-boundary equippd with an exact b-metric $g$. Let $\iota$ be the linearization associated to $g$, and let $\psi$ be a cut-off function for $\iota$. If $A \in \Psi_b^m(M)$, we define the $\psi$-cut b-**normal symbol** of $A$, $\sigma_{A,\psi}$, by*

$$\sigma_{A,\psi} = \mathcal{F}\left[\iota^*(\psi K_A)\right] \tag{5.1}$$

As one would expect, the $b$-normal symbol of $A$ truly is a symbol, of the same order as $A$. Moreover, the choice of cut-off function only affects the symbol up to order $-\infty$.

**Theorem 43.** *Let $A \in \Psi_b^m(M)$ be a scalar b-pseudodifferential operator on $M$, and let $\sigma_{A,\psi}$ be the $\psi$-cut b-normal symbol of $A$. Then*

*(1) $\sigma_{A,\psi} \in S^m({}^bT^*M)$*

*(2) If $\psi'$ is another cut-off function associated to $\iota$, then $\sigma_{A,\psi} - \sigma_{A,\psi'} \in S^{-\infty}({}^bT^*M)$*

*Proof.* In order to prove that $\sigma_{A,\psi}$ is a symbol of order $m$, we must show that, around any point of $M$, there exists a local trivialization of ${}^bT^*M$ with respect to which $\sigma_{A,\psi}$ satisfies the symbol estimates. Now, if the chosen point lies in the interior of $M$, then the argument given for the boundaryless case carries through, so we only need to consider the case when the chosen point lies on $\partial M$. By choosing a collar neighborhood of $Y = \partial M$, we will identify the chosen point with the pair $(0, r) \in [0, 1) \times Y$. We wil then choose a normal neighborhood, $U$, of $r$ in $Y$ and trivialize ${}^bT^*M$ over $[0, 1) \times U$. (Actually, we will trivialize over a smaller set, $[0, 1) \times \tilde{U}$. $\tilde{U} \subseteq U$ is a small enough that the difference of any pair of points of $\tilde{U}$ lies in $U$. The reason we do this will become clear later.) This trivialization is where we will carry out our computation.

This proof is very similar to the proof given for theorem (25). Therefore, we will adopt similar notational conventions, which we summarize here:

$U \subseteq \partial M$ **:** Uniformly normal neighborhood of $r$, with respect to the metric $h$

$p, q, r$ **:** Points of $U$ ($r$ is the fixed point, $p$ and $q$ are arbitrary.)

$\partial_{y^i}$ **:** Coordinate trivialization for $TY$ over $U$, chosen so that $\partial_{y^i}\big|_r$ forms an orthonormal basis for $T_rY$.

$\alpha_p : T_pY \to \mathbb{R}^n$ **:** isomorphism induced by choosing the basis $\partial_{y^i}\big|_p$

$\alpha_{(x,p)} : {}^bT_{(x,p)}M \to \mathbb{R} \times \mathbb{R}^n$ **:** isomorphism induced by choosing the basis $[x\frac{\partial}{\partial x}]_{(x,p)}, \partial_{y^i}\big|_p$

$\mathbf{y}_p = \alpha_p \circ \nu_p : U \to \mathbb{R}^n$ **:** Coordinates based at $p$, with respect to the basis $\partial_{y^i}\big|_p$

$u, w$ **:** Points of $\mathbb{R}^n$

$\beta_p : T_p^*Y \to \mathbb{R}^n$ **:** isomorphism induced by choosing the basis $dy_p^i$

$\beta_{(x,p)} : {}^bT^*_{(x,p)}M \to \mathbb{R} \times \mathbb{R}^n$ **:** isomorphism induced by choosing the basis $\frac{dx}{x}, dy_p^i$

Here is the triviallization, written down explicitly:

$$\begin{array}{rcl} {}^bT^*M|_{[0,1)\times U} & \to & [0,1) \times \mathbf{y}_r(\tilde{U}) \times \mathbb{R} \times \mathbb{R}^n \\ \xi' = \lambda\frac{dx}{x} + \xi & \mapsto & \big(x, \mathbf{y}_r(\pi(\xi)), \lambda, \beta_{\pi(\xi)}(\xi)\big) \\ \lambda\frac{dx}{x} + \beta^{-1}_{\mathbf{y}_r^{-1}y}(u) & \leftarrow\!\shortmid & (x, y, \lambda, u) \end{array}$$

(We are using the natural identification of ${}^bT^*M$ with ${}^bT^*[0,1) \oplus T^*Y$. $\xi'$ belongs to ${}^bT^*_{(x,p)}M$, and $\xi$ belongs to $T_p^*Y$.) We will let $\sigma(x, y, \lambda, u) \in C^\infty([0,1) \times \mathbf{y}_r(U) \times \mathbb{R} \times \mathbb{R}^n)$ be the composition of $\sigma_{A,\psi}$ with this trivialization:

$$\sigma(x, y, \lambda, u) = \sigma_{A,\psi}\left(\lambda\frac{dx}{x} + \beta^{-1}_{\mathbf{y}_r^{-1}y}(u)\right)$$

Our aim is to show that $\sigma(x, y, \lambda, u) \in S^m([0,1) \times \mathbf{y}_r(\tilde{U}) \times (\mathbb{R} \times \mathbb{R}^n))$. To do this, it will be useful to write out the dual trivialization for ${}^bTM$:

$$\begin{array}{rcl} {}^bTM|_{[0,1)\times U} & \to & [0,1) \times \mathbf{y}_r(\tilde{U}) \times \mathbb{R} \times \mathbb{R}^n \\ v' = \tau x\frac{\partial}{\partial x} + v & \mapsto & \big(x, \mathbf{y}_r(\rho(v)), \tau, \alpha_{\rho(v)}(v)\big) \\ \tau x\frac{\partial}{\partial x} + \alpha^{-1}_{\mathbf{y}_r^{-1}y}(w) & \leftarrow\!\shortmid & (x, y, \tau, w) \end{array}$$

Now we're ready to get into the proof. By definition, we can write $\sigma_{A,\psi}$ formally as the following oscillatory integral:

$$\sigma_{A,\psi}(\xi') = \int_{{}^bT_{(x,p)}M} e^{-i\langle v',\xi'\rangle}[\psi K_A](\iota(v'))\, dv'$$

By expressing everything in terms of the trivializations, we see that

$$\sigma(x,y,\lambda,u) = \int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-i(\tau\lambda + w\cdot u)}[\psi K_A]\left(\iota\left(\tau x\partial_x + \alpha^{-1}_{\mathbf{y}_r^{-1}y} w\right)\right)(\det h)^{1/2}\, dw d\tau$$

(Here, $y = \mathbf{y}_r(p)$, where $p$ is the basepoint of $\xi$.)

The next step is to express $\iota\left(\tau x\partial_x + \alpha_y^{-1} w\right)$ in terms of the logarithmic projective coordinates. Begin by recalling the definition of the $\Phi$-coordinate system on $[0,1)^2_b$:

$$\begin{aligned}\Phi : [0,1)^2_b \setminus (lb \cup rb) &\to [0,1)\times\mathbb{R}\\ (x,x') &\mapsto \left(x, \ln\left(\frac{x'}{x}\right)\right)\\ [x(t),x'(t)] &\mapsto \left(0, \ln\left(\frac{\dot{x}'(0)}{\dot{x}(0)}\right)\right)\end{aligned}$$

Let $V_r \subseteq M^2_b$ be the neighborhood of $\Delta_b$ given by $V_r = \left([0,1)^2_b \setminus (lb\cup rb)\right)\times \tilde{U}^2$. Then using $\Phi$ and normal coordinates $\mathbf{y}_r$, we can define coordinates $\Psi_r$ on $V_r$ as follows:

$$\Psi_r : V_r \to [0,1)\times \mathbf{y}_r(\tilde{U})\times\mathbb{R}\times\mathbf{y}_r(U)$$

$$\Psi_r\left(\Phi^{-1}(x,\tau), p, q\right) = (x, \mathbf{y}_r(p), \tau, \mathbf{y}_r(q) - \mathbf{y}_r(p))$$

Combining this with the definition of $\iota$, we see that

$$\begin{aligned}\Psi_r\left(\iota(\tau x\partial_x + \alpha_p^{-1}w)\right) &= \left(x, y, \tau, \mathbf{y}_r(\exp(\alpha_p^{-1}w)) - y\right)\\ &= \left(x, y, \tau, \mathbf{y}_r\mathbf{y}_p^{-1}w - y\right)\end{aligned}$$

which can be inserted into our original integral to obtain

$$\sigma(x,y,\lambda,u)=\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-i(\tau\lambda+w\cdot u)}[\psi K_A]\circ\Psi^{-1}\left(x,y,\tau,\mathbf{y}_r\mathbf{y}_p^{-1}w-y\right)(\det h)^{1/2}\,dwd\tau$$

By the definition of $b$-pseudodifferential operators, $[\psi K_A]\circ\Psi^{-1}\left(x,y,\tau,\mathbf{y}_r\mathbf{y}_p^{-1}w-y\right)$ can be expressed as:

$$[\psi K_A]\circ\Psi^{-1}\left(x,y,\tau,\mathbf{y}_r\mathbf{y}_p^{-1}w-y\right)=\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{i\left(\tau\mu+(\mathbf{y}_r\mathbf{y}_p^{-1}w-y)\cdot\theta\right)}a(x,y,\tau,w,\mu,\theta)\,d\theta d\mu$$

where $a\in S^m\left[[0,1)\times\mathbf{y}_r(U)\times\mathbb{R}\times\mathbf{y}_p(U)\times(\mathbb{R}\times\mathbb{R}^n)\right]$. Applying the Kuranishi trick, using the path $\gamma_w(t)=(1-t)w$, we see that we can write

$$[\psi K_A]\circ\Psi^{-1}\left(x,y,\tau,\mathbf{y}_r\mathbf{y}_p^{-1}w-y\right)=\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{i(\tau\mu+w\cdot\theta)}\hat{a}(x,y,\tau,w,\mu,\theta)\,d\theta d\mu$$

where the $a$ appearing is a different symbol as before, but still of order $m$. We insert this into our original integral, and obtain

$$\sigma(x,y,\lambda,u)=\iiiint e^{i(\tau(\mu-\lambda)+w\cdot(\theta-u))}\hat{a}(x,y,\tau,w,\mu,\theta)(\det h)^{1/2}\,d\mu d\theta d\tau dw$$

Of course, $(\det h)^{1/2}$ only depends on $x$ and $y$, so we can absorb it into the symbol without affecting its order:

$$\sigma(x,y,\lambda,u)=\iiiint e^{i(\tau(\mu-\lambda)+w\cdot(\theta-u))}\hat{a}(x,y,\tau,w,\mu,\theta)\,d\mu d\theta d\tau dw \tag{5.2}$$

Once we transform this integral into the proper form, we will appeal to the estimation theorem to obtain an asymptotic expansion which allows us to conclude that $\sigma$ is indeed a symbol of order $m$. The required transformation is as follows:

$$\delta = |(\lambda, u)|$$
$$\epsilon = \frac{\mu - \lambda}{\delta}$$
$$\nu = \frac{\theta - u}{\delta}$$

Under these new coordinates, the integral (5.2) transforms to

$$\sigma(x, y, \lambda, u) = \delta^{n+1} \iint e^{i\delta(\tau,w)\cdot(\epsilon,\nu)} \hat{a}(x, y, \tau, w, \delta\epsilon + \lambda, \delta\nu + u)\, d(\epsilon, \nu) d(\tau, w) \tag{5.3}$$

This integral is in the exact same form as equation (2.7), so we can apply the estimation lemma to obtain the following asymptotic expansion, which holds uniformly on compact $x$-sets:

$$\sigma(x, y, \lambda, u) \sim (2\pi)^{n+1} \sum \frac{\delta^{-k}}{2^k k!} \left[\partial_{(\tau,w)} \cdot \partial_{(\epsilon,\nu)}\right]^k \hat{a}(x, y, \tau, w, \delta\epsilon + \lambda, \delta\nu + u)\big|_{\epsilon=\nu=\tau=w=0}$$

After cleaning it up with the chain rule, we arrive at

$$\sigma(x, y, \lambda, u) \sim (2\pi)^{n+1} \sum \frac{1}{2^k k!} \left[\partial_{(\tau,w)} \cdot \partial_{(\lambda,u)}\right]^k \hat{a}(x, y, \tau, w, \lambda, u)\big|_{\tau=w=0} \tag{5.4}$$

We recognize this as an asymptotic summation of symbols, whose orders are decreasing to $-\infty$. The $k = 0$ term is a symbol of order $m$, so we conclude that $\sigma(x, y, \lambda, u)$ is also a symbol of order $m$, as required.

$\square$

As before, we can follow the proof of the Kuranishi trick to keep track of what the symbol $\hat{a}$ really is:

$$\hat{a}(x, y, \tau, w, \mu, \theta) = \frac{a\left(x, y, \tau, \mathbf{y}_r \mathbf{y}_p^{-1}(w), {}^tG(x, w)^{-1}\theta\right)}{|\det G(x, w)|} (\det h)^{1/2} \tag{5.5}$$

Here, $G(x, w)$ is the transformation from the Kuranishi trick given by integrating the Jacobian $D\left[\mathbf{y}_r \mathbf{y}_p^{-1}\right]$ over the path $(1-t)w$.

The quantization map is defined as follows:

**Definition 44.** *Let $M$ be a compact manifold-with-boundary, endowed with an exact $b$-metric $g$. Let $\nu$ be the linearization induced by $g$, and $\psi$ a cut-off function for $\nu$. For $a \in S_b^m(M)$, the $\psi$-**cut** $b$-**normal quantization** of $a$ is the operator $Op_\psi(a)$ whose Schwartz kernel is given as follows:*

$$K_{Op_\psi(a)} = \nu^* \left(\psi \mathcal{F}^{-1}(a)\right)$$

Of course, the $b$-normal quantization of $a$ is a $b$-pseudodifferential operator of the same order as $a$, and only depends on the cut-off function up to smoothing operators.

**Theorem 45.** *Let $a \in S^m({}^bT^*M)$. Then*

*(1) $Op_\psi(a) \in \Psi_b^m(M)$.*

*(2) If $\psi'$ is another cut-off function associated to $\iota$, then $\nu^*(\psi\mathcal{F}^{-1}(a)) - \nu^*(\psi'\mathcal{F}^{-1}(a))$ is smooth, and hence $Op_\psi(a) - Op_{\psi'}(a) \in \Psi_b^{-\infty}(M)$.*

*Proof.* By now, this is routine. We begin by writing $K_{Op_\psi(a)} \circ \Psi_r^{-1}(x, y, \tau, z)$ as follows:

$$K_{Op_\psi(a)} \circ \Psi_r^{-1}(x, y, \tau, z) = [\psi\mathcal{F}^{-1}(a)] \left(\nu(\Psi_r^{-1}(x, y, \tau, z)\right)$$

Using the $\alpha$ and $\beta$ trivializations to express this as an oscillatory integral over $\mathbb{R} \times \mathbb{R}^n$, we obtain

$$K_{Op_\psi(a)} \circ \Psi_r^{-1}(x, y, \tau, z) = \iint e^{i(\tau\lambda + \mathbf{y}_p \mathbf{y}_r^{-1}(y+z)\cdot u)} \bar{\psi}(x, y, \tau, z) \bar{a}(x, y, \lambda, u) (\det h)^{-1/2}\, d\lambda du$$

Here, $\bar{a}$ is the composition of $a$ with the trivialization, $\bar{\psi}$ is the composition of $\psi$ with the coordinate chart, and $p = \mathbf{y}_r^{-1}(y)$. Using the Kuranishi trick, with $\gamma(t) = y + (1-t)z$, and absorbing everything into the amplitude, we can transform this to:

$$K_{Op_\psi(a)} \circ \Psi_r^{-1}(x,y,\tau,z) = \iint e^{i(\tau\lambda + z\cdot u)} \hat{a}(x,y,\tau,z,\lambda,u)\, d\lambda du \tag{5.6}$$

This is in the form that we want. That $K_{Op_\psi(a)}$ vanishes to infinite order at the left and right boundaries is a consequence of the fact that $\psi$ vanishes identically outside of a neighborhood of the diagonal, and so $K_{Op_\psi(a)}$ is indeed a $b$-pseudodifferential operator. The proof of the second statement is exactly the same as the proof given for the boundaryless case.

□

The symbol $\hat{a}$ appearing in the transformed integral (5.6) can be computed by following the proof of the Kuranishi trick, as usual. It turns out to be

$$\hat{a}(x,y,z,\lambda,u) = \frac{\bar{\psi}(x,y,\tau,z)\bar{a}\left(x,y,\lambda,{}^tF(y,z)u\right)}{|\det F(y,z)||\det h|^{1/2}} \tag{5.7}$$

Of course, $F(y,z)$ is given by integrating the Jacobian $D\left[\mathbf{y}_p\mathbf{y}_r^{-1}\right]$ over the path $\gamma(t) = y + (1-t)z$. Using (5.7), we can get explicit formulas for the asymptotic expansions of normal symbols.

**Example 5.** The $b$-normal symbol of the $b$-Laplacian on a compact manifold-with boundary.

By the $b$-Laplacian, we mean the operator naturally associated to the $b$-metric $g = \frac{dx^2}{x^2} + h$:

$$\Delta = \left(x\frac{\partial}{\partial x}\right)^2 + \Delta_h$$

.

The local symbol of this operator is

$$a(x,y,\lambda,\xi) = -\lambda^2 + a_h$$

where $a_h$ is the local symbol for the Laplacian associated to $h$. Since the $\lambda^2$ coefficient is constant, the global symbol can be computed by computing the normal symbol of $\Delta_h$, as in Example (1), and then adding on $-\lambda^2$.

Finally, we remark that, just as in the boundaryless case, the symbol and quantization maps are indeed inverses of one another modulo elements of order$-\infty$.

**Theorem 46.** *Let*

$$Op_\psi : S^m({}^bT^*M) \to \Psi_b^m(M)$$

$$\sigma_\psi : \Psi_b^m(M) \to S^m({}^bT^*M)$$

*be the $\psi$-cut normal symbol and quantization maps, as defined above. Then for any $a \in S^m(T^*M)$, and for any $A \in \Psi^m(M)$,*

*(1)* $a - \sigma_{Op_\psi(a),\psi} \in S^{-\infty}({}^bT^*M)$

*(2)* $A - Op_\psi(\sigma_{A,\psi}) \in \Psi_b^{-\infty}(M)$

*Proof.* (1) This works exactly as it did in the boundaryless case. We note that $\tilde{a} = a - \sigma_{Op_\psi(a),\psi}$ can be written as an oscillatory integral whose amplitude contains a factor of $1-\psi^2$. In particular, the amplitude vanishes identically in a neighborhood of the $b$-diagonal. Since the diagonal is also the critical set of the phase, $\tilde{a}$ must rapidly decrease, by the principle of stationary phase.

(2) This also works the same way as the boundaryless case. The Schwartz kernel of $A - Op_\psi(\sigma_{A,\psi})$ is an oscillatory integral whose amplitude contains a factor of $1-\psi^2$, so it vanishes on the $b$-diagonal. The $b$-diagonal is the only possible place where one can find a singularity, so the Schwartz kernel is smooth, and hence $A - Op_\psi(\sigma_{A,\psi}) \in \Psi_b^{-\infty}(M)$.

□

We have shown that the cut-off function only affects the symbol and quantization maps modulo order $-\infty$. So, if we mod this order out, then we obtain maps $Op$ and $\sigma$ which are independent of the cut-off function:

$$Op : S^m({}^bT^*M)/S^{-\infty} \to \Psi_b^m(M)/\Psi^{-\infty}$$

$$\sigma : \Psi_b^m(M)/\Psi^{-\infty} \to S^m({}^bT^*M)/S^{-\infty}$$

They will be called the *b-normal symbol* and *b-normal quantization*, and due to theorem (46), these maps are both isomorphisms.

## 5.3 A new formula for the $b$-Trace

Now we're ready to express the $b$-Trace in terms of the global symbol.

**Theorem 47.** *Let $A \in \Psi_b^{-\infty}(M)$. For any cut-off function $\psi$, the $b$-trace of $A$ can be computed as follows:*

$${}^b\operatorname{Tr}(A) = {}^b\int_M \int_{{}^bT_p^*M} \sigma_{A,\psi}(\xi)\, d\xi dV$$

*Proof.* This is a straightforward consequence of the Fourier inversion theorem. Let $0_p$ denote the zero vector of ${}^bT_p^*M$.

$$\begin{aligned}
\int_{{}^bT_p^*M} \sigma_{A,\psi}(\xi)\, d\xi &= \int_{{}^bT_p^*M} e^{i\langle 0_p,\xi\rangle}\sigma_{A,\psi}(\xi)\, d\xi \\
&= \left[\mathcal{F}^{-1}\sigma_{A,\psi}\right](0_p) \\
&= \left[\mathcal{F}^{-1}\mathcal{F}\iota^*(\psi K_A)\right](0_p) \\
&= \left[\iota^*(\psi K_A)\right](0_p) \\
&= \psi(\iota(0_p))K_A(\iota(0_p)) \\
&= K_A|_{\Delta_b}(p)
\end{aligned}$$

□

# Chapter 6

## Possible Applications and Generalizations

With a global symbol at our disposal, it seems like the natural next step would be to prove an asymptotic product formula. Such a formula could be used to show that the $b$-quantization map $Op$ gives rise to a formal deformation of the cotangent bundle, in the sense of Nest and Tsygan. This could lead to a connection between their algebraic index theorems and the Atiyah, Patodi, Singer index theorem.

[LMP09] contains a relative McKean-Singer formula for the index of a $b$-Dirac operator in terms of $b$-traces. Equipped with this formula and our new formula for the $b$-trace in terms of global symbols, one might obtain new methods for calculating the indices of elliptic $b$-pseudodifferential operators.

The global symbol could potentially be generalized to the large $b$-calculus. When working with $\iota$, we noticed that it is actually a global difeo on ${}^bT[0,1)$. Therefore, when lifting Schwartz kernels to the stretched product, there is actually no need to cut anything off in the $\tau$-direction. This is important because the behavior of $K_A$ at the left and right boundaries, though negligible for the small $b$-calculus, is essential for the large $b$-calculus. Since we have no need to cut this information off, it can be preserved in the symbol.

Within Alain Connes' groupoid framework for pseudodifferential calculi, there is the notion of a "tangent groupoid" and a "product groupoid" which every pseudodifferential calculus must possess. By defining an appropriate notion of linearization as some sort of map between these two groupoids, one might be able to obtain global symbols at a very high level of generality.

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